# The Robustness of Equilibrium Analysis: The Case of Undominated Nash Equilibrium *

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**Abstract.** I consider a strategic game form with a finite set of payoff states and employ undominated Nash equilibrium (UNE) as a solution concept under complete information. I propose notions of the proximity of information according to which the continuity of UNE concept is considered as the robustness criterion. I identify a topology (induced by what I call $d^*$) with respect to which the undominated Bayesian Nash equilibrium (UBNE) correspondence associated with any game form is upper hemi-continuous at any complete information prior. I also identify a slightly coarser topology (induced by what I call $d^{**}$) with respect to which the UBNE correspondence associated with *some* game form exhibits a failure of the upper hemi-continuity at any complete information prior. In this sense, the topology induced by $d^*$ is the coarsest one. The topology induced by $d^{**}$ is also used in both Kajii and Morris (1998) and Monderer and Samet (1989, 1996) with some additional restriction. I apply this robustness analysis to the UNE implementation. Appealing to Palfrey and Srivastava's (1991) canonical game form, I show, as a corollary, that almost any social choice function is robustly UNE implementable relative to $d^*$. I show, on the other hand, that only monotonic social choice functions can be robustly UNE implementable relative to $d^{**}$. This clarifies when Chung and Ely's Theorem 1 (2003) applies.

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# 1 Introduction

Modern economic theory relies heavily on the language of game theory and its solution concepts. We often formalize an economic problem as a "game" and apply some solution concept, such as Nash equilibrium, to the game in order to obtain the predictions. A game consists of three components: the set of players; the set of strategies for each player; and the payoff function for each player. We assume, often implicitly, that all the ingredients of the game and each other's rationality are "common knowledge," which implies the assumption that all players know all players know... all players know about the game and about each other's rationality, actions, knowledge, and beliefs. This common knowledge requirement seems to be very stringent. Thus, we take the requirement as at best a simplifying assumption. The question now arises how crucial this informational assumption is. I shall clarify if there is any assumption weaker than common knowledge that is sufficient to approximate the previously given predictions.

The objective of this paper is to propose notions of the "proximity" of information. By the proximity of the information I mean the coarsest detail of the information so that much finer details of the information do not matter for the prediction in terms of outcomes. Throughout I assume that the analyst is certain both the physical rules of the game (henceforth the *game form*) and the set of payoff types. In this paper, I focus on complete information games and employ undominated Nash equilibrium (UNE) as a solution concept. To scrutinize the robustness of UNE, I fix a game form and a finite set of payoff types and "perturb" only its information structure about the set of payoff types. By perturb the information structure I mean that the analyst *casts doubt* into his specification of the information structure. Thus, the perturbation is the way the analyst is able to check to what extent his hypothesis is innocuous for the conclusions he drew.

Let me illustrate the logic of my main results within the context of a simple example. [3] The two men are *players*. I call them Andy and Bob. There are three possible *outcomes*, $a, b$, and $c$. There are two possible *payoff states*, called $\alpha$ and $\beta$. I assume that the men's preferences over the three outcomes are strict and state dependent as follows:

Andy: $b \succ_A^\alpha a \succ_A^\alpha c$ and $b \succ_A^\beta a \succ_A^\beta c$
Bob: $a \succ_B^\alpha c \succ_B^\alpha b$ and $a \succ_B^\beta b \succ_B^\beta c$

I read $a \succ^\theta a'$ as "$a$ is preferred to $a'$ in state $\theta$." The crucial assumption I make here is that the analyst knows that there is common knowledge between both Andy and Bob about the payoff state. This is called an environment with *complete information.* The game form $\Gamma^* = (M, g)$ (shown below) is given, in which Andy

[3] The example is adapted from Jackson and Srivastava (1996, Example 5).

chooses the row and Bob chooses the column. The analyst is concerned with the set of UNE outcomes of the game. Here $M = M_A \times M_B = \{m_A, m_A^{'}\} \times \{m_B, m_B^{'}\}$ refers to the set of action profiles. The outcome function $g : M \to \{a, b, c\}$ assigns to each action profile $m$ an alternative $g(m) \in \{a, b, c\}$.

| $\Gamma^* = (M, g)$ | | Bob | |
|---|---|---|---|
| | | $m_B$ | $m_B^{'}$ |
| Andy | $m_A$ | $a$ | $a$ |
| | $m_A^{'}$ | $c$ | $b$ |

The profile $(m_A, m_B)$, leading to outcome $a$, is the unique UNE of the game $\Gamma^*(\alpha)$ and the only UNE of the game $\Gamma^*(\beta)$ is $(m_A^{'}, m_B^{'})$ leading to the outcome $b$. Given the complete information assumption and the use of UNE as a solution concept, the analyst will be able to predict the outcome of the game if he knows the true game. Namely, the analyst predicts outcome $a$ if he knows the game is $\Gamma^*(\alpha)$ and he predicts outcome $b$ if he knows the game is $\Gamma^*(\beta)$. Now, I am concerned with the analyst who acknowledges that complete information is an idealization and that in the true environment Andy and Bob may be uncertain about the payoff state.

I shall construct the "nearby" environments in which Andy and Bob may be uncertain about the payoff state. Consider the following story which is adapted from the simplest finite truncated version of the email game of Rubinstein (1989). With probability $p$, the game $\Gamma^*(\alpha)$ is played and the game $\Gamma^*(\beta)$ is played with probability $1 - p$. Which is the true game is known initially only to Andy. The players can communicate, but the means open to them do not allow the game to become common knowledge. Specifically, the players are restricted to communicate via computers under the following protocol. If the game is $\Gamma^*(\alpha)$, then Andy's computer *automatically* sends a message to Bob's computer; if the game is $\Gamma^*(\beta)$, then no message is sent. The technology has the property that there is a small probability $\varepsilon > 0$ that the message does not arrive at Bob's computer. If the message does not arrive, then the communication stops. This process is summarized in the following matrix:

| | | Bob's signal | |
|---|---|---|---|
| | | 0 | 1 |
| Andy's signal | 0 | $1 - p$ | 0 |
| | 1 | $p\varepsilon$ | $p(1 - \varepsilon)$ |

There are three states of the world in the nearby environments parameterized by $\varepsilon$: $(0, 0), (1, 0)$, and $(1, 1)$. Here $(0, 0)$ stands for the state where Andy does not send a message and therefore, Bob does not receive the message. $(1,0)$ stands for the state where Andy sends a message to Bob but Bob does not receive the message. Finally $(1,1)$ stands for the state where Andy sends a message to Bob and Bob receives the message. The row is Andy's signal and the column is Bob's signal. It is important to note that the original complete information assumption is embedded in this set of nearby environments. Because $(0,0)$ corresponds to $\beta$ and $(1,1)$ corresponds to $\alpha$ when $\varepsilon = 0$. Assume now that the analyst knows the game is $\Gamma^*(\beta)$ (i.e., the state is $(0,0)$). However, the analyst acknowledges that Andy and Bob might face the nearby environments described above. Then, the analyst knows that Andy and Bob receive signal 0, respectively, but he also knows that this fact is *not* common knowledge between Andy and Bob. Suppose, under this noisy communication, that Bob has the belief about Andy's strategy of the game as follows: "Andy plays $m_A^{'}$ when he receives signal 1 and plays $m_A$ when he receives signal 0."

Given Bob's belief specified above, playing $m_B$ after receiving signal 0 gives the following lottery:

- $a = g(m_A, m_B)$ in $\Gamma^*(\beta)$ with probability $(1-p)/(1-p+p\varepsilon)$;
- $c = g(m_A^{'}, m_B)$ in $\Gamma^*(\alpha)$ with probability $p\varepsilon/(1-p+p\varepsilon)$.

Given Bob's belief specified above, playing $m_B^{'}$ after receiving signal 0 gives the following lottery:

- $a = g(m_A, m_B^{'})$ in $\Gamma^*(\beta)$ with probability $(1-p)/(1-p+p\varepsilon)$;
- $b = g(m_A^{'}, m_B^{'})$ in $\Gamma^*(\alpha)$ with probability $p\varepsilon/(1-p+p\varepsilon)$.

It turns out that $m_B$ can be a strict best response for *any* $\varepsilon > 0$. Then, the analyst is no longer confident that $m_B$ is dominated in the game $\Gamma^*(\beta)$. Rather, the analyst thinks that even a dominated Nash equilibrium $(m_A, m_B)$, leading to $g(m_A, m_B) = a$, is likely to be played in the game $\Gamma^*(\beta)$ if he casts doubts into the complete information assumption. Note that at state $(1,0)$, Andy and Bob have a completely different opinion about the game being played. That is, at $(1,0)$, Andy knows the game is $\Gamma^*(\alpha)$ but Bob believes with high probability that the game is $\Gamma^*(\beta)$. In section 3, I will characterize this noisy communication as a perturbation of the probability distributions. Using the same game form in this simple example, Kunimoto (2005) also explicitly construct a perturbation of the probability distributions under which dominated strategies continue to be dominated. The basic idea for this perturbation is rather simple: I construct a perturbation of the probability distributions so that it is common knowledge at *any* state that Andy and Bob believe with high probability that the same game is played. In particular, Andy and Bob then might believe with high probability that the false game is to be played. However, the important thing for the domination argument to go through continuously is that their misperception about the game must be *common.* The reader is referred to Kunimoto (2005) for the detail.

In order to perturb the information structure, I keep track of the original information structure and let it be embedded in a measurable space. I translate the original complete information structure into a probability distribution. Mathematically, I define each such proximity in terms of topology over the set of probability distributions. In defining the topology, I will distinguish two probability distributions in the realm of the corresponding *conditional* probabilities. Each proximity of information is aimed at characterizing a class of nearby games within which a given equilibrium concept is robust. I say that the undominated Nash equilibrium "concept" is *robust* within a class of nearby games if *any* undominated Bayesain Nash equilibrium (UBNE) outcome of any game within the class of games is approximated by some UNE outcome of the original complete information game. Then, I am able to reduce my robustness requirement to the upper hemi-continuity of the UBNE correspondence with respect to any topology with which I am concerned.

While this looks a very similar argument in Fudenberg, Kreps, and Levine (1988), Dekel and Fudenberg (1990), and Kajii and Morris (1997), I make a very distinct argument. In this paper, I fix the set of payoff types from the complete information and perturb only the information structure. These authors are, on the other hand, concerned with the situation in which there is no common knowledge among the players about the set of payoff types. These papers rather ask which solution concepts (including non-equilibrium concepts) are immune to payoff uncertainty so that they are able to identify robust solution concepts. The objective of this paper is different from those authors'. Here I take a solution concept as given. I relegate more discussion on the related literature to section 6.

I identify a topology (induced by what I call $d^*$) with respect to which the UBNE correspondence associated with *any* game form is upper hemi-continuous at any complete information prior. This is my main result (Theorem 1). Within the nearby games characterized by $d^*$, the analyst can be confident about his prediction by the UNE outcomes. Those nearby games are the situations in which the analyst is certain that there is *approximate* common knowledge in the sense of Monderer and Samet (1989) (which I will carefully define later). I also identify a slightly coarser topology (induced by what I call $d^{**}$) than that induced by $d^*$ with respect to which the UBNE correspondence associated with *some* game form exhibits a failure of the upper hemi-continuity at any complete information prior. Within the nearby games characterized by $d^{**}$, the analyst might have second thought about rejecting the outcomes. This is the situation in which the analyst is nearly (but not quite) certain that there is approximate common knowledge. In this particular sense, the topology induced by $d^*$ is the coarsest one so as to sustain the upper hemi-continuity of the UBNE correspondence for "any" game form.

I apply the robustness analysis to implementation theory. Using UNE as a solution concept, Palfrey and Srivastava (1991) showed that almost any social choice function is implementable. As a corollary combined with my Theorem 1, I show that almost any social choice function is *robustly* undominated Nash implementable relative to $d^*$. Therefore, my robustness analysis gives us a way of robustifying the permissive result of Palfrey and Srivastava (1991). Nevertheless, appealing to the robustness result relative to $d^{**}$, the topology induced by $d^*$ is the best possibility of making the permissive result more robust . Thus, I give a precise sense in which Palfrey and Srivastava's permissive result is dubious if we believe that the robustness relative to $d^*$ is very restrictive. In a related fashion, Chung and Ely (2003) show that only monotonic social choice functions can be robustly undominated Nash implementable. Then, I show that Chung and Ely's robustness requirement is indeed the robustness relative to $d^{**}$ and clarify when Chung and Ely's Theorem 1 (2003) applies.

The rest of the paper is organized as follows. In section 2, I formalize the setup and definitions and show the main result. In section 3, I show the tightness of the main result through the same example. In section 4, I apply our robust equilibrium analysis to implementation theory. In section 5, I relate my results to the literature. Section 6 concludes.

## 2 Formalities and the Main Result

### 2.1 The Environment

There is a finite set $N = \{1, \dots, n\}$ of players. Let $\Theta$ denotes the set of a finite number of *payoff* states, and $A$ denotes the set of *pure* outcomes. Associated with each state $\theta$ is a preference profile $\succeq^\theta$, which is a list $(\succeq_1^\theta, \dots, \succeq_n^\theta)$ where $\succeq_i^\theta$ is player $i$'s state $\theta$ preference relation over $A$. I read $a \succeq^\theta a'$ as "$a$ is at least as good as $a'$ in state $\theta$." I assume that players have strict preferences over $A$. [4]

Players do not observe the state directly but are informed of the state via payoff types. Player $i$'s set of payoff types is $S_i$ which we set $|S_i| = |\Theta|$ for each $i \in N$. A payoff type profile is an element $s = (s_1, \dots, s_n) \in S = \times_{i \in N} S_i$. Let $\mu$ be a prior probability over $\Theta \times S$. I designate $s^\theta$ to be the payoff type profile in which each player's payoff type corresponds to the state $\theta$. Complete information refers to the environment in which $\mu(\theta, s) = 0$ whenever $s \neq s^\theta$. This specification of the payoff type space and the common prior over it is without loss of generality as long as we only consider environments with complete information.

Given a game form $\Gamma = (M, g)$, an analyst is interested in the set of UNE

[4] All the results in the paper can be extended to hednic preferences. See Chung and Ely (2003) for its definition.

outcomes under complete information. [5] Here $M \equiv \times_{i \in N} M_i$, $M_i$ is player $i$'s pure action space and $g : M \to A$ is the outcome function. Following the literature in game theory, I can instead define $u_i : M \times \Theta \to \mathbb{R}$ and I take as given the von Neumann-Morgenstern representation associated with $u_i$. Since I am also concerned with the minimally necessary representation for preferences under uncertainty, the only part of players' preferences that is taken as exogenously given is their ordinal preferences over pure outcomes. This justifies, to some extent, that I take the basic structure of the environment ($A$, $\Theta$, and $\succeq^\theta$) given and thereafter consider a game form. Aumann and Brandenburger (1995) identified sufficient "epistemic" conditions for Nash equilibrium to result in strategic form games. [6] I take the position that these sufficient conditions are fulfilled in games. Later, I come back to one of the sufficient conditions, the common prior assumption. Dekel and Fudenberg (1990), for instance, provide an argument in favor of "one" round deletion of weakly dominated strategies by introducing small uncertainties about the payoffs. [7] In doing so, Dekel and Fudenberg applied the notion of iterated deletion of weakly dominated strategies as a solution concept to the nearby games. Combining with Proposition 7 of Fudenberg, Kreps, and Levine (1988), Dekel and Fudenberg concluded that "a refinement is robust if and only if it is contained in the closure (with respect to convergence in payoffs) of the set of Nash equilibria in the game remaining after weakly dominated strategies are deleted." [8] In sum, I take UNE as a reasonably robust solution concept for games with complete information.

Even when the analyst knows things about the structure of game $\Gamma$, he entertains the possibility that players face uncertainty about payoffs. Then, the planner has to take into account the set of "nearby" incomplete information structures in which the original complete information structure is subsumed. The analyst employs undominated Bayesian Nash equilibrium (UBNE) as a solution concept for those nearby incomplete information games.

[5] I conjecture that all the analyses can be generalized into games with incomplete information. This is expected to be done for my future work.

[6] Players' epistemic state dictates what they know or believe about the game and about each other's actions, knowledge, and beliefs.

[7] Within their class of perturbations of payoffs, Dekel and Fudenberg (1990) assume that each player is informed of his/her payoff functions. On the other hand, they allow for excessively rich perturbations of the payoffs of *other players*. In fact, each player is aware that any action of any other players' can be a dominant action under some perturbation. This perturbation corresponds to a sequence of the full support trembles in the refinement literature.

[8] Alternatively, Börgers (1994) showed that one round deletion of weakly dominated strategies can be justified in *any* situation where it is approximate common knowledge in the sense of Monderer and Samet (1989) that players do not use weakly dominated strategies. Note that there is a situation in which it cannot be exact common knowledge that players do not use weakly dominated strategies, while the common knowledge of rationality is simultaneously satisfied. The reader is refereed to Samuelson (1992) for the detail.

## 2.2 Definitions and Notations

Mathematically, the coarser the topology chosen, the larger the set of continuous correspondences with respect to it and therefore, the harder the achievement of robustness of a given equilibrium concept with respect to it. By the same token, the finer the topology chosen, the less it demands that the set of correspondences be continuous with respect to it. One of my objectives is to choose a coarsest possible topology with respect to which the UBNE correspondence associated with any game form is continuous. In order to talk about the topology, I explicitly expound the topological structure of the domain of the UBNE correspondence into which the players' belief structure is embedded. Let $(\Theta \times S, \mu)$ be a complete information structure. I shall construct the set of states of the world that is consistent with a given complete information environment. I denote by $\Omega$ the set of states of the world. I keep track of the complete information environment and let it be embedded in a measurable space $(\Omega, \Sigma)$ so that I am able to coherently discuss the epistemic states of all players when the environment is subject to incomplete information.

**Definition 1** *A payoff type space* $\Theta \times S$ ***is (algebraically) immersed in*** $\Omega$ *if there exists a one-to-one correspondence* $h : \Theta \times S \to \Omega$.

$h$ is said to be an immersion of $\Theta \times S$ into $\Omega$. I assume that $\Omega$ is a countable (possibly infinite) set. Let $(\Omega, \Sigma)$ be a measurable space.

**Definition 2** *Assume that* $\Theta \times S$ *is immersed in* $\Omega$*. A probability space* $(\Omega, \Sigma, P^*)$ *is a* ***consistent extension*** *from the complete information structure* $(\Theta \times S, \mu)$ *if* $(\Omega, \Sigma, P^*)$ *is equivalent to the probability space* $\left(\Theta \times S, 2^{\Theta \times S}, \mu\right)$*, where* $P^* \equiv \mu \circ h^{-1}$*. Moreover, a generic complete information prior is denoted as* $P^*$*.*

Let $(\Omega, \Sigma, P^*)$ be a probability space consistently extended from the complete information structure $(\Theta \times S, \mu)$. I fix a measurable space $(\Omega, \Sigma)$ throughout the argument while I change the probability distributions. In particular, I am interested in a net of probability distributions $\{P^k\}_{k=1}^{\infty}$ for which each $P^k$ is a perturbation of $P^*$ and more importantly, $P^k \to P^*$ as $k \to \infty$ in a certain property preserving way. Note that the way I introduce incomplete information into the analysis reveals that the common prior assumption (CPA) continues to be satisfied. One of the most important questions is whether the CPA is necessary for obtaining my main result (Theorem 1). [9] We enlarge on this issue.

As I already discussed in Section 2.1, we assume that Aumann and Brandenburger's (1995) sufficient conditions are fulfilled all the time. Their conditions include a common prior when the number of players is at least three. Since a Bayesian Nash equilibrium is a Nash equilibrium of the Bayesian game which is a game of complete information, we consider the CPA as a reasonable assumption to make the

[9] I am grateful to an anonymous referee for pointing this out.

use of Nash equilibrium appropriate. However, when the number of players is two, the common prior is not necessary to guarantee the appropriateness of Nash equilibrium. Relying on my example with two players, an anonymous referee constructed a nearby information structure without common prior which satisfies all sufficient conditions in my Theorem 1 but the UBNE correspondence exhibits a failure of upper hemi-continuity at any complete information prior. Thus, the CPA seems to be very crucial for my result.

Lipman (2003) characterizes the finite order implications of the CPA - that is, what restrictions it imposes for beliefs about the beliefs about... the beliefs of others, where "beliefs about" is repeated only a finite number of times. He show that in models with finite number of states, the only finite order implications of the CPA are those stemming from the weaker assumption that priors have a common support. [10] It turned out that the counterexample without common prior to my Theorem 1 does not satisfy the common support assumption. Therefore, I conclude that the CPA is crucial for my results to the extent that the common support assumption is indispensable, as long as I am concerned with finite models. In Section 2.4 after the proof of Theorem 1, I argue that there is essentially no loss of generality to assume that $\Omega$ is finite *if* $\Theta$ is finite. With this remark, the CPA is used throughout the paper.

Let $\mathscr{P}$ be the space of all probability distributions over $\Omega$. Let $\mathscr{F}$ be the space of all partitions of $\Omega$, the elements of which are in $\Sigma$. For each $\Pi \in \mathscr{F}$ and $\omega \in \Omega$, I denote by $\Pi(\omega)$ the element of $\Pi$ which contains $\omega$. An element of the form $\Pi = (\Pi_i)_{i \in N} \in \mathscr{F}^n$ is called a *partition structure*. I follow the results of Aumann (1976) here. I define $K_i(E) = \{\omega \in \Omega | \Pi_i(\omega) \subset E\}$ be the set of states in which player $i$ knows event $E$ obtains. I call $K_i : \Omega \to \Sigma$ player $i$'s knowledge operator. An event $E$ is said to be *self-evident* if $E \subset K_i(E)$ for each $i \in N$. This means that whenever $E$ is true, everyone knows $E$ is true. An event $E$ is *common knowledge* at $\omega$ if there exists a self-evident event $F$ such that $\omega \in F \subset \bigcap_{i \in N} K_i(E)$. Let $\Pi^* : \Omega \to \Sigma$ be the finest possibility correspondence that is coarser than $\Pi_i$ for each $i \in N$. Put differently, an event $E$ is common knowledge at $\omega$ if $\omega \in \Pi^*(\omega) \subset E$.

I take for granted Nash equilibrium as a reasonable solution under complete information. Therefore, players are assumed to choose their strategy *independently* of other players' choice provided the common knowledge of the payoffs is satisfied. To elaborate on this strategic independence, suppose, for example, that there are just two players $i$ and $j$, the choice of strategy by $i$ will depend on what $i$ believes $j$'s choice will be, which in turn will depend on what $i$ believes $j$ believes $i$'s choice will be and so on. An infinite regress of this kind underlies the idea of "rationalizable" strategies. In particular, Brandenburger and Dekel (1987) have established

[10] The common support assumption means that if some agent assigns zero probability to a certain event, then all other agents also assign zero probability to the same event.

an equivalence between the set of correlated rationalizable strategies and the set of subjective correlated equilibria. In games with complete information, this regress in beliefs can be generally "cut through" by the imposition of Nash equilibrium. If the analyst believes that the requirement of "being cut through" is too restrictive, his choice of Nash equilibrium as a solution is far from appropriate "*from the beginning of the analysis.*" Then, all the analyst has to do is either (1) to employ much less demanding solution concept, such as rationalizability, from the beginning or (2) to include this (possibly infinite) regress as an explicit correlation and/or communication and thereafter employ (Bayesian) Nash equilibrium in the appropriately extended (Bayesian) game. This paper takes the idea (2). The idea of (2) implies that the amount of incomplete information I allow for the robustness analysis should not contradict the use of solution concept in the beginning.

The formalization of this is summarized by the following measurability condition on possible strategy profiles. In doing so, I fix a partition structure $\Pi = (\Pi_i)_{i \in N} \in \mathscr{F}^n$ for the moment. Player $i$'s strategy in the game $\Gamma(P)$ is a function $\sigma_i : \Omega \to M_i$ which is $\Pi_i$-measurable. Let $\sigma$ be a strategy profile in a game $\Gamma(P)$. This construction of the strategy reveals that I focus only on *pure* strategies when defining Bayesian Nash equilibrium and dominations. [11]

**Definition 3** *Let a partition structure* $\Pi \in \mathscr{F}^n$ *and a game* $\Gamma(P)$ *associated with* $\Pi$ *be given. A strategy profile* $\sigma$ *is* ***consistent*** *with the complete information structure if, for any* $\omega, \omega' \in \Omega$*, whenever there exists a profile* $(\theta, s^\theta) \in \Theta \times S$ *such that* $h(\theta, s^\theta) = \tilde{\omega}$ *for any* $\tilde{\omega} \in \Pi^*(\omega) \cup \Pi^*(\omega')$*, then we have* $\sigma_i(\omega) = \sigma_i(\omega')$ *for each* $i \in N$*. When* $\sigma$ *is consistent with the complete information structure, we simply say that it is a consistent strategy profile.*

Remember that $s^\theta$ denotes the payoff type profile in which each player's payoff type corresponds to the state $\theta$. By the focus only on consistent strategy profiles, I exclude the correlation of equilibrium strategies which invalidates the original equilibrium analysis under complete information. This is my stance consistent with the use of Nash equilibrium for the original complete information game.

An act is a mapping $\alpha : \Omega \to A$. A belief is a probability distribution $\beta$ on $\Omega$. The notation $C(\beta)$ denotes the support of $\beta$. I assume that for any given belief $\beta$ each player $i$ has a preference relation $\succeq_i^\beta$ over acts. I also make the following two requirements about this family of preference relations under uncertainty [12]:

[11] In fact, mixed strategies can easily be incorporated into the definition of Bayesian Nash equilibrium. However, this incorporation can be a non-trivial task for dominations in order to obtain the same results we currently pursue.

[12] These two requirements are also used by Chung and Ely (2003).

**Assumption 1** *Let $\alpha$ and $\hat{\alpha}$ be two acts, and $\beta$ a belief. Then*

$$\left[\alpha(\omega) \succeq_i^{\theta} \hat{\alpha}(\omega) \ \ \forall \omega \in C(\beta)\right] \Longrightarrow \alpha \succeq_i^{\beta} \hat{\alpha},$$

*where $h(\theta, s) = \omega$ for some $(\theta, s) \in \Theta \times S$, and if one of the preferences on the left-hand-side is strict, then the preference on the right-hand-side is strict.*

The following is a continuity property for preferences over lotteries:

**Assumption 2** *For every pair of acts $\alpha$ and $\hat{\alpha}$, the set $\left\{\beta \middle| \ \alpha \succ_i^{\beta} \hat{\alpha}\right\}$ is open relative to the weak topology.*

A characterization of the type of preferences under uncertainty assumed here can be obtained from Theorem 1 of Myerson (1979). I need a variant of the substitution axiom to obtain the subjective expected utility representation. Hence, what I need here is something weaker than Anscombe and Aumann's (1963) type of subjective expected utility representation. Because I take ordinal preferences over pure outcomes as given, it is important to know what is the minimally needed representation for preferences under uncertainty as far as my robustness analysis is concerned. The act $\alpha_{\sigma}^{\Gamma}$ induced by $\sigma$ under $\Gamma$ is defined by $\alpha_{\sigma}^{\Gamma}(\omega) = g(\sigma(\omega))$ for any $\omega \in \Omega$. With these notations, I shall define Nash equilibrium (NE) and Bayesian Nash equilibrium (BNE), respectively.

**Definition 4** *Let a partition structure $\Pi = \times_{i \in N} \Pi_i \in \mathscr{F}^n$ and a game $\Gamma(P)$ associated with $\Pi$ be given. A consistent strategy profile $\sigma$ is a **Bayesian Nash equilibrium** (BNE) of $\Gamma(P)$ if $\sigma_i$ is $\Pi_i$-measurable for each $i \in N$, and for each $i \in N$, state $\omega$ with $P(\Pi_i(\omega)) > 0$, and strategy $\sigma_i^{'}$ which is $\Pi_i$-measurable, we have $\alpha_{\sigma}^{\Gamma} \succeq_i^{P(\cdot|\Pi_i(\omega))} \alpha_{\sigma_i^{'}, \sigma_{-i}}^{\Gamma}$.*

The above definition suffices for $\sigma$ to be a Nash equilibrium if $P = P^*$. Because $\sigma$ is consistent. This paper studies undominated Nash equilibrium (UNE) and undominated Bayesian Nash equilibrium (UBNE) in a given game in which no player uses a dominated strategy. The following is a definition of *interim* weak dominance for this setting. [13]

**Definition 5** *Let a partition structure $\Pi = \times_{i \in N} \Pi_i \in \mathscr{F}^n$ and a game $\Gamma(P)$ associated with $\Pi$ be given. A strategy $\sigma_i$ which is $\Pi_i$-measurable is **dominated** for some $\omega$ with $P(\Pi_i(\omega)) > 0$ if there exists a strategy $\sigma_i^{'}$ which is $\Pi_i$-measurable such that for every strategy profile $\sigma_{-i}$ for which $\sigma_j$ is $\Pi_j$-measurable for $j \neq i$, $\alpha_{\sigma_i^{'}, \sigma_{-i}}^{\Gamma} \succeq_i^{P(\cdot|\Pi_i(\omega))} \alpha_{\sigma}^{\Gamma}$ with strict preference for at least one $\sigma_{-i}$. A strategy $\sigma_i$ is **undominated** if it is not dominated for any $\omega$ with $P(\Pi_i(\omega)) > 0$.*

[13] As I mentioned when defining strategy, I define dominations in terms of *pure* strategies. To my knowledge, there is one paper by Börgers (1993) which advocates the idea that dominations should be defined in terms of pure strategies.

If I choose a generic payoff structure in a strategic game form, weak dominations are always strong dominations. My rational for taking weak dominance seriously is summarized as follows: I take an extensive form game as a primitive, reduce it into an equivalent strategic game form, and then ask the effects of weak dominations in the reduced strategic game. Furthermore, in the context of implementation theory, a game form is the designer's construction so that weak dominations are often explicitly utilized. Finally I shall define UNE and UBNE, respectively.

**Definition 6** *Let a partition structure* $\Pi = \times_{i \in N} \Pi_i \in \mathscr{F}^n$ *and a game* $\Gamma(P)$ *associated with* $\Pi$ *be given. A consistent strategy profile* $\sigma$ *is an* ***undominated Bayesian Nash equilibrium*** *(UBNE) of* $\Gamma(P)$ *if it is a Bayesian Nash equilibrium (BNE) of* $\Gamma(P)$ *for which* $\sigma_i$ *is undominated for each* $i \in N$.

The same definition suffices for UNE if $P = P^*$. The set of acts is denoted as $\mathscr{A} \equiv A^{\Omega}$. I define $\psi_{\Gamma}^{UBNE} : \mathscr{P} \to \mathscr{A}$ as the UBNE correspondence associated with the game form $\Gamma$. Here, I endow $\mathscr{A}$ with product topology. I shall introduce a topology which enables us to determine how close any two probability distributions are. To define such topologies, I need some definitions and notations.

Monderer and Samet (1989) introduced the concept of "*common p-belief*" as an approximation to common knowledge, which is common 1-belief. Let $B_i^q(E) \equiv \{\omega \in \Omega | P(E|\Pi_i(\omega)) \geq q\}$ denote the set of states in which player $i$ assigns probability at least $q$ to the event $E$. I call this player $i$'s *q-belief operator*. In particular, when $q = 1$, I call $B_i^1$ player $i$'s 1-*belief operator* corresponding to player $i$'s knowledge operator. [14] An event $E$ is said to be *q-evident* if $E \subset B_i^q(E)$ for all $i \in N$. This means that whenever $E$ is true, everyone believes with probability at least $q$ that $E$ is true. An event $E$ is said to be *common q-belief* at $\omega$ if there exists a $q$-evident event $F$ such that $\omega \in F \subset \bigcap_{i \in N} B_i^q(E)$. I will loosely say that an event $E$ is *approximate common knowledge* at $\omega$ if $E$ is common $q$-belief at $\omega$, for $q$ close to 1. I consider the notion of the closeness of probability distributions. Define $d_0$ by the rule

$$d_0(P, P') = \sup_{E \subset \Omega} |P(E) - P'(E)|.$$

Note that $d_0(P, P') = 0$ if and only if $P = P'$. Let $P^*$ be the complete information prior. I will require extra conditions on conditional probabilities. Define as $\mathscr{G}(\eta)$ the set of all states in which there is a common $(1-\eta)$-belief about what game being played as follows:

$$\mathscr{G}(\eta) = \left\{ \omega \in \Omega \middle| \ \exists \theta \in \Theta \text{ such that } \Gamma(\theta) \text{ is common } (1-\eta)\text{-belief at } \omega \right\}.$$

Let

$$\begin{aligned} d_1(P) &= \inf\{\eta \mid P(\mathscr{G}(\eta)) = 1\}, \text{ and} \\ d^*(P, P') &= \max\{d_0(P, P'), d_1(P), d_1(P')\}. \end{aligned}$$

[14] Here I define "knowledge" as belief with probability 1.

Note that $d_1(P^*) = 0$ by definition. By construction, $d^*$ is non-negative and symmetric, and $d^*(P, P^{'}) = 0$ if and only if $P = P^{'}$ and both $P$ and $P^{'}$ are complete information priors. I find it very convenient to define a topology by specifying what nets converge to which points. I use Theorem 9 of Kelly (1955) (in p74 ) which shows that every convergence class is actually derived from a topology. Then, it remains to prove that any convergent net according to $d^*$ belongs to some convergence class.

**Definition 7** *Let $\mathscr{C}$ be a class consisting of pairs $(S, s)$, where $S$ is a net in $X$ and $s$ a point. $\mathscr{C}$ is a convergence class for $X$ if it satisfies the conditions listed below. We say that $S$ converges $(\mathscr{C})$ to $s$ or that $\lim_k S_k \equiv s$ $(\mathscr{C})$ if and only if $(S, s) \in \mathscr{C}$.*

1. *If $S$ is a net such that $S_n = s$ for each $n$, then $S$ converges $(\mathscr{C})$ to $s$.*
2. *If $S$ converges $(\mathscr{C})$ to $s$, then so does each subnet of $S$.*
3. *If $S$ does not converge $(\mathscr{C})$ to $s$, then there is a subnet of $S$, no subnet of which converges $(\mathscr{C})$ to $s$.*
4. *Let $D$ be a directed set, let $E_m$ be a directed set for each $m \in D$, let $F$ be the product $D \times \prod_{m \in D} E_m$ and for $(m, f) \in F$, let $R(m, f) = (m, f(m))$. If $\lim_m \lim_n S(m, n) \equiv s$ $(\mathscr{C})$, then $S \circ R$ converges $(\mathscr{C})$ to $s$. Here, $S(m, n)$ is a member of a topological space for each $m$ in $D$ and each $n$ in $E_m$.*

Let $\mathscr{C}^*$ be a class consisting of all pairs of a net $\{P^k\}_{k=1}^{\infty}$ with $d^*(P^k, P^*) \to 0$ as $k \to \infty$ for some complete information prior $P^*$ and a complete information prior.

**Proposition 1** *Let $\mathscr{C}^*$ be a class given above. Then, $\mathscr{C}^*$ is a convergence class for $\mathscr{P}$, where $\mathscr{P}$ is the space of all probability distributions over $\Omega$.*

**Proof of Proposition 1**: We must check four properties for the convergence class. Let $P^*$ be a complete information prior. Set $\{P^k\}_{k=1}^{\infty}$ as $P^k = P^*$ for each $k$. Then, we have that $d^*(P^k, P^*) = 0$ for each $k$, therefore, $(\{P^k\}, P^*) \in \mathscr{C}^*$. Thus, $\mathscr{C}^*$ satisfies property 1. Let $P^k \to P^*$ and $d^*(P^k, P^*) \to 0$ as $k \to \infty$ for some complete information prior $P^*$, that is, $P^k \to P^*$ $(\mathscr{C}^*)$. It is straightforward to see that any subnet of $\{P^k\}$ also converges to $P^*$ $(\mathscr{C}^*)$. Hence, property 2 is satisfied for $\mathscr{C}^*$. Suppose that $P^k$ does not converge to $P^*$ as $k \to \infty$ according to $\mathscr{C}^*$. Then, there exists $\delta > 0$ for which there exists $\bar{k}$ such that $d^*(P^k, P^*) \geq \delta$ for each $k \geq \bar{k}$. Consider a subnet $\{P^l\}_{l=1}^{\infty} \equiv \{P^k\}_{k=\bar{k}}^{\infty}$. By construction, there is $\delta > 0$ such that $d^*(P^l, P^*) \geq \delta$ for each $l$. Then, it is straightforward to see that no subnet of $\{P^l\}_{l=1}^{\infty}$ converges to $P^*$ according to $\mathscr{C}^*$. Thus, property 3 is satisfied for $\mathscr{C}^*$. Let a double indexed net $S(k, l) \equiv \{\{P^{k_l}\}_{l=1}^{\infty}\}_{k=1}^{\infty}$. Now we know by our hypothesis that for any $\varepsilon > 0$, there exist $\bar{k}$ and $\bar{l}$ such that $d^*(P^{k_l}, P^*) < \varepsilon$ for any $k \geq \bar{k}$ and any $l \geq \bar{l}$. Then, in order to check if property 4 is satisfied, it remains to show that, for any $\varepsilon$, we are able to find a member $(k, f) \in F$ such that, if $(n, g) \geq (k, f)$, then $d^*(P^{n_{g(n)}}, P^*) < \varepsilon$. By our hypothesis we can choose $k \in D$ so that $d^*(\lim_l P^{n_l}, P^*) < \varepsilon$ for any $n \geq k$.

For each such $n$, choose a member $f(n) \in E_n$ such that $d^*(P^{n_l}, P^*) < \varepsilon$ for any $l \geq f(n)$. If $n$ is a member of $D$ which does not follow $k$, let $f(n)$ be an arbitrary member of $E_n$. If $(n, g) \geq (k, f)$, then $n \geq k$, hence $d^*(P^{n_l}, P^*) < \varepsilon$, and since $g(n) \geq f(n)$, we have that $d^*(P^{n_{g(n)}}, P^*) < \varepsilon$. Thus, property 4 is satisfied. ■

**Theorem 9 of Kelley (1955)**: Let $\mathscr{C}$ be a convergence class for a set $X$, and for each subset $A$ of $X$, let $A^c$ be the set of all points $s$ such that, for some net $S$ in $A$, $S$ converges ($\mathscr{C}$) to $s$. Then $^c$ is a closure operator, and $(S, s) \in \mathscr{C}$ if and only if $S$ converges to $s$ relative to the topology associated with $^c$.

Note that $d^*(P^k, P^*) \to 0$ as $k \to \infty$ if and only if $P^k \to P^*$ as $k \to \infty$ and there exists $\varepsilon^k \to 0$ such that $P^k(\mathscr{G}(\varepsilon^k)) = 1$ for each $k$. The topology induced by $d^*$ is associated with a closure operator which is applied only to the set of complete information priors. We shall define the upper hemi-continuity of the UBNE correspondence with respect to the topology induced by $d^*$.

**Definition 8** *Let $\Gamma$ be a game form. $\psi_\Gamma^{UBNE}$ is* ***upper hemi-continuous*** *at a complete information prior $P^*$ with respect to the topology induced by $d^*$ if, $\psi_\Gamma^{UBNE}(P^k) \to \psi_\Gamma^{UNE}(P^*)$ as $k \to \infty$ whenever $d^*(P^k, P^*) \to 0$ as $k \to \infty$. Here $\psi_\Gamma^{UNE}(P^*)$ denotes the set of UNE outcomes of the game $\Gamma(P^*)$.*

## 2.3 The Main Theorem

I shall show that the topology induced by $d^*$ guarantees the upper hemi-continuity of the UBNE correspondence associated with *any* game form at any complete information prior.

**Theorem 1** *Suppose that preferences are strict. Let $\Gamma$ be a game form. Then, the UBNE correspondence associated with $\Gamma$, $\psi_\Gamma^{UBNE}$ is upper hemi-continuous at any complete information prior with respect to the topology induced by $d^*$.*

**Proof of Theorem 1**: Let $\{P^k\}_{k=1}^\infty$ be a net of probability distributions converging to a complete information prior $P^*$ with the property that there exists the corresponding sequence $\{q^k\}_{k=1}^\infty$ converging to 1 for which there is a common $q^k$-belief at any $\omega \in \Omega$ with $P^k(\omega) > 0$ about which game being played for each $k$. Thus, $\{P^k\}_{k=1}^\infty$ is a convergent net according to $d^*$. We have the corresponding sequence of partition structures, $\{\Pi^k\}_{k=1}^\infty$ along the sequence. If $\Omega$ is finite, we can always decompose the sequence $\{\Pi^k\}_{k=1}^\infty$ into the finite set of subsequences for each of which the corresponding partition structure is fixed. After the proof of this theorem, we will establish a result (Proposition 2) saying that there is no loss of generality to assume that $\Omega$ is finite as long as $\Theta$ is finite. We focus on such an arbitrary subsequence of $\{\Pi^k\}_{k=1}^\infty$ and call it $\Pi$. We fix this $\Pi$ throughout the argument. Let $\{\sigma^k\}_{k=1}^\infty$ be the corresponding sequence of UBNE strategy profiles such that $\sigma_i^k$ is $\Pi_i$-measurable for any $k \in \mathbb{N}$ and any $i \in N$. Suppose, by way of contradiction, that the UBNE correspondence associated with $\Gamma$ is *not* upper hemi-continuous at the complete information prior $P^*$ with respect to the topology induced by $d^*$. Thus, we assume that $\sigma^k \to \sigma$ as $k \to \infty$ in terms of outcomes with product topology. Then, there exists $\tilde{\omega} \in \Omega$ such that $h(\theta, s^\theta) = \tilde{\omega}$ and $\sigma^*(\tilde{\omega})$ with $P^k(\tilde{\omega}) > 0$ for any $k$ big enough. Remember that $s^\theta$ denotes the payoff type profile in which each player's payoff type corresponds to the state $\theta$.

Let $E_\theta^k$ be the maximal set of states in which it is a common $q^k$-belief that the game $\Gamma(\theta)$ is to be played. If $\Omega$ is finite, there exists $\bar{K} \in \mathbb{N}$ and $E_\theta$ such that $E_\theta^k = E_\theta$ for any $k \geq \bar{K}$. Therefore, we can, without loss of generality, assume that $E_\theta^k = E_\theta$ for $k$ big enough. First, we show the following claim. We aim at showing that $E_\theta$ is the largest possible neighborhood that ought to be considered when we talk about interim dominations and equilibrium of strategies in nearby incomplete information games.

**Claim 1** $\Pi_j(\omega) \subset E_\theta$ *for any* $j \in N$ *and any* $\omega \in E_\theta$.

**Proof of Claim 1**: Suppose not. That is, there exist $\omega \in E_\theta$ and $j \in N$ for which there exists $\tilde{\omega} \in \Omega$ with $P^k(\tilde{\omega}) > 0$ for $k$ big enough such that $\tilde{\omega} \in \Pi_j(\omega)$ but $\tilde{\omega} \notin E_\theta$. Since $\Pi_j$ is partitional and $\tilde{\omega} \in \Pi_j(\omega)$, we have that $\Pi_j(\omega) = \Pi_j(\tilde{\omega})$. Thus, player $j$ cannot distinguish between $\omega$ and $\tilde{\omega}$. Because $\omega \in E_\theta$, player $j$ believes at $\omega$ with high probability that the game $\Gamma(\theta)$ being played. Since $\Pi_j(\omega) = \Pi_j(\tilde{\omega})$, player $j$, at state $\tilde{\omega}$, must also believe with high probability that the game $\Gamma(\theta)$ is to be played. Recall that we require that there be a common $q^k$-belief at $\tilde{\omega}$ about what game being played for each $k$. This implies that there must be a common $q^k$-belief at $\tilde{\omega}$ about the game $\Gamma(\theta)$ being played for each $k$. Due to the maximality of $E_\theta$, we conclude that $\tilde{\omega} \in E_\theta$, which is a contradiction. ■

We must consider two cases: (1) $m^*$ is a weakly dominated strategy profile of the game $\Gamma(\theta)$; and (2) $g(m^*)$ is a non-Nash equilibrium outcome of the game $\Gamma(\theta)$.

## (1) $m^*$ is a weakly dominated strategy profile of $\Gamma(\theta)$

By our hypothesis, there exists a nonempty subset of players $I \subseteq N$ such that for any $i \in I$, there exists $m_i^{'}$ with the following two properties:

- $\alpha^\Gamma_{m_i^{'}, \tilde{m}_{-i}} \succeq_i^\theta \alpha^\Gamma_{m_i^*, \tilde{m}_{-i}}$ for any $\tilde{m}_{-i} \in M_{-i}$;
- $\alpha^\Gamma_{m_i^{'}, \hat{m}_{-i}} \succ_i^\theta \alpha^\Gamma_{m_i^*, \hat{m}_{-i}}$ for some $\hat{m}_{-i} \in M_{-i}$.

Choose one player $i \in I$. We construct a sequence of player $i$'s strategies $\{\hat{\sigma}_i^k\}_{k=1}^\infty$ such that for any $k \in \mathbb{N}$, $\hat{\sigma}_i^k$ is $\Pi_i$-measurable with the following properties:

- $\hat{\sigma}_i^k(\tilde{\omega}) = m_i^{'}$ for any $\tilde{\omega} \in \Pi_i(\omega)$ whenever $\omega \in E_\theta$;
- $\hat{\sigma}_i^k(\tilde{\omega}) = \sigma_i^k(\tilde{\omega})$ for any $\tilde{\omega} \notin \Pi_i(\omega)$ whenever $\omega \in E_\theta$;
- $\hat{\sigma}_i^k(\tilde{\omega}) = \sigma_i^k(\tilde{\omega})$ for any $\tilde{\omega} \notin E_\theta$.

We shall show the following lemma.

**Lemma 1** *There exists $\bar{K} \in \mathbb{N}$ such that for any $k \geq \bar{K}$, $\hat{\sigma}_i^k$ dominates $\sigma_i^k$ in the neighborhood $E_\theta$. That is, the following two conditions hold:*

1. $\alpha^\Gamma_{\hat{\sigma}_i^k, \tilde{\sigma}_{-i}} \succeq_i^{P^k(\cdot|\Pi_i(\omega))} \alpha^\Gamma_{\sigma_i^k, \tilde{\sigma}_{-i}}$ *for any $\tilde{\sigma}_{-i}$ and any $\omega \in E_\theta$;*
2. $\alpha^\Gamma_{\hat{\sigma}_i^k, \tilde{\sigma}_{-i}} \succ_i^{P^k(\cdot|\Pi_i(\omega))} \alpha^\Gamma_{\sigma_i^k, \tilde{\sigma}_{-i}}$ *for some $\tilde{\sigma}_{-i}$ and some $\omega \in E_\theta$.*

By Lemma 1 we aim at showing that $\sigma^k$ is not an UBNE of the game $\Gamma(P^k)$ for any $k \geq \bar{K}$, which contradicts the hypothesis that $\{\sigma^k\}_{k=1}^\infty$ is a sequence of UBNE strategy profiles.

**Proof of Lemma 1**: Since $\sigma_i^k \to \sigma_i$ as $k \to \infty$ in terms of outcomes with product topology, we know that $\sigma_i^k(\omega) = m_i^*$ for $k$ big enough if $\omega \in E_\theta$. Set $\tilde{\sigma}_{-i}(\omega) = \hat{m}_{-i}$ for any $\omega \in E_\theta$. By Assumption 1, condition (2) in Lemma 1 is satisfied. Hence, that condition (1) in Lemma 1 is also satisfied remains to be checked. To continue the argument, we need the following ordering over all players other than $i$.

**Claim 2** *Assume that $\Pi_i(\omega) = E_\theta$ for any $\omega \in E_\theta$. For any $\omega \in E_\theta$ and any permutation $\tau : N \setminus \{i\} \to N \setminus \{i\}$, we can construct an order $\{j_{\tau(1)}, \cdots, j_{\tau(n-1)}\}$ over $N \setminus \{i\}$.*

**Proof of Claim 2**: Fix any $\omega \in E_\theta$. By Claim 1 and the assumption that $\Pi_i(\omega) = E_\theta$, we have that $\Pi_i(\omega) \supseteq \Pi_j(\omega)$ for any $j \neq i$. If we define $\Pi_{j_h}(\omega) \equiv \bigcap_{j=\tau(1)}^{\tau(h)} \Pi_j(\omega)$, by construction, $\Pi_{j_h}(\omega)$ is monotonically non-increasing in $h$ for any permutation $\tau$. ■

We argue that there is no loss of generality to assume that $\Pi_i(\omega) = E_\theta$ for any $\omega \in E_\theta$. Suppose not. That is, suppose, by Claim 1, there exists $\omega \in E_\theta$ such that $\Pi_i(\omega) \subsetneqq E_\theta$. Then we can eliminate the set of agents $J$ from $N \backslash \{i\}$ for whom $\Pi_i(\omega) \subsetneqq \Pi_j(\omega)$ for $j \in J$. When we employ $\tilde{E}_\theta = \Pi_i(\omega)$ as a smaller neighborhood in Lemma 1, instead of $E_\theta$, the same argument goes through. Besides, we can make the identical argument in Claim 2 after eliminating $J$ from our consideration. Hence we may assume that $\Pi_i(\omega) = E_\theta$ for any $\omega \in E_\theta$.

Fix any permutation $\tau$. Note that, for $k$ big enough, it is common $q^k$-belief at any $\omega \in E_\theta$ that the game $\Gamma(\theta)$ is to be played. Fix any such $\omega \in E_\theta$. Then, for $\varepsilon^0 > 0$

small enough and $k$ big enough, there exists a minimal set of states $E^0 \subseteq \Pi_i(\omega)$ such that the game $\Gamma(\theta)$ is to be played at any $\tilde{\omega} \in E^0$ with the property that:

- $P^k\left(E^0 \middle| \Pi_i(\omega)\right) \geq 1 - \varepsilon^0$;
- $P^k\left(\Pi_i(\omega) \backslash E^0 \middle| \Pi_i(\omega)\right) \leq \varepsilon^0$.

Since preferences over pure outcomes are strict, we use the following important fact repeatedly:

**Fact 1** *For any $\tilde{m}_{-i} \in M_{-i}$, the following two must hold:*

1. $g(m_i^{'}, \tilde{m}_{-i}) \neq g(m_i^{'}, \tilde{m}_{-i}) \Longrightarrow g(m_i^{'}, \tilde{m}_{-i}) \succ_i^{\theta} g(m_i^*, \tilde{m}_{-i})$;
2. $g(m_i^{'}, \tilde{m}_{-i}) = g(m_i^{'}, \tilde{m}_{-i}) \Longrightarrow g(m_i^{'}, \tilde{m}_{-i}) \sim_i^{\tilde{\theta}} g(m_i^*, \tilde{m}_{-i})$ *for any* $\tilde{\theta}$.

Observe that players other than $i$ may be conditioning their behavior upon information not available to player $i$. In that case, other players' behavior appears random to player $i$. Thus, we must ensure that $\hat{\sigma}_i^k$ is undominated against such possibly correlated strategy profiles. If $\omega \in E^0$, by Assumption 1 and Fact 1, we have that, for any $\tilde{\sigma}_{-i}$ and $k$ big enough,

$$g\left(\hat{\sigma}_i^k(\omega), \tilde{\sigma}_{-i}(\omega)\right) \succeq_i^{P^k(\cdot|\Pi_i(\omega))} g\left(\sigma_i^k(\omega), \tilde{\sigma}_{-i}(\omega)\right).$$

That is, $\hat{\sigma}_i^k$ dominates $\sigma_i^k$ for $k$ big enough. Because we already set $\hat{\sigma}_i^k(\omega) = m_i^{'}$ for any $k$ by construction and $\sigma_i^k(\omega) = m_i^*$ for $k$ big enough. If $\omega \notin E^0$, then we have that $\omega \in \Pi_i(\omega) \backslash E^0$. By Claim 2, we can choose the smallest integer $h$ with $1 \leq h \leq n-1$ such that $\Pi_i(\omega) \backslash E^0 \supseteq \Pi_{\tau(h)}(\omega)$.

If there is no such $h$, we have that, for any $h$, there exists $\tilde{\omega} \in E^0$ such that $\tilde{\omega} \in \Pi_{\tau(h)}(\omega)$. Suppose that player $i$ hypothesizes $\omega \notin E^0$ despite the fact that he cannot distinguish between $\omega \in E^0$ and $\omega \notin E^0$. Then, due to the minimality of $E^0$, and conditional on his hypothesis that $\omega \notin E^0$, player $i$ still knows that all other players believe with high probability that $E^0$ obtains. In other words, conditional on his hypothesis, player $i$ knows that all other players believe with high probability that the game $\Gamma(\theta)$ is to be played. Then, by Assumption 2, Fact 1, and consistency of strategy profiles, $\hat{\sigma}_i^k$ dominates $\sigma_i^k$ for $k$ big enough.

Assume that there is such an $h$. Since it is common $q^k$-belief at any $\omega \in E_\theta$ for $k$ big enough that the game $\Gamma(\theta)$ is to be played, player $\tau(h)$ believes with high probability that the game $\Gamma(\theta)$ is to be played. Then, for $\varepsilon^h > 0$ small enough and $k$ big enough, there exists a minimal set of states $E^h \subseteq \Pi_{\tau(h)}(\omega)$ such that the game $\Gamma(\theta)$ is to be played at any $\tilde{\omega} \in E^h$ with the property that:

- $P^k\left(E^h \middle| \Pi_{\tau(h)}(\omega)\right) \geq 1 - \varepsilon^h$;
- $P^k\left(\Pi_{\tau(h)}(\omega) \backslash E^h \middle| \Pi_{\tau(h)}(\omega)\right) \leq \varepsilon^h$.

Recall that players $\tau(h)$ through $\tau(n-1)$ may be conditioning their behavior upon information not available to player $i$. In that case, their behavior appears random to players $i, \tau(1), \ldots, \tau(h-1)$. Again, we must ensure that $\hat{\sigma}_i^k$ is undominated against such correlated strategy profiles. Since preferences are strict, if $\omega \in E^h$, by Assumption 1 and Fact 1, we have that, for any $\tilde{\sigma}_{-i}$ and $k$ big enough,

$$g\left(\hat{\sigma}_i^k(\omega), \tilde{\sigma}_{-i}(\omega)\right) \succeq_i^{P^k(\cdot|\Pi_{\tau(h)}(\omega))} g\left(\sigma_i^k(\omega), \tilde{\sigma}_{-i}(\omega)\right).$$

That is, even conditional upon the hypothesis that $\omega \in E^h$ but $\omega \notin E^0$, $\hat{\sigma}_i^k$ dominates $\sigma_i^k$ for $k$ big enough. Because we already set $\hat{\sigma}_i^k(\omega) = m_i^{'}$ for any $k$ by construction and $\sigma_i^k(\omega) = m_i^*$ for $k$ big enough. If $\omega \notin E^h$, then we have that $\omega \in \Pi_{\tau(h)}(\omega) \backslash E^h$. By Claim 2, we can choose the smallest integer $h^{'}$ with $h < h^{'} \leq n-1$ such that $\Pi_{\tau(h)}(\omega) \backslash E^h \supseteq \Pi_{\tau(h^{'})}(\omega)$.

If there is no such $h^{'}$, we have that, for any $h^{'}$ with $h < h^{'} \leq n-1$, there exists $\tilde{\omega} \in E^h$ such that $\tilde{\omega} \in \Pi_{\tau(h^{'})}(\omega)$. Suppose that player $i$ hypothesizes $\omega \in \Pi_{\tau(h)}(\omega) \backslash E^h$ although he cannot distinguish between $\omega \in E^h$ and $\omega \in \Pi_{\tau(h)}(\omega) \backslash E^h$. Then, due to the minimality of $E^h$, and conditional upon his hypothesis that $\omega \in \Pi_{\tau(h)}(\omega) \backslash E^h$, player $i$ knows that all players $\tau(1)$ through $\tau(h-1)$ believe with high probability that $E^0$ obtains and that all players $\tau(h)$ through $\tau(n-1)$ believe with high probability that $E^h$ obtains. Accordingly, conditional on his hypothesis, player $i$ still knows that all other players believe with high probability that the game $\Gamma(\theta)$ is to be played. Therefore, by Assumption 2, Fact 1, and consistency of strategy profiles, $\hat{\sigma}_i^k$ dominates $\sigma_i^k$ for $k$ big enough.

Assume that there is such an $h^{'}$. Since it is common $q^k$-belief at any $\omega \in E_\theta$ for $k$ big enough that the game $\Gamma(\theta)$ is to be played, player $\tau(h^{'})$ believes with high probability that the game $\Gamma(\theta)$ is to be played. Then, for $\varepsilon^{h^{'}} > 0$ small enough and $k$ big enough, there exists a minimal set of states $E^{h^{'}} \subseteq \Pi_{\tau(h^{'})}(\omega)$ such that the game $\Gamma(\theta)$ is to be played at any $\tilde{\omega} \in E^{h^{'}}$ with the property that:

- $P^k\left(E^{h^{'}} \middle| \Pi_{\tau(h^{'})}(\omega)\right) \geq 1 - \varepsilon^{h^{'}}$;
- $P^k\left(\Pi_{\tau(h^{'})}(\omega) \backslash E^{h^{'}} \middle| \Pi_{\tau(h^{'})}(\omega)\right) \leq \varepsilon^{h^{'}}$.

Here the identical argument goes through. That is, if $\omega \in E^{h^{'}}$, by Assumption 1 and Fact 1, $\hat{\sigma}_i^k$ dominates $\sigma_i^k$ for $k$ big enough. If not, the iteration of the same type of the above argument can be carried out inductively with respect to the number of players and therefore, it must stop in finite steps. Furthermore, the argument does not depend upon $\tau$ or $\omega$. Hence, we show that, for $k$ big enough, $\hat{\sigma}_i^k$ dominates $\sigma_i^k$ for any $\omega \in E_\theta$. We complete the proof of Lemma 1. ■

### (2) $g(m^*)$ is a non-Nash equilibrium outcome of $\Gamma(\theta)$

By our hypothesis, there exist a player $i$ and a message $m_i^{'}$ with the following property:

$$\alpha^{\Gamma}_{m_i^{'}, m_{-i}^*} \succ_i^{\theta} \alpha^{\Gamma}_{m^*}.$$

Let $E_\theta^k$ be the maximal set of states in which it is common $q^k$-belief that the game $\Gamma(\theta)$ is to be played. If $\Omega$ is finite, $E_\theta^k = E_\theta$ for $k$ big enough. We construct a sequence of player $i$'s strategies of $\{\hat{\sigma}_i^k\}_{k=1}^{\infty}$ such that for any $k \in \mathbb{N}$, $\hat{\sigma}_i^k$ is $\Pi_i$-measurable with the following properties:

- $\hat{\sigma}_i^k(\tilde{\omega}) = m_i^{'}$ for any $\tilde{\omega} \in \Pi_i(\omega)$ whenever $\omega \in E_\theta$;
- $\hat{\sigma}_i^k(\tilde{\omega}) = \sigma_i^k(\tilde{\omega})$ for any $\tilde{\omega} \notin \Pi_i(\omega)$ whenever $\omega \in E_\theta$;
- $\hat{\sigma}_i^k(\tilde{\omega}) = \sigma_i^k(\tilde{\omega})$ for any $\tilde{\omega} \notin E_\theta$.

As we argued in the previous section, we may, without loss of generality, assume that $\Pi_i(\omega) = E_\theta$ for any $\omega \in E_\theta$. In this case, we shall show the following lemma.

**Lemma 2** *There exists $\bar{K} \in \mathbb{N}$ such that for any $k \geq \bar{K}$ and any $\omega \in E_\theta$, we have*

$$\alpha^{\Gamma}_{\hat{\sigma}_i^k, \sigma_{-i}^k} \succ_i^{P^k(\cdot|\Pi_i(\omega))} \alpha^{\Gamma}_{\sigma_i^k, \sigma_{-i}^k}.$$

By Lemma 2 we aim to show that $\hat{\sigma}_i^k$ is a strictly better reply to $\sigma_{-i}^k$ than $\sigma_i^k$ for any $k \geq \bar{K}$ and any $\omega \in E_\theta$. Thus, this contradicts the hypothesis that $\{\sigma^k\}_{k=1}^{\infty}$ is a sequence of UBNE strategy profiles.

**Proof of Lemma 2**: Note that it is common $q^k$-belief at any $\omega \in E_\theta$ for $k$ big enough that the game $\Gamma(\theta)$ is to be played. Fix any permutation $\tau$ and any such $\omega \in E_\theta$. Then, for $\varepsilon^0 > 0$ small enough and $k$ big enough, there exists a minimal set $E^0 \subseteq \Pi_i(\omega)$ such that the game $\Gamma(\theta)$ is to be played at any $\tilde{\omega} \in E^0$ with the property that:

- $P^k\left(E^0 \middle| \Pi_i(\omega)\right) \geq 1 - \varepsilon^0$;
- $P^k\left(\Pi_i(\omega) \backslash E^0 \middle| \Pi_i(\omega)\right) \leq \varepsilon^0$.

We know that $\hat{\sigma}_i^k(\omega) = m_i^{'}$ for any $k$ by construction, and $\sigma_i^k(\omega) = m_i^*$ for $k$ big enough. Observe that players other than $i$ may be conditioning their behavior upon information not available to player $i$. In that case, other players' behavior appears random to player $i$. Conditional on $\omega \in E^0$, player $i$ knows that all other players believe with high probability that the game $\Gamma(\theta)$ is to be played. Therefore, by consistency of strategy profiles $\{\sigma^k\}_{k=1}^{\infty}$, player $i$ believes with high probability that $\sigma_{-i}^k(\omega) = m_{-i}^*$ for $k$ big enough. By Assumption 1 and 2, we have that, for $k$ big enough,

$$\alpha^{\Gamma}_{\hat{\sigma}_i^k, \sigma_{-i}^k} \succ_i^{P^k(\cdot|\Pi_i(\omega))} \alpha^{\Gamma}_{\sigma^k}.$$

That is, $\hat{\sigma}_i^k$ is a strictly better reply to $\sigma_{-i}^k$ than $\sigma_i^k$ if $\omega \in E^0$. If $\omega \notin E^0$, then we have that $\omega \in \Pi_i(\omega) \backslash E^0$. By Claim 2, we can choose the smallest integer $h$ with $1 \leq h \leq n-1$ such that $\Pi_i(\omega) \backslash E^0 \supseteq \Pi_{\tau(h)}(\omega)$.

If there is no such $h$, we have that, for any $h$, there exists $\tilde{\omega} \in E^0$ such that $\tilde{\omega} \in \Pi_{\tau(h)}(\omega)$. Suppose that player $i$ hypothesizes $\omega \notin E^0$ despite the fact that he cannot distinguish between $\omega \in E^0$ and $\omega \notin E^0$. Then, due to the minimality of $E^0$, and conditional upon his hypothesis that $\omega \notin E^0$, player $i$ knows that all other players believe with high probability that $E^0$ obtains. As a result, conditional on his hypothesis, player $i$ knows that all other players believe with high probability that the game $\Gamma(\theta)$ is to be played. Then, by consistency of strategy profiles $\{\sigma^k\}_{k=1}^{\infty}$, player $i$ believes with high probability that $\sigma_{-i}^k(\omega) = m_{-i}^*$ for $k$ big enough. Therefore, by Assumption 2, $\hat{\sigma}_i^k$ is a strictly better reply to $\sigma_{-i}^k$ than $\sigma_i^k$ for $k$ big enough even if $\omega \notin E^0$.

Assume that there is such an $h$. Since it is common $q^k$-belief at any $\omega \in E_\theta$ for $k$ big enough that the game $\Gamma(\theta)$ is to be played, player $\tau(h)$ believes with high probability that the game $\Gamma(\theta)$ is to be played. Then, for $\varepsilon^h > 0$ small enough and $k$ big enough, there exists a minimal set of states $E^h \subseteq \Pi_{\tau(h)}(\omega)$ such that the game $\Gamma(\theta)$ is to be played at any $\tilde{\omega} \in E^h$ with the property that:

- $P^k\left(E^h \middle| \Pi_{\tau(h)}(\omega)\right) \geq 1 - \varepsilon^h$;
- $P^k(\Pi_{\tau(h)}(\omega) \backslash E^h | \Pi_{\tau(h)}(\omega)) \leq \varepsilon^h$.

We know that $\sigma_i^k(\omega) = m_i^{'}$ for any $k$ by construction, and $\sigma_i^k(\omega) = m_i^*$ for $k$ big enough. Recall that players $\tau(h)$ through $\tau(n-1)$ may be conditioning their behavior upon information not available to players $i, \tau(1), \ldots, \tau(h-1)$. In that case, their behavior appears random to players $i, \tau(1), \ldots, \tau(h-1)$. Conditional on $\omega \in E^h$, player $i$ knows that all other players believe with high probability that the game $\Gamma(\theta)$ is to be played. Therefore, by consistency of strategy profiles $\{\sigma^k\}_{k=1}^{\infty}$, player $i$

believes with high probability that $\sigma^k_{-i}(\omega) = m^*_{-i}$ for $k$ big enough. By Assumption 1 and 2, we have that, for $k$ big enough,

$$\alpha^{\Gamma}_{\hat{\sigma}^k_i, \sigma^k_{-i}} \succ_i^{P^k(\cdot|\Pi_{\tau(h)}(\omega))} \alpha^{\Gamma}_{\sigma^k}.$$

Therefore, $\hat{\sigma}^k_i$ is a strictly better reply to $\sigma^k_{-i}$ than $\sigma^k_i$ for $k$ big enough if $\omega \in E^h$. If $\omega \notin E^h$, then we have that $\omega \in \Pi_{\tau(h)}(\omega)\backslash E^h$. By Claim 2, we can choose the smallest integer $h'$ with $h < h' \leq n-1$ such that $\Pi_{\tau(h)}(\omega)\backslash E^h \supseteq \Pi_{\tau(h')}(\omega)$.

If there is no such $h'$, we have that, for any $h'$ with $h < h' \leq n-1$, there exists $\tilde{\omega} \in E^h$ such that $\tilde{\omega} \in \Pi_{\tau(h')}(\omega)$. Suppose that player $i$ hypothesizes $\omega \notin E^h$ although he cannot distinguish between $\omega \in E^h$ and $\omega \in \Pi_{\tau(h)}(\omega)\backslash E^h$. Then, due to the minimality of $E^h$, and conditional on his hypothesis that $\omega \in \Pi_{\tau(h)}\backslash E^h$, player $i$ knows that all players $\tau(1)$ through $\tau(h-1)$ believe with high probability that $E^0$ obtains and that all players $\tau(h)$ through $\tau(n-1)$ believe with high probability that $E^h$ obtains. As a result, conditional on his hypothesis, player $i$ knows that all other players believe with high probability that the game $\Gamma(\theta)$ is to be played. Then, by consistency of strategy profiles $\{\sigma^k\}_{k=1}^{\infty}$, player $i$ believes with high probability that $\sigma^k_{-i}(\omega) = m^*_{-i}$ for $k$ big enough. By Assumption 2, $\hat{\sigma}^k_i$ is a strictly better reply to $\sigma^k_{-i}$ than $\sigma^k_i$ for $k$ big enough even if $\omega \notin E^h$.

Assume that there is such an $h'$. Since it is common $q^k$-belief at any $\omega \in E_\theta$ for $k$ big enough that the game $\Gamma(\theta)$ is to be played, player $\tau(h')$ believes with high probability that the game $\Gamma(\theta)$ is to be played. Then, for $\varepsilon^{h'} > 0$ small enough and $k$ big enough, there exists a minimal set $E^{h'} \subseteq \Pi_{\tau(h')}(\omega)$ such that the game $\Gamma(\theta)$ is to be played at any $\tilde{\omega} \in E^{h'}$ with the property that for $k$ big enough,

- $P^k\left(E^{h'} \middle| \Pi_{\tau(h')}(\omega)\right) \geq 1 - \varepsilon^{h'}$;
- $P^k\left(\Pi_{\tau(h')}(\omega)\backslash E^{h'} \middle| \Pi_{\tau(h')}(\omega)\right) \leq \varepsilon^{h'}$.

Once again we can continue the identical argument. This induction argument on $\{\tau(1), \ldots, \tau(n-1)\}$ necessarily stops in finite steps. Also, the argument does not rely upon $\tau$ or $\omega$. Thus, we show that, for $k$ big enough, $\hat{\sigma}^k_i$ is a strictly better reply to $\sigma^k_{-i}$ than $\sigma^k_i$ for any $\omega \in E_\theta$. We complete the proof of Lemma 2. ■

With Lemma 1 and 2, we complete the proof of Theorem 1. ■

### 2.4 How Restrictive is the Finiteness of $\Omega$?

One important assumption I make in the proof of Theorem 1 is the finiteness of $\Omega$. In this section, I argue that there is essentially no loss of generality to assume that $\Omega$ is finite in our setup. The formalization of this is given as the proposition below.

**Proposition 2** *Suppose that* $\Theta$ *is finite. Let* $\sigma$ *be a consistent strategy profile. For any countably infinite* $\Omega$*, there exists a finite partition* $\hat{\Pi}$ *over* $\Omega$ *such that, for any* $\omega \in \Omega$*, we have that* $\sigma(\omega^{'}) = \sigma(\omega^{''})$ *for any* $\omega^{'}, \omega^{''} \in \hat{\Pi}(\omega)$*.*

**Proof of Proposition 2**: Let $\Pi^0$ be a partition over $\Omega$ with the property that for any $\theta \in \Theta$, we have that $\Pi^0(\omega) = \Pi^0(\omega^{'})$ for any $\omega, \omega^{'} \in \Omega$ for which $\omega = h(\theta, s)$ for some $s \in S$ and $\omega^{'} = h(\theta, s^{'})$ for some $s^{'} \in S$. In other words, $\Pi^0$ induces an equivalence class over $\Omega$ with respect to $\Theta$. Since $\Theta$ is finite, $\Pi^0$ is a finite partition over $\Omega$. The proof consists of two steps: In step 1, we construct the desired $\hat{\Pi}$ by minimally refining $\Pi^0$ inductively (i.e., $\Pi^0 \subset \Pi^1 \subset \Pi^2 \subset \cdots \subset \hat{\Pi}$) and show that each refinement of the partition in the induction argument is finite (i.e., $\Pi^k$ is finite for each $k \geq 0$); and in step 2, we argue that the number of steps of this induction needed for constructing $\hat{\Pi}$ is at most finite. (i.e., $\Pi^K = \hat{\Pi}$ for some finite $K$.)

**Step 1**: Suppose that for any $\omega \in \Omega$, we have $\sigma(\omega^{'}) = \sigma(\omega^{''})$ for any $\omega^{'}, \omega'' \in \Pi^0(\omega)$. Then, set $\hat{\Pi} = \Pi^0$ and we are done. Suppose, on the other hand, that there exists $\omega_0 \in \Omega$ for which there exist $\omega_0^{'}, \omega_0^{''} \in \Pi^0(\omega_0)$ such that $\sigma(\omega_0^{'}) \neq \sigma(\omega_0^{''})$. Because of consistency of $\sigma$, there are two distinct $\theta$ and $\theta^{'}$, a sequence of states $\tilde{\omega}_0, \ldots, \tilde{\omega}_L$ and a sequence of players $i_1, \ldots, i_L$, for some finite $L$ with $\tilde{\omega}_0 = \omega_0^{'}$ and $\tilde{\omega}_L = \omega_0^{''}$, and

$$\tilde{\omega}_\ell \in \Pi_{i_\ell}(\tilde{\omega}_{\ell-1}), \;\; \ell = 1, \ldots, L,$$

where $\tilde{\omega}_0 = h(\theta, s)$ for some $s \in S$ and $\tilde{\omega}_L = h(\theta^{'}, s^{'})$ for some $s^{'} \in S$. Then, set $\Pi^1 = \Pi^0 \vee \bigvee_{\ell=1}^{L} \Pi_{i_\ell}(\tilde{\omega}_\ell)$. Note that $\Pi^0 \vee \Pi_{i_\ell}(\tilde{\omega}_\ell)$ is equivalent to $\Pi^0$ at all states except $\tilde{\omega}_\ell$ at which the coarsest common refinement of $\Pi^0$ and $\Pi_{i_\ell}$ is taken. Suppose that for any $\omega \in \Omega$, we have $\sigma(\omega^{'}) = \sigma(\omega^{''})$ for any $\omega^{'}, \omega^{''} \in \Pi^1(\omega)$. Then, set $\hat{\Pi} = \Pi^1$ and we are done. Suppose, on the other hand, that there exists $\omega_1 \in \Omega$ for which there exist $\omega_1^{'}, \omega_1^{''} \in \Pi^1(\omega_1)$ such that $\sigma(\omega_1^{'}) \neq \sigma(\omega_1^{''})$. Because of consistency of $\sigma$, we make the same argument so that we can define $\Pi^2$ analogously from $\Pi^1$. Note that $\Pi^2$ is finite. Now, suppose that for any $\omega \in \Omega$, we have $\sigma(\omega^{'}) = \sigma(\omega^{''})$ for any $\omega^{'}, \omega^{''} \in \Pi^2(\omega)$. Then, set $\hat{\Pi} = \Pi^2$ and we are done. If not, we continue the identical argument so that we can inductively define $\Pi^3, \Pi^4, \cdots$ as we wish.

**Step 2**: Let $\Pi^m$ be a finite partition over $\Omega$ which is defined inductively in $m$ steps as described in Step 1. Suppose that there exists $\omega^m \in \Omega$ for which there exist two states, $\omega, \omega^{'} \in \Pi^m(\omega^m)$ such that $\sigma(\omega) \neq \sigma(\omega^{'})$. By consistency of $\sigma$, there must be two distinct payoff states $\theta$ and $\theta^{'}$) for which $(\theta, s) = h^{-1}(\omega)$ for some $s \in S$ and $(\theta^{'}, s^{'}) = h^{-1}(\omega^{'})$ for some $s^{'} \in S$. By consistency of $\sigma$, there exist a sequence, $\{\omega_0, \ldots, \omega_L\}$ with $\omega_0 = \omega$ and $\omega_L = \omega^{'}$ for some finite $L$ and a pair of sequence of players $\{i_1, \ldots, i_L\}$ such that

$$\omega_\ell \in \Pi_{i_\ell}(\omega_{\ell-1}) \;\text{ for } \ell = 1, \ldots, L$$

Set $\Pi^{m+1} = \Pi^m \vee \bigvee_{\ell=1}^{L} \Pi_{i_\ell}(\omega_\ell)$. By construction, we have $\Pi^{m+1}(\omega) \cap \Pi^{m+1}(\omega') = \emptyset$.

Assume that there are two other states $\tilde{\omega}, \tilde{\omega}' \in \Pi^m(\omega^m)$ with $\sigma(\tilde{\omega}) \neq \sigma(\tilde{\omega}')$. Assume further that $(\theta, s) = h^{-1}(\omega) = h^{-1}(\tilde{\omega})$ and $(\theta', s') = h^{-1}(\omega') = h^{-1}(\tilde{\omega}')$ for which there is a sequence of states $(\tilde{\omega}_0, \dots \tilde{\omega}_L)$ with $\tilde{\omega}_0 = \tilde{\omega}$ and $\tilde{\omega}_L = \tilde{\omega}'$ with the property that

$$\tilde{\omega}_\ell \in \Pi_{i_\ell}(\tilde{\omega}_{\ell-1}) \text{ for } \ell = 1, \dots, L.$$

Then, we have $\tilde{\omega} \in \Pi^{m+1}(\omega)$ and $\tilde{\omega}' \in \Pi^{m+1}(\omega')$. This implies that $\Pi^{m+1}(\tilde{\omega}) \cap \Pi^{m+1}(\tilde{\omega}') = \emptyset$. Since the number of players, $L$, $\Theta$ and $S$ are all finite, we do not have to treat $(\omega, \omega')$ and $(\tilde{\omega}, \tilde{\omega}')$ independently. If we distinguish between $\omega$ and $\omega'$ with $\Pi^{m+1}$, we necessarily distinguish between $\tilde{\omega}$ and $\tilde{\omega}'$ with $\Pi^{m+1}$, as well. In other words, we can define an equivalence class over $\Omega$ by $\Pi^{m+1}$ within which $\omega$ is equivalent to $\tilde{\omega}$ and $\omega'$ is equivalent to $\tilde{\omega}'$. This implies that we really need at most finite $K$ steps to exhaust all countably infinite number of states so that for any $\omega \in \Omega$, we have $\sigma(\omega') = \sigma(\omega'')$ for any $\omega', \omega'' \in \Pi^K(\omega)$. ■

# 3 Tightness of the Main Result

## 3.1 The Example Revisited

I show in the previous section that the upper hemi-continuity of the UBNE correspondence at any complete information prior is obtained if *with probability 1*, there is approximate common knowledge about what game being played. I shall argue that the main result (Theorem 1) cannot be improved in the following sense: If the perturbed probability distributions only satisfy the property that, with sufficiently high probability (not probability 1), there is approximate common knowledge about what game being played, the upper hemi-continuity of the UBNE correspondence at any complete information prior is *not* preserved for *some* game form. I build on the example discussed in the introduction. The noisy communication between Andy and Bob is summarized in the following matrix:

| | | Bob's signal | |
|---|---|---|---|
| | | 0 | 1 |
| Andy's signal | 0 | $1-p$ | 0 |
| | 1 | $p\varepsilon$ | $p(1-\varepsilon)$ |

Therefore, I have the following perturbed probability distribution $P^\varepsilon$ over $\{(0,0),(1,0),(1,1)\}$.

$$P^\varepsilon(0,0) = 1-p; \quad P^\varepsilon(1,0) = p\varepsilon; \quad \text{and } P^\varepsilon(1,1) = p(1-\varepsilon).$$

Denote by $P^* \equiv P^0$ the complete information prior. Let $\Omega = \{(0,0),(1,0),(1,1)\}$. Andy's knowledge is characterized by the partitional correspondence: $\Pi_A(0,0) = \{(0,0)\}$ and $\Pi_A(1,0) = \Pi_A(1,1) = \{(1,0),(1,1)\}$. Similarly, Bob's knowledge is characterized by the following partitional correspondence: $\Pi_B(0,0) = \Pi_B(1,0) = \{(0,0),(1,0)\}$ and $\Pi_B(1,1) = \{(1,1)\}$. Let $S_i = \{s_i^\alpha, s_i^\beta\}$ for each $i = A, B$ and $S = S_A \times S_B$. Note that $(s_A^\theta, s_B^\theta)$ is the payoff type profile in which each player's payoff type corresponds to the state $\theta$. Define $h$ as a mapping from $\Theta \times S$ to $\Omega$ with the following property:

$$h^{-1}(0,0) = (\beta, s_A^\beta, s_B^\beta); \;\; h^{-1}(1,0) = (\alpha, s_A^\alpha, s_B^\beta); \;\text{ and }\; h^{-1}(1,1) = (\alpha, s_A^\alpha, s_B^\alpha).$$

Therefore, $h$ is an immersion of $\Theta \times S$ into $\Omega$. Note that there is no common knowledge at *any* state about which game being played. I shall claim that there is *no* approximate common knowledge at $(1,0)$ about which game being played.

**Lemma 3** *There is* ***no*** *approximate common knowledge at* $(1,0)$ *about which game being played. On the contrary, there is approximate common knowledge at* $(0,0)$ *and* $(1,1)$ *about which game being played.*

**Proof**: Fix $\varepsilon > 0$ sufficiently small so that we can take $q$ sufficiently close to 1. Since Andy always knows which game is to be played, we consider two events, $\{(0,0)\}$ and $\{(1,0),(1,1)\}$ and apply each player's $q$-belief operator to those events. When we apply Andy's $q$-belief operator to $\{(0,0)\}$, we have $B_A^q(\{(0,0)\}) = \{(0,0)\}$. When we apply Bob's $q$-belief operator to $\{(0,0)\}$, we have $B_B^q(\{(0,0)\}) = \{(0,0),(1,0)\}$. Since $\{(0,0)\}$ is $q$-evident but $\{(0,0),(1,0)\}$ is not $q$-evident, it is approximate common knowledge at $(0,0)$ that the game $\Gamma^*(\beta)$ is to be played but it is not approximate common knowledge at $(1,0)$ that the game $\Gamma^*(\beta)$ is to be played. When we apply Andy's $q$-belief operator to the event $\{(1,0),(1,1)\}$, we have $B_A^q(\{(1,0),(1,1)\}) = \{(1,0),(1,1)\}$. When we apply Bob's $q$-belief operator to $\{(1,0),(1,1)\}$, we have $B_B^q(\{(1,0),(1,1)\}) = \{(1,1)\}$. Since $\{(1,1)\}$ is $q$-evident, it is approximate common knowledge at $(1,1)$ that the game $\Gamma^*(\alpha)$ is to be played. Since $\{(1,0),(1,1)\}$ is not $q$-evident, it is not approximate common knowledge at $(1,0)$ that the game $\Gamma^*(\alpha)$ is to be played. ■

At state $(1,0)$, Andy knows the true state is $\alpha$, while Bob believes with probability sufficiently close to 1 that the state is $\beta$. Hence, at state $(1,0)$, there is completely asymmetric information about the game being played between Andy and Bob. Here, I consider a slightly coarser topology than that induced by $d^*$. Let

$$\tilde{d}_1(P) = \inf\left\{\eta \middle| P\left(\mathscr{G}(\eta)\right) \geq 1 - \eta\right\}.$$

Define $d^{**}(P, P')$ as follows:

$$d^{**}(P, P') = \max\left\{d_0(P, P'), \tilde{d}_1(P), \tilde{d}_1(P')\right\}.$$

Note that $d^{**}$ is non-negative, symmetric, and $d^{**}(P, P^{'}) = 0$ if and only if $P = P^{'}$ and both $P$ and $P^{'}$ are complete information priors. Clearly, any convergent net according to $d^*$ is always a convergent net according to $d^{**}$. But the converse is not generally true. Let $\mathscr{C}^{**}$ be a class consisting of all pairs of a net $\{P^k\}_{k=1}^{\infty}$ with $d^{**}(P^k, P^*) \to 0$ as $k \to \infty$ for some complete information prior $P^*$ and a complete information prior. In Section 2.2, I establish the equivalence between the topology induced by $d^*$ and the corresponding convergence class $\mathscr{C}^*$. Analogously, I can show that $\mathscr{C}^{**}$ is a convergence class for the set of probability distributions, $\mathscr{P}$. Applying Theorem 9 of Kelly (1955), I conclude that the convergence class $\mathscr{C}^{**}$ indeed generates a topology over $\mathscr{P}$. Note that $d^{**}(P^k, P^*) \to 0$ as $k \to \infty$ if and only if $P^k \to P^*$ as $k \to \infty$ and there exists $\varepsilon^k \to 0$ such that $P^k(\mathscr{G}(\varepsilon^k)) \geq 1 - \varepsilon^k$ for each $k$. I shall define the upper hemi-continuity of the UBNE correspondence at any complete information prior with respect to the topology induced by $d^{**}$.

**Definition 9** *Let $\Gamma$ be a game form. $\psi_\Gamma^{UBNE}$ is* ***upper hemi-continuous*** *at a complete information prior $P^*$ with respect to the topology induced by $d^{**}$ if, $\psi_\Gamma^{UBNE}(P^k) \to \psi_\Gamma^{UNE}(P^*)$ as $k \to \infty$ whenever $d^{**}(P^k, P^*) \to 0$ as $k \to \infty$. Here $\psi_\Gamma^{UNE}(P^*)$ denotes the set of UNE outcomes of the game $\Gamma(P^*)$.*

**Corollary 1** *There exists a sequence of priors $\{P^k\}_{k=1}^{\infty}$ converging to the complete information prior $P^*$ such that $d^{**}(P^k, P^*) \to 0$ as $k \to \infty$.*

**Proof**: Let us define $q_A(\varepsilon) = 1 - \varepsilon$. Then, there is a common $q_A(\varepsilon)$-belief at $(1, 1)$ that the game $\Gamma^*(\alpha)$ is to be played. Let us define

$$q_B(\varepsilon) = \frac{1-p}{1-p+p\varepsilon}.$$

Then, there is a common $q_2(\varepsilon)$-belief at $(0, 0)$ that the game $\Gamma^*(\beta)$ is to be played. Define

$$q^*(\varepsilon) = \min\{1 - p\varepsilon, q_A(\varepsilon), q_B(\varepsilon)\}.$$

We take a sequence $\{\varepsilon^k\}_{k=1}^{\infty}$ converging to 0. We denote $P^{\varepsilon^k} \equiv P^k$ and $q^*(\varepsilon^k) \equiv q^k$ for each $k$, respectively. Then, we know that $q^k$ goes to 1 as $k \to \infty$. Therefore, with probability at least $q^k$, there is a common $q^k$-belief about which game being played. This completes the proof. ■

I already confirmed in the introduction that the UBNE correspondence associated with the game form in the example displays a failure of upper hemi-continuity at any complete information prior. The following corollary attributes the failure of the upper hemi-continuity to the topology induced by $d^{**}$.

**Corollary 2** *The UBNE correspondence associated with the game form $\Gamma^*$ is* ***not*** *upper hemi-continuous at any complete information prior $P^*$ with respect to the topology induced by $d^{**}$.*

In this particular sense, the above corollary shows that $d^*$ is the coarsest possible topology with respect to which the UBNE correspondence associated with *any* game form is upper hemi-continuous at any complete information prior.

# 4 An Application to Implementation Theory

## 4.1 Robust UNE Implementation

I show that the UBNE correspondence associated with any game form is upper hemi-continuous at any complete information prior with respect to the topology induced by $d^*$. I shall apply this result to the concept of robust UNE implementation. First, I formalize UNE implementation under complete information. A mapping $f : \Theta \to A$ is said to be a *social choice function*.

**Definition 10** *A game form* $\Gamma$ *UNE implements a social choice function* $f : \Theta \to A$ *under complete information prior* $P^*$ *if (1)* $\psi_\Gamma^{UNE}(P^*) \neq \emptyset$*, and (2)* $\alpha(\omega) = f(\theta)$ *for each* $\alpha \in g\left(\psi_\Gamma^{UNE}(P^*)\right)$ *and* $\omega$ *with* $h(\theta, s^\theta) = \omega$*.*

Condition (1) in the above definition states that there always exists an UNE in the game $\Gamma(P^*)$. Condition (2) in the above definition says that every UNE outcome coincides with that of $f$. Second, I shall define the *implementability* in undominated Nash equilibrium.

**Definition 11** *A social choice function* $f$ *is UNE* ***implementable*** *under complete information if there exists a game form* $\Gamma$ *that UNE implements* $f$ *under any complete information prior* $P^*$*.*

Finally I can define robust UNE implementability.

**Definition 12** *A social choice function* $f$ *is* ***robustly*** *UNE implementable relative to* $d^*$ *if (1)* $f$ *is UNE implementable under complete information and (2) the UBNE correspondence associated with some implementing game form is upper hemi-continuous at any complete information prior with respect to the topology induced by* $d^*$*.*

I collect a set of UNE implementable mechanisms and eliminate each game form if it is not robust relative to $d^*$. The remaining set of implementable mechanisms is the set of robust UNE implementable mechanisms relative to $d^*$. To state a permissive robust implementation result, I need one requirement called "no-veto-power."

**Definition 13** *A social choice function* $f$ *satisfies* ***no-veto-power*** *if, for any* $\theta \in \Theta$*, whenever* $a \succeq_i^\theta b$ *for any* $b \in A$ *and for at least* $n-1$ *players,* $f(\theta) = a$*.*

**Theorem 2 (Palfrey and Srivastava (1991))** *Suppose that there are at least three players and preferences are strict. Then any social choice function satisfying no-veto-power is implementable in undominated Nash equilibrium.*

I shall establish a permissive robust UNE implementation result relative to $d^*$.

**Corollary 3** *Suppose that there are at least three players and preferences are strict. Then, any social choice function $f$ satisfying no-veto-power is robustly UNE implementable relative to $d^*$.* [15]

**Proof**: Theorem 2 enables us to construct a canonical game form that UNE implements any social choice function $f$ satisfying no-veto-power under complete information. Theorem 1 has shown that the UBNE correspondence associated with the canonical game form is upper hemi-continuous at any complete information prior with respect to the topology induced by $d^*$. Therefore, any social choice function $f$ satisfying no-veto power is robustly UNE implementable relative to $d^*$. ■

This corollary clarifies the extent to which Palfrey and Srivastava's permissive implementability result can be robustified. In the next subsection, I will argue that the topology induced by $d^*$ characterizes the *maximal* extent to which the permissive UNE implementability is robustified. Because I shall show that only monotonic social choice functions can be robustly UNE implementable relative to $d^{**}$. I do not think that the permissive robust UNE implementation result necessarily lend much support to the use of Palfrey and Srivastava's game form. Rather, this robustification gives us a precise sense in which the Palfrey and Srivastava's game form is not robust if we believe that the robustness relative to $d^*$ is very restrictive. I come back to this point shortly in the next subsection.

Jackson, Palfrey, and Srivastava (1994) characterize "*separable*" environments within which they can design a mechanism that UNE implements *any* social choice function. Most importantly, this mechanism also works with two players. Hence, in separable environments with strict preferences, I conclude that any social choice function is robustly undominated Nash implementable relative to $d^*$ irrespective of the number of players.

## 4.2 When is Monotonicity Necessary?

In the previous section, I introduce the topology induced by $d^{**}$ with respect to which the UBNE correspondence is *not* upper hemi-continuous at any complete information prior for *some* game form. In what follows, I am going to show that *any* game form that UNE implements a non-monotonic social choice function exhibits a failure of upper hemi-continuity of the UBNE correspondence at any complete

[15]This result can be extended to social choice correspondences

information prior with respect to the topology induced by $d^{**}$. That is, monotonicity is a necessary condition for robust UNE implementation relative to $d^{**}$. First, I give the formal definition of monotonicity.

**Definition 14** *A social choice function $f$ is* ***monotonic*** *if for every pair of states $\theta$ and $\theta'$ such that for each player $i$ and for each $a \in A$,*

$$a \succ_i^{\theta'} f(\theta) \Longrightarrow a \succ_i^{\theta} f(\theta),$$

*we have $f(\theta) = f(\theta')$.*

Monotonicity was first introduced in Maskin (1999) and identified as a necessary condition for Nash implementation. Before stating the formal result (Theorem 3), we shall illustrate our argument concerning monotonicity again through the same example. Let $f^*$ be defined as a social choice function such that $f^*(\alpha) = a$ and $f^*(\beta) = b$. Then, it is easy to see that the mechanism $\Gamma^*$ (shown in section 2) UNE implements $f^*$ under complete information. Next I claim that $f^*$ is *not* monotonic. Suppose, by way of contradiction, that $f^*$ is monotonic. Since $f^*(\alpha) \neq f^*(\beta)$, by monotonicity, there must exist player $i$ and outcome $y$ such that $y \succ_i^{\beta} f^*(\alpha)$ and $f^*(\alpha) \succeq_i^{\alpha} y$. Such a player must be Bob. Because Andy has the state uniform preference. Because $f^*(\alpha) = a$ and $a$ is the best outcome for Bob, there is no such better $y$, which is a contradiction. Thus, I can attributes the failure of the upper hemi-continuity of the UBNE correspondence to non-monotonicity of the social choice function which is UNE implemented by $\Gamma^*$.

I shall show that monotonicity is a necessary condition for robust UNE implementation relative to $d^{**}$.

**Theorem 3** *Let $f$ be a social choice function. Suppose that a game form $\Gamma$ UNE implements $f$ under complete information. Assume that preferences are strict. Assume further that $\psi_{\Gamma}^{UBNE}$ is upper hemi-continuous at any complete information prior with respect to the topology induced by $d^{**}$. Then, $f$ is necessarily monotonic.*

**Proof of Theorem 3**: We build on the argument of Chung and Ely's Theorem 1. Let complete information prior $\mu$ be given, and let $f$ be a UNE implementable social choice function with implementable game form $\Gamma = (M, g)$.

Suppose that $\theta$ and $\theta'$ are two possible states satisfying a monotonic transformation between $\theta$ and $\theta'$. Since the game form $\Gamma$ implements $f$ in undominated Nash equilibrium by our hypothesis, there exists an undominated Nash equilibrium (UNE), $m^*$ of $\Gamma(\theta)$. We claim that $m^*$ is a NE of $\Gamma(\theta')$. If not, there must exist a player $i$ and a message $m_i$ such that $g(m_i, m_{-i}^*) \succ_i^{\theta'} g(m^*)$. But by monotonic transformation, this implies that $g(m_i, m_{-i}^*) \succ_i^{\theta} g(m^*)$, which is a contradiction since $m^*$ is presumed to be a NE of $\Gamma(\theta)$.

To avoid a trivial case where we can automatically conclude that $f(\theta) = f(\theta')$, we must assume that $m^*$ is dominated in $\Gamma(\theta')$. Then let $I \subset N$ be the nonempty set of players for whom $m_i^*$ is dominated in $\Gamma(\theta')$ for each $i \in I$. With abuse of notations, we use the expression that $|I| = I \geq 1$. Chung and Ely construct the following family of information structures $\nu^\varepsilon$, parameterized by $\varepsilon > 0$. Let $\tau^i$ represent the profile of signals $(s_1, \ldots, s_n)$ defined by $s_i = \theta'$ for some $i \in I$ and $s_j = \theta$ for all $j \neq i$. The perturbed information structure is given below:

$$\begin{aligned}
\nu^\varepsilon(\theta, \tau^i) &= \frac{\varepsilon}{I}\mu(\theta, s^\theta) \text{ for all } i \in I \\
\nu^\varepsilon(\theta, s^\theta) &= (1-\varepsilon)\mu(\theta, s^\theta), \\
\nu^\varepsilon(\tilde{\theta}, s^{\tilde{\theta}}) &= \mu(\tilde{\theta}, s^{\tilde{\theta}}), \text{ for all } \tilde{\theta} \neq \theta.
\end{aligned}$$

In this information structure, when the state is anything other than $\theta$ or $\theta'$, the state is common knowledge. Furthermore, when a player observes the signal $\theta$, that player *knows* that the state is $\theta$. Therefore, without loss of generality, we may make the rest of the argument as if there were only two states, $\theta, \theta'$. Define as follows:

$$p = \mu(\theta, s^\theta | \{\theta, \theta'\}) \text{ and } 1 - p = \mu(\theta', s^{\theta'} | \{\theta, \theta'\})$$

We decompose the set of players of $N$ into the following:

$$N = I \cup J = \{i_1, \ldots, i_I, j_1, \ldots, j_J\},$$

where $I = \{i_1, \ldots, i_I\}, J = \{j_1, \ldots, j_J\}$ and $I \cap J = \emptyset$. Again with abuse of notation, we use the expression that $|J| = J$. Let us denote $\theta = 1$ and $\theta' = 0$. There are $2 + I$ possible states, $\omega_0, \omega_1, \omega_2, \ldots, \omega_{1+I}$.

- $\omega_0 = (\underbrace{0, \ldots, 0}_{n})$ with probability $1 - p$;
- $\omega_1 = (\underbrace{1, \ldots, 1}_{n})$ with probability $p(1 - \varepsilon)$;
- $\omega_2 = (\overbrace{0}^{i_1}, \underbrace{1, \ldots, 1}_{i_2, \ldots, i_I}, \overbrace{1, \ldots, 1}^{J})$ with probability $p\varepsilon/I$ if $s_{i_1} = \theta'$ and $s_k = \theta$ for any $k \neq i_1$;
- $\omega_3 = (\underbrace{1}_{i_1}, \overbrace{0}^{i_2}, \underbrace{1, \ldots, 1}_{i_3, \ldots, i_I}, \overbrace{1, \ldots, 1}^{J})$ with probability $p\varepsilon/I$ if $s_{i_2} = \theta'$ and $s_k = \theta$ for any $k \neq i_2$;
- $\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

- $\omega_{1+I} = (\underbrace{1,\dots,1}_{i_1,\dots,i_{I-1}}, \overbrace{0}^{i_I}, \underbrace{1,\dots,1}_{J})$ with probability $p\varepsilon/I$ if $s_{i_I} = \theta'$ and $s_k = \theta$ for any $k \neq i_I$

Denote $\Omega^* = \{\omega_0, \omega_1, \dots, \omega_{1+I}\}$. Consider the information structure from the viewpoint of any player $i \in I$:

| | | $i$'s signal | |
|---|---|---|---|
| | | 0 | 1 |
| | $(\overbrace{0,\dots,0}^{N\backslash\{i\}})$ | 1 | 0 |
| signals of | $(\overbrace{1,\dots,1}^{N\backslash\{i\}})$ | $\varepsilon/I$ | $1-\varepsilon$ |
| $N\backslash\{i\}$ | $(\overbrace{0}^{i_1}, 1, \dots, 1)$ | 0 | $\varepsilon/I$ |
| players | $(1, \overbrace{0}^{i_2}, 1, \dots, 1)$ | 0 | $\varepsilon/I$ |
| | $\vdots$ | $\vdots$ | $\vdots$ |
| | $(1, \dots, 1, \underbrace{0}_{i_I}, \overbrace{1,\dots,1}^{J})$ | 0 | $\varepsilon/I$ |

Consider the information structure from the viewpoint of any player $j \in J$:

| | | $j$'s signal | |
|---|---|---|---|
| | | 0 | 1 |
| | $(\overbrace{0,\dots,0}^{N\backslash\{j\}})$ | 1 | 0 |
| signals of | $(\overbrace{1,\dots,1}^{N\backslash\{j\}})$ | 0 | $1-\varepsilon$ |
| $N\backslash\{j\}$ | $(\overbrace{0}^{i_1}, \underbrace{1,\dots,1}_{i_2,\dots,i_I}, \overbrace{1,\dots,1}^{J\backslash\{j\}})$ | 0 | $\varepsilon/I$ |
| players | $(\overbrace{1}^{i_1}, \underbrace{0}_{i_2}, \overbrace{1,\dots,1}^{i_3,\dots,i_I}, \underbrace{1,\dots,1}_{J\backslash\{j\}})$ | 0 | $\varepsilon/I$ |
| | $\vdots$ | $\vdots$ | $\vdots$ |
| | $(\overbrace{1,\dots,1}^{i_1,\dots,i_{I-1}}, \underbrace{0}_{i_I}, \overbrace{1,\dots,1}^{J\backslash\{j\}})$ | 0 | $\varepsilon/I$ |

We can define the partitional function for each player $i \in I$ as follows:

$$\begin{aligned}\Pi_i(\omega_0) &= \Pi_i(\omega_{1+i}) = \{\omega_0, \omega_{1+i}\} \\ \Pi_i(\omega_1) &= \Pi_i(\omega_{1+k}) \text{ for all } k \in I \backslash \{i\} \\ &= \{\omega_1, \ldots, \omega_{1+I}\} \backslash \{\omega_{1+i}\}\end{aligned}$$

We can also define the partitional function for each player $j \in J$ as follows:

$$\begin{aligned}\Pi_j(\omega_0) &= \{\omega_0\} \\ \Pi_j(\omega_1) &= \Pi_j(\omega_2) = \cdots = \Pi_j(\omega_{1+I}) \\ &= \{\omega_1, \ldots, \omega_{1+I}\}\end{aligned}$$

We will show that with small probability, there is *no* approximate common knowledge at $\omega_2, \ldots, \omega_{1+I}$ about what game is to be played.

**Claim 3** *It is approximate common knowledge at $\omega_0$ that the game $\Gamma(\theta')$ is to be played. Besides, it is approximate common knowledge at $\omega_1$ that the game $\Gamma(\theta)$ is to be played. Moreover, there is no approximate common knowledge at $\omega_2, \ldots, \omega_{1+I}$ about what game is to be played.*

**Proof of Claim 3**: Since each player $j \in J$ always knows what game is to be played, we consider two events, $\{\omega_0\}$ and $\{\omega_1, \ldots, \omega_{1+I}\}$ and apply each player's $q$-belief operator to those events. When we apply player $j$'s $q$-belief operator to $\{\omega_0\}$, we have $B_j^q(\{\omega_0\}) = \{\omega_0\}$ for each $j \in J$. Let us define

$$q_I(\varepsilon) = \frac{1-p}{1-p+p\varepsilon/I}.$$

When we apply player $i$'s $q$-belief operator to $\{\omega_0\}$, we have $B_i^q(\{\omega_0\}) = \{\omega_0, \omega_{1+i}\}$ for $q = q_I(\varepsilon)$. Therefore, we have, for $q = q_I(\varepsilon)$,

$$\bigcap_{i \in N} B_i^q(\{\omega_0\}) = \bigcap_{i \in I} B_i^q(\{\omega_0\}) \bigcap_{j \in J} B_j^q(\{\omega_0\}) = \{\omega_0\}.$$

Since $\{\omega_0\}$ is $q_I(\varepsilon)$-evident, it is a common $q_I(\varepsilon)$-belief at $\omega_0$ that the game $\Gamma(\theta')$ is to be played. Besides, it is *not* approximate common knowledge at $\omega_2, \ldots, \omega_{1+I}$ that the game $\Gamma(\theta')$ is to be played.

Let us define

$$E_1 \equiv \{\omega_1, \omega_{1+1}, \ldots, \omega_{1+I}\}.$$

When we apply each player $j$'s $q$-belief operator to the event $E_1$, we have $B_j^q(E_1) = E_1$ for each $j \in J$. Let us define

$$q_J(\varepsilon) = 1 - \varepsilon.$$

When we apply each player $i$'s $q$ belief operator to the event $E_1$, we have,

$$B_i^q(E_1) = E_1 \backslash \{\omega_{1+i}\}.$$

Thus, we have, for $q = q_J(\varepsilon)$,

$$\bigcap_{i \in N} B_i^q(E_1) = \bigcap_{i \in I} B_i^q(E_1) \bigcap_{j \in J} B_j^q(E_1) = \{\omega_1\}.$$

Since $\{\omega_1\}$ is $q_J(\varepsilon)$-evident, it is a common $q_J(\varepsilon)$-belief at $\omega_1$ that the game $\Gamma(\theta)$ is to be played. Besides, it is *not* approximate common knowledge at $\omega_2, \ldots, \omega_{1+I}$ that the game $\Gamma(\theta)$ is to be played. Therefore, there is no approximate common knowledge at $\omega_2, \ldots, \omega_{1+I}$ about which game is to be played. ■

Define $q^*(\varepsilon) = \min\{1 - p\varepsilon, q_I(\varepsilon), q_J(\varepsilon)\}$. Consider a sequence of $\{\varepsilon^k\}_{k=1}^{\infty}$ converging to 0. Set $q^k \equiv q^*(\varepsilon^k)$ for each $k$. Then, the above claim shows that, with probability at least $q^k$, there is a common $q^k$-belief about what game is to be played for each $k$. Therefore, $\nu^\varepsilon$ converges to $\mu$ as $\varepsilon \to 0$ according to the topology induced by $d^{**}$. We define $\sigma_i(\omega) = m_i^*$ for all $i \in N$ and all $\omega \in \Omega^*$. Chung and Ely have shown that this strategy profile $\sigma$ constitutes an undominated Bayesian Nash equilibrium of the game $\Gamma(\nu^\varepsilon)$ for any $\varepsilon > 0$. Because of upper hemi-continuity of $\psi_\Gamma^{UBNE}$ at any complete information prior with respect to the topology induced by $d^{**}$, we must have $f(\theta) = g(\sigma(\omega_0)) = g(m^*) = g(\sigma(\omega_1)) = f(\theta')$. Thus, we complete the proof. ■

Now, we can restate Theorem 1 of Chung and Ely (2003) in terms of our terminology.

**Corollary 4 (Theorem 1 of Chung and Ely (2003))** *Suppose that preferences are strict. If a social choice function is robustly UNE implementable relative to $d^{**}$, it is necessarily monotonic.*

In the proof of Theorem 3, I construct a state space $\Omega$ which is finite. [16] In my setup, the finiteness of $\Omega$ is already justified by Proposition 2. On a finite state space, Fudenberg and Tirole (1991) showed that if event $E$ has probability close to 1, there is high probability that it is common $q$-belief, for $q$ close to 1. This is indeed the topology induced by $d^{**}$. Kajii and Morris (1997) instead considered the situation in which even if event $E$ has probability close to 1, there is high probability that it is common $q$-belief for some $q$ which is *not* close to 1. In order for this situation to be realized, Kajii and Morris (1997) explicitly needed a countably infinite state space which cannot be reduced to the finite state space. Of this type of situations, the Rubinstein's email game (1989) is the most famous one. Therefore, Kajii and Morris (1997) needed the topology much coarser than that induced by $d^{**}$ as their robustness criterion.

[16] I am grateful to an anonymous referee for pointing out the reliance of the finiteness on the proof of Theorem 3.

## 5 Related Literature

Fudenberg, Kreps, and Levine (1988), Dekel and Fudenberg (1990), and Kajii and Morris (1997) are all concerned with payoff uncertainty in considering the nearby games. That is, these authors slightly but *arbitrarily* – though the extent of arbitrariness varies across the papers – perturb the payoff functions. This is very different from my analysis. In contrast, I fix the set of payoff types from the beginning and slightly perturb players' beliefs over the fixed set of payoff types. In the same vein, Monderer and Samet (1989) considered a class of nearby games such that any Nash equilibrium of the complete information game is approximated by an $\varepsilon$-Bayesian Nash equilibrium of the nearby Bayesian games. In this paper, I explicitly topologize the Monderer and Samet (1989)'s class of nearby games and identify it as the topology induced by $d^{**}$. Regarding this topology, I consider the upper hemi-continuity of the UBNE correspondence, while Monderer and Samet (1989) consider the lower hemi-continuity of the $\varepsilon$-Bayesian Nash equilibrium correspondence.

Monderer and Samet (1996) go further and generalize the idea of Monderer and Samet (1989) into the general incomplete information game and characterize the relevant topology. If I appropriately extend my original information structure from complete to general incomplete information, the topology Monderer and Samet (1996) found is the same as the topology induced by $d^{**}$ with the additional requirement that the prior probability measure of the original incomplete information is fixed throughout. Kajii and Morris (1998) also provided another topology which turned out to be the same topology induced by $d^{**}$ with the additional requirement that the original partition structure is fixed throughout. The Kajii and Morris (1998)'s restricted topology induced by $d^{**}$ is exactly the same as the one used by Chung and Ely (2003). In sum, Monderer and Samet (1996) and Kajii and Morris (1998) focus on the same topology but require a different restriction to be imposed on the set of limit points of the probability distributions when checking its robustness.

This paper rather restricts possible strategy profiles to be consistent with the equilibrium concept (Definition 3), while it does not restrict at all the set of limit points of the probability distributions. Furthermore, as long as the state space $\Omega$ is finite, Kajii and Morris (1997)'s robustness requirement is equivalent to the lower hemi-continuity of the Bayesian Nash equilibrium correspondence with respect to the topology induced by $d^{**}$. [17] Kajii and Morris (1997), however, do not consider the finiteness assumption of $\Omega$ as justifiable. Once I explicitly allow for $\Omega$ to be countably infinite, Kajii and Morris (1997)'s robustness induces a much coarser topology than that induced by $d^{**}$ with respect to which the Bayesian Nash equilibrium correspondence is required to be lower hemi-continuous. This is well illustrated in the literature using the Rubinstein's (1989) email game.

[17] I am grateful to an anonymous referee for drawing my attention to this connection.

In this paper, I show that Chung and Ely's robust UNE implementation is equivalent to the robust UNE implementation relative to $d^{**}$. Chung and Ely (2003) show that only monotonic social choice functions can be robustly UNE implementable relative to $d^{**}$. Thus, Chung and Ely's contribution now can be interpreted as follows: they exploited a non-trivial structure of the game form (through monotonicity) in which the UBNE correspondence is required to be upper hemi-continuous at any complete information prior with respect to the topology induced by $d^{**}$. This is also consistent with the tightness of my main result, because the upper hemi-continuity result cannot be extended from $d^*$ to $d^{**}$ for "any" game form.

# 6 Concluding Remarks

I conclude this paper with possible future works. Following "Wilson's doctrine," the recent works of Bergemann and Morris (2005a,b) address some of the issues of robust mechanism design. [18] In particular, they show that interim (i.e., Bayesian) implementation on *all* type spaces is possible if and only if it is possible to implement the social choice function using an iterative deletion procedure. If, however, one can characterize a class of type spaces over which robust interim implementation is defined, I conjecture that it is possible to clarify the extent to which the classical implementation results can be robustified. Indeed, the methodology developed in this paper helps us characterize such a class of type spaces via topology. By clarifying exactly how much structure we need in terms of topology for robustifying the results of interim implementation, I will be able to argue that many of the permissive results are very sensitive to the specification of the original information structure. This can be considered as a complementary approach to Bergemann and Morris (2005a,b). While this paper rather focus on complete information and undominated Nash equilibrium, I believe that it has the potential for addressing more general questions in the context of robust mechanism design.

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[18] Wilson's doctrine is summarized as the following comment by himself (1987): "*I foresee the progress of game theory as depending on successive reductions in the base of common knowledge required to conduct useful analyses of practical problems. Only by repeated weakening of common knowledge assumptions will the theory theory approximate reality.*"

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