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FIXED POINTS THEOREMS VIA NASH EQUILIBRIA

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Abstract. In this note we show that the fixed points of a continuous function (or of an upper semi-continuous correspondence, with non-empty and convex values) can be attained as Nash Equilibria of a game with finitely many players.

1. Introduction

In the non-cooperative game theory, the proofs of existence of a Nash Equilibrium for static games, with convex sets of strategies for the players, were uniformly obtained by the application of a fixed point theorem.

In fact, in seminal works, Nash [5,6] show that the existence of equilibria for non-cooperative static games is a direct consequence of Kakutani [4] (or Brouwer [2]) Fixed Point Theorem. That is, he shows that, given a game, there always exists a upper semi-continuous correspondence such that, all the equilibriums points for the game are fixed points for the correspondence and vice versa. In a extension of the Nash Equilibrium Theorem, Debreu [3] shows that for *social systems*[1] (i.e. games when not only the objective function, but also the strategies available to a player depend on the choices of the other agents) the equilibrium existence is a consequence of Kakutani Theorem too.

Thus it is natural to ask whether the fixed points arguments are in fact necessary tools to guarantee the Nash equilibrium existence.[2]

So in this note we study the conditions that guarantee that the fixed points of a continuous function (or of an upper semi-continuous correspondence) can be attained as Nash Equilibria of a non-cooperative game.

We show that every fixed point of a continuous function is a Nash Equilibrium of a static game, in which the objective functions of the players are strictly quasi-concave. Moreover, every fixed point of an upper semi-continuous correspondence that satisfy certain regularity conditions (as non-empty and convex values), is a Nash Equilibrium of a game with quasi-concave players' objective functions.

As a corollary, we obtain the equivalence between the Nash Equilibrium Existence Theorem [5,6] and the Debreu's Social Equilibrium Existence Theorem [3], and therefore the dependence of the admissible strategies of the agents, on the strategies choices for the other players can be avoided, that is, every equilibrium point of a social system can be attained as an equilibrium point for a game, with only the players' objective functions depend on the strategies of the other players.

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[1]Also called "abstract economies" or "generalized games"

[2]In the general equilibrium theory, the proofs of existence of economic equilibrium also depend on the application of fixed point theorems of Brouwer or Kakutani type. Usawa [7] showed that the Walrasian Equilibrium Existence Theorem implies the Kakutani theorem (and therefore, implies the Brouwer fixed point theorem too). For more details about Usawa's result, and other interconnections between economic theorems and fixed point theorems, see Border [1].

In a recent work, Zhao [8] shows the equivalence between Nash Equilibrium Existence Theorem and Kakutani (or Brouwer) Fixed Point Theorem in an indirect way. As he points out, a constructive proof is preferable.

2. Results

For a game in normal form $\mathcal{G} = \{I, S_i, v^i\}$, where each player $i \in I = \{1, 2, \dots, n\}$ is characterized by a non-empty set of strategies $S_i \subset \mathbb{R}^{n_i}$, and by a objective function $v^i : \Pi_{j=1}^n S_j \to \mathbb{R}$, a *Nash Equilibrium* is a vector of strategies $\overline{s} = (\overline{s}_1, \overline{s}_2, \dots, \overline{s}_n) \in \Pi_{i=1}^n S_i$, such that,

$$v^i(\overline{s}_i, \overline{s}_{-i}) \geq v^i(s, \overline{s}_{-i}), \quad \forall s \in S_i, \ \forall i \in I, \tag{1}$$

where $\overline{s}_{-i} = (\overline{s}_1, \dots, \overline{s}_{i-1}, \overline{s}_{i+1}, \dots, \overline{s}_n)$.

In order to allow greater applicability to economic problems, Debreu extended the concept of Nash Equilibria to *social systems*, where not only the objective functions, but also the admissible strategies of the players depend on the choices of the other players in the game.

Thus, given a social system $\Gamma = \{I, S_i, v^i, \Gamma_i\}$ where, for each player $i \in I$, the correspondence $\Gamma_i : \Pi_{j \neq i} S_j \twoheadrightarrow S_i$ determines the admissible strategies in S_i, given the actions of the other players in $\Pi_{j \neq i} S_j$, a *Social Equilibrium* is a vector of actions $\overline{s} = (\overline{s}_1, \overline{s}_2, \dots, \overline{s}_n) \in \Pi_{i=1}^n S_i$, such that, $\overline{s}_i \in \Gamma_i(\overline{s}_{-i})$ and

$$v^i(\overline{s}_i, \overline{s}_{-i}) \geq v^i(s, \overline{s}_{-i}), \quad \forall s \in \Gamma_i(\overline{s}_{-i}), \ \forall i \in I.$$

We suppose the the sets of strategies and the objective functions satisfy the following conditions,

(H-1) The sets S_i are non-empty, compact and convex subsets of $\mathbb{R}^{n_i}$. The objective functions for the players $\{v^i\}_{i \in I}$ are continuous in $\Pi_{i=1}^n S_i$.

(H-2) The correspondences of admissible strategies, $\{\Gamma_i\}_{i \in I}$ are continuous and have non-empty and convex values.

Now, we state the equivalence between the Nash Equilibrium Existence Theorem and Brouwer Fixed Point Theorem:

The following statements are equivalent:

i. *(Brouwer [2]) Given a non-empty, compact and convex set $X \subset \mathbb{R}^m$, every continuous function $f : X \to X$ have at least a fixed point in X.*

ii. *(Nash [6]) Given a game $\mathcal{G} = \{I, S_i, v^i\}$ that satisfies hypothesis (H-1), if the objective functions v^i are* **strictly quasi-concave** *in s_i, then there exists a Nash Equilibrium for the game.*

iii. *(Debreu [3]) Given a social system $\Gamma = \{I, S_i, v^i, \Gamma_i\}$ that satisfy hypotheses (H-1) and (H-2), if the objective functions v^i are* **strictly quasi-concave** *in s_i, then there exists a Social Equilibrium.*

In 1952, Gerard Debreu show that the Kakutani theorem guarantees the existence of a Social Equilibrium in a game where the objective functions of the agents are quasi-concave. In the case that the objective functions are strictly quasi-concave, the correspondence of optimal strategies for the players are uni-valued, and therefore the Brouwer Fixed Point Theorem is sufficient to ensure the result. Thus *(i.)* implies *(iii.)*.

Its clear that *(iii.)* implies *(ii.)*. Therefore, it is sufficient to show that the existence of Nash Equilibrium, in a game that satisfies hypothesis in *(ii.)* guarantees the Brouwer Fixed Point Theorem.

Now, given a non-empty, convex and compact set $X \subset \mathbb{R}^m$ and a continuous function $f : X \to X$, consider a game $\mathcal{G}$ with two players $I = \{A, B\}$, that are characterized by the set of strategies:

$S_A = S_B = X$ and by the objective functions,

$$v^A(x_A, x_B) = -\sum_{i=1}^{n} ((x_A)_i - (x_B)_i)^2\,, \quad v^B(x_A, x_B) = -\sum_{i=1}^{n} (f_i(x_A) - (x_B)_i)^2\,, \tag{2}$$

where $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$.

The sets S_A and S_B satisfy the hypothesis (H-1) and the objective functions are continuous in $S_A \times S_B$ and strictly quasi-concave in own strategy.

Therefore, there exists a Nash Equilibria $(\overline{x}_A, \overline{x}_B)$. Moreover, it follows from the optimality of $\overline{x}_A$ for the player A (given $\overline{x}_B$) that $\overline{x}_A = \overline{x}_B$; and the optimality of strategy $\overline{x}_B$ for the player B (given the strategy $\overline{x}_A$ of the agent A) implies that $\overline{x}_B = f(\overline{x}_A)$. This completes the proof that *(ii.)* implies *(i.)*.

We now show that given an upper semi-continuous correspondence $\Phi : X \twoheadrightarrow X$, that satisfies some regularity conditions, there is a non-cooperative static game, with a finite number of players, whose Nash Equilibria coincides with the fixed points of Φ. Thus, we show the equivalence between the Nash Equilibrium Existence Theorem (in the case that the objective functions are quasi-concave and the strategy sets are intervals) and a weak version of Kakutani Fixed Point Theorem,

The following statements are equivalent:

i'. *Given a non-empty, compact and convex set $X \subset \mathbb{R}^m$, every upper semi-continuous correspondence $\Phi : X \twoheadrightarrow X$, that has non-empty and convex values, and that satisfies*[3]

$$\Phi(x) = \pi_1^m(\Phi(x)) \times \pi_2^m(\Phi(x)) \times \ldots \times \pi_m^m(\Phi(x)), \tag{3}$$

has a fixed point. That is, there is $\overline{x} \in X$ such that $\overline{x} \in \Phi(\overline{x})$.

ii'. *Given a game $\mathcal{G} = \{I, S_i, v^i\}$ that satisfies hypothesis (H-1), if the objective functions v^i are* **quasi-concave** *in $s_i \in S_i \subset \mathbb{R}$, then there exists a Nash Equilibrium for the game.*

iii'. *Given a social system $\Gamma = \{I, S_i, v^i, \Gamma_i\}$ that satisfies hypotheses (H-1) and (H-2), if the objective functions v^i are* **quasi-concave** *in $s_i \in S_i \subset \mathbb{R}$, then there exists a Social Equilibrium.*

Notice that in the proof of the Equilibrium Existence Theorem (Debreu [3]) for social systems, when the sets of strategies are subset of real line it is only necessary the weak version of Kakutani theorem given by *(i'.)*. Thus, analogous to the equivalence result between Nash's and Brouwer's Fixed Point Theorems, it is sufficient to show that *(ii'.)* implies *(i'.)*.

Given a non-empty, convex and compact set $X \subset \mathbb{R}^m$, consider an upper semi-continuous correspondence $\Phi : X \twoheadrightarrow X$, that has non-empty and convex values, and that satisfies the condition given by equation (3).

For each $j = 1, 2, \ldots, m$, define the functions $p_j^m : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^{m+1}$ as $p_j^m(x; y) = (x; y_j)$, and consider a game $\mathcal{G}$ with $(m+1)$ players, characterized by the sets of strategies $S_0 = X$, $S_i = \pi_i^m(X)$; and by the objective functions given by

$$v^0(x_0, x_1, \ldots, x_n) = -\sum_{i=1}^{n} ((x_0)_i - x_i)^2\,, \tag{4}$$

$$v^i(x_0; x_1, \ldots, x_n) = -\min_{(y_0, y_i) \in p_i^m(Gr[\Phi])} \|(x_0; x_i) - (y_0; y_i)\|_{\max}, \quad \forall i \in \{1, 2, \ldots, m\}. \tag{5}$$

where $\|\cdot\|_{\max}$ denotes the max-norm in $\mathbb{R}^{m+1}$ (see Appendix), and $Gr[\Phi] \equiv \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m\,;\ y \in \Phi(x)\}$ denotes the graph of the correspondence Φ, which is a non-empty and compact subset of $\mathbb{R}^m \times \mathbb{R}^m$. Thus, the set $p_i^m(Gr[\Phi])$ is a non-empty and compact subset of $\mathbb{R}^m$.

[3] For each $j \in \{1, 2, \ldots, m\}$, the projection $\pi_j^m : \mathbb{R}^m \to \mathbb{R}$ is given by, $\pi_j^m(x_1, x_2, \ldots, x_m) = x_j$.

Now, it is clear that the set of strategies and the objective functions of the players satisfy the hypothesis (H-1). Moreover, the function v^0 is quasi-concave, and follows from Lemma 1 (see Appendix) that the functions v^i are quasi-concave, for each $i \in \{1, 2, \ldots, m\}$.

Therefore, there is a Nash equilibrium $(\overline{x}_0; \overline{x}_1, \ldots, \overline{x}_m)$ for the game $\mathcal{G}$. Since $\overline{x}_i$ is a optimal strategy for the player i, given the strategy $\overline{x}_0$, it follows that $\overline{x}_i \in \pi_i^m(\Phi(\overline{x}_o))$, for each $i \in \{1, 2, \ldots, m\}$.

Thus, from equation (3) it follows that $\overline{y} \equiv (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_m) \in \Phi(\overline{x}_0)$, and, since $\overline{x}_0$ is optimal given $\overline{y} \in X$, we have that $\overline{x}_0 = \overline{y}$. This concludes the proof.

Finally, it is important to remark that, given a social system Γ where either the objective functions of the players are strictly quasi-concave or the set of strategies are subset of real line, there is always a game $\mathcal{G}$ such that all social equilibrium points of Γ are Nash Equilibria of $\mathcal{G}$, and vice versa. Thus, the dependence of the strategies of the players on the actions of the other agents can be avoided, at least in these cases.

Appendix

Let $\|\cdot\|$ be the max-norm in $\mathbb{R}^{m+1}$, i.e., $\|(x_1, \ldots, x_{m+1})\| = \max_{1 \leq i \leq m+1} |x_i|$. Given $x \in \mathbb{R}^{m+1}$ and a non-empty set $Z \subset \mathbb{R}^{m+1}$, the distance from x to Z is $d(x, Z) = \inf_{z \in Z} \|x - z\|$.

The function $x \mapsto d(x, Z)$ is continuous, because $|d(x_1, Z) - d(x_2, Z)| \leq \|x_1 - x_2\|$.

Lemma 1. *Let $\mathbb{R}^{m+1} = \mathbb{R}^m \times \mathbb{R}$ and let $\pi : \mathbb{R}^{m+1} \to \mathbb{R}^m$ be the associated projection. Let $Z \subset \mathbb{R}^{m+1}$ be a non-empty compact set such that $\pi(Z)$ is convex and $Z \cap \pi^{-1}(x)$ is convex for every $x \in \mathbb{R}^m$. Let $f : \mathbb{R}^{m+1} \to \mathbb{R}$ be defined by $f(x, t) = d((x, t), Z)$, $x \in \mathbb{R}^m$, $t \in \mathbb{R}$. Then, for every x, $f(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is a quasi-convex function.*

Proof: Fix an arbitrary $x_0 \in \mathbb{R}^{m+1}$. We have to show that $L_c = \{t \in \mathbb{R};\ d((x_0, t), Z) \leq c\}$ is a convex set for every $c \geq 0$. So, fix $c \geq 0$ and assume, by contradiction, that there exists $t_1 < t_* < t_2$ such that t_1, $t_2 \in L_c$ and $t_* \notin L_c$.

Consider the following subsets of $\mathbb{R}^n$:

$$\begin{aligned} A &= \{x \in \pi(Z);\ \|x - x_0\| \leq c\}, \\ A_1 &= \{x \in A;\ \exists t \in \mathbb{R} \text{ s.t. } (x, t) \in Z \text{ and } t \leq t_* - c\}, \\ A_2 &= \{x \in A;\ \exists t \in \mathbb{R} \text{ s.t. } (x, t) \in Z \text{ and } t \geq t_* + c\}. \end{aligned}$$

Since $d((x_0, t_*), Z) > c$, we have $A = A_1 \cup A_2$. Moreover, $A_1 \cap A_2 = \emptyset$. (Because if $x \in A_1 \cap A_2$ existed then $\|x - x_0\| \leq c$ and, by the convexity of $Z \cap \pi^{-1}(x)$, $(x, t_*) \in Z$, contradicting $d((x_0, t_*), Z) > c$.) Since Z is compact, A_1 and A_2 are compact as well. And since $\pi(Z)$ is convex, so is A. In particular, A is connected. Therefore, $A_1 = \emptyset$ or $A_2 = \emptyset$.

On the other hand, $d((x_0, t_1), Z) \leq c$, so there exists a point $(x', t') \in Z$ c-close to (x_0, t_1). Then $\|x' - x_0\| \leq c$ and $t' \leq t_1 + c < t_* + c$, therefore $x' \in A_1$, proving that $A_1 \neq \emptyset$. Analogously, it follows from $d((x_0, t_2), Z) \leq c$ that $A_2 \neq \emptyset$. We have obtained a contradiction. □

Given the non-empty and compact set $Z = p_i\left(Gr[\Phi]\right)$, we have that the sets $\pi(Z) = X$ and $Z \cap \pi^{-1}(x)$ are convex, because the correspondence Φ have convex domain and convex values. Therefore, follows from the former lemma that the function $v^i(x, y) = -d\left((x, y_i); p_i^m\left(Gr[\Phi]\right)\right)$ is quasi-concave in y_i.

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References

[1] Border, K. (1985),"Fixed point theorems with aplications to economics and game theory", *Cambridge University Press,* Cambridge, UK;
[2] Brouwer, L. (1912),"Uber abbildungen von Mannigfaltigkeiten", *Mathematische Annalen* 71, 97-115;
[3] Debreu, G. (1952), "A Social Equilibrium Existence Theorem," *Procedings of the National Academy of Science* 38, 886-893;
[4] Kakutani, S. (1941), "A generalization of Brouwer's Fixed Point Theorem," *Duke Mathematical Journal* 8, 457-459;
[5] Nash, J.F. (1950), "Equilibrium Points in N-Person Games," *Procedings of the National Academy of Science* 36, 48-49;
[6] Nash, J.F. (1951), "Non-cooperative Games," *Annals of Mathematics* 54, 289-295;
[7] Uzawa, H. (1962), "Walras' Existence Theorem and Brouwer's Fixed Point Theorem", *Economic Studies Quartely*, 13, 59-62;
[8] Zhao, J. (2002), "The Equivalence between Four Economic Theorems and Brouwer's Fixed Point Theorem", *working paper*, Departament of Economics, Iowa State University.

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