# Characterization of the Support of the Mixed Strategy Price Equilibria in Oligopolies with Heterogeneous Consumers

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**Abstract**

This paper revisits the theory of oligopoly pricing and shows that for a large class of demand and cost functions, a mixed strategy equilibrium necessarily implies that each firm's equilibrium strategy is a discrete distribution over a finite number of prices.

Keywords: Price Competition; Price Dispersion; Mixed Equilibrium; Heterogeneous Consumers

JEL classification: D43; L13

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# 1 Introduction

Consider a market where a particular good is produced by $K$ firms. The demands are given by functions $G_i(p_i, \mathbf{p_{-i}})$, and firms compete in prices. How can we find the firms' optimal prices in a static setting? Surprisingly, the answer to this very basic question of oligopoly theory is yet incomplete.

Faced with such a question, one can first search for a pure strategy equilibrium. It is relatively straightforward to write out the profit functions, differentiate them, and try to solve the resulting system of equations. There exists a large body of literature examining restrictions on demand or best reply functions that ensure the existence and uniqueness of a pure strategy price equilibrium (see Vives (1999) for an overview of the latest results in this area). However, as is well-known, not all demand functions result in a pure strategy price equilibrium. In such cases, one must proceed to search for mixed strategy equilibria.

The use of mixed strategy equilibria in oligopoly pricing models has also been extensive. Some examples that result in price dispersion include literature on sales (Shilony (1977), Varian (1980), Gal-Or (1982), Narasimhan (1988) and Baye et al. (1992)), capacity constraints (Beckmann (1965), Levitan and Shubik (1972), Kreps and Scheinkman (1983), Osborne and Pitchik (1986), Davidson and Deneckere (1986), and Allen and Hellwig (1993)), and consumer search (Burdett and Judd (1983), Rob (1985), Stahl (1988), Dana (1994), and McAfee (1995)). An important feature of these models is that they all use discontinuous demands and predict that the firms pick their prices from an interval according to some continuous probability distribution function. This is not a coincidence, as Dasgupta and Maskin (1986) have shown that in the symmetric case, for the type of discontinuities encountered in the above models, the equilibrium mixed strategies are atomless on the set of discontinuities. The technique for computing these equilibria is by now standard.

But what happens when $G_i$ do not have singularities and do not result in a pure strategy equilibrium? As an example, take any of the price dispersion models and "smooth" the demand discontinuity by adding some heterogeneity into consumer preferences.[1] As shown by Benessy (1989) for the model of price competition with capacity constraints, if this heterogeneity is small enough, a pure strategy equilibrium does not exist. The technique for computing an atomless mixed strategy equilibrium does not work for such demands. Thus, we know that the mixed strategy equilibrium exists[2], but do not know anything else about it.

The purpose of this work is to fill in this gap in the oligopoly pricing theory. I prove that for analytic demand functions a mixed strategy price equilibrium is characterized by each firm charging a finite number of prices. This characterization covers the functions typically used to model the demand for differentiated products,

[1] For example, instead of consumers purchasing the product with the lowest price, assume that their tastes are heterogeneous, and the price premium they are willing to pay for the firm's product is distributed according to some CDF. In consumer search models assume that all consumers have differing search costs, thus there will be no mass of consumers with zero search costs.

[2] This existence result is provided by Glicksberg (1952).

for example, logit, probit, or CES. The finiteness result gives a theoretical foundation for the computational technique used to find the mixed strategy equilibria when demands are analytic.

To illustrate an application of the theory developed in the first part of the paper, I compute the optimal pricing strategies for a simple example of a duopoly with some loyal consumers and some switchers, whose preferences for the goods are distributed normally.

## 2 Main Result

Consider a market where a particular good is produced by $K$ firms at a zero marginal cost. Firms compete in prices $p$ that range between 0 and $r$, where $r$ is the maximum price any consumer is willing to pay for the product. The set of consumers has measure 1. Consumers have heterogeneous tastes for the goods produced by the firms. These tastes result in the demand functions $G_i(\mathbf{p})$ that show the percent of consumers captured by firm $i$ as a function of prices of all firms $\mathbf{p}$, where $\mathbf{p} = (p_1, p_2, ..., p_K)$. The demand functions $G_i(\mathbf{p})$ are analytic, which means that each function has a Taylor series about each point $x$ that converges to this function in an open neighborhood of $x$.

Since all analytic functions are continuous, the demand functions $G_i$ are continuous, and the profit function of each firm is also continuous. Glicksberg (1952) proved that for the case with continuous payoffs and a non-empty and compact set of actions there exists a mixed strategy equilibrium. Thus, the following proposition holds:

**Proposition 1** *For the setting described above there exists a mixed strategy Nash equilibrium.*

Now, in Theorem 1, I will show that the support of the mixed strategy Nash equilibrium consists of a finite number of prices.

**Theorem 1** *Let the demand function $G_i(\mathbf{p})$ be an analytic function on $[0,r]^K$. In addition, $G_i(\mathbf{p}) > 0$ for any $\mathbf{p} = (p_1, ..., p_{i-1}, 0, p_{i+1}, ..., p_K)$. Then the support of the price distribution of firm $i$ in any mixed strategy Nash equilibrium has a finite number of points.*

**Proof.** Without loss of generality I will prove the theorem for $i = 1$. Assume that firms $2, ..., K$ are charging the prices from the set $\Lambda = [0,r]^{K-1}$ according to some probability distribution function $F(\mathbf{p_{-1}})$, where $\mathbf{p_{-1}} = (p_2, ..., p_K)$. Then, the expected profit function of firm 1 is

$$\pi_1(p_1) = \int\limits_{\Lambda} p_1 G_1(p_1, \mathbf{p}_{-1}) dF(\mathbf{p_{-1}}) = \int\limits_{\Lambda} p_1 G_1(\mathbf{p}) dF(\mathbf{p_{-1}}). \tag{1}$$

Assume that the support of the price distribution of firm 1 contains an infinite number of points. $p_1 G_1(\mathbf{p})$ is an analytic function since it is a product of analytic functions. Then, by Theorem 7.1.1 from Karlin (1959), $\pi_1(p_1) = c$ for all $p_1 \in [0,r]$.[3]

[3]This theorem is stated as follows. If the kernel is analytic and if one of the players has an optimal strategy whose support contains an infinite number of points, then every strategy that is optimal for the other player is an equalizing strategy (Karlin (1959)).

However, $\pi_1(0) = 0$, therefore the profit function must be equal to zero at every point of the price support.

$G_1(\mathbf{p}) > 0$ for any $\mathbf{p} = (0, p_2, ..., p_K)$. Since $G_1$ is a continuous function, then $G_1(\mathbf{p}) > 0$ for any $\mathbf{p} = (\varepsilon, p_2, ..., p_K)$ provided that $\varepsilon$ is small enough. It means that firm 1 charging price $\varepsilon$ will have a positive demand regardless of the competitors' prices. Thus $\pi_1(\varepsilon) > 0$ – contradiction. ■

**Remark 1** *Theorem 1 easily extends to the case of general cost functions. First, assume that the cost function $C_1(\cdot)$ is everywhere analytic. Then $\psi(\mathbf{p}) = p_1 G_1(p_1, \mathbf{p_{-1}}) - C_1(G_1(p_1, \mathbf{p_{-1}}))$ is analytic since the sums, products, and compositions of analytic functions are analytic. Thus, using the same argument as above, $\pi_1(p_1) = \int\limits_{\Lambda} \psi(\mathbf{p}) dF(\mathbf{p_{-1}}) = c$ for all $p_1 \in [0, r]$ if the support of the price distribution of the first firm contains an infinite number of points. Now, in addition to the previous assumption that the firm sells a positive amount at price zero, also assume that $C_1(\cdot)$ is nondecreasing and that $G_1(p_1, \mathbf{p_{-1}})$ is nonincreasing in $p_1$. Since $\pi_1$ is a constant, $\pi_1$ must be equal to $\pi_1(0)$ everywhere on $[0; r]$. Now, consider charging a price $\varepsilon$. Since $G_1(p_1, \mathbf{p_{-1}})$ is nonincreasing in $p_1$, by charging $\varepsilon$ the firm will produce at most the amount that it produces at price $0$. Since $C_1(\cdot)$ is nondecreasing, the firm's costs when charging $\varepsilon$ will be at most the costs that it has when charging $0$. If $\varepsilon$ is small enough, the firm is assured of a positive demand, thus, its revenues are positive. Therefore, by charging $\varepsilon$ instead of zero the firm increases its revenues and does not increase the costs, so $\pi_1(\varepsilon) > \pi_1(0)$ – contradiction.*

**Remark 2** *The relation between the analyticity of the payoff functions and the finiteness of the support of a mixed strategy equilibrium was also addressed by Karlin (1959). In Theorem 7.1.2 he proved this result for the analytic payoff functions that are strictly decreasing at some point $\xi_0$ for any strategy of the rival. This requirement is rather restrictive for the pricing games, as it implies that the profit function $p_1 G_1(p_1, \mathbf{p_{-1}}) - C_1(G_1(p_1, \mathbf{p_{-1}}))$ is strictly decreasing at some $p_1$ for all possible $\mathbf{p_{-1}}$.*

Proposition 1 and Theorem 1 together imply that for the demand functions typically encountered in practice (such as logit, probit, or CES) a mixed strategy Nash equilibrium exists and, moreover, the solution must involve a finite number of prices.

Knowing that each firm charges a finite number of prices, it is possible to write out a standard set of equations, the solution to which gives the mixed strategy equilibrium.[4] It is beyond the scope of this work to perform a technical analysis of how to compute these mixed strategy equilibria or to examine their properties. Instead, to illustrate the application of Theorem 1, in the next section I will provide a simple example of a duopoly model and calculate mixed strategy equilibria for a few values of the parameters of the demand function.

[4] These equations include the first order conditions – the profit function must be maximized at each point of the support of the price distribution. Also, the profits at all the points of the support of the price distribution have to be equal to each other.

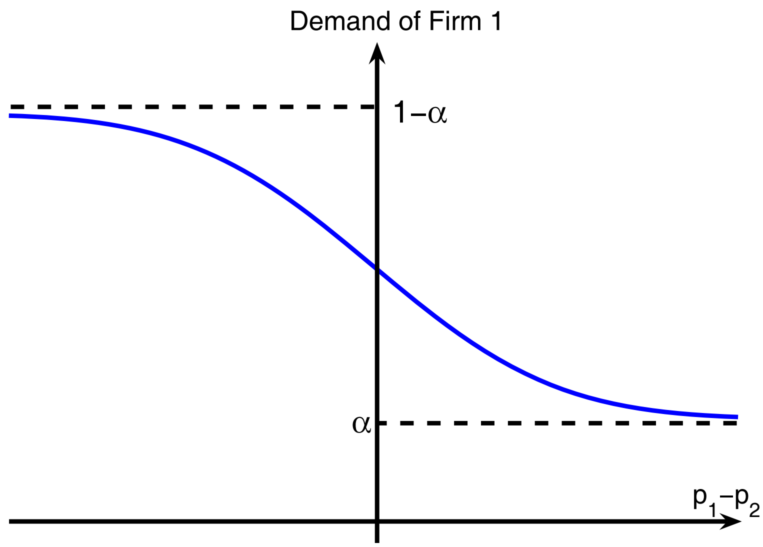
# 3 Application

Consider a market where a particular good is produced by two firms. Assume that prices range between 0 and 1 (1 is the maximum price any consumer is willing to pay for the product). Marginal cost is equal to zero. Each consumer buys one unit per period. There are two types of consumers: loyals and switchers. Each firm has share $\alpha$ (between 0 and $\frac{1}{2}$) of loyal consumers – those who buy only from their favorite firm, provided that the price is lower than 1. The remaining $(1-2\alpha)$ are switchers with heterogeneous tastes for the products that are distributed normally with mean 0 and variance $\sigma^2$ ($\sigma$ shows the degree of consumer heterogeneity). Thus, the demand is $G_i(p_i,p_j) = \alpha + (1-2\alpha)\left(1 - \frac{1}{\sigma\sqrt{2\pi}}\int\limits_{-\infty}^{p_i-p_j} e^{-\frac{1}{2}(\frac{x}{\sigma})^2}dx\right)$. The shape of this function is shown in Figure 1.

This demand function satisfies the conditions of Proposition 1 and Theorem 1, thus there exists a mixed strategy Nash equilibrium that must have a finite number of prices. Figure 2 presents the equilibrium pricing strategies for the different values of the parameters of the demand function.

I fix $\alpha$ to be 0.25 and let $\sigma$ change from 0.3 (where a pure strategy equilibrium exists) to 0.068 (where in the mixed strategy equilibrium each firm mixes between five prices). For each value of $\sigma$ in that range I plot the optimal pricing strategies of the firms. For example, for $\sigma = 0.1$, each firm uses four prices, and the mixed strategy equilibrium involves the firms charging prices $\{p_1 = 0.4673;\ p_2 = 0.7307;\ p_3 = 1;\ p_4 = 0.9384\}$ with corresponding probabilities $\{\gamma_1 = 0.6284;\ \gamma_2 = 0.2296;$

Figure 1: Percent of consumers buying good 1 as a function of $p_1 - p_2$

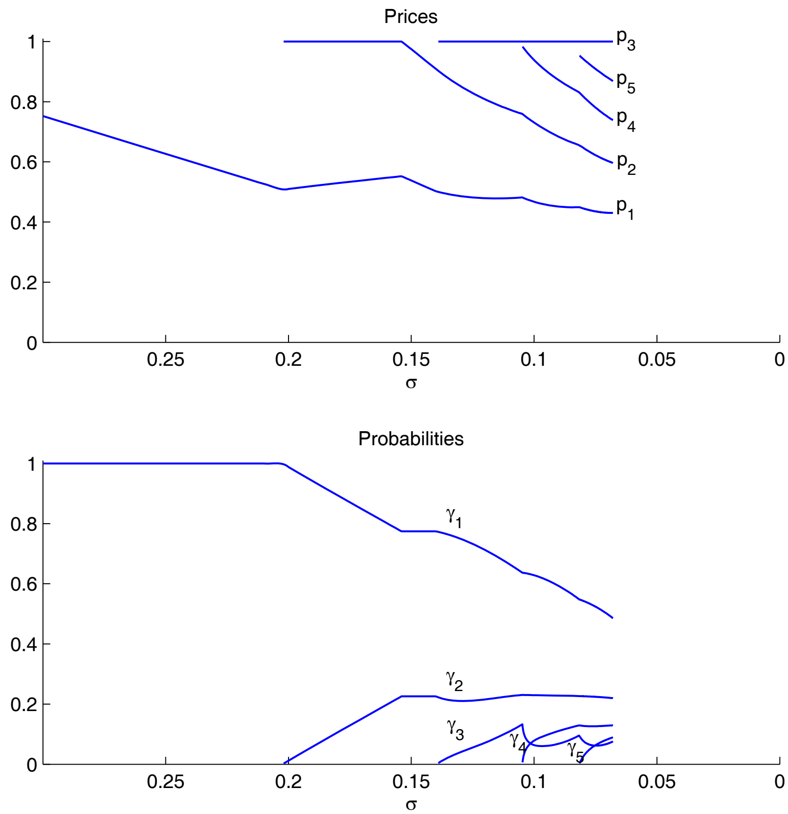



$\gamma_3 = 0.0636$; $\gamma_4 = 0.0784\}$.

# 4 Conclusion

This work extends the theory of static price competition to the case of analytic demands and costs functions, for which no pure strategy price equilibrium exists. I show that the mixed strategy equilibrium is characterized by each firm charging a finite number of prices. This is in contrast with the existing literature on mixed equilibria in prices, which uses discontinuous demands and, thus, has firms mixing over a continuum of prices. This characterization provides researchers with a tool to calculate mixed strategy equilibria in situations for which previously it was unclear

Figure 2: Optimal pricing strategies for $\alpha = 0.25$




how to find a solution.

**Acknowledgement 1** *This work is based on the first chapter of my Ph.D. dissertation (Northwestern University, 2005). I have benefited from discussions with Jim Dana, Rob Porter, Ngo Van Long, and Vicky Zinde-Walsh. I appreciate the valuable comments that advanced the paper from Igor Sinitsyn and Tymon Tatur.*

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