Bargaining among groups: an axiomatic viewpoint

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Abstract. We introduce a model of bargaining among groups, and characterize a family of solutions using a Consistency axiom and a few other invariance and monotonicity properties. For each solution in the family, there exists some constant $\alpha \geq 0$ such that the "bargaining power" of a group is proportional to c^{α}, where c is the cardinality of the group.

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1. Introduction

In many concrete bargaining situations, the actors are not individual agents but groups of agents, represented by a single agent at the bargaining table. Examples include labor disputes between "management" representing stockholders, and a union representing workers. The negotiation of an international treaty is carried out by a few individuals representing a complex mix of political and business interests. In fact, a survey at economics journals reveals that most applied bargaining papers deal with group bargaining problems. In a group bargaining problem, the bargaining power of a group vis-à-vis other groups depends in general on the group's internal composition as well as those of the other groups. Often the literature sidesteps the need for group bargaining solutions by assuming that the payoffs are linear in a physical good or monetary unit so that the group can be regarded as a bargaining party with a well-defined payoff. (See, for example, Jun [6] and Horn and Wolinsky [5]).

Harsanyi's joint bargaining paradox ([4]) is the observation that several parties to a negotiation may not find it advantageous to "join," namely to mandate a representative to negotiate on their behalf.

In this paper we take an axiomatic view of joint bargaining that does not preclude, nor imply, the paradox. We start in Section 2 with a simple model where groups of different sizes divide a dollar. We impose a version of the celebrated Consistency axiom (Aumann and Maschler [1], Moulin [10], Young [15]; Thomson [12] surveys its applications to various fair division problems), as well as a handful of milder axioms on which more below. We characterize the following one-dimensional family of division rules (Theorem 1 in Section 2). If K groups of respective sizes $c_k, k = 1, ..., K$ bargain over a dollar, the total share to the k-th group is proportional to c_k^{α}, where the parameter α is a nonnegative real number. Thus if the intragroup division rule is equal split, the share of an agent in the k-th group is $\frac{c_k^{\alpha-1}}{\sum_{j=1}^{K} c_j^{\alpha}}$.

The joint bargaining paradox occurs when the merging of two disjoint groups of sizes $c_1$and c_2 results in a net loss for the union of these two groups

$$\frac{(c_1+c_2)^{\alpha}}{(c_1+c_2)^{\alpha}+\sum_{k\geq 3} c_k^{\alpha}} < \frac{c_1^{\alpha}+c_2^{\alpha}}{\sum_{k\geq 1} c_k^{\alpha}}$$

namely when $0 \leq \alpha < 1$. Symmetrically, merging is advantageous when $\alpha > 1$. The two extreme rules are $\alpha = 0$, where all groups receive the same share of the dollar, irrespective of their size, and $\alpha = +\infty$ where the largest groups split the dollar and leave no crumbs for smaller groups. With the neutral rule corresponding to $\alpha = 1$, the formation of groups has no effect on individual shares.

Next we transport our axiomatic discussion to generalized bargaining problems, specifying a partition of the agents in groups (coalitions), in addition to a set of feasible utility profiles and a disagreement utility profile. The Consistency axiom becomes a group version of the classic separability property studied by Harsanyi [3], [4], Lensberg [9] and Thomson and Lensberg [13]. The other axioms are

- symmetry of the solution with respect to permutations of the agents that respect the group structure: *Anonymity*
- invariance with respect to the replication of the group structure and the feasible utility set or with respect to the rescaling of the utility set: *Replication Invariance* and *Common Scale Invariance*
- monotonicity of group share in the size of the group, when the bargaining problem is symmetric across groups (as in the divide-the-dollar example): *Group Size Monotonicity*

While Common Scale and Replication invariances are classic properties in the axiomatic bargaining literature (see Thomson and Lensberg [13], Chapter 9), Group Size Monotonicity is genuine to our model. It rules out "trivial" versions of the joint bargaining paradox where a larger coalition is systematically discriminated against, as when we take $\alpha < 0$ in the formulas above.

We show in Section 3 that our axioms are met by a rich family of (generalized) bargaining solutions, that can not be parameterized with a finite number of variables. But when in Section 4 we add to the group properties above the familiar requirements of the Nash solution - Contraction Independence and Invariance to Affine Transformations - we are left with the one-dimensional family of asymmetric Nash solutions where the bargaining power of an agent in a group of size c_k is $c_k^{\alpha-1}$; the parameter α varies as before between 0 and $+\infty$.

Similarly when we add in Section 5 the requirements of the path monotone solutions - Issue Monotonicity - the resulting solutions select a utility profile u by equalizing $\frac{u_i}{c_k^{\alpha-1}}$ across all agents.

2. Divide-the-dollar model

The set $\mathcal{N}$ of potential agents is infinite, and each problem specifies first a finite subset N of $\mathcal{N}$ of concerned agents. We write $P = \{G_k; k \in K\}$ a partition of N in *nonempty* groups (or coalitions) G_k. Here K is an arbitrary finite set of indices. We write the cardinality of G_k as $c_k = |G_k|$, and set $|P| = \{c_k; k \in K\}$. Note that $c_k \in \mathbb{N}_* = \{1, 2, ...\}$.

A *division problem* is a triple (N, P, s) where s, a nonnegative real number, is the amount of money to be divided. A *solution* to this problem is a vector $y \in \mathbb{R}_+^N$ such that $\sum_N y_i = s$. We write $A(N, s)$ for the set of such solutions. A *division rule* is a mapping φ associating to every problem (N, P, s) a solution $\varphi(N, P, s) \in A(N, s)$.

We write $\sum(\mathcal{N})$ for the set of permutations of $\mathcal{N}$ with a generic element σ. For any $y \in A(N, s)$ we define $\sigma(y) \in A(\sigma(N), s)$ by $[\sigma(y)]_i = y_{\sigma^{-1}(i)}$. For any partition $P = \{G_k; k \in K\}$ of N, $\sigma(P) = \{\sigma(G_k); k \in K\}$ defines a partition of $\sigma(N)$. Hence the notation $\sigma(N, P, s) = (\sigma(N), \sigma(P), s)$. We are ready to define the basic horizontal equity requirement:

Anonymity: for all $\sigma \in \sum(\mathcal{N})$, and all (N, P, s) :

$$\varphi(\sigma(N, P, s)) = \sigma[\varphi(N, P, s)]$$

By taking for σ the permutation exchanging two agents in G_k, we see that these two agents must receive the same share. Moreover, two coalitions $G_k, G_{k'}$ of the same size also receive the same share (choose a permutation exchanging G_k and $G_{k'}$). And two problems (N, G, s) and (N', G', s) where $|N| = |N'|$ and $|P'| = |P|$ must give the same share to $G_k, G'_{k'}$ if $|G_k| = |G'_{k'}|$.

In the next axiom we use the notation P_{-k} for the partition of $N \backslash G_k$ induced by P.

Group Consistency (GCSY): for all $(N, P, s), k \in K$ and $i \in N \backslash G_k$

$$y = \varphi(N, P, s) \Longrightarrow y_i = \varphi_i(N \backslash G_k, P_{-k}, s - \sum_{G_k} y_i)$$

The interpretation is standard: the shares allocated in the N-problem to the coalitions in P_{-k} are a fair division of $\sum_{N/G_k} y_i$ under the restricted partition.

Scale Invariance (SI): for all (N, P, s) and $\lambda \in \mathbb{R}_{++} : \varphi(N, P, \lambda \cdot s) = \lambda \varphi(N, P, s)$

Scale Invariance rules out the "wealth effect" justifying different divisions for \$10 or \$10,000. We are primarily interested in the impact of the group structure, and will not challenge the SI requirement. It is an interesting and open question to generalize Theorem 1 and its corollary when this property is removed.

Our next axiom conveys the simple yet crucial idea that a larger group size per se is not harmful:

Group Size Monotonicity (GSM) for all (N, P, s), all $k, l \in K$:

$$|G_k| \geq |G_l| \Longrightarrow \sum_{G_k} y_i \geq \sum_{G_l} y_i, wherey = \varphi(N, P, s)$$

Violation of Group Size Monotonicity is a coarse version of the joint bargaining paradox that is of little practical relevance. Instead the presence or absence of the paradox is captured by the next two properties.

Given (N, P) and $k, l \in K$, we write $P[k, l]$ for the partition of N obtained by merging G_k and G_l into a single coalition $G_k \cup G_l$.

Joint Bargaining Harmful (JBH): for all (N, P, s) and all $k, l \in K$

$$y = \varphi(N, P, s), y' = \varphi(N, P[k, l], s) \Longrightarrow \sum_{G_k \cup G_l} y'_i < \sum_{G_k \cup G_l} y_i$$

Joint Bargaining Beneficial (JBB) is the statement obtained by switching to the opposite inequality.

Our last axiom, Replication Invariance, requires identical divisions of the dollar between two coalitions with respective sizes 2 and 5, or with respective sizes 200 and 500. This may or may not make sense from the point of view of the relative bargaining power of groups, therefore we present below two sets of results where this axiom is or is not present.

Replication Invariance (RI): for any $r \in \mathbb{N}_*$, any $(N^t, P^t, s), t = 1, 2$ s. t. $|N^1| = r \cdot |N^2|$ and $|P^1| = r \cdot |P^2|$:

$$\{|G^1_k| = r \cdot |G^2_l|\} \Longrightarrow \{\sum_{G^1_k} y^1_i = \sum_{G^2_l} y^2_j\} \text{ where } y^t = \varphi(N^t, P^t, s)$$

Taking $r = 1$ in the above property, we see that RI implies a weaker form of Anonymity, namely the total share of a coalition only depends on the cardinality profile $|P|$ of the partition. In the sequel we only consider anonymous rules anyway.

Each rule described in our next result is constructed from two ingredients. First a priority ranking of coalitions based on their sizes, and captured in the statement below by the preordering $\succsim$ of $\mathbb{N}$, coarser than its natural ordering: if the partition P contains two coalitions of respective sizes c_k, c_l and $c_k \succ c_l$, then G_l receives no resources: they are shared among the coalitions whose size is in the highest indifference class of $\succsim$. The second ingredient is a weight function determining the relative shares of the latter coalitions.

Theorem 1

Fix an arbitrary preording (complete, transitive) $\succsim$ of $\mathbb{N}_*$ compatible with the natural order of $\mathbb{N}_*$ ($a \geq b \Longrightarrow a \succsim b$), and a positive function w on $\mathbb{N}_*$. Assume also that w is nondecreasing on each indifference class $I(t)$ of $\succsim$.

For each problem (N, P, s), let $I(t^*)$ be the highest indifference class reached by $|P|$, namely $c_k \in I(t^*) \Longleftrightarrow c_k \succsim c_l$ for all $l \in K$. The formula

$$\varphi_i(N, P, s) = \frac{1}{c_k} \cdot \frac{w(c_k)}{\sum_{l \in I(t^*)} w(c_l)} \cdot s \quad \text{if} \quad i \in G_k \quad \text{and} \quad c_k \in I(t^*) \tag{1}$$

$$\varphi_i(N,P,s)=0 \qquad\qquad \textit{if} \;\; i\in G_k \;\; \textit{and} \;\; c_k\notin I(t^*)$$

defines an anonymous division rule satisfying Group Consistency, Scale Invariance, and Group Size Monotonicity.

Conversely, every anonymous rule satisfying these three axioms is constructed in this fashion.

When the preordering $\succsim$ is complete indifference, the share of a coalition of size c_k is always proportional to $w(c_k)$. If, on the other hand, $\succsim$ is the natural ordering of $\mathbb{N}$, the rule divides s equally among all largest coalitions of P.

Corollary to Theorem 1

Within anonymous division rules, the four properties Group Consistency, Scale Invariance, Group Size Monotonicity and Replication Invariance, characterize the following one-dimensional family of rules:

$$\varphi_i^{\alpha}(N,P,s)=\frac{c_k^{\alpha-1}}{\sum_{l\in K}c_l^{\alpha}}\cdot s \;\; \textit{for all} \;\; (N,P,s), \;\; \textit{all} \;\; i\in G_k$$

where $0\leq\alpha\leq+\infty$. *Here* φ^{∞} *is the rule dividing* s *equally among all agents in the largest coalitions of* P.

A benchmark rule in the family φ^{α} is φ^{1}, sharing s equally among all agents, and ignoring P entirely. For this rule joint bargaining is "neutral." For $0\leq\alpha<1$, the rule φ^{α} exhibits the joint bargaining paradox (axiom JBH), with the extreme rule φ^{0} dividing s equally among all coalition, irrespective of their sizes. For $1<\alpha<+\infty$, the rule φ^{α} meets JBB, and φ^{∞} meets the weak form of JBB with a weak inequality in lieu of a strict one.

Proof of Theorem 1

The first statement - the rule φ defined by (1) meets ANO, GCSY, SI and GSM - is straightforward, hence we omit its proof.

To prove the converse statement, we observe that under an anonymous rule φ, the share y_i of an agent i only depends upon $|P|=(c_k,k\in K)=c$, and the membership of agent i; in particular all members of a given coalition G_k get the same share, and two coalitions G_k,G_l of identical size receive the same total share as well. Thus an anonymous rule φ can be written in the reduced form $\psi(K,c,s)$, where K is a finite set, $c\in\mathbb{N}_*^K$ and $s\in\mathbb{R}_+$. The reduced rule ψ is a symmetric function of all variables c_k, and $\psi(K,c,s)=x$ is a vector of shares in $\mathbb{R}_+^K$ such that $\sum_K x_k=s$. We interpret x_k as the *total* share of the k-th coalition, so that φ is related to ψ as follows:

$$\varphi_i(N,P,s)=\frac{1}{c_k}\psi_k(K,c,s) \text{ if } c=|P| \text{ and } i\in G_k$$

For the anonymous rule φ, GCSY, SI and GSM imply respectively for ψ :

$$CSY\text{: for all } (K,c,s) \text{ and all } k,l \in K, k \neq l$$
$$x = \psi(K,c,s) \Longrightarrow x_l = \psi_l(K \backslash \{k\}, c_{-k}, s - x_k)$$
$$SI\text{: for all } (K,c,s) \text{ and } \lambda \geq 0 : \psi(K,c,\lambda s) = \lambda\psi(K,c,s)$$
$$\text{GSM: for all } (K,c,s) \text{ and } k,l \in K :$$
$$c_k \geq c_l \Longrightarrow x_k \geq x_l, \text{ where } x = \psi(K,c,s)$$

We fix now a reduced rule ψ meeting these three properties and use the simplifying notation $f(k;c) = \psi(\{1,...,k\},c,1)$. Define the binary relation $\succsim$on $\mathbb{N}_*$ as follows:

$$c_1 \succsim c_2 \iff f_1(2;c_1,c_2) > 0$$

Its associated strict and indifference parts are:

$$c_1 \succ c_2 \iff f(2;c_1,c_2) = (1,0);\ c_1 \sim c_2 \iff f(2;c_1,c_2) \gg 0$$

This relation is complete, and by GSM it is compatible with the natural order of $\mathbb{N}_*$.We claim that it is transitive. We check first that $\succ$ is transitive. Assume $c_1 \succ c_2$ and $c_2 \succ c_3$ and consider $x = f(3;c_1,c_2,c_3)$. CSY and SI imply:

$$x_3 = (1-x_1)f_3(2;c_2,c_3) = 0 \Longrightarrow x_1 = (1-x_3)f_1(2;c_1,c_2) = 1$$

By CSY and SI again, $x_3 = (1-x_2)f_3(2;c_1,c_3)$, implying $f(2;c_{1,}c_3) = (1,0)$ as was to be proved.

Next we check that $\sim$ is transitive. Assume $c_1 \sim c_2$ and $c_2 \sim c_3$, and invoke CSY and SI again:

$$(x_2,x_3) = (1-x_1)f(2;c_2,c_3); (x_1,x_2) = (1-x_3)f(2;c_1,c_2)$$

If $x_3 = 1$, the r.h.s. equality gives $x_1 = x_2 = 0$, hence a contradiction in the l.h.s. one. Thus $x_3 < 1$, and the r.h.s. equality implies x_1, x_2 are both positive, hence $x_1, x_2 < 1$ as well. Then $x_3 > 0$ from the l.h.s. equality, and finally $0 < x_k < 1$ for all k. From CSY + SI again, $(x_1,x_3) = (1-x_2)f(2;c_1,c_3)$ gives $c_1 \sim c_3$ as desired.

A similar argument completes the proof of the claim. For instance suppose $c_1 \succ c_2$ and $c_2 \sim c_3$. If $x_1 < 1$, the l.h.s. equality implies $0 < x_2, x_3 < 1$, and a contradiction in the r.h.s. Thus $x_1 = 1$ and $(x_1,x_3) = f(2;c_1,c_3)$ gives $c_1 \succ c_3$ as desired.

Given a profile $(k;c)$ of arbitrary size and $x = f(k;c)$, repeated applications of CSY and SI give

$$(x_k, x_{k'}) = (1 - \sum_{l \neq k,k'} x_l) \cdot f(2,(c_k,c_{k'})) \qquad (2)$$

implying that $x_k = 0$ if and only if c_k is not in the highest indifference class reached by c. It remains to show that within any given indifference class $I(t)$, our method divides s in proportion to $w(c_k)$, for some positive function w, nondecreasing on each indifference class $I(t)$. If $I(t)$ is a singleton there is nothing to prove so we assume $|I(t)| \geq 2$.

Take a set $K, |K| \geq 3$, and a profile $c \in I(t)^K$. Set $x = f(k; c)$ and apply (2), recalling $x_k > 0$ for all $k \in K$:

$$\frac{x_k}{x_l} = \frac{f_k(2; (c_k, c_l))}{f_l(2; (c_k, c_l))} = h(c_k, c_l) \text{ for all } k, l \in K$$

where h does not depend on k, l by Anonymity. Therefore for all $j, k, l \in K$:

$$h(c_j, c_k) \cdot h(c_k, c_l) \cdot h(c_l, c_j) = 1$$

A standard argument shows that h can be written $h(c_k, c_l) = \frac{w(c_k)}{w(c_l)}$ for some positive function w, and x now takes the form given in (1). That w is nondecreasing on $I(t)$ follows at once from GSY. This concludes the proof of Theorem 1.

Remark 1

Given a reduced rule ψ as described in the theorem, consider the Joint Bargaining Harmful property. First of all, JBH implies that $\succsim$ is the full indifference relation on $\mathbb{N}_*$. Suppose not. Then we can find c_1 such that $c_1 + 1 \succ c_1$ and we have

$$f(3; c_1, 1, c_1 + 1) = (0, 0, 1) \text{ and } f(2; c_1 + 1, c_1 + 1) = (\tfrac{1}{2}, \tfrac{1}{2})$$

contradicting JBH.

Thus ψ is entirely described by a nonincreasing and positive function w defined on $\mathbb{N}_*$:

$$\psi_k(K, c, s) = \frac{w(c_k)}{\sum_K w(c_l)} \cdot s \text{ for all } k, all c, s \tag{3}$$

Now JBH requires w to be strictly subadditive.

Turning to the Joint Bargaining Beneficial property, we note that it does not place any restriction on the relation $\succsim$, and simply requires w to be strictly superadditive within each class $I(t)$. We omit the easy proof.

Proof of Corollary

It is easy to check that each rule $\varphi^\alpha, 0 \leq \alpha \leq +\infty$, meets Replication Invariance. Conversely, let φ be a rule as in the statement of the theorem. If φ is replication invariant, its reduced form ψ satisfies:

$$\text{RI: for all } (K,c,s) \text{ and all } r\in\mathbb{N}_*: \psi(K,r\cdot c,s)=\psi(K,c,s).$$

We check first that $\succsim$ must be the natural order of $\mathbb{N}_*$ or the full indifference.

Suppose $1\backsim 2$. Then RI implies $r\backsim 2r$ for all $r\in\mathbb{N}_*$, hence $[r,2r]$ is contained in one indifference class ($\succsim$ is compatible with the natural order). Therefore $\succsim$ is the full indifference. Suppose next $2\succ 1$ and $2\backsim 3$. Then RI implies $2r\succ r$ and $2r\backsim 3r$. The latter implies $4\backsim 6$ and $6\backsim 9$, therefore $4\backsim 8$, a contradiction. Thus $2\succ 1$ implies $3\succ 2$. Repeat the argument to show that $3\succ 2$ and $3\backsim 4$ is impossible: the latter would imply $9\backsim 12$ and $12\backsim 16$ hence $10\backsim 15$, contradiction. An obvious induction shows that $\succsim$ must be the natural order of $\mathbb{N}_*$.

If $\succ$ is the natural order of $\mathbb{N}_*,\psi$ divides s equally among all largest coalitions, namely $\varphi=\varphi^\infty$. Assume now that $\succsim$ is full indifference, so that ψ is represented as in (3) by a simple positive and nondecreasing function w. We apply RI to an arbitrary profile c_1,c_2 :

$$\frac{w(rc_1)}{w(rc_1)+w(rc_2)}=\frac{w(c_1)}{w(c_1)+w(c_2)}\Longleftrightarrow\frac{w(rc_1)}{w(c_1)}=\frac{w(rc_2)}{w(c_2)}$$

Without loss of generality, we can choose $w(1)=1$, and the above property now implies

$$w(a\cdot b)=w(a)\cdot w(b) \text{ for all } a,b\in\mathbb{N}_*$$

Let $p_1=2,p_2,p_3,...$be the increasing sequence of prime numbers. Set $a=\Pi p_i^{r_i},b=\Pi p_i^{r'_i}$, with $r_i,r'_i\in\mathbb{N}$, almost all zero. The multiplicative property of w implies

$$\frac{w(a)}{w(b)}=\Pi w(p_i)^{r_i-r'_i}$$

That w is nondecreasing implies, for any two sequences r_i,r'_i in $\mathbb{N}$ with only finitely many non-zero terms:

$$\{\Pi {p_i}^{r_i-r'_i}\geq 1\Longrightarrow \Pi w(p_i)^{r_i-r'_i}\geq 0\}$$

$$\Longleftrightarrow\{\sum(r_i-r'_i)\log p_i\geq 0\Longrightarrow\sum(r_i-r'_i)\log w(p_i)\geq 0\}$$

The implication $\sum\{\log\ p_i\}z_i\ \geq\ 0\Longrightarrow\sum\{\log w(p_i)\}\cdot z_i\ \geq 0$ thus holds for any sequence z_i in $\mathbb{Z}$ with finitely many non-zero terms. Likewise for any sequence in $\mathbb{Q}$, and even in $\mathbb{R}$ by continuity. Farkas' Lemma now implies the existence of $\alpha\in\mathbb{R}_+$ such that

$$\log w(p_i)=\alpha\log p_i \text{ for all } i\Longrightarrow w(a)=a^\alpha \text{ for all } a\in\mathbb{N}_*$$

Thus $\psi=\psi^\alpha$, as desired. The proof of the Corollary is complete.

3. Axiomatic bargaining with a coalition structure

Given the set $\mathcal{N}$ of potential agents, a classic *bargaining problem* is a pair (N, S) where N is a finite subset of $\mathcal{N}$ and S is a convex, compact, comprehensive subset of $\mathbb{R}^N_*$ containing at least one point $u \gg 0$. The profile 0 is the disagreement utility profile. Note that our results extend easily to the model where the disagreement point d is arbitrary in $\mathbb{R}^N$.

A *group bargaining problem* is a triple (N, P, S) where (N, S) is a classic problem and $P = \{G_k; k \in K\}$ is a partition of N as in Section 2. A *bargaining solution* F associates to every group bargaining problem (N, P, S) a utility profile $u = F(N, P, S)$ that is efficient ($\{u' \geq u$ and $u' \in S\} \Longrightarrow u' = u$) and strictly positive ($u \gg 0$). Thus for each partition P, the restriction of F is a classic bargaining solution, not necessarily symmetric with respect to N: when S is symmetric (e.g., S corresponds to a "divide the dollar" problem) but P is not, the solution u is typically not egalitarian.

We now adapt the five axioms introduced in Section 2 to the bargaining context. For a permutation $\sigma \in \sum(\mathcal{N})$, we write $\sigma(S) = \{\sigma(u) | u \in S\}$, and $\sigma(N, P, S) = (\sigma(N), \sigma(P), \sigma(S))$.

Anonymity (ANO) : for all $\sigma \in \sum(\mathcal{N})$, all (N, P, S), $F(\sigma(N, P, S)) = \sigma(F(N, P, S))$

If $P_0 = \{\{i\}, i \in N\}$ denotes the finest partition of N, and $P^0 = \{N\}$ its coarsest partition, Anonymity of F implies that $F(N, P_0, S)$ and $F(N, P^0, S)$ are two anonymous classic bargaining solutions.

In the statement of the group consistency axiom, we use the same notation P_{-k} as in Section 2. In addition, given $u \in \mathbb{R}^N_+$, and S, we write $S_{-k}(u)$ for the restriction of S to $N \backslash G_k$ when agents in G_k receive u_{G_k}, the projection of u on $\mathbb{R}^{G_k}$: $S_{-k}(u) = \{v \in \mathbb{R}^{N \backslash G_k}_+ | (v, u_{G_k}) \in S\}$.

Group Consistency (GCSY): for all (N, P, S) and all $k \in K$

$$u_i = F_i(N \backslash G_k; P_{-k}; S_{-k}(u)) \text{ for all } i \in N \backslash G_k, \text{ where } u = F(N, P, S)$$

If F is group-consistent, its restriction to P_0 is consistent in the usual sense of this term for bargaining solutions: Lensberg [9], Thomson and Lensberg [13]. But when P is arbitrary, group consistency bears only on the elimination of entire coalitions of agents. It requires intergroup consistency, but says nothing of intragroup consistency.

Common Scale Invariance (CSI): for any (N, P, S) and any $\lambda > 0$: $F(N, P, \lambda \cdot S) = \lambda \cdot F(N, P, S)$.

Before defining Group Size Monotonicity and Replication Invariance, we need a few more definitions. Given (N, P), we call a utility profile u *group symmetric* if for all $\sigma \in \sum(N)$ (a permutation of N) we have $\sigma(P) = P \Longrightarrow \sigma(u) = u$. Equivalently, u is group symmetric if and only if

$$u_i = u_j \text{ for all } i \in G_k, j \in G_l \text{ such that } k = l \text{ or } |G_k| = |G_l|.$$

We denote $U(N,P)$ the set of such utility profiles. We call a problem (N,P,S) *group-symmetric* if we have

$$\sigma(P)=P \Longrightarrow \sigma(S)=S \text{ for all } \sigma\in \textstyle\sum(N)$$

For both the coarsest and the finest partitions of N, group symmetry is simply the symmetry of S in all coordinates. For other partitions, the condition imposes fewer constraints on S.

An important consequence of Anonymity follows. If F is an anonymous solution and (N,P,S) a group symmetric problem, then $F(N,P,S)=u$ is itself group symmetric.

We define next the *trace* operator t from $\mathbb{R}^N$ into $\mathbb{R}^K$, for a given partition P of $N:\{t[N,P,u]\}_k=\sum_{G_k}u_i$ for all $k\in K$. Finally, the trace of the group bargaining problem (N,P,S) is the following classic K-bargaining problem

$$t[N,P,S]=\{t[N,P,u]|u\in U(N,P)\cap S\}$$

We let the reader check that $t[N,P,S]$ is indeed convex, compact, comprehensive, and contains some $v\gg 0$. Note that adding utilities accross agents would not make necessarily make sense in an arbitrary bargaining context. But in the sequel we are only using the trace operation for group symmetric problems (N,P,S) and anonymous solutions F, that always select a utility profile in $U(N,P)\cap S$. In this case the trace problem $t[N,P,u]$ collects all profiles of "group utilities" that can feasibly be distributed in a group symmetric way. We are now ready to state our last two axioms.

Group Size Monotonicity (GSM): for any group symmetric (N,P,S) s. t. $t[N,P,S]$ is symmetric as well, for all $k,l\in K$:

$$|G_k|\geq|G_l| \Longrightarrow \textstyle\sum_{G_k}u_i\geq\sum_{G_l}u_j \text{ where } u=F(N,P,S)$$

The comparison of group utilities in the above statement applies only to those problems (N,P,S) that are both intra-group and inter-group symmetric: the former is the group symmetry property of (N,P,S), the latter is the assumption that the trace problem $t[N,P,S]$ is itself symmetric in $\mathbb{R}^K$. Under those circumstances, Group Size Monotonicity requires group shares to be at least weakly increasing in group size.

Replication Invariance (RI): for any $r\in\mathbb{N}_*$ and any two group symmetric problems $(N^\varepsilon,P^\varepsilon,S^\varepsilon),\varepsilon=1,2$, such that $|N^1|=r\cdot|N^2|,|P^1|=r\cdot|P^2|$ and $t[N^1,P^1,S^1]=t[N^2,P^2,S^2]$:

$$|G_k^1|=r\cdot|G_l^2| \Longrightarrow \textstyle\sum_{G_k^1}u_i^1=\sum_{G_k^2}u_j^2, \text{ where } u^\varepsilon=F(N^\varepsilon,P^\varepsilon.S^\varepsilon)$$

We abuse notation slightly by equating the two traces $t[N^\varepsilon, P^\varepsilon.S^\varepsilon]$ in the above definition. They are in fact two subsets in $\mathbb{R}_+^{K_1}$ and $\mathbb{R}_+^{K_2}$ respectively, but the assumptions on the two problems $(N^\varepsilon, P^\varepsilon, S^\varepsilon)$ allow us to identify $k_1 \in K_1$ and $k_2 \in K_2$ whenever $c_{k_1} = c_{k_2}$.

To interpret RI, observe that the replication (from $\varepsilon = 2$ to 1) "spreads" r copies of $U(N^2, P^2) \cap S^2$ to obtain $U(N^1, P^1) \cap S^1$. The axiom requires our solution to spread similarly the selected utility profile. Naturally, this property is only meaningful for anonymous solutions F.

We conclude this section by explicitely relating the two models, divide the dollar and bargaining, developed in this section and the previous one. We identify the division of \$$s$ and the bargaining set $A^-(N,s) = \{u \in \mathbb{R}_+^N | \sum_N u_i \leq s\}$. And we project the group bargaining solution F into the following division rule φ :

$$\varphi(N,P,s) = F(N,P,A^-(N,s))$$

Lemma 1

The above projection operator transports the five properties of F - Anonymity, Group Consistency, Group Size Monotonicity, Common Scale Invariance and Replication Invariance-, into the properties with the same name for φ.

We omit the straightforward proof. The key observation is that for any partition of N, $A^-(N,s)$ is group symmetric and its trace $t[N,P,A^-(N,s)] = A^-(K,s)$ is symmetric as well.

Combining Lemma 1 with Theorem 1 and its Corollary in Section 2, we see that if a bargaining solution F meets ANO, GCSY, RI, CSI and GSM, its projection φ is determined up to a single parameter α, therefore so is $F(N,P,A^-(N,s))$. But we still have a great many ways to define F for general problems (N,P,S), in particular when S is not group symmetric. For instance, we can choose for each $c = 1, 2, ...$ a collective utility function W_c on $\mathbb{R}_+^c$ with the following properties:

$$W_c(a \cdot \mathbb{I}) = c^\alpha \cdot a^p \text{ for all } c \in \mathbb{N}_*, a \in \mathbb{R}_+$$

where $\mathbb{I} = (1, ..., 1)$ is the diagonal unit vector in $\mathbb{R}_+^c$. Then the group bargaining solution

$$F(N,P,S) = \arg\max_{u \in S} \sum_{k \in K} W_{|G_k|}(u_{G_k})$$

does meet all five axioms. Notice that F is narrowly constrained for group symmetric problems, but unconstrained for other problems.

<u>Remark 2</u>

There is an alternative formulation for the two axioms of group size monotonicity and replication invariance. Let

$$\hat{S} \equiv \{\hat{u} \in \mathbb{R}^K : \hat{u}_k = u_i \text{ for } i \in G_k, \text{ for some } u \in U(N,P) \cap S\},$$

and consider the projection of a group symmetric bargaining problem (N, P, S) onto the classic $K-$bargaining problem $(K, \widehat{S})$. This projection differs from- but is related to- the trace operator.

Group Size Monotonicity (GSM): for any group symmetric (N, P, S) s. t. $(K, \hat{S})$ is symmetric as well, for all $k, l \in K$:

$$|G_k| \geq |G_l| \Longrightarrow u_i \geq u_j \text{ for } i \in G_k \text{ and } j \in G_l \text{ where } u = F(N,P,S)$$

Replication Invariance (RI): Let $(N^\varepsilon, P^\varepsilon, S^\varepsilon), \varepsilon = 1, 2$, be two group symmetric problems. Suppose that $|N^1| = r \cdot |N^2|, |P^1| = r \cdot |P^2|$ and $(K^1, \hat{S}^1) = (K^2, \hat{S}^2)$:

$$|G_k^1| = r \cdot |G_l^2| \Longrightarrow u_i^1 = u_j^2 \text{ for } i \in G_k \text{ and } j \in G_l \text{ where } u^\varepsilon = F(N^\varepsilon, P^\varepsilon, S^\varepsilon)$$

We let the reader check the equivalence of these two formulations.

Remark 3

The joint bargaining paradox, or its absence, cannot be easily formulated when two coalitions of different sizes merge: as utility is not transferable, comparing the total utility over these two coalitions before and after the merging has no clear interpretation. However it is a simple matter to express JBB and JBH when two coalitions *of identical size* merge, because anonymous solutions do not discriminate between the merging agents before or after the merging.

Given a partition $P = \{G_k, k \in K\}$ of N and two coalitions G_k, G_l of identical size, recall that $P[k,l]$ is the partition resulting from merging these two coalitions. Then JBH requires the following from an anonymous bargaining solution F. For all S such that *both* (N, P, S) and $(N, P[k,l], S)$ are group symmetric:

$$F_i(N, P[k,l], S) < F_i(N, P, S) \text{ for all } i \in G_k \cup G_l$$

4. Nash bargaining solution with a coalition structure

The most common axiomatization of the Nash bargaining solution relies on the following two properties:

Contraction Independence (CI): for all (N, P) and all S, S' :

$$\{S' \subseteq S \text{ and } F(N,P,S) = u \in S'\} \Longrightarrow F(N,P,S') = u$$

Individual Scale Invariance (ISI): for all (N, P, S) and all $\beta \in \mathbb{R}^N_{++}$:

$$F(N, P, \beta \bullet S) = \beta \bullet F(N, P, S)$$

where $\beta \bullet u = (\beta_i u_i)_{i \in N}$.

The interpretation of these two axioms is standard: see for instance Thomson and Lensberg [13] or Peters [11]. In the classic bargaining problem (without a coalition structure), the two axioms characterize the weighted Nash solutions: Kalai [7]. Therefore a group bargaining solution F meets CI and ISI if and only if there exists for all pairs (N, P) a profile of "weights" $\theta(N, P) \in \mathbb{R}^N_{++}$ such that

$$F^\theta(N, P, S) = \arg\max_{u \in S} \sum_{i \in N} \theta_i(N, P) \cdot \log u_i, \text{ for all } S.$$

Note that the profile of weights is unique for a given solution, up to a multiplicative constant. When we combine the group sensitive axioms of Section 3 with the two properties above, we obtain a simple family of weighted Nash solutions.

Theorem 2

Fix an arbitrary positive and nondecreasing function w on $\mathbb{N}_*$, and for all pair (N, P) with $c = |P|$ define the following profile of weights

$$\theta_i(N, P) = \frac{w(c_k)}{c_k} \text{ for all } k \text{ and all } i \in G_k$$

The corresponding weighted Nash solution F^w meets the three axioms Anonymity, Group Consistency, and Group Size Monotonicity.

Conversely, this family of solutions exhausts all group bargaining solutions meeting these three axioms, as well as Contraction Independence and Individual Scale Invariance.

Corollary to Theorem 2

The four group sensitive axioms - Anonymity, Group Consistency, Replication Invariance and Group Size Monotonicity-, together with Contraction Independence and Individual Scale Invariance, characterize the one-dimensional family of weighted Nash solutions $F^\alpha, 0 \leq \alpha < +\infty$:

$$\theta_i(N, P) = c_k^{\alpha - 1} \text{ for all } k \text{ and all } i \in G_k$$

In the group bargaining solutions identified by Theorem 2 and its Corollary, the weight of an individual agent may actually decrease with respect to the size of the coalition of which he is a member: this holds true whenever $0 \leq \alpha < 1$ for the solutions described in the Corollary, and more generally if $w(z)/z$ decreases in z. This corresponds to the joint bargaining paradox in the sense of Remark 3, Section 3.

One benchmark member of the family F^α is the group-insensitive solution F^1 : this solution is the ordinary symmetric Nash bargaining solution, ignoring the partition

P altogether. Another benchmark is the solution F^0, treating all groups equally - irrespective of their size - in the divide-the-dollar game, and exhibiting the strongest form of joint bargaining paradox compatible with Group Size Monotonicity. This solution was introduced and axiomatically characterized by Chae and Heidhues [2].

Note that the family F^α does not allow $\alpha = +\infty$, because we require from a group bargaining solution that it gives a positive share to every participant. The limit of F^α when α goes to infinity applies the symmetric Nash solution among all agents in the largest coalitions, while keeping all other agents at their disagreement level.

Proof of Theorem 2

First Statement.

For any choice of w, the solution F^w is clearly Anonymous because the permutation σ respects the size of coalitions, hence their weight $w(|G_k|)$.

The solution F^w meets GCSY because the relative weight of two agents $i \in G_k$ and $j \in G_l$ only depends upon the relative sizes of G_k and G_l, hence is not affected when other coalitions are removed.

To check that F^w meets GSM, pick a problem (N, P, S) where S is group symmetric and $T = t[N, P, S]$ is symmetric as well. Because the weights $\theta_i(N, P)$ are constant within each coalition, and equal in two coalitions of equal size, the utility profile $u^* = F^w(N, P, S)$ is group symmetric. Setting $z^* = t(N, P, u^*)$ we have

$$u^* = \arg\max_{u \in S} \sum_{k \in K} \frac{w(c_k)}{c_k} \sum_{i \in G_k} \log u_i \Longleftrightarrow z^* = \arg\max_{z \in T} \sum_{k \in K} w(c_k) \log z_k$$

The GSM property now follows from the fact that over a symmetric utility set T, a weighted Nash solution is monotonic in weights: $w(c_k) \geq w(c_l) \Longrightarrow z_k^* \geq z_l^*$. This fact is clear for two coalition problems ($|K| = 2$). For an arbitrary $|K|$, and two coalitions k, l, notice that (z_k^*, z_l^*) is the corresponding weighted Nash solution in the $\{k, l\}-$ slice of T at z^* - namely the set $\{(z_k, z_l)/(z_k, z_l, z_{-k,l}^*) \in T\}$-, and the latter set is symmetric.

Second Statement.

If F is a solution meeting CI and ISI, it is a weighted Nash solution F^θ, where the weight profile depends arbitrarily upon (N, P). The only restriction is $\theta_i(N, P) > 0$ because we insist that a solution to a group bargaining problem give a positive utility to each agent (see at the beginning of Section 3).

Let φ be the projection of F^θ into a divide-the-dollar solution. By Lemma 1, φ meets ANO, CSY, CSI, and GSM. Therefore φ is represented as in Theorem 1 by a preordering $\succsim$ of N and a weight function w. As φ gives a positive share to every coalition, $\succsim$ is the full indifference relation and w is nondecreasing over all $\mathbb{N}_*$. For all (N, P, s) :

$$\varphi_i(N,P,s) = \frac{1}{c_k}\frac{w(c_k)}{\sum_{l\in K} w(c_l)} \text{ for all } k, \text{ all } i \in G_k$$

On the other hand, the utility profile $u^* = F^\theta(N, P, A^-(N,s))$ is easily computed

$$u^* = \arg\max\{\sum_{i\in N}\theta_i(N,P)\log u_i | \sum_{i\in N} u_i = 1\} \Longrightarrow u_i^* = \frac{\theta_i(N,P)}{\sum_{j\in N}\theta_j(N,P)}\}$$

From $u^* = \varphi(N,P,s)$, we deduce first that $\theta_i(N,P)$ is constant within each coalition G_k, next that up to a multiplicative rescaling, θ is as stated in Theorem 2.

Proof of the Corollary

Immediate in view of the Corollary to Theorem 1, and the fact that Replication Invariance of F is respected in the projection of F into φ.

5. Egalitarian Solution with a Coalition Structure

The egalitarian solution for a classic bargaining problem (N,S) picks the highest point in S along the diagonal of $\mathbb{R}_+^N$. In order to guarantee that this defines an efficient point of S, we require the feasible utility set to meet

Minimal Transferability: for all $u \in S$, all $i \in N : \{u_i > 0\} \Longrightarrow \{\exists v \in S', v_i < u_i$ and $v_j > u_j$, all $j \neq i\}$.

This additional requirement does not affect the discussion of Section 3.

The asymmetric generalizations of the egalitarian solution are called the *path monotone* solutions (Thomson and Myerson [14]). Fix a monotone and continuous path in $\mathbb{R}_+^N$, starting at 0 and strictly increasing in all coordinates. The intersection of this path with the Pareto frontier of S defines a solution F satisfying the following property

Issue Monotonicity (IM): for all N, all $S,T : \{S \subseteq T\} \Longrightarrow \{F(N,S) \leq F(N,T)$.

Issue Monotonicity is a much stronger requirement than Contraction Independence, and is enough to characterize the path monotone solutions. Upon adding Common Scale Invariance, the path must now be a straight line borne by a line $\delta \gg 0$.See Kalai [8].

Turning now to our group bargaining model, we see that the combination of IM and CSI implies that for all $S, F(N,P,S) = u$ is the efficient utility profile in S borne by a strictly positive vector $\delta(N,P)$, namely $\frac{u_i}{\delta_i(N,P)}$ is independent of $i \in N$. Note that the set of weights $\delta(N,P)$ is defined up to a multiplicative constant. In our last result we show that the group sensitive axioms ANO, GCSY, GSM and RI force precisely the same structure on the weights δ as on the weights θ of the Nash solution in the previous section.

Theorem 3

Fix an arbitrary positive and nondecreasing function w on $\mathbb{N}_*$ and for all pairs (N,P) with $c=|P|$ define the following profile of weights

$$\delta_i(N,P)=\frac{w(c_k)}{c_k} \text{ for all } k \text{ and all } i\in G_k$$

The group bargaining solution borne by the ray $\delta(N,P)$, for all (N,P), meets Anonymity, Group Consistency, and Group Size Monotonicity.

Conversely, this family exhausts all group bargaining solutions meeting these three axioms, as well as Issue Monotonicity and Common Scale Invariance.

Corollary to Theorem 3

Adding Replication Invariance to the axioms above selects the one-dimensional family of weights

$$\delta_i(N,P)=c_k^{\alpha-1} \text{ for all } k,\ \text{all } i\in G_k$$

where $0\leq\alpha<+\infty$.

Proof of Theorem 3

In the direct statement, checking ANO and GCSY is straightforward. Only GSM requires some attention. Pick a group symmetric problem (N,P,S) and let u be the utility profile equalizing $\frac{c_k u_i}{w(c_k)}$ across all $i\in N$. Then $u\in U(N,P)$, therefore $z=t[N,P,u]$ equalizes $\frac{z_k}{w(c_k)}$ across $k\in K$, hence $c_k\geq c_l\Longrightarrow z_k\geq z_l$, establishing GSM.

Conversely, let F be a solution satisfying IM and CSI. Then $F(N,P,\cdot)$ is the path solution borne by a ray $\delta(N,P)\gg 0$, equalizing $\frac{u_i}{\delta_i(N,P)}$ across all i.

By Lemma 1 and Theorem 1, the divide-the-dollar solution φ associated with F takes the form

$$\varphi_i(N,P,s)=\frac{1}{c_k}\frac{w(c_k)}{\sum_K w(c_l)}\cdot s \text{ all } k,\ \text{all } i\in G_k$$

for some positive and nondecreasing function w. Recall that $\succsim$ is the full indifference because $\varphi(N,P,s)\gg 0$ for all problems. The desired conclusion follows.

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