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INFINITE HORIZON ECONOMIES WITH BORROWING CONSTRAINTS

EMMA MORENO-GARCÍA AND JUAN PABLO TORRES-MARTÍNEZ

Abstract. We consider infinite horizon economies with incomplete financial markets. Securities are in positive net supply and may be infinite-lived. We establish existence of equilibria by requiring borrowing constrains instead of portfolio restrictions.

Keywords: Equilibrium, Infinite horizon incomplete markets, Infinite-lived real assets.

1. Introduction

Ponzi schemes need to be avoided to obtain existence of equilibria in infinite horizon incomplete markets. Thus, debt constraints or transversality conditions have been required to assure that agents do not postpone, *ad infinitum*, the payments of their commitments. Within this context, Magill and Quinzii (1994) and Levine and Zame (1996) show that equilibrium exists when financial markets are composed by short-lived numeraire or nominal assets. Also, Hernandez and Santos (1996) prove the existence of equilibrium when only one infinite-lived real asset, in positive net supply, is available for trade.

Since conventional debt constraints bound the portfolio market value, agents can have more access to credit in any asset just by increasing their investment in the other securities. Thus, it becomes difficult to bound the amount of borrowing. As a consequence, when assets live for more than one period, finite horizon economies, that are obtained by truncating the infinite horizon economy in order to prove equilibrium existence, may not have equilibrium. For this reason, when long-lived or infinite lived assets are available, equilibrium existence has been guaranteed at most generically (see, for instance, Magill and Quinzii (1996) and Hernandez and Santos (1996)).

The aim of this paper is to show the existence of equilibrium in a market where infinite-lived assets can be traded. To prevent Ponzi schemes, the amount of borrowing that each agent is able to get becomes dependent on the market value of her physical endowments.

We follow the classical approach that finds an equilibrium as a limit of equilibria corresponding to a sequence of finite horizon economies. As a first step, we show a result of equilibrium existence for truncated economies by defining associated generalized games and showing that equilibrium asset prices are uniformly bounded. Note that, a positive lower bound for asset prices leads to short-sales constraints induced by borrowing restrictions. Moreover, as the amounts of borrowing will be bounded by the market value of commodity a bundle, an upper bound for the asset prices leads

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to a natural restriction on the set of prices that is selected in the generalized game. Thus, we can guarantee the non-emptiness of the interior of the budget constraint correspondences.

In a second step, we check the asymptotic properties of individual debt, namely, transversality conditions, which are actually obtained as a consequence of the structure of restrictions on borrowing. Indeed, since under Kuhn-Tucker multipliers the discounted value of individual wealth will be finite, borrowing constraints will prevent agents to be borrowers at infinity.

We remark that economies where physical endowments have no strictly positive lower bound are included within the framework stated in this paper. Furthermore, although utilities are separable in time and states of nature, hyperbolic discounting is also compatible with our assumptions. In addition, when at each node of the economy there is only one asset to be traded, we can go further and assure that borrowing constraints become non-binding.

The remainder of the paper is organized as follows. Section 2 presents the model. In Section 3 we state our main result of equilibrium existence. Section 4 is devoted to discuss some particular cases of our analysis. Finally, an Appendix includes the proof of the equilibrium existence result.

2. Model

We consider a discrete time economy with infinite horizon. Let S denote the non-empty set of states of nature. At each date, individuals have common information about the realization of the uncertainty. Let $\mathcal{F}_t$ be the information available at date $t \in \{0, 1, \ldots\}$, which is given by a finite partition of S. For simplicity, we assume that there is no loss of information along the event-tree, i.e. $\mathcal{F}_{t+1}$ is finer than $\mathcal{F}_t$, for each $t \geq 0$. Moreover, no information is available at $t = 0$, i.e. $\mathcal{F}_0 = S$.

A pair $\xi = (t, \sigma)$, where $t \geq 0$ and $\sigma \in \mathcal{F}_t$, is called a *node* of the economy. The date associated to ξ is denoted by $t(\xi)$. The set of all nodes, called the *event-tree*, is denoted by D. Given $\xi = (t, \sigma)$ and $\mu = (t', \sigma')$, we say that μ is a *successor* of ξ, and we write $\mu \geq \xi$, if $t' \geq t$ and $\sigma' \subset \sigma$. Let ξ^+ be the set of immediate successors of ξ, that is, the set of nodes $\mu \geq \xi$, where $t(\mu) = t(\xi) + 1$. The (unique) predecessor of ξ is denoted by ξ^- and ξ_0 is the node at $t = 0$. Let $D(\xi) := \{\mu \in D : \mu \geq \xi\}$, $D^T(\xi) := \{\mu \in D(\xi) : t(\mu) \leq T + t(\xi)\}$ and $D_T(\xi) := \{\mu \in D(\xi) : t(\mu) = T + t(\xi)\}$.

At each $\xi \in D$ there is a finite ordered set, L, of perishable commodities that can be traded in spot markets. Let $p(\xi) = (p_l(\xi); l \in L) \in \mathbb{R}^L_+$ be the price system of goods at ξ. Also, the process of commodity prices is denoted by $p = (p(\xi); \xi \in D)$.

There is an ordered set J of long-lived real assets that can be negotiated in the economy. Each asset $j \in J$ is characterized by the node in which it is issued, $\xi_j \in D$, by the maximum number of period in which it can be negotiated, $T_j \in \mathbb{N} \cup \{+\infty\}$, and by (unitary) real payments, $A(\mu, j) \in \mathbb{R}^L_+$, where $\mu \in D^{T_j}(\xi_j) \setminus \{\xi_j\}$. We assume that, for each $j \in J$, $(A(\mu, j), \mu \in D^{T_j}(\xi_j) \setminus \{\xi_j\}) \neq 0$. Thus, by construction, we avoid *fiat money* in our economy.

At each node the number of issued asset is finite. Therefore, the set $J(\xi) = \{j \in J : (\xi \in D^{T_j - 1}(\xi_j)) \wedge (\exists \mu > \xi, A(\mu, j) \neq 0)\}$, formed by the assets that can be negotiated at ξ, is either empty or finite. Note that if $j \in J$ is an infinite-lived asset, then for every $T > 0$, there exists $\xi \in D_T(\xi_j)$ such that $j \in J(\xi)$.

Let $q(\xi) = (q_j(\xi);\, j \in J(\xi))$ be the vector of asset prices at ξ. Analogously, $q = (q(\xi); \xi \in D)$ denotes the process of asset prices in the economy. Define $D(J) = \{(\xi, j) \in D \times J : j \in J(\xi)\}$.

A finite number of agents, $h \in H$, can trade securities and buy commodities at each node in the event-tree. Each $h \in H$ is characterized by her physical and financial endowments, $(w^h(\xi), e^h(\xi)) \in \mathbb{R}^L_{++} \times \mathbb{R}^{J(\xi)}_+$, at each $\xi \in D$, and by her preferences on consumption, which are represented by an utility function $U^h : \mathbb{R}^{D\times L}_+ \to \mathbb{R}_+ \cup \{+\infty\}$.

For each $j \in J(\xi)$, $\overline{e}^h_j(\xi) = \sum_{\xi_j \le \mu \le \xi} e^h_j(\mu)$ denotes the vector of aggregated financial endowments received by agent h up to node ξ, where $e^h_j(\mu)$ is the quantity of asset j received by agent h at μ. Essentially, we assume that assets' net supply does not disappear or depreciate, before its terminal nodes.

We denote by $W^h(\xi) = w^h(\xi) + \sum_{j\in J(\xi^-)} A(\xi, j)\overline{e}^h_j(\xi^-)$ the agent h's aggregated physical endowments up to node $\xi \in D$, where $A(\xi_0, j) = 0$, for each $j \in J(\xi_0)$. Also, we write $W(\xi) = \sum_{h\in H} W^h(\xi)$.

Now, given prices (p, q), each $h \in H$ maximizes her preferences by choosing an allocation,

$$(x^h, \theta^h, \varphi^h) := \big((x^h(\xi), \theta^h(\xi), \varphi^h(\xi)); \xi \in D\big) \in \mathbb{E} := \mathbb{R}^{D\times L}_+ \times \mathbb{R}^{D(J)}_+ \times \mathbb{R}^{D(J)}_+,$$

which belongs to her budget set,

$$B^h(p,q) = \left\{ \begin{array}{ll} (x,\theta,\varphi) \in \mathbb{E} : & \text{For each } \xi \in D,\;\; q(\xi)\varphi(\xi) \le \kappa p(\xi) w^h(\xi) \quad \text{ and} \\ & p(\xi)\left(x(\xi) - w^h(\xi)\right) + q(\xi)\left(\theta(\xi) - \varphi(\xi) - e^h(\xi)\right) \\ & \qquad\qquad \le \sum_{j\in J(\xi^-)} (p(\xi)A(\xi, j) + q_j(\xi))\,(\theta_j(\xi^-) - \varphi_j(\xi^-)) \end{array} \right\},$$

where $\kappa > 0$ and $(\theta(\xi_0^-), \varphi(\xi_0^-)) = 0$.

It follows that $x^h(\xi) = (x^h_l(\xi); l \in L)$ is the consumption bundle of agent h at ξ. Analogously, $\theta^h_j(\xi)$ and $\varphi^h_j(\xi)$ denote, respectively, the quantity of asset $j \in J(\xi)$ that agent h buy and sell at ξ.

It is important to remark that we introduce a *borrowing constraint* in order to prevent agents from entering into Ponzi schemes. In fact, at each $\xi \in D$, agent h is restricted to choose a short-position $\varphi^h(\xi)$ in order to maintain an amount of borrowing which is less than or equal to a fixed proportion $\kappa > 0$ of her initial wealth.

Definition. An *equilibrium* for our economy is given by a vector of prices (p, q) jointly with allocations $\big((x^h, \theta^h, \varphi^h); h \in H\big)$, such that,

(a) For each agent $h \in H$, $(x^h, \theta^h, \varphi^h) \in \text{argmax}_{(x,\theta,\varphi)\in B^h(p,q)}\, U^h(x)$.

(b) At each $\xi \in D$, physical and asset markets clear,

$$\sum_{h\in H} x^h(\xi) = W(\xi); \qquad \sum_{h\in H} \theta^h_j(\xi) = \sum_{h\in H} \overline{e}^h_j(\xi) + \sum_{h\in H} \varphi^h_j(\xi), \quad \forall j \in J(\xi).$$

3. Existence of Equilibrium

In this section we formalize our main results which assures that equilibrium exists in our economy. For this, we state the following assumptions on endowments and preferences,

Assumption E1. For each $(\xi, h) \in D \times H$, $w^h(\xi) \gg 0$.

Assumption E2. Assets have positive net supply. That is, $\sum_{h\in H} \overline{e}_j^h(\xi_j) > 0,\ \ \forall j \in J.$

Assumption P1. For each $h \in H$, $U^h(x) = \sum_{\xi\in D} u^h(\xi, x(\xi))$, where $u^h(\xi,\cdot) : \mathbb{R}^L_+ \to \mathbb{R}_+$ is continuous, concave and strictly increasing. Moreover, $U^h(W) < +\infty$.

Assumption P2. Let $L(J) := \{l \in L : \exists(\mu, j) \in D \times J,\ \ A_l(\mu, j) > 0\}$ and define $\|x\|_{L(J)} = \max_{l\in L(J)} |x_l|$. For each $(\xi, h) \in D \times H$, we assume that,

$$\lim_{x\in\mathbb{R}^L_{++};\|x\|_{L(J)}\to+\infty} u^h(\xi, x) = +\infty.$$

Hypotheses E1 and P1, which require positivity of physical endowments and separability of preferences, are commonly established in the related literature. The objective of Assumptions E2 and P2 is just to get bounds for equilibrium asset prices.

Precisely, we prove that, if inter-temporal utilities go to infinity as consumption increases, assets prices are bounded away from zero. Moreover, when assets have positive net supply, Assumption P2 will allow us to assure that assets prices have an upper bound as well.

Note that, when we consider numeraire assets, P2 simply states that inter-temporal utilities go to infinity as the consumption of the commodity in which assets pay increases.

Theorem. *Under Assumptions E1, E2, P1 and P2 our economy has an equilibrium.*

We remark that impatience properties, as those imposed by Magill and Quinzii (1994, 1996) or Hernandez and Santos (1996), are not required in this paper to prove equilibrium existence.[1]

Indeed, our financial constraints allow us to establish a link between the asymptotic amount of borrowing and the asymptotic value of initial endowments. Thus, to prove optimality of individual allocations, that will be obtained as limit of optimal allocations in finite horizon economies, it is enough to assure that the discounted value of individual wealth is finite (using as deflators the cluster point of the Kuhn-Tucker multipliers corresponding to finite horizon economies). This will be the case, as it is proved in the Appendix (see discussion after Lemma A2).

In the other side, note that agents are not restricted to select *bounded* consumption plans. However, if we suppose, as in Magill and Quinzii (1996), that consumers can only choose plans $x^h = (x^h(\xi); \xi \in D)$ in

$$l^\infty_+(L \times D) := \{y \in \mathbb{R}^{L\times D}_+ : \max_{(l,\xi)\in L\times D} y_l(\xi) < \infty\},$$

then hypothesis P2 can be removed when both aggregated endowments are bounded and P1 is strengthened by requiring also separability on the commodities in $L(J)$. That is, we will require,

Assumption E3. $W = (W(\xi); \xi \in D) \in l^\infty_+(L \times D)$.

[1]The impatience properties used in the literature are joint requirements in preferences and endowments (see, for instance, Assumptions B2 and B4 in Magill and Quinzii (1996)). Therefore, although our utilities can be separable with a constant inter-temporal factor (as in Assumption P4 below) if endowments are not *uniformly* bounded away from zero then Magill and Quinzii's impatience property is not necessarily satisfied.

Assumption P3. Given $(\xi, l) \in D \times L(J)$, there are functions $v^h(\xi, \cdot) : \mathbb{R}_+^{L \setminus L(J)} \to \mathbb{R}_+$ and $f_l^h(\xi, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ such that,

$$u^h(\xi, x) = v^h\left(\xi, (x_l)_{l \in L \setminus L(J)}\right) + \sum_{l \in L(J)} f_l^h(\xi, x_l), \quad \forall h \in H,\ \forall\, x = (x_l; l \in L) \in \mathbb{R}_+^L.$$

Corollary. *Suppose that Assumptions E1-E3, P1 and P3 hold and that agents are restricted to select bounded consumption plans. Then, there exists an equilibrium for our economy.*

In particular, when there is only one commodity to be traded, E1-E3 and P1 are sufficient to assure existence of equilibria.

4. Final Remarks

In this section we present comments which allow us thoroughly to provide subtle interpretations and readings of our results. Consider the following hypotheses:

Assumption E4. $\exists \underline{w} \in \mathbb{R}_{++}^L : w^h(\xi) \geq \underline{w}, \quad \forall\, (\xi, h) \in D \times H.$

Assumption E5. $\exists \rho \in (0,1)$ such that, $\rho W(\xi) \leq w^h(\xi), \quad \forall (\xi, h) \in D \times H.$

Assumption P4. For each $h \in H$, $U^h(x) = \sum_{\xi \in D} \beta_h^{t(\xi)} \rho^h(\xi) u^h(x(\xi))$, where $u^h : \mathbb{R}_+^L \to \mathbb{R}_+$ is a continuous, concave and strictly increasing function. Moreover, $\beta_h \in (0,1)$, $\rho^h(\xi_0) = 1$ and, for each $\xi \in D$, $\sum_{\mu \in \xi^+} \rho^h(\mu) = \rho^h(\xi) > 0$.

Bounded debts and non-binding borrowing constraints. Assume that Assumptions E1-E3, P2 and P4 hold. Then, equilibrium exists and *individual debts are uniformly bounded along the event-tree*. In fact, under these hypotheses, there exists an scalar $a > 0$ such that,

$$\min_{h \in H} u^h(a, \ldots, a) > \max_{h \in H} \frac{u^h(\overline{W})}{1 - \beta_h},$$

where the vector $\overline{W}$ is an upper bound for the plan $W = (W(\xi); \xi \in D)$. Thus, given an equilibrium $\left[(p,q), \left((x^h, \theta^h, \varphi^h); h \in H\right)\right]$ we have that, for any $\xi \in D$, the amount of debt $q(\xi)(\theta^h(\xi) - \varphi^h(\xi)) \leq a\|p(\xi)\|_\Sigma$, provided that $q(\xi)(\theta^h(\xi) - \varphi^h(\xi)) \geq 0$.[2]

Therefore, financial market feasibility implies that, for each $h \in H$, we have that

$$-a(\#H - 1)\|p(\xi)\|_\Sigma \leq q(\xi)(\theta^h(\xi) - \varphi^h(\xi)) \leq a\|p(\xi)\|_\Sigma.$$

In particular, when there is only one security to be negotiated, the uniform bound on debt above induces an uniform bound on borrowing. Within this context, for proportions κ large enough, our borrowing constraints are not binding at equilibrium.[3]

[2] Given $z = (z_1, \ldots, z_n) \in \mathbb{R}_+^n$, $\|z\|_\Sigma = \sum_{i=1}^n z_i$.

[3] Previously, Hernandez and Santos (1996) have shown equilibrium existence in an economy with debt constraints, when only one infinite-lived asset in positive net supply is traded. We assure more when agents are burden by borrowing constraints, namely, restrictions on the amount of borrowing became non-binding.

Alternative borrowing constraints. Assume, as in Hernandez and Santos (1997), that E5 holds. Then we can bound the growth of borrowing by requiring that, at each node ξ, $q(\xi)\varphi(\xi) \leq \kappa p(\xi)W(\xi)$. Thus, borrowing constraints may depends on the value of the *aggregated* amount of commodities.

Alternatively, the constraint $q(\xi)\varphi(\xi) \leq p(\xi)M$, where $M \in \mathbb{R}^L_+ \setminus \{0\}$, can be implemented provided that initial endowments, as in Magill and Quinzii (1996), satisfy Assumption E4.

Actually, in both cases the same technique of proof will operate: Under hypotheses E2, E4 (or E5), P1 and P2, truncated economies will also have equilibrium, given that asset prices will be bounded away from zero and from above, node by node. The main point is that transversality condition will also hold (see equations (6), (7) and (8) in the Appendix).

Rational asset pricing bubbles. Suppose that hypotheses E1-E3, P2 and P4 hold. It follows from the previous comments that, for the equilibrium allocation we construct, (i) marginal rates of substitution will be summable (see equation (8) below), and (ii) individual debt will be uniformly bounded along the event-tree. In particular, as assets have positive net supply, their prices will be uniformly bounded along the event-tree. Therefore, the discounted value of asset prices, using the marginal rates of substitution as deflators, goes to zero as time goes to infinity. That is, analogous to Magill and Quinzii (1996), assets are free of bubbles.

Appendix

To prove our main result we show, firstly, that there exist equilibria in finite horizon truncated economies. Then, an equilibrium for the original economy will be found as a limit of equilibria corresponding to a sequence of truncated economies, when the time horizon increases.

Truncated economies. For each $T \in \mathbb{N}$, we define a truncated economy, $\mathcal{E}^T$, in which agents consume commodities and trade assets in the restricted event-tree $D^T(\xi_0)$.

Let $J^T(\xi) = \{j \in J(\xi) : \exists \mu \in D^{T-t(\xi)}(\xi),\, \mu \neq \xi,\, A(\mu, j) \neq 0\}$ be the set of available securities at $\xi \in D^{T-1}(\xi_0)$. At each $\xi \in D_T(\xi_0)$, we define $J^T(\xi) = \emptyset$. It follows that, given $\xi \in D$, $J^T(\xi) = J(\xi)$ for every T large enough. Let $D^T(J) = \{(\xi, j) \in D^T(\xi_0) \times J : j \in J^T(\xi)\}$.

Each individual $h \in H$ is characterized by her physical, $(w^h(\xi); \xi \in D^T(\xi_0))$, and financial, $(e^h(\xi); \xi \in D^{T-1}(\xi_0))$, endowments. Also, when agent h chooses a consumption plan $(x(\xi))_{\xi \in D^T(\xi_0)}$, her utility is given by $U^{h,T}(x) = \sum_{\xi \in D^T(\xi_0)} u^h(\xi, x(\xi))$.

For each truncated economy $\mathcal{E}^T$, we can consider, without loss of generality, prices (p, q) in

$$\mathbb{P}^T := \prod_{\xi \in D^{T-1}(\xi_0)} \left(\Delta^L_+ \times \mathbb{R}^{J^T(\xi)}_+\right) \times \prod_{\xi \in D_T(\xi_0)} \Delta^L_+,$$

where $\Delta_+^L := \{p \in \mathbb{R}_+^L : \|p\|_\Sigma = 1\}$. Then, given $(p,q) \in \mathbb{P}^T$, agent $h \in H$ solves the following optimization problem:

$$(P^{h,T}) \qquad \begin{array}{ll} \max & U^{h,T}(x) \\ \text{s.t.} & \begin{cases} y(\xi) = (x(\xi), \theta(\xi), \varphi(\xi)) \geq 0, & \forall \xi \in D^T(\xi_0), \\ g_\xi^{h,T}(y(\xi), y(\xi^-); p, q) \leq 0, & \forall \xi \in D^T(\xi_0), \\ q(\xi)\varphi(\xi) - \kappa p(\xi) w^h(\xi) \leq 0, & \forall \xi \in D^{T-1}(\xi_0), \\ (\theta(\xi), \varphi(\xi)) = 0, & \forall \xi \in D_T(\xi_0), \end{cases} \end{array}$$

where $y(\xi_0^-) = 0$ and, for each $\xi \in D^T(\xi_0)$,

$$\begin{aligned} g_\xi^{h,T}(y(\xi), y(\xi^-); p, q) := p(\xi)\left(x(\xi) - w^h(\xi)\right) &+ \sum_{j \in J^T(\xi)} q_j(\xi)\left(\theta_j(\xi) - \varphi_j(\xi) - e_j^h(\xi)\right) \\ &- \sum_{j \in J^T(\xi^-)} (p(\xi)A(\xi,j) + q_j(\xi))\left(\theta_j(\xi^-) - \varphi_j(\xi^-)\right). \end{aligned}$$

Let $B^{h,T}(p,q)$ be the truncated budget set of agent h, i.e. the set of plans $(y(\xi))_{\xi \in D^T(\xi_0)}$ that satisfy the restrictions of the problem $P^{h,T}$ above.

Definition A1. An *equilibrium* for the economy $\mathcal{E}^T$ is given by prices $(p^T, q^T) \in \mathbb{P}^T$ and individual allocations $(y^{h,T}(\xi))_{\xi \in D^T(\xi_0)} \in \mathbb{E}^T := \mathbb{R}_+^{D^T(\xi_0) \times L} \times \mathbb{R}_+^{D^T(J)} \times \mathbb{R}_+^{D^T(J)}$, such that:

(1) For each $h \in H$, $(y^{h,T}(\xi))_{\xi \in D^T(\xi_0)}$ is an optimal solution for $P^{h,T}$ at prices (p^T, q^T);
(2) Physical and financial markets clear at each $\xi \in D^T(\xi_0)$.

Equilibrium existence in truncated economies. In order to show the existence of equilibria in $\mathcal{E}^T$ we follow a generalized game approach. For each $(\mathcal{X}, \Theta, \Psi, M) \in \mathbb{F}^T := \mathbb{E}^T \times \mathbb{R}_{++}^{D^T(J)}$, consider the convex and compact set $\mathcal{K}(\mathcal{X}, \Theta, \Psi) = [0, \mathcal{X}] \times [0, \Theta] \times [0, \Psi] \subset \mathbb{E}^T$ and define,

$$\mathbb{P}_M^T = \prod_{\xi \in D^{T-1}(\xi_0)} \left(\Delta_+^L \times [0, M_\xi]\right) \times \prod_{\xi \in D_T(\xi_0)} \Delta_+^L.$$

Let $\mathcal{G}^T(\mathcal{X}, \Theta, \Psi, M)$ be a generalized game where each consumer is represented by a player $h \in H$ and, at each $\xi \in D^T(\xi_0)$, there is also a player who behaves as an auctioneer.

More precisely, in $\mathcal{G}^T(\mathcal{X}, \Theta, \Psi, M)$ each player $h \in H$ behaves as price-taker and, given $(p,q) \in \mathbb{P}_M^T$, she chooses strategies in the truncated budget set $B^{h,T}(p,q) \cap \mathcal{K}(\mathcal{X}, \Theta, \Psi)$ in order to maximize the function $U^{h,T}$. Also, at each $\xi \in D^{T-1}(\xi_0)$ (resp. $\xi \in D_T(\xi_0)$) the corresponding auctioneer chooses commodity and asset prices $(p(\xi), q(\xi)) \in \Delta_+^L \times [0, M_\xi]$ (resp. just commodity prices $p(\xi) \in \Delta_+^L$) in order to maximize the function $\sum_{h \in H} g_\xi^{h,T}(y^h(\xi), y^h(\xi^-); p, q)$, where $y^h = (y^h(\xi))_{\xi \in D^T(\xi_0)}$ are the strategies selected by player $h \in H$.

Definition A2. A strategy profile $\left[(p^T(\xi), q^T(\xi)); (y^{h,T}(\xi))_{h \in H}\right]_{\xi \in D^T(\xi_0)} \in \mathbb{P}_M^T \times (\mathcal{K}(\mathcal{X}, \Theta, \Psi))^H$ is a *Nash equilibrium* for $\mathcal{G}^T(\mathcal{X}, \Theta, \Psi, M)$ if each player maximizes her objective function, given the strategies chosen by the other players, i.e. no player has an incentive to deviate.

Lemma A1. Let $T \in \mathbb{N}$ and $(\mathcal{X},\Theta,\Psi,M) \in \mathbb{F}^T$. Under Assumptions E1 and P1 the set of Nash equilibria for the game $\mathcal{G}^T(\mathcal{X},\Theta,\Psi,M)$ is non-empty.

Proof. Note that each player's strategy set is non-empty, convex and compact. Further, it follows from Assumption P1 that the objective function of each player is continuous and quasi-concave in her own strategy. Assumption E2 assures that the correspondences of admissible strategies are continuous, with non-empty, convex and compact values. Therefore, we can find an equilibrium of the generalized game by applying Kakutani Fixed Point Theorem to the correspondence defined as the product of the optimal strategy correspondences. □

Lemma A2. Let $T \in \mathbb{N}$. Under Assumptions E1, E2, P1 and P2, there exists (Θ^T,Ψ^T) such that, if $(\Theta,\Psi) \gg (\Theta^T,\Psi^T)$, then every Nash equilibrium of the game $\mathcal{G}^T(\mathcal{X},\Theta,\Psi,M)$ is an equilibrium of the economy $\mathcal{E}^T$ whenever $\mathcal{X}$ and M are large enough.

Proof. Let $\left[(p^T(\xi),q^T(\xi));(y^{h,T}(\xi))_{h\in H}\right]_{\xi\in D^T(\xi_0)}$ be a Nash equilibrium for $\mathcal{G}^T(\mathcal{X},\Theta,\Psi,M)$, with allocations given by $y^{h,T}(\xi) = (x^{h,T}(\xi),\theta^{h,T}(\xi),\varphi^{h,T}(\xi))$. Note that, for each $h \in H$,

$$(y^{h,T}(\xi))_{\xi\in D^T(\xi_0)} \in \operatorname{argmax}_{B^{h,T}(p^T,q^T)\cap\mathcal{K}(\mathcal{X},\Theta,\Psi)}\ U^{h,T}(x).$$

Then, as each auctioneer maximizes his objective function, we have that, at each $\xi \in D^T(\xi_0)$,

$$\sum_{h\in H} x^{h,T}(\xi) \le \Upsilon^T(\Theta,\xi) := \sum_{h\in H}\left(w^h(\xi) + \sum_{j\in J^T(\xi^-)} A(\xi,j)\Theta(\xi^-,j)\right).$$

It follows from Assumptions P1 and P2 that, for each $\xi \in D^T(\xi_0)$, there exists a real number $a^T_\Theta(\xi) > 0$ such that,

$$\min_{h\in H}\ u^h\left(\xi,(a^T_\Theta(\xi),\dots,a^T_\Theta(\xi))\right) > \max_{h\in H}\ U^{h,T}(\Upsilon^T(\Theta)),$$

where $\Upsilon^T(\Theta) := (\Upsilon^T(\Theta,\xi);\xi\in D^T(\xi_0))$.

Suppose that $\mathcal{X}(\xi,l) > a^T_\Theta(\xi)$, for every $(\xi,l) \in D^T(\xi_0)\times L$. As $\|p^T(\xi)\|_\Sigma = 1$, it follows from individual optimality that the value of accumulated individual financial endowments, at any $\xi \in D^T(\xi_0)$, is necessarily less than $p^T(\xi)(a^T_\Theta(\xi),\dots,a^T_\Theta(\xi)) = a^T_\Theta(\xi)$. Therefore, for each $j \in J^T(\xi)$,

$$q^T_j(\xi) \le M^T_\Theta(\xi,j) := \frac{a^T_\Theta(\xi)\,\#H}{\sum_{h\in H}\overline{e}^h_j(\xi)}.$$

Let $M^T_\Theta = (M^T_\Theta(\xi,j);(\xi,j)\in D^T(J))$. We conclude that if $M \gg M^T_\Theta$, then in any Nash equilibrium of $\mathcal{G}^T(\mathcal{X},\Theta,\Psi,M)$ the upper bounds of asset prices, which were previously imposed, are non-binding. Along the rest of this proof we assume that this property holds.

Step 1. Physical markets clear. For each $\xi \in D^T(\xi_0)$, let

$$\Gamma(\xi) = \sum_{h\in H} x^{h,T}(\xi) - W(\xi), \qquad \Omega(\xi) = \sum_{h\in H}\theta^{h,T}(\xi) - \sum_{h\in H}\overline{e}^h(\xi) - \sum_{h\in H}\varphi^{h,T}(\xi).$$

Summing up the budget constraints at ξ_0 we have $p^T(\xi_0)\Gamma(\xi_0) + q^T(\xi_0)\Omega(\xi_0) \le 0$. Since the auctioneer at ξ_0 maximizes $p(\xi_0)\Gamma(\xi_0) + q(\xi_0)\Omega(\xi_0)$, we obtain that $\Gamma(\xi_0) \le 0$. Assume now that

$\Omega(\xi_0, j) > 0$, for some $j \in J^T(\xi_0)$. By the construction of the plan M, we know that $q_j^T(\xi_0) < M_{\xi_0,j}$, which leads us to obtain a contradiction with the optimal behaviour of the auctioneer at ξ_0. Thus $\Omega(\xi_0) \leq 0$. Hence, if $\mathcal{X}(\xi_0, l) > \max\{W(\xi_0, l), a_\Theta^T(\xi_0)\}$ for each $l \in L$, then the upper bound on consumption is not binding at ξ_0, allowing us to conclude, as a consequence of the monotonicity of preferences, that commodity markets clear at the initial node ξ_0, i.e. $\Gamma(\xi_0) = 0$. Moreover, $q^T(\xi_0)\Omega(\xi_0) = 0$.

Consider now a node ξ with $t(\xi) = 1$, and recall that the corresponding auctioneer at ξ chooses prices in $\Delta_+^L \times [0, M_\xi]$ in order to maximize the function $\sum_{h \in H} g_\xi^{h,T}(y^{h,T}(\xi), y^{h,T}(\xi_0); p, q)$. Using the fact that $\Omega(\xi_0) \leq 0$, we can deduce that $p^T(\xi)\Gamma(\xi) + q^T(\xi)\Omega(\xi) \leq 0$, for every ξ with $t(\xi) = 1$. As before, $\Gamma(\xi) \leq 0$ and $\Omega(\xi) \leq 0$. Furthermore, if $\mathcal{X}(\xi) > \max\{W(\xi, l), a_\Theta^T(\xi)\}$ for every $l \in L$, then the upper bound on consumption is not binding at ξ, which implies that $\Gamma(\xi) = 0$.

By applying successively analogous arguments to the nodes with periods $t = 2, \ldots, T$, we conclude that $\Gamma(\xi) = 0$ for every $\xi \in D^T(\xi_0)$, provided that, for each $l \in L$, $\mathcal{X}(\xi, l) > \max\{W(\xi, l),\, a_\Theta^T(\xi)\}$. That is, physical markets clear in the economy $\mathcal{E}^T$. Furthermore, there is no excess of demand for financial markets, i.e. $\Omega(\xi) \leq 0$, for every $\xi \in D^{T-1}(\xi_0)$.

Step 2. Lower bounds of asset prices. Given $(\xi, j) \in D^T(J)$, fix a node $\mu(\xi, j)$ that belongs to the non-empty set $\mathit{argmin}\{t(\mu) : \mu \in D^{T-t(\xi)}(\xi),\, \mu \neq \xi,\, A(\mu, j) \neq 0\}$.

By Assumptions E1, P1 and P2, there exists $b(\xi, j) \in (0, 1)$, independent of T, such that, for every $h \in H$, the following inequality holds,

$$u^h\left(\mu(\xi, j), w^h(\mu(\xi, j)) + \frac{A(\mu(\xi, j), j)\min_{l \in L} w_l^h(\xi)}{b(\xi, j)}\right) > U^h(W). \tag{1}$$

Suppose that,

$$\Theta(\xi, j) > \widehat{\Theta}(\xi, j) := \max_{h \in H} \frac{\min_{l \in L} w_l^h(\xi)}{b(\xi, j)},$$

and for every $\mu \in D^{T-t(\xi)}(\xi)$ with $j \in J^T(\mu)$,

$$\min_{l \in L} \mathcal{X}(\mu, l) > \mathcal{X}_{\Theta,\xi}^T(\mu, j) := \max_{(l,h) \in H \times L} \left\{W(\mu, l),\, a_\Theta^T(\mu),\, w_l^h(\mu) + \frac{A_l(\mu, j)\min_{l' \in L} w_{l'}^h(\xi)}{b(\xi, j)}\right\}.$$

We claim that $q_j^T(\xi) > b(\xi, j)$. In fact, if $q_j^T(\xi) \leq b(\xi, j)$ then, as by Step 1 $x^{h,T}(\mu) \leq W(\mu)$ for every $\mu \in D^T(\xi_0)$, it follows from Assumption P1 and equation (1) that any agent $h \in H$ has an incentive to deviate by choosing any budget feasible strategy $(x^h, \theta^h, \varphi^h)$ that satisfies,

$$\begin{aligned} \theta_j^h(\xi) &= \frac{\min_{l \in L} w_l^h(\xi)}{b(\xi, j)}, \\ x^h(\mu) &= w^h(\mu) + A(\mu, j)\theta_j^h(\xi), \quad \text{if } \mu = \mu(\xi, j). \end{aligned}$$

Therefore, if for each $\eta \in D^T(\xi_0)$,

$$\begin{aligned} \Theta(\eta, j) &> \widehat{\Theta}(\eta, j),\ \forall j \in J^T(\eta), \\ \mathcal{X}(\eta, l) &> \mathcal{X}_\Theta^T(\eta) := \max_{\substack{(\xi,j) \in D^T(J): \\ \eta > \xi,\, j \in J^T(\eta)}} \mathcal{X}_{\Theta,\xi}^T(\eta, j),\ \forall l \in L, \end{aligned}$$

then equilibrium asset prices have a positive lower bound away from zero. In fact, for each $(\eta, j) \in D^T(J)$, we have that $q_j^T(\eta) > b(\eta, j)$.

Step 3. Non-binding short-sales constraints. Define $\widehat{\Theta}^T = (\widehat{\Theta}(\eta, j); (\eta, j) \in D^T(J))$ and $\mathcal{X}_\Theta^T = (\mathcal{X}_\Theta^T(\eta); \eta \in D^T(\xi_0))$. If $\Theta \gg \widehat{\Theta}^T$ and $\mathcal{X} \gg \mathcal{X}_\Theta^T$, asset prices are bounded away from zero. Thus, using the borrowing constraints, we conclude that, for every player $h \in H$,

$$\varphi_j^{h,T}(\xi) < \widehat{\Psi}_j(\xi) := \kappa \frac{\max_{(h,l)\in H\times L} w_l^h(\xi)}{b(\xi, j)}, \qquad \forall (\xi, j) \in D^T(J).$$

Let $\Psi^T = (\widehat{\Psi}_j(\xi); (\xi, j) \in D^T(J))$. If $\Psi \gg \Psi^T$ then short-sales restrictions induced by $\mathcal{K}(\mathcal{X}, \Theta, \Psi, M)$ are not binding.

Step 4. Financial markets clear and upper bounds for long-positions are non-binding. Suppose that $(\Theta, \Psi) \gg (\widehat{\Theta}^T, \Psi^T)$ and $\mathcal{X} \gg \mathcal{X}_\Theta^T$. Now, by Step 1 we have that $q^T(\xi)\Omega(\xi) = 0$ and $\Omega(\xi) \leq 0$, for each $\xi \in D^{T-1}(\xi_0)$. Thus, if for some $(\xi, j) \in D^T(J)$, $\Omega_j(\xi) < 0$, then $q_j^T(\xi) = 0$, which is in contradiction with the lower bound on asset prices find in Step 2.

On the other hand, for each $\xi \in D^{T-1}(\xi_0)$, $(\varphi^{h,T}(\xi))_{h\in H}$ is bounded. Thus, as $\Omega(\xi) \leq 0$, $\sum_{h\in H} \theta^{h,T}(\xi)$ is also bounded. We conclude that there exists $\Theta^T \geq \widehat{\Theta}^T$ such that, if $\Theta \gg \Theta^T$ then upper bounds on long positions are non-binding.

Step 5. Individual optimality. As a consequence of all previous steps, if $(\Theta, \Psi) \gg (\Theta^T, \Psi^T)$ and $(\mathcal{X}, M) \gg (\mathcal{X}_\Theta^T, M_\Theta^T)$ then, for each $h \in H$, the optimal allocation $y^{h,T}$ belongs to the interior of $\mathcal{K}(\mathcal{X}, \Theta, \Psi, M)$ (relative to $\mathbb{E}^T$). As budget correspondences has finite-dimensional convex values, we conclude that,

$$(y^{h,T}(\xi))_{\xi\in D^T(\xi_0)} \in \operatorname{argmax}_{B^{h,T}(p^T,q^T)} \sum_{\xi\in D^T(\xi_0)} u^h(\xi, x(\xi)).$$

Therefore, any Nash equilibrium of $\mathcal{G}^T(\mathcal{X}, \Theta, \Psi, M)$ is an equilibrium of $\mathcal{E}^T$, provided that $(\Theta, \Psi) \gg (\Theta^T, \Psi^T)$ and $(\mathcal{X}, M) \gg (\mathcal{X}_\Theta^T, M_\Theta^T)$. □

Recall that, given $\xi \in D$, $J^T(\xi) = J(\xi)$ for T large enough. Thus, by construction, the upper bounds $(\Theta^T(\xi), \Psi^T(\xi))$ are independent of $T > t(\xi)$, when T is large enough. Therefore, node by node, independently of the truncated horizon T, individual equilibrium allocations are uniformly bounded and commodity prices belong to the simplex.

Moreover, under Assumptions E2, P1 and P2, asset prices are uniformly bounded by above, node by node. In fact, as consumption allocations are bounded by the aggregated resources, by analogous arguments to those made in the proof of Lemma A2, we can conclude that,

$$q_j^T(\xi) \leq \frac{a(\xi)\,\#H}{\sum_{h\in H} \bar{e}_j^h(\xi)}, \qquad \forall j \in J^T(\xi),$$

where $a(\xi) > 0$ is independent of $T > t(\xi)$ and is defined implicitly by

$$\min_{h\in H}\; u^h\left(\xi, (a(\xi), \ldots, a(\xi))\right) > \max_{h\in H} U^h(W).$$

Asymptotic equilibria. In order to find an equilibrium of our original economy, we look for an uniform bound (node by node) for the Kuhn-Tucker multipliers associated to the truncated individual problems.

To attempt this aim, for each $T \in \mathbb{N}$, consider an equilibrium $\left[p^T(\xi), q^T(\xi); (y^{h,T}(\xi))_{h\in H}\right]_{\xi\in D^T(\xi_0)}$ for the economy $\mathcal{E}^T$. Then. there exist *non-negative* multipliers $\left((\gamma_\xi^{h,T})_{\xi\in D^T(\xi_0)};\, (\rho_\xi^{h,T})_{\xi\in D^{T-1}(\xi_0)}\right)$ such that,

$$\gamma_\xi^{h,T} g_\xi^{h,T}(y^{h,T}(\xi),\, y^{h,T}(\xi^-); p^T, q^T) = 0, \quad \forall \xi \in D^T(\xi_0); \tag{2}$$

$$\rho_\xi^{h,T}\left(\kappa p^T(\xi) w^h(\xi) - q^T(\xi)\varphi^{h,T}(\xi)\right) = 0, \quad \forall \xi \in D^{T-1}(\xi_0). \tag{3}$$

Moreover, for each plan $(x(\xi), \theta(\xi), \varphi(\xi))_{\xi\in D^T(\xi_0)} \geq 0$, with $(\theta(\eta), \varphi(\eta))_{\eta\in D_T(\xi_0)} = 0$, the following saddle point property is satisfied (see Rockafellar (1997), Section 28, Theorem 28.3),

$$U^{h,T}(x) - \sum_{\xi\in D^T(\xi_0)} \gamma_\xi^{h,T} g_\xi^{h,T}(y(\xi), y(\xi^-);\, p^T, q^T) + \sum_{\xi\in D^{T-1}(\xi_0)} \rho_\xi^{h,T}(\kappa p^T(\xi) w^h(\xi) - q^T(\xi)\varphi(\xi)) \leq U^{h,T}(x^{h,T}). \tag{4}$$

Let us take $(x(\xi), \theta(\xi), \varphi(\xi))_{\xi\in D^T(\xi_0)} = (0,0,0)$ to obtain,

$$\sum_{\xi\in D^{T-1}(\xi_0)} p^T(\xi) w^h(\xi)\left[\gamma_\xi^{h,T} + \rho_\xi^{h,T}\kappa\right] \leq U^h(W) < +\infty. \tag{5}$$

As commodity prices are in the simplex, node by node, for every $\xi \in D$ and for all $T > t(\xi)$, we conclude that,

$$0 \leq \gamma_\xi^{h,T} \leq \frac{U^h(W)}{\underline{w}_\xi^h}, \qquad 0 \leq \rho_\xi^{h,T} \leq \frac{U^h(W)}{\kappa\, \underline{w}_\xi^h},$$

where, by Assumption E1, $\underline{w}_\xi^h := \min_{l\in L} w_l^h(\xi) > 0$.

In short, for each $\xi \in D$, the sequence formed by equilibrium prices, equilibrium allocations and Kuhn-Tucker multipliers, $((p^T(\xi), q^T(\xi)); (y^{h,T}(\xi), \gamma_\xi^{h,T}, \rho_\xi^{h,T})_{h\in H})_{T>t(\xi)}$, is bounded. Applying Tychonoff Theorem we can find a common subsequence $(T_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that, for each $\xi \in D$,

$$\lim_{k\to+\infty} ((p^{T_k}(\xi), q^{T_k}(\xi)); (y^{h,T_k}(\xi), \gamma_\xi^{h,T_k}, \rho_\xi^{h,T_k})_{h\in H}) = \left((\overline{p}(\xi), \overline{q}(\xi)); (\overline{y}^h(\xi), \overline{\gamma}_\xi^h, \overline{\rho}_\xi^h)_{h\in H}\right).$$

Hence, for each $h \in H$, $\left(\overline{y}^h(\xi)\right)_{\xi\in D} \in B^h(\overline{p}, \overline{q})$. As limit allocations are cluster points, node by node, of equilibria in truncated economies, market clearing also follows. Therefore, in order to conclude that $\left[(\overline{p}(\xi), \overline{q}(\xi)); (\overline{y}^h(\xi))_{h\in H}\right]_{\xi\in D}$ is an equilibrium it remains to show that, for each agent $h \in H$, $(\overline{y}^h(\xi))_{\xi\in D}$ is an optimal choice when prices are $(\overline{p}, \overline{q})$.

Lemma A3. Under Assumptions E1, E2, P1 and P2, $U^h(\tilde{x}) \leq U^h(\overline{x})$, for every $\tilde{y} := (\tilde{x}, \tilde{\theta}, \tilde{\varphi}) \in B^h(\overline{p}, \overline{q})$.

Proof. Fix a node $\xi \in D$. Let us take $T > t(\xi)$ large enough, to assure that $J^T(\mu) = J(\mu)$ for each $\mu \leq \xi$ and consider the allocation,

$$(x(\mu), \theta(\mu), \varphi(\mu)) = \begin{cases} (x^{h,T}(\mu), \theta^{h,T}(\mu), \varphi^{h,T}(\mu)), & \text{if } \mu \neq \xi, \\ (\tilde{x}(\xi), \tilde{\theta}(\xi), \tilde{\varphi}(\xi)), & \text{if } \mu = \xi. \end{cases}$$

Then, it follows from inequality (4) that, under Assumption P1,

$$u^h(\xi,\tilde{x}(\xi)) - u^h(\xi,x^{h,T}(\xi)) \leq -\rho_\xi^{h,T}\left(\kappa p^T(\xi)w^h(\xi) - q^T(\xi)\tilde{\varphi}(\xi)\right)$$
$$+\gamma_\xi^{h,T} g_\xi^h(\tilde{y}(\xi), y^{h,T}(\xi^-);\, p^T, q^T) + \sum_{\mu\in\xi^+} \gamma_\mu^{h,T} g_\mu^h(y^{h,T}(\mu), \tilde{y}(\xi);\, p^T, q^T),$$

where $g_\xi^h \leq 0$ denotes the budget constraint at $\xi \in D$. As $\tilde{y}$ is budget feasible at prices $(\overline{p},\overline{q})$, taking the limit as $T = T_k$ goes to infinity, we obtain that,

$$u^h(\xi,\tilde{x}(\xi)) - u^h(\xi,\overline{x}(\xi)) \leq \overline{\gamma}_\xi^h g_\xi^h(\tilde{y}(\xi), \overline{y}^h(\xi^-);\, \overline{p},\overline{q}) + \sum_{\mu\in\xi^+} \overline{\gamma}_\mu^h g_\mu^h(\overline{y}^h(\mu), \tilde{y}(\xi);\, \overline{p},\overline{q}).$$

As $\tilde{y}$ and $(\overline{y}^h(\xi))_{\xi\in D}$ belongs to $B^h(\overline{p},\overline{q})$, adding previous inequality over the nodes in $D^N(\xi_0)$, with $N\in\mathbb{N}$, it follows that,

$$U^{h,N}(\tilde{x}) - U^{h,N}(\overline{x}) \leq \sum_{\mu\in D_{N+1}(\xi_0)} \overline{\gamma}_\mu^h g_\mu^h(\overline{y}^h(\mu), \tilde{y}(\mu^-);\, \overline{p},\overline{q}).$$

Thus, as $\tilde{y}$ is budget feasible, borrowing constraints imply that,

$$U^{h,N}(\tilde{x}) - U^{h,N}(\overline{x}) \leq \sum_{\mu\in D_{N+1}(\xi_0)} \overline{\gamma}_\mu^h \left(\overline{p}(\mu)\overline{x}^h(\mu) + \overline{q}(\mu)(\overline{\theta}^h(\mu) - \overline{\varphi}^h(\xi)) + \kappa\overline{p}(\mu)w^h(\mu)\right). \tag{6}$$

In the other side, define $L_\xi^{h,T} = p^T(\xi)x^{h,T}(\xi) + q^T(\xi)(\theta^{h,T}(\xi) - \varphi^{h,T}(\xi))$. Consider the allocation,

$$(x(\mu),\theta(\mu),\varphi(\mu)) = \begin{cases} (x^{h,T}(\mu),\theta^{h,T}(\mu),\varphi^{h,T}(\mu)), & \text{if } \mu\neq\xi\,, \\ (0,0,0), & \text{if } \mu=\xi. \end{cases}$$

Using inequality (4), Assumption P1 assures that,

$$\begin{aligned} \gamma_\xi^{h,T} L_\xi^{h,T} &\leq u^h(\xi, x^{h,T}(\xi)) + \sum_{\mu\in\xi^+} \gamma_\mu^{h,T} L_\mu^{h,T}, \quad \forall \xi\in D^{T-1}(\xi_0); \\ \gamma_\xi^{h,T} L_\xi^{h,T} &\leq u^h(\xi, x^{h,T}(\xi)), \quad \forall \xi\in D_T(\xi_0). \end{aligned}$$

Thus, by monotonicity of preferences,

$$\sum_{\xi\in D_{N+1}(\xi_0)} \gamma_\xi^{h,T} L_\xi^{h,T} \leq \sum_{\mu\in D\setminus D^N(\xi_0)} u^h(\mu, W(\mu)), \quad \forall T > N+1.$$

Taking the limit as T goes to infinity we obtain,

$$\sum_{\xi\in D_{N+1}(\xi_0)} \overline{\gamma}_\xi^h \left(\overline{p}(\xi)\overline{x}^h(\xi) + \overline{q}(\xi)(\overline{\theta}^h(\xi) - \overline{\varphi}^h(\xi))\right) \leq \sum_{\mu\in D\setminus D^N(\xi_0)} u^h(\mu, W(\mu)).$$

Thus, it follows from inequality (6) that,

$$U^{h,N}(\tilde{x}) - U^{h,N}(\overline{x}) \leq \sum_{\mu\in D\setminus D^N(\xi_0)} u^h(\mu, W(\mu)) + \kappa \sum_{\mu\in D_{N+1}(\xi_0)} \overline{\gamma}_\mu^h \overline{p}(\mu) w^h(\mu).$$

Now, inequality (5) assures that,

$$\sum_{\xi\in D} \overline{\gamma}_\xi^h \overline{p}(\xi) w^h(\xi) < +\infty. \tag{7}$$

Therefore, it follows from Assumption P1 that: For each $\varepsilon > 0$ there exists $\overline{N}_\varepsilon > 0$ such that,

$$\sum_{\xi\in D^N(\xi_0)} u^h(\xi,\tilde{x}(\xi)) < \varepsilon + U^h(\overline{x}), \quad \forall N > \overline{N}_\varepsilon$$

Finally, we conclude that, for each $\varepsilon > 0$, $U^h(\tilde{x}) \leq \varepsilon + U^h(\overline{x})$, which ends the proof. $\square$

Proof of the Corollary. Given $(\xi, h) \in D \times H$, define

$$\tilde{u}^h(\xi, x) = v^h\left(\xi, (x_l)_{l \in L \setminus L(J)}\right) + \sum_{l \in L(J)} \left(f_l^h\left(\xi, \min\{x_l, 2W_l(\xi)\}\right) + \rho(\xi, l) \max\{x_l - 2W_l(\xi), 0\}\right),$$

where $x = (x_l; l \in L) \in \mathbb{R}_+^L$ and $\rho(\xi, l) \in \partial f_l^h(\xi, 2W_l(\xi))$.[4] It follows from the separability of the inter-temporal utilities on commodities in $L(J)$ that the functions,

$$\tilde{U}^h(x) := \sum_{\xi \in D} \tilde{u}^h(\xi, x(\xi)),$$

satisfy Assumptions P1 and P2. Therefore, there exists an equilibrium $\left[(\overline{p}(\xi), \overline{q}(\xi)); (\overline{y}^h(\xi))_{h \in H}\right]_{\xi \in D}$, being $\overline{y}^h(\xi) = (\overline{x}^h(\xi), \overline{\theta}^h(\xi), \overline{\varphi}^h(\xi))$, for the economy in which each $h \in H$ has preferences represented by the function $\tilde{U}^h$ instead of U^h. Moreover, this equilibrium is an equilibrium for the original economy. In fact, since agents are restricted to choose bounded consumption plans, if there exists a budget feasible allocation $(x^h, \theta^h, \varphi^h)$ such that $U^h(x^h) > U^h(\overline{x}^h)$ then there is $\lambda \in (0, 1)$ such that, the consumption plan $x(\lambda) := \lambda x^h + (1 - \lambda)\overline{x}^h$, with $x(\lambda) = (x_l(\lambda, \xi); \xi \in D)$, satisfies $x_l(\lambda, \xi) < 2W_l(\xi)$, $\forall l \in L(J)$. Thus,

$$\tilde{U}^h(x(\lambda)) = U^h(x(\lambda)) > \lambda U^h(x^h) + (1 - \lambda)U^h(\overline{x}^h) > U^h(\overline{x}^h) = \tilde{U}^h(\overline{x}^h),$$

which is a contradiction. $\square$

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[4] We denote by $\partial f_l^h(\xi, x)$ the *super-gradient* of a concave function $f_l^h(\xi, \cdot)$ at point x. That is, $z \in \partial f_l^h(\xi, x)$ iff $f_l^h(\xi, y) - f_l^h(\xi, x) \leq z(y - x)$ for every $y \in \mathbb{R}_+$. Recall that, given $l \in L(J)$, $\partial f_l^h(\xi, x) \neq \emptyset$ at any point $x > 0$.

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