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Unconditionally Optimal Monetary Policy *

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ABSTRACT

We develop a simple and intuitive approach for analytically deriving unconditionally optimal (UO) policies, a topic of enduring interest in optimal monetary policy analysis. The approach can be employed to both general linear-quadratic problems and to the underlying non-linear environments. We provide a detailed example using a canonical New Keynesian framework.

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1. Introduction

In this paper we take up a theme from Taylor (1979), who proposes adopting a monetary policy, under rational expectations, which is optimal "on average". That is, given a model of the economy, including knowledge of the time series properties of the underlying shocks, and assuming rational expectations, Taylor proposes that optimal monetary policy optimize the unconditional expectation of the policymaker's objective function. That approach to policy evaluation has been adopted many times since; for example, Rotemberg and Woodford (1998), Woodford (1999), Clarida, Gali and Getler (1999), Erceg, Henderson and Levin (2000), Kollman (2002) and Schmitt-Grohe and Uribe (2007), to name but a few. More recently, Blake (2001) and Jensen and McCallum (2002, 2006) also suggest a procedure for determining optimal, time-invariant monetary policy based on optimization of the unconditional value of the criterion function. However, these analyses employ numerical approaches to recover the unconditionally optimal monetary policy. An exception to that is Whiteman (1986). In a simple linear, rational expectations model with endogenous variables which are partly a function of their own expected future values, he derives a closed-form solution for optimal policy. However, Whiteman's proof of optimality is algebraically intensive.

In this paper we devise a straightforward, intuitive and easy-to-implement approach to deriving policies that are unconditionally optimal in a general setting which we lay out in Section 2. The key technical challenge involves constructing an optimal policy program taking expectations over all feasible initial conditions. In Section 2.1 we derive these optimal continuation policies, to use Jensen and McCallum's terminology, in a way that is applicable to both linear-quadratic (LQ) and non-linear models. In Section 2.2 we demonstrate the approach in the simplest LQ New Keynesian monetary policy model (whilst a general LQ problem is set out in the appendix). In Section 3 we then apply the approach to the underlying non-linear New Keynesian model. We show that linear approximation is possible around the "unconditionally optimal" deterministic steady state, analogous to the approach adopted by Khan, King and Wolman (2003) in the context of (conditionally) optimal monetary policy under commitment. We linearize the optimality conditions of the non-linear model and indicate how one can obtain a LQ framework and the same optimal policy as the simple LQ set-up of Section 2.2.

In Section 4, we discuss briefly the two defining characteristics of unconditionally optimal policies. The first issue is the treatment of initial conditions. The second is the sense in which consumers' discount rates do not matter for unconditionally optimal policies, an observation going back to Taylor's (1979) contribution. We conclude in Section 5.

2. The framework

Consider a discounted loss function of the form

$$L_t = (1-\beta)\, E_t \sum_{j=0}^{\infty} \beta^j l(x_{t+j}, \mu_{t+j}), \tag{2.1}$$

where E_t is the expectations operator *conditional on information up through date* t, β is the time discount factor, $l(x_{t+j}, \mu_{t+j})$ is the period loss function and x_t is a vector of target variables. Specifically, we define

$$x_t = \begin{bmatrix} Z_t \\ z_t \\ i_t \end{bmatrix}.$$

Z_t is a vector of predetermined endogenous variables (lags of variables that are included in z_t and i_t), z_t is a vector of non-predetermined endogenous variables, the value of which may depend upon both policy actions and exogenous disturbances at date t, and i_t is a vector of policy instruments, the value of which is chosen in period t. μ_t denotes a vector of exogenous disturbances. We will assume that μ_t is a function of primary i.i.d. shocks, $(e_i)_{-\infty}^{t}$.

We further assume that the evolution of the endogenous variables z_t and Z_t is determined by a system of simultaneous equations

$$F\left(E_t x_{t+1,} x_t, \mu_t\right) = 0. \tag{2.2}$$

We assume that the policy maker minimizes the unconditional expectation of the loss function (2.1) subject to constraint (2.2). That is he searches for a policy rule

$$\varphi\left(Z_{t+1}, E_t z_{t+1,} Z_t, z_t, i_t, \mu_t\right) = 0 \tag{2.3}$$

such that

$$\varphi = \arg\min EL_t(\varphi), \tag{2.4}$$

where E denotes the unconditional expectations operator. We call such a policy "Unconditionally Optimal" and denote it 'UO-policy'.

2.1. Solution

Formally, the unconditional expectation of any function $u(x)$ can be represented in Lebesgue integral form as

$$Eu_t(x_t(\varphi)) = \int u_t(x_t(\varphi, e))de,$$

where de is the Cartesian product probability measure of i.i.d. primary shocks with history, $(de_{t-k})_{k=0}^{\infty}$. We emphasize that de is given exogenously and does not change with policy. To optimize the integral we need to optimize the corresponding Hamiltonian, which is the expression *under* the integral, $u_t(x_t(\varphi, e))$. Intuitively this is plausible as the policy which minimizes a loss function in every state of nature (the components of the sum), will also minimize the expectation (i.e., the sum or integral). With these observations in mind, we employ standard methods of stochastic Lagrange multipliers to solve for unconditionally optimal policy:

- Step 1: Write the Lagrangian function[1]:

$$J = E\left[(1-\beta)\,E_t \sum_{j=0}^{\infty} \beta^j \left[l\left(x_{t+j}, \mu_{t+j}\right) + \xi_{t+j} F\left(E_{t+j} x_{t+1+j,} x_{t+j}, \mu_{t+j}\right)\right]\right].$$

- Step 2: Using the property of unconditional expectations, such that $Ey_t = Ey_{t+j}$, re-formulate this as

$$J = E\left[l\left(x_t, \mu_t\right) + \xi_t F\left(E_t x_{t+1,} x_t, \mu_t\right)\right],$$

  which corresponds to the Hamiltonian

$$H = l\left(x_t, \mu_t\right) + \xi_t F\left(E_t x_{t+1,} x_t, \mu_t\right).$$

- Step 3: Write the necessary first-order conditions for the unconditionally optimal policy with respect to all endogenous variables;

$$\frac{\partial H}{\partial x_t} = \frac{\partial l\left(x_t, \mu_t\right)}{\partial x_t} + \xi_t \frac{\partial F\left(E_t x_{t+1,} x_t, \mu_t\right)}{\partial x_t} + \xi_{t-1} \frac{\partial F\left(x_{t,} x_{t-1}, \mu_{t-1}\right)}{\partial E_t x_{t+1}} = 0. \quad (2.5)$$

The necessary conditions for the optimality of policy φ is that it implies this path for the endogenous variables, x_t, and that there exists Lagrange multipliers, ξ_t, that together satisfy the first order conditions (2.5) and constraints (2.2).[2]

[1] In order to reduce notation when we write ξF we refer to the tensor product, $\sum_{i=1}^{n} \xi_i F_i$.

[2] Step 3 may require further explanation. To obtain it we introduce a new set of variables $y_t = E_t x_{t+1}$. Then the Lagrangian can be written as

$$\begin{aligned} J_1 &= E\left[l\left(x_t, \mu_t\right) + \xi_t F\left(y_t, x_t, \mu_t\right) + \eta_t (y_t - E_t x_{t+1})\right]; \\ &= E\left[l\left(x_t, \mu_t\right) + \xi_t F\left(y_t, x_t, \mu_t\right) + \eta_t y_t - \eta_{t-1} x_t\right]. \end{aligned}$$

The corresponding Hamiltonian will be

$$H_1 = l\left(x_t, \mu_t\right) + \xi_t F\left(y_t, x_t, \mu_t\right) + \eta_t y_t - \eta_{t-1} x_t.$$

We now provide an example of this approach in a very simple LQ New Keynesian model. This is a useful example, however, because the simplicity of the model notwithstanding, analytical derivations of UO policy have not been presented so far.

2.2. Linear-Quadratic Example[3]

We search for an unconditionally optimal policy given the loss function,

$$L_t = (1-\beta) E_t \sum_{j=0}^{\infty} \beta^j \frac{1}{2} \left\{ \pi_{t+j}^2 + \upsilon y_{t+j}^2 \right\}, \tag{2.6}$$

and the Phillips Curve

$$\pi_t = \beta E_t \pi_{t+1} + \gamma y_t + \mu_t. \tag{2.7}$$

Here π_t denotes inflation, y_t is a measure of the output gap and μ_t is a cost-push shock. Following the above steps, we formulate the unconditional Lagrangian:

$$\begin{aligned} J &= E(1-\beta) E_t \sum_{j=0}^{\infty} \beta^j \left(\frac{\pi_{t+j}^2 + \upsilon y_{t+j}^2}{2} + \xi_{t+j} \left(\pi_{t+j} - \beta E_t \pi_{t+j+1} - \gamma y_{t+j} - \mu_{t+j} \right) \right); \\ &= E \left[\frac{\pi_t^2 + \upsilon y_t^2}{2} + \xi_t (\pi_t - \gamma y_t - \mu_t) - \beta \pi_t \xi_{t-1} \right]. \end{aligned}$$

The corresponding Hamiltonian is

$$H_t = \left(\frac{\pi_t^2 + \upsilon y_t^2}{2} \right) + \xi_t (\pi_t - \gamma y_t - \mu_t) - \beta \pi_t \xi_{t-1}.$$

The first order conditions are

$$\begin{aligned} \frac{\partial H_1}{\partial x_t} &= \frac{\partial l(x_t, \mu_t)}{\partial x_t} + \xi_t F'(y_t, x_t, \mu_t) - \eta_{t-1} = 0; && \text{(F1.1)} \\ \frac{\partial H_1}{\partial y_t} &= \xi_t F'(y_t, x_t, \mu_t) + \eta_t = 0. && \text{(F1.2)} \end{aligned}$$

Combining (F1.1) and (F1.2) we receive the first order conditions (2.5).

[3] The extension to general LQ problems is set out in the Appendix.

The first order conditions $\partial H_t/\partial \pi_t$ and $\partial H_t/\partial y_t$ imply

$$\pi_t = \frac{\upsilon}{\gamma}\left(-y_t + \beta y_{t-1}\right). \tag{2.8}$$

This is the unconditionally optimal program proposed by Blake-Jensen-McCallum and proved to be unconditionally optimal by Whiteman (1986).

3. Non-linear Application[4]

The model of Section 2.2 represents an approximation to an underlying, non-linear model. However, some researchers, such as Khan, King and Wolman (2003), prefer to solve a non-linear Ramsey problem and analyze the resulting linearized first-order conditions. In this section we solve for the UO monetary policy of the non-linear model underlying the set-up of Section 2.2. For an appropriate choice of steady state, around which linearization of the first-order necessary conditions takes place, we derive the same optimal policy as in Section 2.2. We also recover a LQ formulation of the model.

In terms of the framework of Section 2, to find a first order approximation to unconditionally optimal policy one log-linearizes the system of first order conditions (2.5) and constraints (2.2) around the deterministic steady state (X, ξ) defined by the system (3.1):

$$\begin{aligned} F(X, X, \mu) &= 0; \\ \frac{\partial l(X,\mu)}{\partial x_t} + \xi \frac{\partial F(X,X,\mu)}{\partial x_t} + \xi \frac{\partial F(X,X,\mu)}{\partial (E_t x_{t+1})} &= 0, \end{aligned} \tag{3.1}$$

where X, ξ and μ indicate the vectors of steady state values of endogenous variables, Lagrange multipliers and the average value of shocks, respectively. We refer to (X, ξ) as the "optimal steady state".

[4] We thank the editor for encouraging us to undertake the analysis in this section.

We can easily derive the first order approximation to (2.5) in the neighborhood of (3.1):

$$\begin{aligned}
F\left(E_t x_{t+1}, x_t, \mu_t\right) &= X\frac{\partial F}{\partial x_t}\widehat{x}_t + X\frac{\partial F}{\partial E_t x_{t+1}}E_t\widehat{x}_{t+1} + \mu\frac{\partial F}{\partial \mu_t}\widehat{\mu}_t + O2; \\
\frac{\partial H}{\partial x_t} &= X\frac{\partial^2 l}{\partial x_t^2}\widehat{x}_t + \mu\frac{\partial^2 l}{\partial x_t \partial \mu_t}\widehat{\mu}_t + \xi\frac{\partial F}{\partial x_t}\widehat{\xi}_t + \xi\frac{\partial F}{\partial E_t x_{t+1}}\widehat{\xi}_{t-1} \\
&\quad +\xi\left[X\frac{\partial^2 F}{\partial x_t^2}\widehat{x}_t + X\frac{\partial^2 F}{\partial x_t \partial E_t x_{t+1}}\left(E_t\widehat{x}_{t+1} + \widehat{x}_{t-1}\right) + \mu\frac{\partial^2 F}{\partial x_t \partial \mu_t}\widehat{\mu}_t\right] \\
&\quad +\xi\left[X\frac{\partial^2 F}{\partial E_t x_{t+1}^2}\widehat{x}_t + \mu\frac{\partial^2 F}{\partial E_t x_{t+1}\partial \mu_t}\widehat{\mu}_{t-1}\right] + O2,
\end{aligned} \tag{3.2}$$

where $O2$ denotes terms of order two, or higher.

3.1. The Model

We now provide a concrete example of this approach. The model can be very briefly laid out as it is developed at length in Woodford (2002) and Damjanovic and Nolan (2006). Household period utility has the form $u_t = \log(Y_t) - \lambda\frac{N_t^{v+1}}{v+1}$, where Y_t is consumption defined over a basket of goods of measure one and indexed by i, in the manner of Spence-Dixit-Stiglitz: $Y_t = \left[\int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}}di\right]^{\frac{\theta}{\theta-1}}$; β is the time discount rate; N_t is labour with $v > 0$. Labour is not firm-specific. The demand for each good is given by $Y_t(i) = \left(\frac{p_t(i)}{P_t}\right)^{-\theta} Y_t$, where $p_t(i)$ is the nominal price of the final good produced by firm i and P_t is the aggregate price level, $P_t = \left[\int_0^1 p_t(i)^{1-\theta}di\right]^{\frac{1}{1-\theta}}$. π_t represents inflation and α is the Calvo parameter and also the fraction of firms with sticky prices; Δ_t is a measure of price dispersion, $\Delta_t = \int_0^1 \left(\frac{p_t(i)}{P_t}\right)^{-\theta\phi} di$. Firms are monopolistic competitors who produce their distinctive goods according to the following technology, $Y_t(i) = A_t\left[N_t(i)\right]^{1/\phi}$, where A_t is a productivity shifter and $\phi > 1$. It follows that the total amount of labour demanded will be $N_t = \int N_t(i)di = \left(\frac{Y_t}{A_t}\right)^{\phi}\int\left(\frac{P_t(i)}{P_t}\right)^{-\theta\phi}di = \left(\frac{Y_t}{A_t}\right)^{\phi}\Delta_t$. Finally μ_t is a stochastic cost-push shock where $\mu \equiv E(\mu_t)$. Parameter τ is the wage tax rate and we define $\Phi := \frac{\theta-1}{\theta}\frac{1-\tau}{\mu}$. So, we are allowing for labour market subsidies. The model can be reduced to three equations: The representative agent's utility (3.3); the pricing equation (3.4) and the law of motion for price dispersion (3.5):

$$E_t \sum_{k=0}^{\infty} \beta^k \left(\log\left(Y_{t+k}\right) - \lambda \Delta_{t+k}^{v+1} \frac{\left(\frac{Y_{t+k}}{A_{t+k}}\right)^{(v+1)\phi}}{v+1} \right); \tag{3.3}$$

$$\left[\frac{1-\alpha\pi_t^{\theta-1}}{1-\alpha}\right]^{\frac{\theta-\theta\phi-1}{\theta-1}} = \frac{E_t \sum_{k=0}^{\infty} (\beta\alpha)^k \frac{\phi\lambda\mu_{t+k}}{\mu} \Delta_{t+k}^v \left(\frac{Y_{t+k}}{A_{t+k}}\right)^{(v+1)\phi} \left(\frac{P_{t+k}}{P_t}\right)^{\theta\phi}}{\Phi E_t \sum_{k=0}^{\infty} (\beta\alpha)^k \left(\frac{P_t}{P_{t+k}}\right)^{1-\theta}}; \tag{3.4}$$

$$\Delta_t - \alpha\Delta_{t-1}\pi_t^{\theta\phi} = (1-\alpha)\left[\frac{1-\alpha\pi_t^{\theta-1}}{1-\alpha}\right]^{\frac{\theta\phi}{\theta-1}}. \tag{3.5}$$

Before proceeding, we need to rewrite the pricing equation (3.4) in canonical form (2.2). For this purpose, the following change of variables proves useful: Let $X_t := E_t \sum_{k=0}^{\infty} (\beta\alpha)^k \left(\frac{P_t}{P_{t+k}}\right)^{1-\theta}$ and rewrite the price-setting equation (3.4) as

$$X_t = 1 + E_t\beta\alpha\pi_{t+1}^{\theta-1}X_{t+1}; \tag{3.6}$$

$$\begin{aligned} & \frac{\phi\lambda}{\Phi}\frac{\mu_t}{\mu}\Delta_t^v \left(A_t^{-1}Y_t\right)^{(v+1)\phi} - \left[\frac{1-\alpha\pi_t^{\theta-1}}{1-\alpha}\right]^{\frac{\theta-\theta\phi-1}{\theta-1}} \\ = \quad & E_t\beta\alpha\left[\left[\frac{1-\alpha\pi_t^{\theta-1}}{1-\alpha}\right]^{\frac{\theta-\theta\phi-1}{\theta-1}} \pi_{t+1}^{\theta-1} - \pi_{t+1}^{\theta\phi}\left[\frac{1-\alpha\pi_{t+1}^{\theta-1}}{1-\alpha}\right]^{\frac{\theta-\theta\phi-1}{\theta-1}}\right] X_{t+1}. \end{aligned}$$

Hence, the unconditional Lagrangian may be written as (3.7)

$$\begin{aligned} L = {} & E\left(\log\left(Y_t\right) - \lambda\Delta_t^{v+1}\frac{\left(A_t^{-1}Y_t\right)^{(v+1)\phi}}{v+1}\right) \\ & + E\rho_t\left(-X_t+1\right) + E\beta\alpha\rho_{t-1}\pi_t^{\theta-1}X_t + E\xi_t\frac{\phi\lambda}{\Phi}\frac{\mu_t}{\mu}\Delta_t^v\left(A_t^{-1}Y_t\right)^{(v+1)\phi} \\ & - E\left[\frac{1-\alpha\pi_t^{\theta-1}}{1-\alpha}\right]^{\frac{\theta-\theta\phi-1}{\theta-1}} \left(\xi_t\left(1+\beta\alpha X_{t+1}\pi_{t+1}^{\theta-1}\right) - \xi_{t-1}\beta\alpha X_t\pi_t^{\theta\phi}\right) \\ & + E\eta_t\left(-\Delta_t + (1-\alpha)\left[\frac{1-\alpha\pi_t^{\theta-1}}{1-\alpha}\right]^{\frac{\theta\phi}{\theta-1}}\right) + E\eta_{t+1}\left(\alpha\Delta_t\pi_{t+1}^{\theta\phi}\right) \end{aligned} \tag{3.7}$$

where ρ_t, ξ_t, and η_t are multipliers for constraints (3.5), and system (3.6).

The first order conditions for the corresponding Hamiltonian are given by system (3.8):

$$\begin{aligned}
Y_t\frac{\partial H}{\partial Y_t} &= \left(1-\phi\lambda\Delta_t^{v+1}\left(A_t^{-1}Y_t\right)^{(v+1)\phi}\right)-\xi_t\left((v+1)\,\phi\frac{\phi\lambda}{\Phi}\frac{\mu_t}{\mu}\Delta_t^{v}\left(A_t^{-1}Y_t\right)^{(v+1)\phi}\right)=0\\
\frac{\partial H}{\partial \Delta_t} &= \lambda\Delta_t^{v-1}\left(A_t^{-1}Y_t\right)^{(v+1)\phi}\left(\Delta_t+v\xi_t\frac{\phi}{\Phi}\frac{\mu_t}{\mu}\right)-\eta_t+\eta_{t+1}\alpha\pi_{t+1}^{\theta\phi}=0\\
\frac{1}{\beta\alpha}\frac{\partial H}{\partial X_t} &= \frac{\rho_{t-1}}{\beta\alpha}\pi_t^{\theta-1}-\rho_t+\xi_{t-1}\left[\pi_t^{\theta-1}\left[\frac{1-\alpha\pi_{t-1}^{\theta-1}}{1-\alpha}\right]^{1-\frac{\theta\phi}{\theta-1}}-\pi_t^{\theta\phi}\left[\frac{1-\alpha\pi_t^{\theta-1}}{1-\alpha}\right]^{1-\frac{\theta\phi}{\theta-1}}\right] \qquad (3.8)\\
\pi_t\frac{\partial H}{\partial \pi_t} &= (\theta-1)\,\beta\alpha\rho_{t-1}\pi_t^{\theta-1}X_t-\eta_t\alpha\theta\phi\left(\left[\frac{1-\alpha\pi_t^{\theta-1}}{1-\alpha}\right]^{\frac{\theta\phi}{\theta-1}-1}\pi_t^{\theta-1}-\Delta_{t-1}\pi_t^{\theta\phi}\right)\\
&\quad-\left[\frac{1-\alpha\pi_t^{\theta-1}}{1-\alpha}\right]^{\frac{\theta\phi}{1-\theta}}\frac{\alpha\left(\theta\phi-\theta+1\right)}{1-\alpha}\pi_t^{\theta-1}\left[\xi_t\left(1+\beta\alpha\pi_{t+1}^{\theta-1}X_{t+1}\right)-\beta\alpha\xi_{t-1}\pi_t^{\theta\phi}X_t\right]\\
&\quad+\beta\alpha\xi_{t-1}X_t\left[\theta\phi\left[\frac{1-\alpha\pi_t^{\theta-1}}{1-\alpha}\right]^{\frac{\theta-\theta\phi-1}{\theta-1}}\pi_t^{\theta\phi}-(\theta-1)\left[\frac{1-\alpha\pi_{t-1}^{\theta-1}}{1-\alpha}\right]^{\frac{\theta-\theta\phi-1}{\theta-1}}\pi_t^{\theta-1}\right]
\end{aligned}$$

The optimal policy rule should solve system $\Upsilon := \{(3.5), (3.6) \text{ and } (3.8)\}$.

3.2. Optimal linear policy and the choice of steady state

We can verify that there is a unique steady state which solves system $\{(3.5), (3.6) \text{ and } (3.8)\}$ for any given level of tax, τ. However price stability, $\pi = 1$, will be optimal steady state policy if and only if the level of subsidies is optimal: $\Phi = 1$, and $\tau = 1 - \mu\frac{\theta}{\theta-1} < 1$. Otherwise, the optimal deterministic steady-state policy would imply a trend in inflation. Log linearization of the model around trend inflation is straightforward but rather messy (For example see Damjanovic and Nolan, 2006). In order to keep things uncluttered, we will consider the case when price stability is optimal. That implies $\pi = 1$, $\Delta = 1$, and $X = \frac{1}{1-\alpha\beta}$. The steady state value of output corresponds to $\phi\lambda\,(Y/A)^{(v+1)\phi} = 1$. The steady state values of the Lagrange multipliers, using (3.8), are $\xi = 0$, $\eta = -\frac{1}{(1-\alpha)\phi}$, and $\rho = 0$.

Further, following conventional notation we define $\widehat{y}_t := \log(Y_t/Y) -$

$\log\left(A_t/A\right)$, $\widehat{\pi}_t := \log \pi_t$, $\widehat{X}_t := \log(X_{t+1}/X)$, $\widehat{\eta}_t := \log\left(\eta_t/\eta\right)$, $\widehat{\mu}_t := \log(\mu_t/\mu)$. We define $\widehat{\xi}_t := \log(1+\xi_t)$, $\widehat{\rho}_t := \log(1+\rho_t)$, which implies that up to second order $\xi_t := \widehat{\xi}_t + O2$, $\rho_t := \widehat{\rho}_t + O2$.

The linearization of the Phillips curve and law of motion of prices yields

$$\begin{aligned}
\widehat{X}_t &= E_t \alpha\beta \left((\theta - 1)\,\widehat{\pi}_{t+1} + \widehat{X}_{t+1} \right) + O2; \\
\widehat{y}_t + \phi\widehat{\Delta}_t &= E_t \frac{1}{1-\beta\alpha}\frac{-\theta+\theta\phi+1}{(v+1)\,\phi}\frac{\alpha}{1-\alpha}\left[\widehat{\pi}_t - \beta\widehat{\pi}_{t+1}\right] - \frac{1}{(v+1)\,\phi}\widehat{\mu} + O2; \\
\widehat{\Delta}_t &= \alpha\widehat{\Delta_{t-1}} + O2,
\end{aligned}$$

while the first order conditions imply:

$$\begin{aligned}
\frac{Y_t}{(v+1)\,\phi}\frac{\partial H}{\partial Y_t} &= \widehat{y}_t - \widehat{\xi}_t + O2 = 0; \\
\frac{\partial H}{\partial \Delta_t} &= \widehat{y}_t - \frac{1}{(1-\alpha)\,\phi}\left(\widehat{\eta}_t - \alpha\left(\widehat{\eta}_{t+1} + \theta\phi\widehat{\pi}_{t+1}\right)\right) + O2; \\
\frac{\partial H}{\partial X_t} &= -\widehat{\rho}_t + \beta\alpha\widehat{\rho}_{t-1} + O2 = 0; \\
\pi_t \frac{\partial H}{\partial \pi_t} &= (\theta-1)\frac{\beta\alpha}{1-\alpha\beta}\widehat{\rho}_{t-1} + \frac{\alpha}{1-\alpha}\frac{(-\theta+\theta\phi+1)}{1-\alpha\beta}\left(\widehat{\xi}_t - \beta\widehat{\xi}_{t-1}\right) \\
&\quad + \frac{\theta\,(-\theta+\theta\phi+1)}{(1-\alpha)}\frac{\alpha}{1-\alpha}\widehat{\pi}_t + \frac{\alpha\theta}{(1-\alpha)}\Delta_{t-1} + O2.
\end{aligned}$$

From these relations we can solve for the optimal policy rule:

$$\widehat{\pi}_t = \frac{(1-\alpha)}{(1-\alpha\beta)\,\theta}\left(-\widehat{y}_t + \beta\widehat{y}_{t-1}\right). \tag{3.9}$$

Thus, for this particular assumption about subsidies, the non-linear problem can be easily nested to the earlier LQ example considered in Section 2.2. The second order approximation to the period objective function is

$$E\left(\log\left(Y_t\right) - \lambda\Delta_t^{v+1}\frac{\left(A_t^{-1}Y_t\right)^{(v+1)\phi}}{v+1}\right) = -E\left(\frac{1}{\phi}\widehat{\Delta}_t + \frac{1}{2}\frac{(v+1)}{\phi}\widehat{\Delta}_t^2 + \frac{1}{2}(v+1)\,\phi\widehat{y}_t^2\right) + O3,$$

and the second order approximation to the law of motion of price dispersion is

$$\widehat{\Delta}_t = \alpha\widehat{\Delta_{t-1}} + \frac{1}{2}\frac{\alpha}{1-\alpha}\theta\phi\left(\theta\phi+1-\theta\right)\widehat{\pi}_t^2 + O3. \tag{3.10}$$

This implies that the expectation of price dispersion is a second order variable: $E\widehat{\Delta_t} = E\frac{1}{2}\frac{\alpha}{(1-\alpha)^2}\theta\phi\left(\theta\phi+1-\theta\right)\widehat{\pi}_t^2$. Therefore, in the case of optimal subsidies the maximization of the unconditional expectation of (3.3), subject to (3.4) and (3.5) can be written in linear quadratic form as in Section 2.2., with $\upsilon := \frac{(1-\alpha)^2}{\alpha}\frac{(v+1)\phi}{\theta(\theta\phi+1-\theta)}$; and $\gamma := \frac{1-\alpha}{\alpha}\left(1-\beta\alpha\right)\frac{(v+1)\phi}{1-\theta+\theta\phi}$. We conclude that solution (3.9) is the same as (2.8).

4. Discussion

Two distinguishing attributes of UO policies are worth commenting on briefly. The first issue is how UO policies deal with initial conditions; and the second issue concerns the impact of discounting.

4.1. The distribution of initial conditions

A key attribute of unconditionally optimal (monetary) policy is how it takes account of the initial conditions that face current and future policymakers. In a sense, one can think of UO policy as internalizing the distribution of initial conditions. And even when models lack 'jump variables' the UO policy still impacts on the distribution of initial conditions. To see this, note that the discounted loss function, $L_t(\cdot)$, very generally depends upon two factors, initial conditions, X_{t-1}, and the policy adopted by the government P, $L_t(X_{t-1}, P)$. The conditionally optimal policy minimizes the loss function

$$P_c = \arg\min L_t(X_{t-1}, P),$$

taking the initial conditions as given. In models without jump variables the same policy will generally be optimal for all initial conditions, (see Walsh 2003). However, the initial conditions depend on the policy of predecessors as well as on the shocks: $X_{t-1} = X\left(P, e_{t-}\right)$, where $e_{t-} := \{e_{t-k}\}_{k=0}^{\infty}$ is the history of primary shocks. For any particular history of shocks, there will generally be a policy which, had the previous policymaker adopted it, would have bequeathed its successor with better (indeed, the best) initial conditions. That is,

$$P_e\left(e_{t-}\right) = \arg\min L_t(X_{t-1}\left(P, e_{t-}\right), P). \tag{4.1}$$

The choice of this policy will depend on the realization of the shock. Since (e_{t-}) is stochastic, the policymaker will generally wish to revise its policy each period. The unconditionally optimal policy minimizes the loss function (4.1) "on average" across all possible histories of shocks

$$P_u = \arg\min \int L_t(X_{t-1}\left(P, e_{t-}\right), P) d\left(e_{t-}\right). \tag{4.2}$$

4.2. Invariance with respect to the social discount rate

Taylor (1979, p.1278-9) suggests that an infinite horizon perspective with no discounting may be hallmarks of optimal (time consistent) policy. That may be contentious (in models with forward-looking constraints, at any rate), but the sense in which discounting is irrelevant is not fully spelled out. However, we can establish that the choice of social discount rate is, in a sense, irrelevant from the perspective of unconditional optimality. Hence, we attribute this proposition to Taylor:

Proposition 4.1. *(Taylor, 1979) The time preference parameter in loss function (2.1) is not important for the UO policymaker. That is, the best UO policy minimizes losses (4.3) for all exogenous discount functions* $\rho = \left\{\rho_j\right\}_{j=0}^{\infty}$, *such that* $\sum_{j=0}^{\infty} \rho_j = \Gamma < \infty$.

$$EL_t\left(\rho\right) = EE_t \sum_{j=0}^{\infty} \rho_j l_{t+j}. \tag{4.3}$$

Here, l_t *denotes the period loss function.*

Proof. *It follows immediately that,*

$$\arg\min_{\varphi'} EL_t(\gamma, \varphi) = \arg\min_{\varphi'} \sum_{j=0}^{\infty} \rho_j El_t(\varphi) = \arg\min_{\varphi'} \Gamma El_t(\varphi)$$

Hence, we have proved that the same policy is unconditionally optimal for any time invariant discounting.

Proposition (4.1) is interesting as it demonstrates that the same policy is unconditionally optimal for all households, regardless of their individual time discount factors. For example, if we assume that the time discount rate does not depend on current welfare, the unconditionally optimal policy would not depend on the time-discounting function. Further, we may consider an overlapping generations economy, or economy with hyperbolic time discounting, or any time and condition invariant mixture of economic agents with different time discounting. The 'best-on-average' criterion avoids the need for one to take a stand on what is the appropriate social discount rate; see the discussions of these issues in Barro (1999) and Somers (1971). Of course, this issue was famously raised by Ramsey (1928).

5. Conclusion

The simple procedure we have presented for uncovering UO policies appears to be useful in a wide variety of environments of practical interest to researchers. An interesting and important question is whether actual monetary (and other) policies are, or should be, optimal from the unconditional perspective.

Appendix: Unconditional optimization for a general LQ problem

- Step 1: Write the 'conditional Lagrangian' for the policy problem:

$$J_t = E_t \sum_{j=0}^{\infty} \beta^j \left[\frac{1}{2} \left(x_{t+j} - x^*_{t+j} \right)' Q \left(x_{t+j} - x^*_{t+j} \right) + \xi'_{t+j} \left(\widetilde{A} x_{t+j} - \widetilde{I} x_{t+j+1} \right) \right].$$

x^* is a vector of target values which could depend on disturbance terms, and Q is a symmetric, positive definite matrix. x_t is defined as in the main text. The evolution of the endogenous variables z_t and Z_t is determined by a system of simultaneous equations

$$\widehat{I} \begin{bmatrix} Z_{t+1} \\ E_t z_{t+1} \end{bmatrix} = A \begin{bmatrix} Z_t \\ z_t \end{bmatrix} + B i_t + C \mu_t,$$

where $B = \begin{bmatrix} 0 \\ B \end{bmatrix}$, $C = \begin{bmatrix} 0 \\ C \end{bmatrix}$ and μ_t is a vector of exogenous disturbances, all mean zero.

- Step 2: Re-formulate this as an unconditional Lagrangian:

$$J = E J_t.$$

Since, $E x_t = E x_{t+j}$, we can write

$$J = \frac{1}{1-\beta} E \left(\frac{1}{2} \left(x_t - x^*_t \right)' Q \left(x_t - x^*_t \right) + \xi'_t \widetilde{A} x_t - \xi'_{t-1} \widetilde{I} x_t \right),$$

which corresponds to the Hamiltonian

$$H = \frac{1}{1-\beta} \left(\frac{1}{2} \left(x_t - x^*_t \right)' Q \left(x_t - x^*_t \right) + \xi'_t \widetilde{A} x_t - \xi'_{t-1} \widetilde{I} x_t \right).$$

- Step 3: Write the first-order conditions for the optimal policy with respect to all endogenous variables;

$$\frac{\partial H}{\partial x_t} = \frac{1}{1-\beta} \left(\left(x_t - x^*_t \right)' Q + \xi'_t \widetilde{A}_t - \xi'_{t-1} \widetilde{I} \right) = 0. \tag{A}$$

Condition (A) implies the following dynamics for the Lagrange multipliers

$$\left(x_t - x^*_t \right)' Q + \xi'_t \widetilde{A}_t - \xi'_{t-1} \widetilde{I} = 0.$$

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