2
The Centralized Economy

2.1 Introduction

In this chapter we introduce the basic dynamic general equilibrium model for a closed economy. The aim is to explain how the optimal level of output is determined in the economy and how this is allocated between consumption and capital accumulation or, put another way, between consumption today and consumption in the future. We exclude government, money, and financial markets, and all variables are in real, not money, terms. Although apparently very restrictive, this model captures most of the essential features of the macroeconomy. Subsequent chapters build on this basic model by adding further detail but without drastically altering the substantive conclusions derived from the basic model.

Various different interpretations of this model have been made. It is sometimes referred to as the Ramsey model after Frank Ramsey (1928), who first introduced a very similar version to study taxation (Ramsey 1927). The model can also be interpreted as a central (or social) planning model in which the decisions are taken centrally by the social planner in the light of individual preferences, which are assumed to be identical. (Alternatively, the social planner's preferences may be considered as imposed on everyone.) It is also called a representative-agent model when all economic agents are identical and act as both a household and a firm. Another interpretation of the model is that it can be regarded as referring to a single individual. Consequently, it is sometimes called a Robinson Crusoe economy. Any of these interpretations may prove helpful in understanding the analysis of the model. This model has also formed the basis of modern growth theory (see Cass 1965; Koopmans 1967). Our interest in this model, however interpreted, is to identify and analyze certain key concepts in macroeconomics and key features of the macroeconomy. The rest of the book builds on this first pass through this highly simplified preliminary account of the macroeconomy.

2.2 The Basic Dynamic General Equilibrium Closed Economy

The model may be described as follows. Today's output can either be consumed or invested, and the existing capital stock can either be consumed today or used to produce output tomorrow. Today's investment will add to the capital stock and increase tomorrow's output. The problem to be addressed is how best to allocate output between consumption today and investment (i.e., to accumulating capital) so that there is more output and consumption tomorrow.

The model consists of three equations. The first is the national income identity:

$$y_t = c_t + i_t, \tag{2.1}$$

in which total output y_t in period t consists of consumption c_t plus investment goods i_t. The national income identity also serves as the resource constraint for the whole economy. In this simple model total output is also total income and this is either spent on consumption, or is saved. Savings $s_t = y_t - c_t$ can only be used to buy investment goods, hence $i_t = s_t$.

The second equation is

$$\Delta k_{t+1} = i_t - \delta k_t. \tag{2.2}$$

This shows how k_t, the capital stock at the beginning of period t, accumulates over time. The increase in the stock of capital (net investment) during period t equals new (gross) investment less depreciated capital. A constant proportion δ of the capital stock is assumed to depreciate each period (i.e., to have become obsolete). This equation provides the (intrinsic) dynamics of the model.

The third equation is the production function:

$$y_t = F(k_t). \tag{2.3}$$

This gives the output produced during period t by the stock of capital *at the beginning* of the period using the available technology. An increase in the stock of capital increases output, but at a diminishing rate, hence $F > 0$, $F' > 0$, and $F'' \leqslant 0$. We also assume that the marginal product of capital approaches zero as capital tends to infinity, and approaches infinity as capital tends to zero, i.e.,

$$\lim_{k\to\infty} F'(k) = 0 \quad \text{and} \quad \lim_{k\to 0} F'(k) = \infty.$$

These are known as the Inada (1964) conditions. They imply that at the origin there are infinite output gains to increasing the capital stock whereas, as the capital stock increases, the gains in output decline and eventually tend to zero.

If we interpret the model as an economy in which the population is constant through time, then this is like measuring output, consumption, investment, and capital in per capita terms. For example, if there is a constant population N, then $y_t = Y_t/N$ is output per capita, where Y_t is total output for the whole economy.

Output and investment can be eliminated from the subsequent analysis, and the model reduced to just one equation involving two variables. Combining the three equations gives the economy's resource constraint:

$$F(k_t) = c_t + \Delta k_{t+1} + \delta k_t. \tag{2.4}$$

This is a nonlinear dynamic constraint on the economy.

Given an initial stock of capital, k_t (the endowment), the economy must choose its preferred level of consumption for period t, namely c_t, and capital at the start of period $t+1$, namely k_{t+1}. This can be shown to be equivalent to choosing consumption for periods $t, t+1, t+2, \dots$, with the preferred levels of capital, output, investment, and savings for each period derived from the model.

Having established the constraints facing the economy, the next issue is its preferences. What is the economy trying to maximize subject to these constraints? Possible choices are output, consumption, and the utility derived from consumption. We could choose their values in the current period or over the long term. We are also interested in whether a particular choice for the current period is sustainable thereafter. This is related to the existence and stability of equilibrium in the economy. We consider two solutions: the "golden rule" and the "optimal solution." Both of these assume that the aim of the economy (the representative economic agent or the central planner) is to maximize consumption, or the utility derived from consumption. The difference is in attitudes to the future. In the golden rule the future is not discounted whereas in the optimal solution it is. In other words, any given level of consumption is valued less highly if it is in the future than if it is in the present. We can show that, as a result, the golden rule is not sustainable following a negative shock to output but the optimal solution is.

2.3 Golden Rule Solution

2.3.1 The Steady State

Consider first an attempt to maximize consumption in period t. This is perhaps the most obvious type of solution. It would be equivalent to maximizing utility $U(c_t)$. From the resource constraint, equation (2.4), c_t must satisfy

$$c_t = F(k_t) - k_{t+1} + (1-\delta)k_t. \tag{2.5}$$

To maximize c_t the economy must, in period t, consume the whole of current output $F(k_t)$ plus undepreciated capital $(1-\delta)k_t$, and undertake no investment so that $k_{t+1} = 0$. In the following period output would, of course, be zero as there would be no capital to produce it. This solution is clearly unsustainable. It would only appeal to an economic agent who is myopic, or one who has no future.

We therefore introduce the additional constraint that the level of consumption should be sustainable. This implies that in each period new investment is required to maintain the capital stock and to produce next period's output. In effect, we are assuming that the aim is to maximize consumption *in each period*. With no distinction being made between current and future consumption, the problem has been converted from one with a very short-term objective to one with a very long-term objective.

The solution can be obtained by considering just the long run and we therefore omit time subscripts. In the long run the capital stock will be constant and long-run consumption is obtained from equation (2.5) as

$$c = F(k) - \delta k. \tag{2.6}$$

Consumption in the long run is output less that part of output required to replace depreciated capital in order to keep the stock of capital constant. Thus the only investment undertaken is that to replace depreciated capital. The output that remains can be consumed.

The problem now is how to choose k to maximize c. The first-order condition for a maximum of c is

$$\frac{\partial c}{\partial k} = F'(k) - \delta = 0 \tag{2.7}$$

and the second-order condition is

$$\frac{\partial^2 c}{\partial k^2} = F''(k) \leqslant 0.$$

Equation (2.7) implies that the capital stock must be increased until its marginal product $F'(k)$ equals the rate of depreciation δ. Up to this point an increase in the stock of capital increases consumption, but beyond this point consumption begins to decrease. This is because the output cost of replacing depreciated capital in each period requires that consumption be reduced. The solution can be depicted graphically. Figure 2.1 shows straightforwardly that the marginal product of capital falls as the stock of capital increases. Given the rate of depreciation δ, the value of the capital stock can be obtained. The higher the rate of depreciation, the smaller the sustainable size of the capital stock.

We can determine the optimal level of consumption from figure 2.2. The curved line is the production function: the level of output $F(k)$ produced by the capital stock k that is in place at the beginning of the period. The straight line is replacement investment δk. The difference between the two is consumption plus net investment (capital accumulation), i.e.,

$$F(k) - \delta k = c + \Delta k,$$

which is simply equation (2.5). The maximum difference occurs where the lines are furthest apart. This happens when $F'(k) = \delta$, i.e., when the slope of the tangent to the production function—the marginal product of capital $F'(k)$—equals the slope of the line depicting total depreciation, δ. For ease of visibility, in the diagram the size of δ (and hence of depreciated capital) has been exaggerated.

Figure 2.3 provides another way of depicting the solution. The curved line represents consumption plus net investment (i.e., net output or the vertical distance between the two lines in figure 2.2) and is plotted against the capital stock. Points above the line are not attainable due to the resource constraint $F(k) - \delta k \geqslant c + \Delta k$. The maximum level of consumption plus net investment

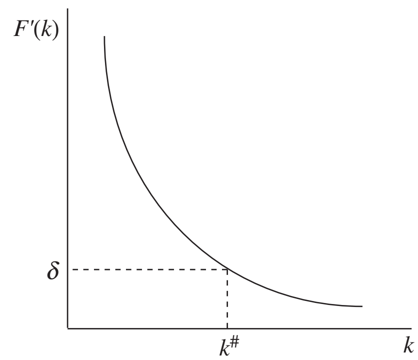

Figure 2.1. The marginal product of capital.

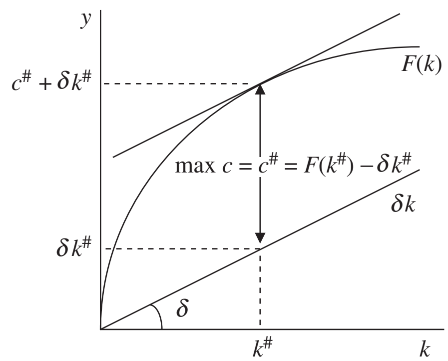

Figure 2.2. Total output, consumption, and replacement investment.

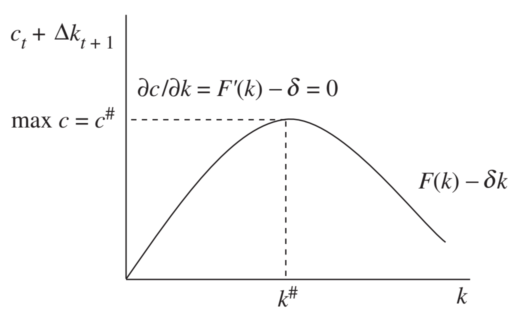

Figure 2.3. Net output.

occurs where the slope of the tangent is zero. At this point net investment $F'(k) - \delta = 0$.

We can now find the sustainable level of consumption. This occurs when the capital stock is constant over time, implying that $\Delta k = 0$ and that net investment is zero. The maximum point on the line is then the maximum sustainable level of consumption $c^{\#}$. This requires a constant level of the capital stock $k^{\#}$. This solution is known as the golden rule.

2.3.2 The Dynamics of the Golden Rule

Due to the constraint that the capital stock is constant, $c^{\#}$ is sustainable indefinitely *provided there are no disturbances to the economy*. If there are disturbances, then the economy becomes dynamically unstable at $\{c^{\#}, k^{\#}\}$. To see why the golden rule is not a stable solution, consider what would happen if the economy tried to maintain consumption at the maximum level $c^{\#}$ even when the capital stock differs from $k^{\#}$ due to a negative disturbance.

If $k < k^{\#}$ then the level of output would be $F(k) < F(k^{\#})$. In order to consume the amount $c^{\#}$, it would then be necessary to consume some of the existing capital stock, with the result that $\Delta k < 0$, and the capital stock would no longer be constant, but would fall. With less capital, future output would therefore be even smaller and attempts to maintain consumption at $c^{\#}$ would cause further decreases in the capital stock. Eventually the economy would no longer be able to consume even $c^{\#}$ as there would be too little capital to produce this amount.

An important implication emerges from this: an economy that consumes too much will, sooner or later, find that it is eroding its capital base and will not be able to sustain its consumption. In practice, of course, it is not possible to switch to consuming capital goods, except in a few special cases. The analysis can, however, be interpreted to mean that switching resources from producing capital goods to consumption goods will eventually undermine the economy, and hence consumption. Thus, the apparently small technical point concerning the stability of the solution turns out to have profound implications for macroeconomics.

There is, however, a simple solution. The economy can reduce its consumption temporally and divert output to rebuilding the capital stock to a level that restores the original equilibrium. This would mean that negative shocks to the system would impact heavily on consumption in the short term. Trying to achieve the maximum level of consumption in each period may not, therefore, result in maximizing consumption in the longer term. The solution is to suspend the consumption objective temporarily.

It may be noted that if, as a result of a positive disturbance, $k > k^{\#}$ and hence output is raised, it would be possible to increase consumption temporarily until the capital stock returns to the lower, but sustainable, level $k^{\#}$. We make further observations on the stability of the economy under the golden rule below after we have considered the optimal solution.

2.4 Optimal Solution

2.4.1 Derivation of the Fundamental Euler Equation

Instead of assuming that future consumption has the same value as consumption today, we now assume that the economy values consumption today more than consumption in the future. In particular, we suppose that the aim is to maximize the present value of current and future utility,

$$\max_{\{c_{t+s},k_{t+s}\}} V_t = \sum_{s=0}^{\infty} \beta^s U(c_{t+s}),$$

where additional consumption increases instantaneous utility $U_t = U(c_t)$, implying $U'_t > 0$, but does so at a diminishing rate as $U''_t \leqslant 0$. Future utility is therefore valued less highly than current utility as it is discounted by the discount factor $0 < \beta < 1$, or equivalently at the rate $\theta > 0$, where $\beta = 1/(1+\theta)$. The aim is to choose current and future consumption to maximize V_t subject to the economy-wide resource constraint equation (2.4).

As the problem involves variables defined in different periods of time it is one of dynamic optimization. This sort of problem is commonly solved using either dynamic programming, the calculus of variations, or the maximum principle. But because, as formulated, it is not a stochastic problem, it can also be solved using the more familiar method of Lagrange multiplier analysis. (See the mathematical appendix for further details of dynamic optimization by these methods.)

First we define the Lagrangian constrained for each period by the resource constraint

$$\mathcal{L}_t = \sum_{s=0}^{\infty} \{\beta^s U(c_{t+s}) + \lambda_{t+s}[F(k_{t+s}) - c_{t+s} - k_{t+s+1} + (1-\delta)k_{t+s}]\}, \tag{2.8}$$

where λ_{t+s} is the Lagrange multiplier s periods ahead. This is maximized with respect to $\{c_{t+s}, k_{t+s+1}, \lambda_{t+s};\ s \geqslant 0\}$. The first-order conditions are

$$\frac{\partial \mathcal{L}_t}{\partial c_{t+s}} = \beta^s U'(c_{t+s}) - \lambda_{t+s} = 0, \qquad s \geqslant 0, \tag{2.9}$$

$$\frac{\partial \mathcal{L}_t}{\partial k_{t+s}} = \lambda_{t+s}[F'(k_{t+s}) + 1 - \delta] - \lambda_{t+s-1} = 0, \quad s > 0, \tag{2.10}$$

plus the constraint equation (2.4) and the transversality condition

$$\lim_{s\to\infty} \beta^s U'(c_{t+s})k_{t+s} = 0. \tag{2.11}$$

Notice that we do not maximize with respect to k_t as we assume that this is predetermined in period t.

To help us understand the role of the transversality condition (2.11) in intertemporal optimization, consider the implication of having a finite capital stock at time $t+s$. If consumed this would give discounted utility of $\beta^s U'(c_{t+s})k_{t+s}$. If the time horizon were $t+s$, then it would not be optimal to have any capital left in period $t+s$; it should have been consumed instead. Hence, as $s \to \infty$, the transversality condition provides an extra optimality condition for intertemporal infinite-horizon problems.

The Lagrange multiplier can be obtained from equation (2.9). Substituting for λ_{t+s} and λ_{t+s-1} in equation (2.10) gives

$$\beta^s U'(c_{t+s})[F'(k_{t+s}) + 1 - \delta] = \beta^{s-1} U'(c_{t+s-1}), \quad s > 0.$$

For $s = 1$ this can be rewritten as

$$\beta \frac{U'(c_{t+1})}{U'(c_t)}[F'(k_{t+1}) + 1 - \delta] = 1. \tag{2.12}$$

Equation (2.12) is known as the Euler equation. It is the fundamental dynamic equation in intertemporal optimization problems in which there are dynamic constraints. The same equation arises using each of the alternative methods of optimization referred to above.

2.4.2 Interpretation of the Euler Equation

It is possible to give an intuitive explanation for the Euler equation. Consider the following problem: if we reduce c_t by a small amount $\mathrm{d}c_t$, how much larger must c_{t+1} be to fully compensate for this while leaving V_t unchanged? We suppose that consumption beyond period $t + 1$ remains unaffected. This problem can be addressed by considering just two periods: t and $t + 1$. Thus we let

$$V_t = U(c_t) + \beta U(c_{t+1}).$$

Taking the total differential of V_t, and recalling that V_t remains constant, implies that

$$0 = \mathrm{d}V_t = \mathrm{d}U_t + \beta\,\mathrm{d}U_{t+1} = U'(c_t)\,\mathrm{d}c_t + \beta U'(c_{t+1})\,\mathrm{d}c_{t+1},$$

where $\mathrm{d}c_{t+1}$ is the small change in c_{t+1} brought about by reducing c_t. Since we are reducing c_t, we have $\mathrm{d}c_t < 0$. The loss of utility in period t is therefore $U'(c_t)\,\mathrm{d}c_t$. In order for V_t to be constant, this must be compensated by the discounted gain in utility $\beta U'(c_{t+1})\,\mathrm{d}c_{t+1}$. Hence we need to increase c_{t+1} by

$$\mathrm{d}c_{t+1} = -\frac{U'(c_t)}{\beta U'(c_{t+1})}\,\mathrm{d}c_t. \tag{2.13}$$

As the resource constraint must be satisfied in every period, in periods t and $t + 1$ we require that

$$\begin{aligned} F'(k_t)\,\mathrm{d}k_t &= \mathrm{d}c_t + \mathrm{d}k_{t+1} - (1-\delta)\,\mathrm{d}k_t, \\ F'(k_{t+1})\,\mathrm{d}k_{t+1} &= \mathrm{d}c_{t+1} + \mathrm{d}k_{t+2} - (1-\delta)\,\mathrm{d}k_{t+1}. \end{aligned}$$

As k_t is given and beyond period $t + 1$ we are constraining the capital stock to be unchanged, only the capital stock in period $t + 1$ can be different from before. Thus $\mathrm{d}k_t = \mathrm{d}k_{t+2} = 0$. The resource constraints for periods t and $t + 1$ can therefore be rewritten as

$$\begin{aligned} 0 &= \mathrm{d}c_t + \mathrm{d}k_{t+1}, \\ F'(k_{t+1})\,\mathrm{d}k_{t+1} &= \mathrm{d}c_{t+1} - (1-\delta)\,\mathrm{d}k_{t+1}. \end{aligned}$$

These two equations can be reduced to one equation by eliminating $\mathrm{d}k_{t+1}$ to give a second connection between $\mathrm{d}c_t$ and $\mathrm{d}c_{t+1}$, namely,

$$\mathrm{d}c_{t+1} = -[F'(k_{t+1}) + 1 - \delta]\,\mathrm{d}c_t. \tag{2.14}$$

This can be interpreted as follows. The output no longer consumed in period t is invested and increases output in period $t+1$ by $-F'(k_{t+1})\,\mathrm{d}c_t$. All of this can be consumed in period $t+1$. And as we do not wish to increase the capital stock beyond period $t+1$, the undepreciated increase in the capital stock, $(1-\delta)\,\mathrm{d}c_t$, can also be consumed in period $t+1$. This gives the total increase in consumption in period $t+1$ stated in equation (2.14). The discounted utility of this extra consumption as measured in period t is

$$\beta U'(c_{t+1})\,\mathrm{d}c_{t+1} = -\beta U'(c_{t+1})[F'(k_{t+1}) + 1 - \delta]\,\mathrm{d}c_t.$$

To keep V_t constant, this must be equal to the loss of utility in period t. Thus

$$U'(c_t)\,\mathrm{d}c_t = \beta U'(c_{t+1})[F'(k_{t+1}) + 1 - \delta]\,\mathrm{d}c_t.$$

Canceling $\mathrm{d}c_t$ from both sides and dividing through by $U'(c_t)$ gives the Euler equation (2.12).

2.4.3 Intertemporal Production Possibility Frontier

The production possibility frontier is associated with a production function that has more than one type of output and one or more inputs. It measures the maximum combination of each type of output that can be produced using a fixed amount of the factor(s). The result is a concave function in output space of the quantities produced. The *intertemporal* production possibility frontier (IPPF) is associated with outputs at different points of time and is derived from the economy's resource constraint. This gives the second relation between c_t and c_{t+1}. It is obtained by combining the resource constraints for periods t and $t+1$ to eliminate k_{t+1}. The result is the two-period intertemporal resource constraint (or IPPF)

$$\begin{aligned} c_{t+1} &= F(k_{t+1}) - k_{t+2} + (1-\delta)k_{t+1} \\ &= F[F(k_t) - c_t + (1-\delta)k_t] - k_{t+2} + (1-\delta)[F(k_t) - c_t + (1-\delta)k_t]. \end{aligned} \tag{2.15}$$

This provides a concave relation between c_t and c_{t+1}.

The slope of a tangent to the IPPF is

$$\frac{\partial c_{t+1}}{\partial c_t} = -[F'(k_{t+1}) + 1 - \delta]. \tag{2.16}$$

As noted previously, this is also the slope of the indifference curve at the point where it is tangent to the resource constraint. Hence, the IPPF also touches the indifference curve at this point. And as

$$\frac{\partial^2 c_{t+1}}{\partial c_t^2} = F''(k_{t+1}) < 0,$$

the tangent to the IPPF flattens as c_t decreases, implying that the IPPF is a concave function. We use this result in the discussion below.

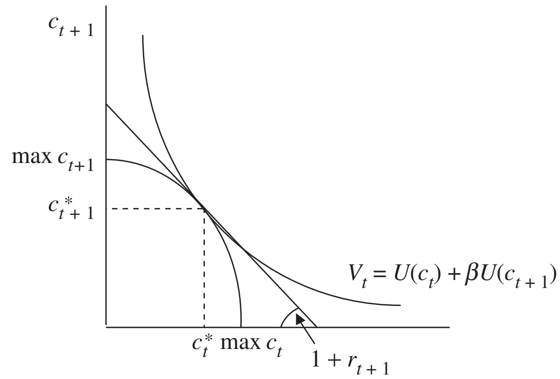

Figure 2.4. A graphical solution based on the IPPF.

2.4.4 Graphical Representation of the Solution

The solution to the two-period problem is represented in figure 2.4. The upper curved line is the indifference curve that trades off consumption today for consumption tomorrow while leaving V_t unchanged. It is tangent to the resource constraint. The lower curved line represents the trade-off between consumption today and consumption tomorrow from the viewpoint of production, i.e., it is the IPPF. It touches the indifference curve at the point of tangency with the budget constraint. This solution arises as in equilibrium equations (2.13) and (2.14), and (2.16) must be satisfied simultaneously so that

$$-\left.\frac{\mathrm{d}c_{t+1}}{\mathrm{d}c_t}\right|_{V_{\text{const.}}} = F'(k_{t+1}) + 1 - \delta = 1 + r_{t+1} = -\left.\frac{\partial c_{t+1}}{\partial c_t}\right|_{\text{IPPF}}.$$

The net marginal product $F'(k_{t+1}) - \delta = r_{t+1}$ can be interpreted as the implied real rate of return on capital after allowing for depreciation. An increase in r_{t+1} due, for example, to a technology shock that raises the marginal product of capital in period $t+1$ makes the resource constraint steeper, and results in an increase in V_t, c_t, and c_{t+1}.

2.4.5 Static Equilibrium Solution

We now return to the full optimal solution and consider its long-run equilibrium properties. The long-run equilibrium is a static solution, implying that in the absence of shocks to the macroeconomic system, consumption and the capital stock will be constant through time. Thus $c_t = c^*$, $k_t = k^*$, $\Delta c_t = 0$, and $\Delta k_t = 0$ for all t. In static equilibrium the Euler equation can therefore be written as

$$\frac{\beta U'(c^*)}{U'(c^*)}[F'(k^*) + 1 - \delta] = 1,$$

implying that

$$F'(k^*) = \frac{1}{\beta} + \delta - 1 = \delta + \theta.$$

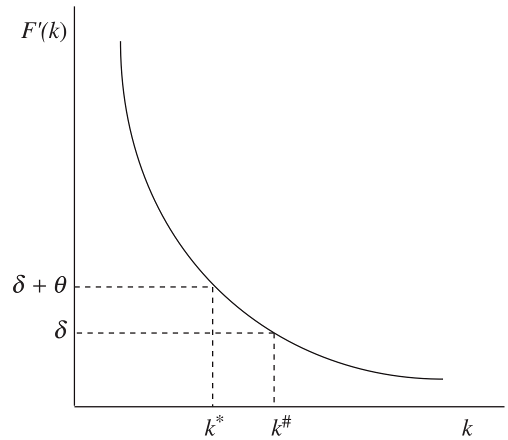

Figure 2.5. Optimal long-run capital.

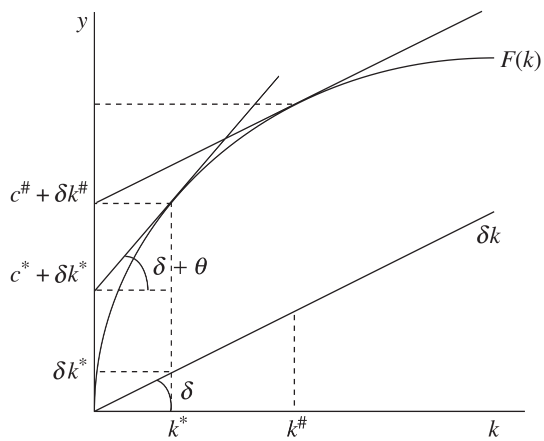

Figure 2.6. Optimal long-run consumption.

The solution is therefore different from that for the golden rule, where $F'(k) = \delta$. Figure 2.1 is replaced by figure 2.5. This shows that the optimal level of capital is less than for the golden rule. The reason for this is that future utility is discounted at the rate $\theta > 0$.

The implications for consumption can be seen in figures 2.5 and 2.6. In figure 2.5 the solution is obtained where the slope of the tangent to the production function is $\delta + \theta$. As the tangent must be steeper than for the golden rule, this implies that the optimal level of capital must be lower. Figure 2.6 shows that this entails a lower level of consumption too. Thus $c^* < c^\#$ and $k^* < k^\#$.

We have shown that discounting the future results in lower consumption. This may seem to be a good reason for not discounting the future. To see what the benefit of discounting is we must analyze the dynamics and stability of this solution.

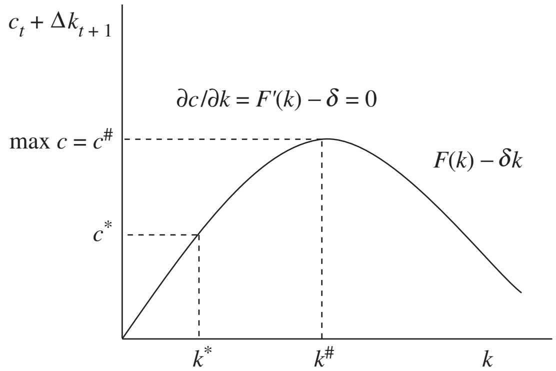

Figure 2.7. Optimal consumption compared.

2.4.5.1 An Example

Suppose that utility is the power function

$$U(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}.$$

It can be shown that $\sigma = -cU''/U'$ is the coefficient of relative risk aversion. Suppose also that the production function is Cobb–Douglas so that

$$y_t = Ak_t^{\alpha}.$$

Then the Euler equation (2.12) is

$$\beta \frac{U'(c_{t+1})}{U'(c_t)}[F'(k_{t+1}) + 1 - \delta] = \beta \left(\frac{c_{t+1}}{c_t}\right)^{-\sigma} [\alpha A k_{t+1}^{-(1-\alpha)} + 1 - \delta] = 1.$$

Hence the steady-state level of capital is

$$k^* = \left(\frac{\alpha A}{\delta + \theta}\right)^{1/(1-\alpha)}$$

and the steady-state level of consumption is

$$\begin{aligned} c^* &= Ak^{*\alpha} - \delta k^* \\ &= \left(\frac{A}{\delta + \theta}\right)^{1-\alpha} \left(\frac{(1-\alpha)\delta + \theta}{\alpha^{\alpha}}\right). \end{aligned}$$

2.4.6 Dynamics of the Optimal Solution

The dynamic analysis that we require uses a so-called phase diagram. This is based on figure 2.7. To construct the phase diagram, we must first consider the two equations that describe the optimal solution at each point in time. These are the Euler equation and the resource constraint. For convenience they are reproduced here:

$$\begin{aligned} \frac{\beta U'(c_{t+1})}{U'(c_t)}[F'(k_{t+1}) + 1 - \delta] &= 1, \\ \Delta k_{t+1} &= F(k_t) - \delta k_t - c_t. \end{aligned} \tag{2.17}$$

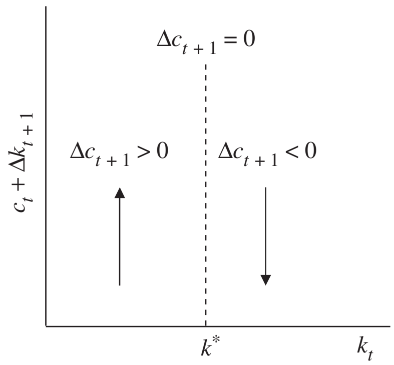

Figure 2.8. Consumption dynamics.

A complication is that both equations are nonlinear. We therefore consider a local solution (i.e., a solution that holds in the neighborhood of equilibrium) obtained through linearizing the Euler equation by taking a Taylor series expansion of $U'(c_{t+1})$ about c_t. This gives

$$U'(c_{t+1}) \simeq U'(c_t) + \Delta c_{t+1} U''(c_t).$$

Hence

$$\frac{U'(c_{t+1})}{U'(c_t)} \simeq 1 + \frac{U''}{U'}\Delta c_{t+1}, \quad \frac{U''}{U'} \leqslant 0,$$

and

$$\Delta c_{t+1} = \frac{U'}{U''}\left[1 - \frac{1}{\beta[F'(k_{t+1}) + 1 - \delta]}\right]. \tag{2.18}$$

Thus we have two equations that determine the changes in consumption and capital: equations (2.17) and (2.18).

These equations confirm the static-equilibrium solution as when $c_t = c^*$ and $k_t = k^*$, we have $\Delta c_{t+1} = 0$, $\Delta k_{t+1} = 0$, and $F'(k^*) = \delta + \theta$. From equation (2.18) we note that when $k > k^*$ we have $F'(k) < F'(k^*)$, and therefore $F'(k) + 1 - \delta < F'(k^*) + 1 - \delta$. It follows that if $k = k^*$ we have $\Delta c = 0$, i.e., consumption is constant, and if $k > k^*$ then $\Delta c < 0$, i.e., consumption must be decreasing. By a similar argument, if $k < k^*$ then $\Delta c > 0$ and consumption is increasing. Thus, $\Delta c \lesseqqgtr 0$ for $k \gtreqqless k^*$. This is represented in figure 2.8.

The dynamic behavior of capital is determined from equation (2.17). When $c_t \gtreqqless F(k_{t+1}) - \delta k_t$ we have $\Delta k_{t+1} \lesseqqgtr 0$. This is depicted in figure 2.9. Above the curve consumption plus long-run net investment exceeds output. The capital stock must therefore decrease to accommodate the excessive level of consumption. Below the curve there is sufficient output left over after consumption to allow capital to accumulate.

Combining figures 2.8 and 2.9 gives figure 2.10, the phase diagram we require. Note that this applies in the general nonlinear case and is not a local approximation. The optimal long-run solution is at point B. The line SS through B is known as the saddlepath, or stable manifold. Only points on this line are attainable. This is not as restrictive as it may seem, as the location of the saddlepath

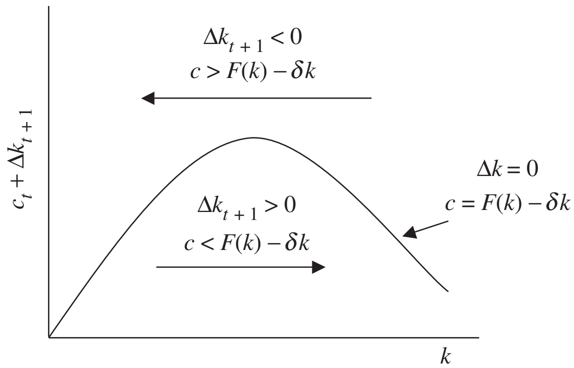

Figure 2.9. Capital dynamics.

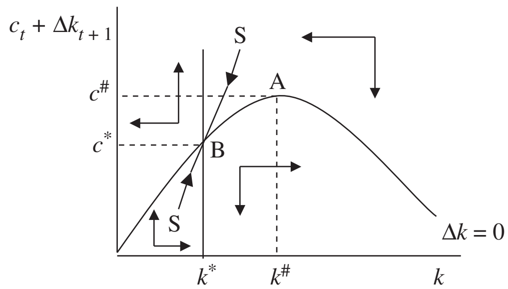

Figure 2.10. Phase diagram.

is determined by the economy, i.e., the parameters of the model, and could in principle be in an infinite number of places depending on the particular values of the parameters. The arrows denote the dynamic behavior of c_t and k_t. This depends on which of four possible regions the economy is in. To the northeast, but on the line SS, consumption is excessive and the capital stock is so large that the marginal product of capital is less than $\delta + \theta$. This is not sustainable and therefore both consumption and the capital stock must decrease. This is indicated by the arrow on SS. The opposite is true on SS in the southwest region. Here consumption and capital need to increase. As the other two regions are not attainable they can be ignored. The economy therefore attains equilibrium at the point B by moving along the saddlepath to that point. At B there is no need for further changes in consumption and capital, and the economy is in equilibrium. Were the economy able to be off the line SS—*which it is not and cannot be*—the dynamics would ensure that it could not attain equilibrium. When there are two regions of stability and two of instability like this the solution is called a saddlepath equilibrium.

2.4.7 Algebraic Analysis of the Saddlepath Dynamics

An algebraic analysis of the dynamic behavior of the economy may be based on the two nonlinear dynamic equations describing the optimal solution, namely, the Euler equation and the resource constraint:

$$\frac{\beta U'(c_{t+1})}{U'(c_t)}[F'(k_{t+1}) + 1 - \delta] = 1, \tag{2.19}$$

$$\Delta k_{t+1} = F(k_t) - \delta k_t - c_t. \tag{2.20}$$

The static (or long-run) equilibrium solutions $\{c^*, k^*\}$ are obtained from

$$F'(k^*) = \delta + \theta, \tag{2.21}$$

$$c^* = F(k^*) - \delta k^*. \tag{2.22}$$

As equations (2.19) and (2.20) are nonlinear in c and k, our analysis is based on a local linear approximation to the full nonlinear model. The linear approximation to equation (2.19) is obtained as a first-order Taylor series expansion about $\{c^*, k^*\}$:

$$\beta\left[F'(k^*) + 1 - \delta + \frac{U''(c^*)}{U'(c^*)}\Delta c_{t+1} + F''(k^*)(k_{t+1} - k^*)\right] \simeq 1.$$

Using the long-run solutions (2.21) and (2.22), this can be rewritten as

$$\frac{U''(c^*)}{U'(c^*)}(c_{t+1} - c^*) + F''(k^*)(k_{t+1} - k^*) \simeq \frac{U''(c^*)}{U'(c^*)}(c_t - c^*). \tag{2.23}$$

The linear approximation to (2.20) is

$$\Delta k_{t+1} \simeq F(k^*) + F'(k^*)(k_t - k^*) - \delta k_t - c_t$$

or

$$\begin{aligned} k_{t+1} - k^* &\simeq -\{c_t - [F(k^*) - \delta k^*]\} + [F'(k^*) + 1 - \delta](k_t - k^*) \\ &= -(c_t - c^*) + \theta(k_t - k^*). \end{aligned} \tag{2.24}$$

We can now write equations (2.24) and (2.23) as a matrix equation of deviations from long-run equilibrium:

$$\begin{bmatrix} c_{t+1} - c^* \\ k_{t+1} - k^* \end{bmatrix} = \begin{bmatrix} 1 + \dfrac{U'F''}{U''} & -(1+\theta)\dfrac{U'F''}{U''} \\ -1 & 1 + \theta \end{bmatrix} \begin{bmatrix} c_t - c^* \\ k_t - k^* \end{bmatrix}.$$

This is a first-order vector autoregression, which has the generic form

$$x_{t+1} = Ax_t,$$

where $x_t = (c_t - c^*, k_t - k^*)'$.

The next step is to determine the dynamic behavior of this system. As shown in the mathematical appendix, this depends on the roots of the matrix A or, equivalently, the roots of the quadratic equation

$$B(L) = 1 - (\operatorname{tr} A)L + (\det A)L^2 = 0.$$

If the roots are denoted $1/\lambda_1$ and $1/\lambda_2$, then they satisfy

$$(1 - \lambda_1 L)(1 - \lambda_2 L) = 0.$$

If the dynamic structure of the system is a saddlepath, then one root, say λ_1, will be the stable root and will satisfy $|\lambda_1| < 1$ and the other root will be unstable and will have the property $|\lambda_2| \geqslant 1$. It is shown in the mathematical appendix that approximately the roots are

$$\begin{aligned}\{\lambda_1, \lambda_2\} &\simeq \left\{\frac{\det A}{\operatorname{tr} A}, \operatorname{tr} A - \frac{\det A}{\operatorname{tr} A}\right\} \\ &= \left\{\frac{1+\theta}{2+\theta+(U'F''/U'')}, 2+\theta+\frac{U'F''}{U''} - \frac{1+\theta}{2+\theta+(U'F''/U'')}\right\}.\end{aligned}$$

Thus, as $U'F''/U'' \geqslant 0$, we have $0 < \lambda_1 < 1$ and $\lambda_2 > 1$. The dynamics of the optimal solution are therefore a saddlepath, as already shown in the diagram. We note that in the previous example

$$\frac{U'F''}{U''} = \frac{\alpha(1-\alpha)cy}{\sigma k^2} > 0.$$

2.5 Real-Business-Cycle Dynamics

2.5.1 The Business Cycle

In practice an economy is continually disturbed from its long-run equilibrium by shocks. These shocks may be temporary or permanent, anticipated or unanticipated. Depending on the type of shock, the equilibrium position of the economy may stay unchanged or it may alter; and optimal adjustment back to equilibrium may be instantaneous or slow. The path followed by the economy during its adjustment back to equilibrium is commonly called the business cycle, even though the path may not be a true cycle. Although the economy will not be in long-run equilibrium during the adjustment, it is behaving optimally during the adjustment back to long-run equilibrium. In effect, it is attaining a sequence of temporary equilibria, each of which is optimal at that time.

The traditional aim of stabilization policy is to speed up the return to equilibrium. This is more relevant when market imperfections due to, for example, monopolistic competition and price inflexibilities have caused a loss of output, and hence economic welfare, than it is in our basic model, where there are neither explicit markets nor market imperfections. We return to these issues in chapters 9 and 13.

Real-business-cycle theory focuses on the effect on the economy of a particular type of shock: a technology (productivity) shock. We already have a model capable of analyzing this. The previous analysis has assumed that the economy is nonstochastic. In keeping with this assumption we presume that the technology shock is known to the whole economy the moment it occurs. A technology shock shifts the production function upwards. Thus for every value of the stock of capital k there is an increase in output y and hence in the marginal product of capital $F'(k)$. We consider both permanent and temporary technology shocks.

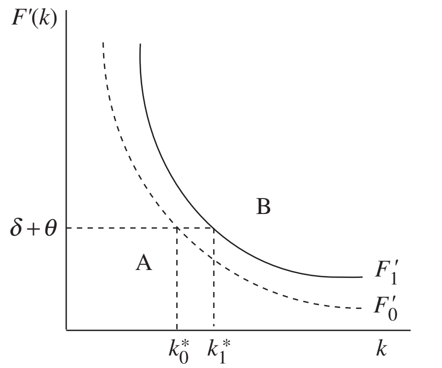

Figure 2.11. The effect on capital of a positive technology shock.

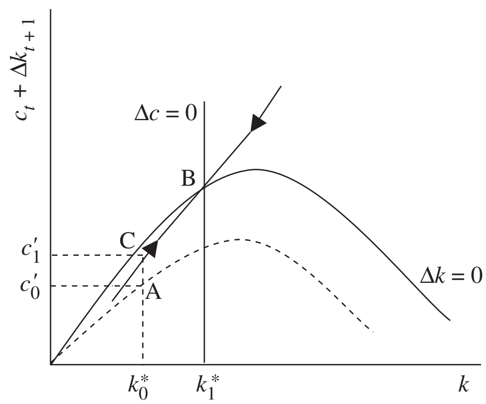

Figure 2.12. The effect on consumption of a positive technology shock.

2.5.2 Permanent Technology Shocks

A positive technology shock increases the marginal product of capital. This is depicted in figure 2.11 as a shift from F_0' to F_1'. As $\delta + \theta$ is unchanged, the equilibrium optimal level of capital increases from k_0^* to k_1^*.

The exact dynamics of this increase and the effect on consumption is shown in figure 2.12. A positive technology shock shifts the curve relating consumption to the capital stock upwards. The original equilibrium was at A, the new equilibrium is at B, and the saddlepath now goes through B. As the economy must always be on the saddlepath, how does the economy get from A to B? The capital stock is initially k_0^* and it takes one period before it can change. As the productivity increase raises output in period t, and the capital stock is fixed, consumption will increase in period t so that the economy moves from A to C, which is on the new saddlepath. There will also be extra investment in period t. By period $t + 1$ this investment will have caused an increase in the stock of capital, which will produce a further increase in output and consumption. In period $t + 1$, therefore, the economy starts to move along the saddlepath—in geometrically declining steps—until it reaches the new equilibrium at B. Thus a permanent positive technology shock causes both consumption and capital to increase, but in the first period—the short run—only consumption increases.

2.5.3 Temporary Technology Shocks

If the positive technology shock lasts for just one period, then there is no change in the long-run equilibrium levels of consumption and capital. The increase in output in period t is therefore consumed and no net investment takes place. In period $t+1$ the original equilibrium level of consumption is restored. If the shock is negative, then consumption would decrease.

This can also be interpreted as roughly what happens when there is a temporary supply shock. Business-cycle dynamics can be explained in a similar way, though in a deep recession there is usually time for the capital stock to change too. As the economy comes out of recession the level of the capital stock is restored.

2.5.4 The Stability and Dynamics of the Golden Rule Revisited

Further understanding of the stability and dynamics of the golden rule solution can now be obtained. The golden rule equilibrium occurs at point A in figure 2.10. It will be recalled that the golden rule does not discount the future and therefore implicitly sets $\theta = 0$. As a result the vertical line dividing the east and west regions now goes through A, which is an equilibrium point.

The model appropriate for the golden rule can be thought of as using a modified version of the Euler equation (2.12) in which the marginal utility functions are omitted. The Euler equation therefore becomes $F'(k) = \delta$, in which there are no dynamics at all. This equation determines k_t. The other equation is the resource constraint, equation (2.17), and this determines c_t. Thus the only dynamics in the model are those associated with equation (2.17), and these concern the capital stock.

At every point on the curved line in figure 2.10—except the point $\{c^{\#}, k^{\#}\}$—we have $\Delta k_{t+1} < 0$. At the point $\{c^{\#}, k^{\#}\}$ we have $\Delta k_{t+1} = 0$. This point is therefore an equilibrium, but, as we have seen, it is not a stable equilibrium because achieving maximum consumption at each point in time requires absorbing all positive shocks through higher consumption and all negative shocks by consuming the capital stock, which reduces future consumption. Thus, after a negative shock the economy is unable to regain equilibrium if it continues to consume as required by the golden rule.

The lack of stability of the golden rule solution can be attributed to the impatience of the economy. By trading off consumption today against consumption in the future, and by discounting future consumption, the optimal solution is a stable equilibrium.

We now consider two extensions to the basic model that involve labor and investment.

2.6 Labor in the Basic Model

In the basic model labor is not included explicitly. Implicitly, it has been assumed that households are involved in production and spend a given fixed amount of time working. In an extension of the basic model we allow people to choose between work and leisure, and how much of their time they spend on each. Only leisure is assumed to provide utility directly; work provides utility indirectly by generating income for consumption. This enables us to derive a labor-supply function and an implicit wage rate. In practice, people can usually choose whether or not to work, but have limited freedom in the number of hours they may choose. We take up this point in chapter 4.

Suppose that the total amount of time available for all activities is normalized to one unit—in effect it has been assumed in the basic model considered so far that labor input is the whole unit. We now assume instead that households have a choice between work n_t and leisure l_t, where $n_t + l_t = 1$. Thus, in effect, in the basic model, $n_t = 1$. We now allow n_t to be chosen by households.

We assume that households receive utility from consumption and leisure and so we rewrite the instantaneous utility function as $U(c_t, l_t)$, where the partial derivatives $U_c > 0$, $U_l > 0$, $U_{cc} \leqslant 0$, and $U_{ll} \leqslant 0$. In other words, there is positive, but diminishing, marginal utility to both consumption and leisure. For convenience, we assume that $U_{cl} = 0$, which rules out substitution between consumption and leisure. We also assume that labor is a second factor of production, so that the production function becomes $F(k_t, n_t)$, with $F_k > 0$, $F_{kk} \leqslant 0$, $F_n > 0$, $F_{nn} \leqslant 0$, $F_{kn} \geqslant 0$, $\lim_{k\to\infty} F_k = \infty$, $\lim_{k\to 0} F_k = 0$, $\lim_{l\to\infty} F_n = \infty$, and $\lim_{l\to 0} F_n = 0$, which are the Inada conditions.

The economy maximizes discounted utility subject to the national resource constraint

$$F(k_t, n_t) = c_t + k_{t+1} - (1 - \delta)k_t$$

and the labor constraint $n_t + l_t = 1$. Often it will be more convenient to replace l_t by $1 - n_t$ but, for the sake of clarity, here we introduce the labor constraint explicitly.

The Lagrangian is therefore

$$\begin{aligned}\mathcal{L}_t = \sum_{s=0}^{\infty} \{\beta^s U(c_{t+s}, l_{t+s}) + \lambda_{t+s}[F(k_{t+s}, n_{t+s}) - c_{t+s} - k_{t+s+1} + (1 - \delta)k_{t+s}] \\ + \mu_{t+s}[1 - n_{t+s} - l_{t+s}]\},\end{aligned}$$

which is maximized with respect to $\{c_{t+s}, l_{t+s}, n_{t+s}, k_{t+s+1}, \lambda_{t+s}, \mu_{t+s};\ s \geqslant 0\}$. The first-order conditions are

$$\frac{\partial \mathcal{L}_t}{\partial c_{t+s}} = \beta^s U_{c,t+s} - \lambda_{t+s} = 0, \quad s \geqslant 0, \tag{2.25}$$

$$\frac{\partial \mathcal{L}_t}{\partial l_{t+s}} = \beta^s U_{l,t+s} - \mu_{t+s} = 0, \quad s \geqslant 0, \tag{2.26}$$

$$\frac{\partial \mathcal{L}_t}{\partial n_{t+s}} = \lambda_{t+s} F_{n,t+s} - \mu_{t+s} = 0, \qquad s \geqslant 0, \tag{2.27}$$

$$\frac{\partial \mathcal{L}_t}{\partial k_{t+s}} = \lambda_{t+s}[F_{k,t+s} + 1 - \delta] - \lambda_{t+s-1} = 0, \quad s > 0. \tag{2.28}$$

From the first-order conditions for consumption and capital we obtain the same solutions as for the basic model. The consumption Euler equation for $s = 1$ is as before:

$$\beta \frac{U_{c,t+1}}{U_{c,t}}[F_{k,t+1} + 1 - \delta] = 1. \tag{2.29}$$

Eliminating λ_{t+s} and μ_{t+s} from the first-order conditions for consumption, leisure, and employment gives, for $s = 0$,

$$U_{l,t} = U_{c,t} F_{n,t}. \tag{2.30}$$

This has the following interpretation. Consider giving up $\mathrm{d}l_t = -\mathrm{d}n_t < 0$ units of leisure. The loss of utility is $U_{l,t}\,\mathrm{d}l_t < 0$, which is the left-hand side of equation (2.30). This is compensated by an increase in utility due to producing extra output of $F_{n,t}\,\mathrm{d}n_t = -F_{n,t}\,\mathrm{d}l_t$. When consumed, each unit of output gives an extra $U_{c,t}$ in utility, implying a total increase in utility of $-U_{c,t}F_{n,t}\,\mathrm{d}l_t > 0$, which is the right-hand side of equation (2.30) when $\mathrm{d}l_t = -1$.

The long-run solution is obtained sequentially. In steady-state equilibrium the long-run solution for capital is obtained from

$$F_k = \theta + \delta,$$

where $\beta = 1/(1 + \theta)$. The long-run solution for consumption is then obtained from the resource constraint. So far this is the same as for the basic model. Given c and k, we solve for l_t and n_t from equation (2.30) and the labor constraint. The short-run solutions for c_t and k_t are the same as before. The short-run dynamics for l_t are similar to those for c_t.

We can now obtain expressions for the wage rate and the total rate of return to capital, which are implicit in the model but have not been defined explicitly. If the production function is a homogeneous function of degree one (implying that the production function has constant returns to scale), then we can show that[1]

$$F(k_t, n_t) = F_{n,t} n_t + F_{k,t} k_t. \tag{2.31}$$

Recalling that the general price level is unity, equation (2.31) says that the total value of output is shared between labor and capital. The first term on the right-hand side is the share of labor and the second is the share of capital. If labor is paid its marginal product, then this is also the implied wage rate, i.e., $F_{n,t} = w_t$, with each unit of labor having a cost (receiving a return) equal to the wage rate. Similarly, if capital is paid its (net) marginal product, then $F_{k,t} - \delta = r_t$ is the

[1] A function $f(x, y)$ that is homogeneous of degree α satisfies $\lambda^\alpha f(x, y) = f(\lambda x, \lambda y)$. Alternatively, we could take a first-order expansion of the production function $F(k_t, n_t)$ about $k_t = n_t = 0$, which would give the approximation $F(k_t, n_t) \simeq F_{n,t} n_t + F_{k,t} k_t$.

return on capital. Thus, $F_{n,t}$ and $F_{k,t} - \delta$ are the implicit wage and rate of return to capital in the basic model.

Consequently, we can write equation (2.31) as

$$F(k_t, n_t) = w_t n_t + (r_t + \delta)k_t.$$

It follows that the real wage can also be expressed as

$$w_t = \frac{F(k_t, n_t) - (r_t + \delta)k_t}{n_t}.$$

In the steady state, when $r_t = \theta$, these become

$$\begin{aligned} F(k^*, n^*) &= wn^* + (\theta + \delta)k^*, \\ w^* &= \frac{F(k^*, n^*) - (\theta + \delta)k^*}{n^*}. \end{aligned}$$

As previously noted, labor was not included explicitly in the basic model. But if we assume that $n_t = 1$, then, in effect, labor was included implicitly. The implied real wage is then

$$\begin{aligned} w_t &= F(k_t, 1) - F_{k,t}k_t \\ &= F(k_t) - (r_t + \delta)k_t. \end{aligned}$$

In equilibrium this is

$$w^* = F(k^*) - (\theta + \delta)k^*.$$

To summarize, we have found that when we allow people to choose how much to work and we determine the wage rate and the rate of return to capital explicitly, the solutions for consumption and capital are virtually unchanged from those of the basic model. This suggests that where appropriate and convenient we may continue to omit labor explicitly from the analysis knowing that it is present implicitly. Moreover, the wage rate and the rate of return to capital, although not explicitly included either, are also defined implicitly.

2.7 Investment

Investment is included explicitly in the basic model, but the emphasis is on the capital stock, not investment. We have assumed previously that there are no costs to installing new capital. We now consider investment and capital accumulation when there are installation costs. Although we focus on Tobin's (1969) q-theory of investment (see also Hayashi 1982), which has the effect of complicating the dynamic behavior of the economy, there are other ways to account for the effects of investment on dynamic behavior. One alternative examined below is to assume that it takes time to install new investment. This is the approach adopted by Kydland and Prescott (1982) and they called it "time to build."

2.7.1 *q*-Theory

In the basic model the focus was on obtaining the optimal levels of consumption and the capital stock. As the change in the stock of capital equals gross investment net of depreciation, this also implies a theory of net investment. We saw that following a permanent change in the long-run equilibrium level of capital, it is optimal if the actual level of capital adjusts to its new equilibrium over time along the saddlepath. The adjustment path for capital implies an optimal level of investment each period. This optimal level of investment will differ each period until the new long-run general equilibrium level of capital is attained. At this point investment is only replacing depreciated capital. Although capital takes time to adjust to its new steady-state level, investment in the basic model adjusts instantaneously to the level that is optimal for each period. In practice, however, due to costs of installation, it is usually optimal to adjust investment more slowly. As a result, the dynamic behavior of capital reflects both adjustment processes.

To illustrate, suppose that new investment imposes an additional resource cost of $\frac{1}{2}\phi i_t/k_t$ for each unit of investment, where $\phi > 0$. In other words, the cost of a unit of investment depends on how large it is in relation to the size of the existing capital stock. We choose this particular functional form due to its mathematical convenience and the consequent ease of interpreting the results. The resource constraint facing the economy now becomes nonlinear in i_t and k_t and is given by

$$F(k_t) = c_t + \left(1 + \frac{\phi}{2}\frac{i_t}{k_t}\right)i_t, \quad \phi \geqslant 0, \tag{2.32}$$

where for simplicity we have reverted to the assumption that capital is the sole factor of production and we ignore leisure. Since our primary interest here is investment we do not combine the resource constraint with the capital accumulation equation, but treat them as two separate constraints.

The Lagrangian for maximizing the present value of utility is therefore

$$\begin{aligned}\mathcal{L}_t = \sum_{s=0}^{\infty}\bigg\{\beta^s U(c_{t+s}) + \lambda_{t+s}\bigg[F(k_{t+s}) - c_{t+s} - i_{t+s} - \frac{\phi}{2}\frac{i_t^2}{k_t}\bigg] \\ + \mu_{t+s}[i_{t+s} - k_{t+s+1} + (1-\delta)k_{t+s}]\bigg\}.\end{aligned}$$

The first-order conditions are

$$\begin{aligned}\frac{\partial\mathcal{L}_t}{\partial c_{t+s}} &= \beta^s U_{c,t+s} - \lambda_{t+s} = 0, && s \geqslant 0,\\ \frac{\partial\mathcal{L}_t}{\partial i_{t+s}} &= -\lambda_{t+s}\left(1 + \frac{\phi}{2}\frac{i_t}{k_t}\right) + \mu_{t+s} = 0, && s \geqslant 0,\\ \frac{\partial\mathcal{L}_t}{\partial k_{t+s}} &= \lambda_{t+s}\left[F_{k,t+s} + \frac{\phi}{2}\left(\frac{i_t}{k_t}\right)^2\right] - \mu_{t+s-1} + (1-\delta)\mu_{t+s} = 0, && s > 0.\end{aligned}$$

The first-order condition for investment implies that

$$i_{t+s} = \frac{1}{\phi}(q_{t+s} - 1)k_{t+s}, \quad s \geqslant 0, \tag{2.33}$$

where the ratio of the Lagrange multipliers

$$q_{t+s} = \frac{\mu_{t+s}}{\lambda_{t+s}} \geqslant 1 \tag{2.34}$$

is called Tobin's q. It follows that investment will take place in period $t + s$ provided $q_{t+s} > 1$.

q can be interpreted as follows. An extra unit of capital raises output, and hence consumption and utility, and λ is the marginal benefit in terms of the utility of sacrificing a unit of current consumption in order to have an extra unit of investment, and hence the extra capital. Similarly, μ is the marginal benefit in terms of utility of an extra unit of investment. Thus q measures the benefit from investment per unit of benefit from capital. Expressing utility in terms of units of output, q can also be interpreted as the ratio of the market value of one unit of investment to its cost.

Combining the three first-order conditions, we obtain the following nonlinear dynamic relation when $s = 1$:

$$F_{k,t+1} = \frac{U_{c,t}}{\beta U_{c,t+1}} q_t - (1 - \delta)q_{t+1} - \frac{1}{2\phi}(q_{t+1} - 1)^2. \tag{2.35}$$

This equation together with equations (2.32)–(2.34) and the capital accumulation equation form a system of four nonlinear dynamic equations that we can solve for the decision variables c_t, k_t, i_t, and q_t.

2.7.1.1 Long-Run Solution

In the steady-state long run we have $\Delta c_t = \Delta k_t = \Delta i_t = \Delta q_t = 0$. In the long run the capital accumulation equation and equation (2.33) imply that

$$\frac{i}{k} = \delta = \frac{1}{\phi}(q - 1).$$

Hence, the long-run value of q is

$$q = 1 + \phi\delta \geqslant 1. \tag{2.36}$$

The long-run level of the capital stock is obtained from the steady-state solution of equation (2.35). From $\beta = 1/(1 + \theta)$, and using the long-run solution for q, equation (2.35) can be written as

$$F_k = \theta + \delta + \phi\delta(\theta + \tfrac{1}{2}\delta) \geqslant \theta + \delta. \tag{2.37}$$

In the absence of costs of installation, $\phi = 0$, and so $q_t = 1$ and $F_k = \theta + \delta$, which is the same result as that obtained in the basic closed-economy model. From figure 2.5, in order for $F_k \geqslant \theta + \delta$, a lower level of capital is required, implying that installation costs reduce the optimal long-run level of the capital stock and hence also the optimal long-run levels of consumption and investment. This is because installation costs reduce the resources available for consumption and investment.

2.7.1.2 Short-Run Dynamics

Introducing installation costs affects the short-run dynamic behavior of the economy as well as its long-run solution. To gain some insight into the effects of installation costs on dynamic behavior we analyze an approximation to equation (2.35) obtained by assuming that consumption is at its steady-state level and using a linear approximation to the quadratic term in q_{t+1} about q, its steady-state level given by equation (2.36). As a result, we are able to approximate equation (2.35) by the forward-looking equation

$$q_t = \beta q_{t+1} + \beta(F_{k,t+1} - \delta - \tfrac{1}{2}\phi\delta^2).$$

Using the steady-state level of $F_{k,t+1}$ given by equation (2.37), this equation can be rewritten in terms of deviations from long-run equilibrium as

$$q_t - q = \beta(q_{t+1} - q) + \beta(F_{k,t+1} - F_k). \tag{2.38}$$

Solving this forwards, the solution for q_t is

$$q_t - q = \sum_{s=0}^{\infty} \beta^{s+1}(F_{k,t+s+1} - F_k).$$

A further interpretation of q_t can now be provided. It is the present value of the extra output produced by undertaking one more unit of investment. The greater this is, the more investment will be undertaken in period t. Since the price of one unit of investment is one, $q_t - 1$ is the increase in the implied value of the firm.

In practice, the measurement of q_t presents a problem. Although q_t can be interpreted as the ratio of the market value of one unit of investment to its cost, it is often estimated by the ratio of the market value of a firm to its book value. This implies using the average value of current and past investment instead of the marginal value of new investment.

Consider next the dynamic interaction between k_t and q_t. Two equations capture this. The first is equation (2.38). The second is obtained by using (2.33) to eliminate i_t from the capital accumulation equation to give

$$i_t = \frac{1}{\phi}(q_t - 1)k_t = k_{t+1} - (1 - \delta)k_t.$$

We therefore have two nonlinear equations:

$$\begin{aligned} q_t - q &= \beta(q_{t+1} - q) + \beta(F_{k,t+1} - F_k), \\ (q_t - q + \phi)k_t &= \phi k_{t+1}. \end{aligned}$$

These can be linearly approximated about the steady-state levels of k_t and q_t as

$$(1-\beta)(q_t - q) - \beta F_{kk}(k_t - k) = \beta\Delta q_{t+1} + \beta F_{kk}\Delta k_{t+1}, \tag{2.39}$$

$$k(q_t - q) = \phi\Delta k_{t+1}, \tag{2.40}$$

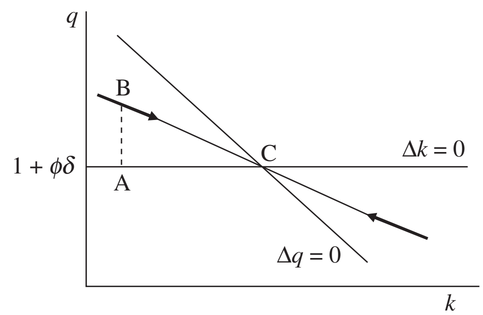

Figure 2.13. Phase diagram for q.

where k is the steady-state level of k_t. Thus, as $F_{kk} < 0$, in steady state k_t is negatively related to q_t through

$$k_t - k = \frac{\theta}{F_{kk}}(q_t - q). \tag{2.41}$$

2.7.1.3 The Effect of a Productivity Increase

The dynamic behavior of k_t and q_t can be illustrated by considering the effect of a permanent increase in capital productivity. In figure 2.13 the line $\Delta q = 0$ depicts the long-run relation between k_t and q_t given by equation (2.41). This was derived from equation (2.39) by setting $\Delta k_{t+1} = \Delta q_{t+1} = 0$. The line $\Delta k = 0$ gives the long-run equilibrium level of k_t and is obtained from equation (2.40) by setting $\Delta k_{t+1} = 0$. Before the productivity increase these two lines intersected at A. Note that at this initial equilibrium $q_t = 1$. Following the productivity increase there is a "jump" increase in q_t so that $q_t > 1$. This induces a rise in investment above its normal replacement level δk. Initially, k_t remains unchanged and so the economy moves to point B. New investment increases the capital stock each period until the economy reaches its new long-run equilibrium at C by moving along the saddlepath from B. At this point q_t is restored to its long-run equilibrium level of one and the equilibrium capital stock, output, and consumption are permanently higher.

2.7.2 Time to Build

An alternative way of reformulating the basic model that results in more general dynamics is to assume that it takes time to install new investment. Kydland and Prescott (1982) were the first to incorporate this idea from neoclassical investment theory into their real-business-cycle DGE macroeconomic model (see also Altug 1989).

Taking account of time-to-build effects results in a respecification of the capital accumulation equation (2.2). Consider two ways of doing this. Suppose, first, that investment expenditures recorded at time t are the result of decisions to invest i_t^s made earlier. Moreover, suppose that a proportion φ_i of recorded investment in period t is investment starts made in period $t - i$, then we can write

$$i_t = \sum_{i=0}^{\infty} \varphi_i i^s_{t-i}. \tag{2.42}$$

If there is a lag in installation, the initial values of φ_i may be zero; and if the lag is finite, then $\varphi_i = 0$ for some $i > J > 0$. The capital accumulation equation (2.2) remains unchanged. We now carry out the optimization of discounted utility with respect to i_t and i^s_t as well as consumption and capital subject to the extra constraint, equation (2.42). This is the Kydland and Prescott approach.

An alternative possible formulation is to assume that a proportion φ_i of investment undertaken in period t is installed and ready for use as part of the capital stock by period $t + i$. Equation (2.2) may therefore be rewritten as

$$\Delta k_{t+1} = \sum_{i=0}^{\infty} \varphi_i i_{t-i} - \delta k_t, \tag{2.43}$$

where $\sum_{i=0}^{J} \varphi_i = 1$. The shape of the distributed lag function will reflect the costs of installation. A modification of this is to incorporate depreciation in φ_i and assume that it reflects the proportion of investment undertaken in period t that contributes to productive capital in period $t + i$. Equation (2.43) can then be rewritten with $\delta = 0$. As a result, using the national income identity (2.1) and the production function, the economy's resource constraint becomes

$$\Delta k_{t+1} = \sum_{i=0}^{\infty} \varphi_i [F(k_{t-i}) - c_{t-i}]. \tag{2.44}$$

We may now maximize $\sum_{s=0}^{\infty} \beta^s U(c_{t+s})$ subject to equation (2.44).

2.8 Conclusions

In this highly simplified account of macroeconomics we have developed a skeleton model that provides the basic framework that will be built on in the rest of the book. The framework itself will need little change, but more detail will be required.

The key features of the macroeconomy that we have represented are the economy's objectives (its preferences), the resource constraint facing the economy (which is derived from the production function, the capital accumulation equation, and the national income identity), and the endowment of the economy (its initial capital stock). We have shown that the central issues are intertemporal: whether to consume today or in the future, and whether to maximize consumption each period or take account of future consumption. Consumption in the future is increased by consuming less today and by saving today's surplus and investing it in additional capital in order to produce, and hence to consume, more in the future. Trying to maximize today's consumption without considering future consumption, or preservation of the capacity to produce in the future, is shown to destabilize the economy, leaving it vulnerable to negative output shocks.

We have found that the dynamics of the basic model derive from just two sources: the intertemporal utility function and the presence of the change in stocks in the resource constraint. An important issue in macroeconomics is the extent to which the dynamic behavior of the macroeconomy can be attributed to these two factors, or whether accounting for business cycles requires additional features.

We have extended the basic model in two ways. One allowed people flexibility in their choice between work and leisure. We were then able to derive a labor-supply function and to obtain an implicit measure of the wage rate. We found that the solutions for consumption and capital were virtually unchanged from those of the basic model. This suggests that, where appropriate and convenient, we may continue to omit labor explicitly, recognizing that it is present implicitly.

The second extension was to take account of the cost of installing capital. As a result, we were able to derive the investment function. In the absence of costs of installing capital, investment takes place instantaneously, even though it takes time for capital to adjust to the desired level. Introducing installation costs for new investment has the effect of delaying the completion of new investment and slowing down the adjustment of the capital stock even more. Having considered the theory of investment that arises from the presence of costs of installing capital and noted the extra complexity it brings to the analysis of the short-run behavior of the economy, for simplicity we will assume hereafter that there are no capital installation costs.

The basic model provides a centralized analysis of the economy. In chapter 4 we decentralize the decisions of households and firms and introduce goods and labor markets to coordinate their decisions.

For further discussion of the basic model see Blanchard and Fischer (1989) and Intriligator (1971).