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DSGE Models, Solutions, and Approximations

This chapter describes some standard dynamic stochastic general equilibrium (DSGE) models and some issues concerning their specification and solution. Such models will be used in examples and exercises throughout the book. It aims to familiarize the reader with such objects rather than providing a fully fledged introduction to DSGE modeling. Since the models we consider do not have a closed-form solution, except in very special circumstances, we also present several methods for obtaining approximate solutions to the optimization problems.

There is a variety of models currently used in macroeconomics. The majority is based on two simple setups: a competitive structure, where allocations are, in general, Pareto optimal; and a monopolistic competitive structure, where one type of agent can set the price of the goods she supplies and allocations are suboptimal. Typically, an expression for the variables of interest in terms of the exogenous forces and the states is found in two ways. When competitive allocations are Pareto optimal, the principle of dynamic programming is typically used and iterations on the Bellman equation are employed to compute the value function and the policy rules, whenever they are known to exist and to be unique. As we will see, calculating the value function is a complicated enterprise except with simple but often economically unpalatable specifications. For general preference and technological specifications, quadratic approximations of the utility function, and discretizations of the dynamic programming problem, are generally employed.

When the equilibrium allocations are distorted, one must alter the dynamic programming formulation and in that case the Bellman equation does not have a hedge over a more standard stochastic Lagrangian multipliers methodology, where one uses the first-order conditions, the constraints, and the transversality condition to obtain a solution. Solutions are also hard to find with the Lagrangian approach since the problem is nonlinear and involves expectations of future variables. Euler equation methods, which approximate the first-order conditions, the expectational equations, or the policy function can be used in these frameworks. Many methods exist in the literature. Here we restrict attention to the three widely used approaches: discretization of the state and shock space; log-linear and second-order approximations; and parametrizing expectations. For a thorough discussion of the various methodologies, see Cooley (1995, chapters 2 and 3) or Marimon and Scott (1999).

The next two sections illustrate features of various models and the mechanics of different solution methods with the aid of examples and exercises. A comparison between various approaches concludes the chapter.

2.1 A Few Useful Models

It is impossible to provide a thorough description of the models currently used in macroeconomics. We therefore focus attention on two prototype structures: one involving only real variables and one also considering nominal ones. In each case, we analyze models with both representative and heterogeneous agents and consider both optimal and distorted setups.

2.1.1 A Basic Real Business Cycle (RBC) Model

Much of the current macroeconomic literature uses versions of the one-sector growth model to jointly explain the cyclical and the long-run properties of the data. In the basic setup we consider there is a large number of identical households that live forever and are endowed with one unit of time, which they can allocate to leisure or to work, and K_0 units of productive capital, which depreciates at the rate $0 < \delta < 1$ every period. The social planner chooses $\{c_t, N_t, K_{t+1}\}_{t=0}^{\infty}$ to maximize

$$E_0 \sum_t \beta^t u(c_t, c_{t-1}, N_t), \tag{2.1}$$

where c_t is consumption, N_t is total hours, K_t is capital, and $E_0 \equiv E[\cdot \mid \mathcal{F}_0]$ is the expectation operator, conditional on the information set $\mathcal{F}_0$, $0 < \beta < 1$. The instantaneous utility function is bounded, twice continuously differentiable, strictly increasing, and strictly concave in all arguments. It depends on c_t and c_{t-1} to account for possible habit formation in consumption. The maximization of (2.1) is subject to the sequence of constraints

$$c_t + K_{t+1} \leqslant (1 - T^y) f(K_t, N_t, \zeta_t) + T_t + (1 - \delta) K_t, \quad 0 \leqslant N_t \leqslant 1, \tag{2.2}$$

where $f(\cdot)$ is a production technology, twice continuously differentiable, strictly increasing, and strictly concave in K_t and N_t; ζ_t is a technological disturbance; T^y is a (constant) income tax rate; and T_t are lump sum transfers.

There is a government which finances a stochastic flow of current expenditure with income taxes and lump sum transfers: expenditure is unproductive and does not yield utility for the agents. We assume a period-by-period balanced budget of the form

$$G_t = T^y f(K_t, N_t, \zeta_t) - T_t. \tag{2.3}$$

The economy is closed by the resource constraint, which provides a national account identity:

$$c_t + K_{t+1} - (1 - \delta) K_t + G_t = f(K_t, N_t, \zeta_t). \tag{2.4}$$

Note that in (2.3) we have assumed that the government balances the budget at each t. This is not restrictive since households in this economy are Ricardian; that is, the addition of government debt does not change optimal allocations. This is because, if debt is held in equilibrium, it must bear the same rate of return as capital, so that $(1 + r_t^{\mathrm{B}}) = E_t[f_k(1 - T^y) + (1 - \delta)]$, where $f_k = \partial f / \partial K$. In other words, debt is a redundant asset and can be priced by arbitrage, once (δ, T^y, f_k) are known. An example where debt matters is considered later on.

Exercise 2.1. Decentralize the RBC model so that there is a representative household and a representative firm. Assume that the household makes the investment decision while the firm hires capital and labor from the household. Is it true that decentralized allocations are the same as those obtained in the social planner's problem? What conditions need to be satisfied? Repeat the exercise assuming that the firm makes the investment decision.

Exercise 2.2. Set $c_{t-1} = 0$ in (2.1) and assume $T^y = 0, \forall t$.

(i) Define the variables characterizing the state of the economy (the states) and the choice variables (the controls) at each t.

(ii) Verify that the problem in (2.1)–(2.4) can be equivalently written as

$$\mathbb{V}(K, \zeta, G) = \max_{\{K^+, N\}} u\{[f(K, N, \zeta) + (1 - \delta)K - G - K^+], N\} + \beta E[\mathbb{V}(K^+, \zeta^+, G^+) \mid K, \zeta, G], \quad (2.5)$$

where the value function $\mathbb{V}$ is the utility value of the optimal plan, given (K_t, ζ_t, G_t), $E(\mathbb{V} \mid \cdot)$ is the expectation of $\mathbb{V}$ conditional on the available information, the superscript "+" indicates future values, and $0 < N_t < 1$.

(iii) Assume that $u(c_t, c_{t-1}, N_t) = \ln c_t + \ln(1 - N_t)$ and that $\mathrm{GDP}_t \equiv f(K_t, N_t, \zeta_t) = \zeta_t K_t^{1-\eta} N_t^{\eta}$. Find values for $(K_t/\mathrm{GDP}_t, c_t/\mathrm{GDP}_t, N_t)$ when ζ_t, G_t are set to their unconditional values (we call this the steady state of the economy).

Note that (2.5) defines the so-called Bellman equation, a recursive functional equation giving the maximum value of the problem for each value of the states and the shocks, given that the next period value of the function is optimally chosen.

There are a few conditions that need to be satisfied for a model to be fitted into a Bellman equation format. First, preferences and technologies must define a convex optimization problem. Second, the utility function must be time separable in the contemporaneous control and state variables. Third, the objective function and the constraints have to be such that current decisions affect current and future utilities but not past ones. While these conditions are typically satisfied, there are situations where the Bellman equation (and its associated optimality principle) may fail to characterize particular economic problems. One is the time inconsistency problem analyzed by Kydland and Prescott (1977), a version of which is described in the next example.

Example 2.1. Suppose a representative household maximizes $E_0 \sum_t \beta^t (\ln c_t + \gamma \ln B_{t+1}/p_t)$ subject to $c_t + B_{t+1}/p_t \leqslant w_t + B_t/p_t + T_t \equiv \mathrm{We}_t$, by choosing sequences for c_t and B_{t+1}, given T_t, p_t, where B_{t+1} are government backed nominal assets, p_t the price level, w_t labor income, T_t lump sum taxes (transfers), and We_t the wealth at t. The government budget constraint is $g_t = B_{t+1}/p_t - B_t/p_t + T_t$, where g_t is random. We assume that the government chooses B_{t+1} to maximize the household's welfare. The household problem is recursive. In fact, the Bellman equation is $\mathbb{V}(\mathrm{We}) = \max_{\{c,B^+\}}(\ln c + \gamma \ln B^+/p) + \beta E \mathbb{V}(\mathrm{We}^+)$ and the constraint is $\mathrm{We} = c + B/p$. The first-order conditions for the problem can be summarized via $1/(c_t p_t) = E_t[\beta/(c_{t+1} p_{t+1}) + \gamma/B_{t+1}]$. Therefore, solving forward and using the resource constraint, we have

$$\frac{1}{p_t} = (w_t - g_t) E_t \sum_{j=0}^{\infty} \beta^j \frac{\gamma}{B_{t+j+1}}. \tag{2.6}$$

The government takes (2.6) as given and maximizes utility subject to the resource constraint. Substituting (2.6) into the utility function we have

$$\max_{B_t} E_0 \sum_t \beta^t \left(\ln c_t + \gamma \ln \left\{ B_t \left[\gamma (w_t - g_t) \sum_{j=0}^{\infty} \frac{\beta^j}{B_{t+j+1}} \right] \right\} \right). \tag{2.7}$$

Clearly, in (2.7) future values of B_t affect current utility. Therefore, the government problem cannot be cast into a Bellman equation.

A solution to (2.5) is typically hard to find since $\mathbb{V}$ is unknown and there is no analytic expression for it. Had the solution been known, we could have used (2.5) to define a function h mapping every (K, G, ζ) into (K^+, N) that gives the maximum.

Since $\mathbb{V}$ is unknown, methods to prove its existence and uniqueness and to describe its properties have been developed (see, for example, Stokey and Lucas 1989). These methods implicitly provide a way of computing a solution to (2.5), which we summarize next.

Algorithm 2.1.

(1) Choose a differentiable and concave function $\mathbb{V}^0(K, \zeta, G)$.

(2) Compute $\mathbb{V}^1(K, \zeta, G) = \max_{\{K^+, N\}} u\{[f(K, \zeta, N) + (1 - \delta)K - G - K^+], N\} + \beta E[\mathbb{V}^0(K^+, \zeta^+, G^+) \mid K, \zeta, G]$.

(3) Set $\mathbb{V}^0 = \mathbb{V}^1$ and iterate on (2) until $|\mathbb{V}^{l+1} - \mathbb{V}^l| < \iota$, ι small.

(4) When $|\mathbb{V}^{l+1} - \mathbb{V}^l| < \iota$, compute $K^+ = h_1(K, \zeta, G)$ and $N = h_2(K, \zeta, G)$.

Hence, $\mathbb{V}$ can be obtained as the limit of $\mathbb{V}^l$ for $l \to \infty$. Under regularity conditions, this limit exists, it is unique, and the sequence of iterations defined by algorithm 2.1 achieves it.

For simple problems algorithm 2.1 is fast and accurate. For more complicated ones, where the combined number of states and shocks is large, it may be computationally demanding. Moreover, unless $\mathbb{V}^0$ is appropriately chosen, the iteration process may be time-consuming. In a few simple cases, the solution to the Bellman equation has a known form and the simpler method of undetermined coefficients can be used. We analyze one of these cases in the next example.

Example 2.2. Assume, in the basic RBC model, that $u(c_t, c_{t-1}, N_t) = \ln c_t + \vartheta_n \ln(1 - N_t)$, $\delta = 1$, the production function is $\text{GDP}_{t+1} = \zeta_{t+1} K_t^{1-\eta} N_t^{\eta}$, the resource constraint is $\text{GDP}_t = K_t + c_t$, $\ln \zeta_t$ is an AR(1) process with persistence ρ, and set $G_t = T^y = T_t = 0$. The states of the problem are GDP_t and ζ_t while the controls are c_t, K_t, N_t. We guess that the value function has the form $\mathbb{V}(K, \zeta) = \mathbb{V}_0 + \mathbb{V}_1 \ln \text{GDP}_t + \mathbb{V}_2 \ln \zeta_t$. Since the Bellman equation maps logarithmic functions into logarithmic ones, the limit, if it exists, will also have a logarithmic form. To find $\mathbb{V}_0, \mathbb{V}_1, \mathbb{V}_2$, we proceed as follows. First, we substitute the constraint into the utility function and use the guess to eliminate future GDP. That is,

$$\begin{aligned} \mathbb{V}_0 + \mathbb{V}_1 \ln \text{GDP}_t + \mathbb{V}_2 \ln \zeta_t = {} & \ln(\text{GDP}_t - K_t) + \vartheta_N \ln(1 - N_t) + \beta \mathbb{V}_0 \\ & + \beta \mathbb{V}_1 (1 - \eta) \ln K_t + \beta \mathbb{V}_1 \eta \ln N_t \\ & + \beta(\mathbb{V}_2 + \mathbb{V}_1) E_t \ln \zeta_{t+1}. \end{aligned} \tag{2.8}$$

Maximizing (2.8) with respect to (K_t, N_t) we have $N_t = \beta \mathbb{V}_1 \eta / (\vartheta_N + \beta \mathbb{V}_1 \eta)$ and $K_t = [\beta(1-\eta)\mathbb{V}_1/(1 + \beta(1-\eta)\mathbb{V}_1)]\,\text{GDP}_t$. Substituting into (2.8) and using the fact that $E_t \ln \zeta_{t+1} = \rho \ln \zeta_t$, we obtain

$$\begin{aligned} & \mathbb{V}_0 + \mathbb{V}_1 \ln \text{GDP}_t + \mathbb{V}_2 \ln \zeta_t \\ & \qquad = \text{const.} + (1 + (1-\eta)\beta \mathbb{V}_1) \ln \text{GDP}_t + \beta\rho(\mathbb{V}_2 + \mathbb{V}_1) \ln \zeta_t. \end{aligned} \tag{2.9}$$

Matching coefficients on the two sides of the equation we have $1 + (1-\eta)\beta\mathbb{V}_1 = \mathbb{V}_1$ or $\mathbb{V}_1 = 1/(1-(1-\eta)\beta)$ and $\beta\rho(\mathbb{V}_2 + \mathbb{V}_1) = \mathbb{V}_2$ or $\mathbb{V}_2 = \rho\beta/(1-(1-\eta)\beta)^2$. Using the solution for $\mathbb{V}_1$ into the expressions for K_t, N_t we have that $K_t = (1-\eta)\beta\,\text{GDP}_t$ and $N_t = \beta\eta/[\vartheta_N(1 - \beta(1-\eta)) + \beta\eta]$. From the resource constraint one has that $c_t = (1 - (1-\eta)\beta)\,\text{GDP}_t$. Hence, with this preference specification, the optimal labor supply decision is very simple: keep hours constant, no matter what the state and the shocks are.

Exercise 2.3. Assume, in the basic RBC model, that $u(c_t, c_{t-1}, N_t) = \ln c_t, \delta = 1$, the production function has the form $\text{GDP}_t = \zeta_t K_t^{1-\eta} N_t^{\eta}$, the resource constraint is $c_t + K_{t+1} + G_t = \text{GDP}_t$, $G_t = T_t$, and that both (ζ_t, G_t) are i.i.d. Guess that the value function is $\mathbb{V}(K, \zeta, G) = \mathbb{V}_0 + \mathbb{V}_1 \ln K_t + \mathbb{V}_2 \ln \zeta_t + \mathbb{V}_3 \ln G_t$. Determine $\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3$. Show the optimal policy for K^+.

Two other cases where a solution to the Bellman equation can be found analytically are analyzed in the next exercise.

Exercise 2.4. (i) Suppose that, in the basic RBC model, $u(c_t, c_{t-1}, N_t) = a_0 + a_1 c_t - a_2 c_t^2$ and that $G_t = T_t = T^y = 0, \forall t$. Show that the value function is of the form $\mathbb{V}(K, \zeta) = [K, \zeta]' \mathbb{V}_2 [K, \zeta] + \mathbb{V}_0$. Find the values of $\mathbb{V}_0$ and $\mathbb{V}_2$. (Hint: use the fact that $E(e_t' \mathbb{V}_2 e_t) = \text{tr}(\mathbb{V}_2) E(e_t' e_t) = \text{tr}(\mathbb{V}_2)\sigma_e^2$, where σ_e^2 is the covariance matrix of e_t and $\text{tr}(\mathbb{V}_2)$ is the trace of $\mathbb{V}_2$.) Show that the decision rule for c and K^+ is linear in K and ζ.

(ii) Suppose $u(c_t, c_{t-1}, N_t) = c_t^{1-\varphi}/(1-\varphi)$, $K_t = 1, \forall t$, and assume that ζ_t can take three values. Let ζ_t evolve according to $P(\zeta_t = i \mid \zeta_{t-1} = i') = p_{ii'} > 0$. Assume that there are claims to the output in the form of stocks S_t, with price p_t^{s} and dividend sd_t. Write down the Bellman equation. Let $\beta = 0.9$, $p_{ii} = 0.8$, $i = 1, \dots, 3$, $p_{i,i+1} = 0.2$, and $p_{ii'} = 0$, $i' \neq i, i+1$. Calculate the first two iterations of the value function. Can you guess what the limit is?

We can relax some of the assumptions we have made (e.g., we can use a more general law of motion for the shocks), but, except for these simple cases, even the most basic stochastic RBC model does not have a closed-form solution. As we will see later, existence of a closed-form solution is not necessary to estimate the structural parameters of the model (here β, δ, η), and the parameters of the process for ζ_t and G_t and to examine its fit to the data. However, a solution is needed when one wishes to simulate the model, compare its dynamics with those of the data, and/or perform policy analyses.

There is an alternative to the Bellman equation approach to solve simple optimization problems. It involves substituting all the constraints in the utility function and maximizing the resulting expression unconstrained or, if this is not possible, using a stochastic Lagrange multiplier approach. We illustrate the former approach next with an example.

Example 2.3. Suppose a representative household obtains utility from the services of durable and nondurable goods according to $E_0 \sum_t \beta^t (\text{cs}_t - \upsilon_t)'(\text{cs}_t - \upsilon_t)$, where $0 < \beta < 1$, υ_t is a preference shock and consumption services cs_t satisfy $\text{cs}_t = b_1 \text{cd}_{t-1} + b_2 c_t$, where cd_{t-1} is the stock of durable goods, accumulated according to $\text{cd}_t = b_3 \text{cd}_{t-1} + b_4 c_t$, where $0 < b_1, b_3 < 1$, and $0 < b_2, b_4 \leqslant 1$ are parameters. Output is produced with the technology $f(K_{t-1}, \zeta_t) = (1-\eta)K_{t-1} + \zeta_t$, where $0 < \eta \leqslant 1$ and ζ_t is a productivity disturbance, and divided between consumption and investment goods according to $b_5 c_t + b_6 \text{inv}_t = \text{GDP}_t$. Physical capital accumulates according to $K_t = b_7 K_{t-1} + b_8 \text{inv}_t$, where $0 < b_7 < 1$, $0 < b_8 \leqslant 1$.

Using the definition of $(\text{cs}_t, \text{cd}_t, K_t)$ and the resource constraint we have

$$\text{cs}_t + \text{cd}_t = (b_1 + b_3)\text{cd}_{t-1} + \frac{b_2 + b_4}{b_5}\left((1-\eta)K_{t-1} + \zeta_t - \frac{b_6}{b_8}(K_t - b_7 K_{t-1})\right). \quad (2.10)$$

Letting $b_9 = b_1 + b_3$, $b_{10} = (b_2 + b_4)/b_5$, $b_{11} = b_{10}b_6/b_8$, $b_{12} = b_{11}b_7$, and using (2.10) in the utility function, the problem can be reformulated as

$$\max_{\{\text{cd}_t, K_t\}} E_0 \sum_t \beta^t \{\mathcal{C}_1[\text{cd}_t, K_t]' + \mathcal{C}_2[\text{cd}_{t-1}, k_{t-1}, \zeta_t, \upsilon_t]'\}' \times \{\mathcal{C}_1[\text{cd}_t, K_t]' + \mathcal{C}_2[\text{cd}_{t-1}, k_{t-1}, \zeta_t, \upsilon_t]'\},$$

where $\mathcal{C}_1 = [-1, -b_{11}]$, $\mathcal{C}_2 = [b_9, b_{12} + b_{10}(1-\eta), b_{10}, -1]$. If $\mathcal{C}_1'\mathcal{C}_1$ is invertible, and the shocks (ζ_t, υ_t) are known at each t, the first-order condition of the model imply $[\text{cd}_t, K_t]' = (\mathcal{C}_1'\mathcal{C}_1)^{-1}(\mathcal{C}_1'\mathcal{C}_2)[\text{cd}_{t-1}, K_{t-1}, \zeta_t, \upsilon_t]'$. Given $(\text{cd}_t, K_t, \zeta_t, \upsilon_t)$, values for cs_t and c_t can be found from (2.10) and from the consumption services constraint.

Economic models with quadratic objective functions and linear constraints can also be cast into a standard optimal control problem format. Such a format allows one to compute the solution with simple and fast algorithms.

Exercise 2.5. Take the model of example 2.3 but let $\upsilon_t = 0$. Cast it into an optimal linear regulator problem of the form $\max_{\{y_{1t}\}} E_0 \sum_t \beta^t (y_{2t}\mathcal{Q}_2 y_{2t}' + y_{1t}\mathcal{Q}_1 y_{1t}' + 2y_{2t}\mathcal{Q}_3 y_{1t}')$ subject to $y_{2t+1} = \mathcal{Q}_4 y_{2t} + \mathcal{Q}_5 y_{1t} + \mathcal{Q}_6 y_{3t+1}$, where y_{3t} is a vector of (serially correlated) shocks, y_{2t} a vector of states, and y_{1t} a vector of controls. Show the form of $\mathcal{Q}_i$, $i = 1, \ldots, 6$.

A stochastic Lagrange multiplier approach works even when the Bellman equation cannot be used but requires a somewhat stronger set of assumptions to be applicable. Basically, we need the objective function to be strictly concave, differentiable, and its derivatives to have finite expectations; the constraints to be convex, differentiable, and their derivatives to have finite expectations; the choice variables to be observable at time t; the utility function to be bounded in expectations and to converge to a limit as $T \to \infty$; and the sequence of multipliers λ_t to be such that at the optimum the Kuhn–Tucker conditions hold with probability 1 (see Sims (2002) for a formal statement of these requirements).

It is straightforward to check that these conditions are satisfied for the simple RBC model we have considered so far. Then, letting $f_N = \partial f/\partial N$, $U_{c,t} = \partial u(c_t, c_{t-1}, N_t)/\partial c_t$, $U_{N,t} = \partial u(c_t, c_{t-1}, N_t)/\partial N_t$, the Euler equation for capital accumulation is

$$E_t \beta \frac{U_{c,t+1}}{U_{c,t}}[(1 - T^y) f_k + (1 - \delta)] - 1 = 0, \tag{2.11}$$

while the intratemporal marginal condition between consumption and labor is

$$\frac{U_{c,t}}{U_{N,t}} = -\frac{1}{(1 - T^y) f_N}. \tag{2.12}$$

Equations (2.11) and (2.12), the budget constraint, and the transversality conditions,

$\lim_{t\to\infty} \sup \beta^t (U_{c,t} - \lambda_t g_{c,t})(c_t - \hat{c}_t) \leqslant 0$, $\lim_{t\to\infty} \sup \beta^t (U_{N,t} - \lambda_t g_{N,t}) \times (N_t - \hat{N}_t) \leqslant 0$, where $g_{j,t}$ is the derivative of the constraints with respect to $j = c, N$, $\hat{j}_t$ is the optimal choice, and j_t any other choice, then need to be solved for (K_{t+1}, N_t, c_t), given (G_t, ζ_t, K_t). This is not easy. Since the system of equations is nonlinear and involves expectations of future variables, no analytical solution exists in general.

Exercise 2.6. Solve the problem of example 2.3 by using a Lagrange multiplier approach. Show that the conditions you need for the solution are the same as in example 2.3.

Versions of the basic RBC model with additional shocks, alternative inputs in the production function, or different market structures have been extensively examined in the macroeconomic literature. We consider some of these extensions in the next four exercises.

Exercise 2.7 (utility producing government expenditure). Consider a basic RBC model and suppose that government expenditure provides utility to the representative household, that private and public consumption are substitutes in the utility function, and that there is no habit in consumption, e.g., $U(c_t, c_{t-1}, G_t, N_t) = (c_t + \vartheta_{\mathrm{G}} G_t)^{\vartheta}(1 - N_t)^{1-\vartheta}$.

(i) Using steady-state relationships describe how private and public consumption are related. Is there some form of crowding out?

(ii) In a cross section of steady states, is it true that countries which have a higher level of government expenditure will also have lower levels of leisure, i.e., is it true that the income effect of distortionary taxation is higher when G is higher?

Exercise 2.8 (noncompetitive labor markets). Assume that, in a basic RBC model, there are one-period labor contracts. The contracts set the real wage on the basis of the expected marginal product of labor. Once shocks are realized, and given the contractual real wage, the firm chooses hours worked to maximize its profits. Write down the contractual wage equation and the optimal decision rule by firms. Compare it with a traditional Phillips curve relationship where $\ln N_t - E_{t-1}(\ln N_t) \propto \ln p_t - E_{t-1}(\ln p_t)$.

Exercise 2.9 (capacity utilization). Assume that $G_t = T_t = T^y = 0$, that the production function depends on capital (K_t) and its utilization (ku_t), and that it is of the form $f(K_t, \mathrm{ku}_t, N_t, \zeta_t) = \zeta_t (K_t \mathrm{ku}_t)^{1-\eta} N_t^{\eta}$. This production function allows firms to respond to shocks by varying utilization even when the stock of capital is fixed. Assume that capital depreciates in proportion to its use. In particular, assume that $\delta(\mathrm{ku}_t) = \delta_0 + \delta_1 \mathrm{ku}_t^{\delta_2}$, where δ_0, δ_1, and δ_2 are parameters.

(i) Write down the optimality conditions of the firm's problem and the Bellman equation.

(ii) Show that, if capital depreciates instantaneously, the solution of this problem is identical to the one of the standard RBC model examined in exercise 2.2.

Exercise 2.10 (production externalities). In a basic RBC model assume that output is produced with firm-specific inputs and the aggregate capital stock, i.e., $f(K_{it}, N_{it}, \zeta_t, K_t) = K_t^{\aleph} K_{it}^{1-\eta} N_{it}^{\eta} \zeta_t$, $\aleph > 0$, and $K_t = \int K_{it}\,\mathrm{d}i$.

(i) Derive the first-order conditions and discuss how to find optimal allocations.

(ii) Can the Bellman equation be used for this problem? What assumptions are violated?

Although it is common to proxy for technological disturbances with Solow residuals, such an approach is often criticized in the literature. The main reason is that such a proxy tends to overstate the variability of these shocks and may capture not only technology but also other sources of disturbances. The example below provides a case where this can occur.

Example 2.4. Suppose that output is produced with part-time hours (N^{PT}) and full-time hours (N^{FT}) according to the technology $\mathrm{GDP}_t = \zeta_t K_t^{1-\eta}(N_t^{\mathrm{FT}})^{\eta} + \zeta_t K_t^{1-\eta}(N_t^{\mathrm{PT}})^{\eta}$. Typically, Solow accounting proceeds by assuming that part-time and full-time hours are perfect substitutes and by using total hours in the production function, i.e., $\mathrm{GDP}_t = \zeta_t K_t^{1-\eta}(N_t^{\mathrm{FT}} + N^{\mathrm{PT}})^{\eta}$. An estimate of ζ_t is obtained via $\widehat{\ln \zeta_t} = \ln \mathrm{GDP}_t - (1-\eta)\ln K_t - \eta \ln(N_t^{\mathrm{FT}} + N_t^{\mathrm{PT}})$, where η is the share of labor income. It is easy to see that $\widehat{\ln \zeta_t} = \ln \zeta_t + \ln[(N_t^{\mathrm{FT}})^{\eta} + (N^{\mathrm{PT}})^{\eta}] - \eta \ln(N_t^{\mathrm{FT}} + N^{\mathrm{PT}})$, so that the variance of $\widehat{\ln \zeta_t}$ overestimates the variance of $\ln \zeta_t$. This is a general problem: whenever a variable is omitted from an estimated equation, the variance of the estimated residuals is at least as large as the variance of the true one. Note also that, if $N_t^{\mathrm{FT}} > N_t^{\mathrm{PT}}$ and if N_t^{FT} is less elastic than N_t^{PT} to shocks (e.g., if there are differential costs in adjusting full- and part-time hours), $\ln[(N_t^{\mathrm{FT}})^{\eta} + (N^{\mathrm{PT}})^{\eta}] - \eta \ln(N_t^{\mathrm{FT}} + N^{\mathrm{PT}}) > 0$. In this situation any (preference) shock which alters the relative composition of N^{FT} and N^{PT} could induce procyclical labor productivity movements, even if $\zeta_t = 0, \forall t$.

Several examples in this book are concerned with the apparently puzzling correlation between hours (employment) and labor productivity. Since with competitive markets labor productivity is equal to the real wage, we will interchangeably use the two, unless otherwise stated. What is puzzling is that the contemporaneous correlation between hours and labor productivity is roughly zero in the data while it is high and positive in an RBC model. We will study later how demand shocks can affect the magnitude of this correlation. In the next example we examine how the presence of government capital alters this correlation when an alternative source of technological disturbances is considered.

Example 2.5 (Finn). Suppose $u(c_t, c_{t-1}, N_t) = [c_t^{\vartheta}(1-N_t)^{1-\vartheta}]^{1-\varphi}/(1-\varphi)$, the budget constraint is $(1-T^y)w_t N_t + [r_t - T^K(r_t - \delta)]K_t^{\mathrm{P}} + T_t + (1+r_t^{\mathrm{B}})B_t = c_t + \mathrm{inv}_t^{\mathrm{P}} + B_{t+1}$, and private capital evolves according to $K_{t+1}^{\mathrm{P}} = (1-\delta)K_t^{\mathrm{P}} + \mathrm{inv}_t^{\mathrm{P}}$, where T^K (T^y) are capital (income) taxes, r^{B} is the net rate on real bonds, and r_t the net return on private capital. Suppose also that the government budget constraint is $T^y w_t N_t + T^K(r_t - \delta)K_t^{\mathrm{P}} + B_{t+1} = \mathrm{inv}_t^{\mathrm{G}} + T_t + (1 + r_t^{\mathrm{B}})B_t$, and government investments increase government capital according to $\mathrm{inv}_t^{\mathrm{G}} = K_{t+1}^{\mathrm{G}} - (1-\delta)K_t^{\mathrm{G}}$. The production function is $\mathrm{GDP}_t = \zeta_t N^{\eta}(K^{\mathrm{P}})^{1-\eta}(K^{\mathrm{G}})^{\aleph}$ and $\aleph \geqslant 0$. Output is used for private consumption and investment.

This model does not have an analytic solution but some intuition on how hours and labor productivity move can be obtained by analyzing the effects of random variations in government investment. Suppose that $\mathrm{inv}_t^{\mathrm{G}}$ is higher than expected. Then, less income is available for private use and, at the same time, more public capital is available in the economy. Which will be the dominant factor depends on the size of the investment increase relative to $\aleph$. If it is small, there will be a positive instantaneous wealth effect so that hours, investment, and output decline while consumption and labor productivity increases. If it is large, a negative wealth effect will result, so hours and output will increase and consumption and labor productivity decrease. In both cases, despite the RBC structure, the contemporaneous correlation between hours and labor productivity will be negative.

2.1.2 Heterogeneous Agent Models

Although representative agent models constitute the backbone of current dynamic macroeconomics, the literature has started examining setups where some heterogeneities in either preferences, the income process, or the type of constraints that agent face are allowed for. The presence of heterogeneities does not change the structure of the problem: it is only required that the sum of individual variables match aggregate ones and that the planner problem is appropriately defined. The solution still requires casting the problem into a Bellman equation or setting up a stochastic Lagrange multiplier structure.

We consider a few of these models here. Since the scope is purely illustrative we restrict attention to situations where there are only two types of agent. The generalization to a larger but finite number of types of agent is straightforward.

Example 2.6 (a two-country model with full capital mobility). Consider two countries and one representative household in each country. The household in country i chooses sequences for consumption, hours, capital, and contingent claim holdings to maximize $E_0 \sum_{t=0}^{\infty} \beta^t [c_{it}^{\vartheta}(1 - N_{it})^{1-\vartheta}]^{1-\varphi}/(1-\varphi)$ subject to the following constraint,

$$c_{it} + \sum_j B_{jt+1} p_{jt}^{\mathrm{B}} \leqslant B_{jt} + w_{it}N_{it} + r_{it}K_{it} - \left(K_{it+1} - (1-\delta)K_{it} - \frac{b}{2}\left(\frac{K_{it+1}}{K_{it}} - 1\right)^2 K_{it}\right), \quad (2.13)$$

where $w_{it}N_{it}$ is labor income, $r_{it}K_{it}$ is capital income, B_{jt} is a set of Arrow–Debreu one-period contingent claims and p_{jt}^{B} is its price, b is an adjustment cost parameter, and δ is the depreciation rate of capital. Since financial markets are complete, the household can insure itself against all forms of idiosyncratic risk.

We assume that factors of production are immobile. The domestic household rents capital and labor to domestic firms which produce a homogeneous intermediate good by using a constant returns-to-scale technology. Domestic markets for factors of production are competitive and intermediate firms maximize profits. Intermediate goods are sold to domestic and foreign final-good-producing firms. The resource constraints are

$$\text{inty}^1_{1t} + \text{inty}^1_{2t} = \zeta_{1t} K^{1-\eta}_{1t} N^{\eta}_{1t}, \tag{2.14}$$

$$\text{inty}^2_{1t} + \text{inty}^2_{2t} = \zeta_{2t} K^{1-\eta}_{2t} N^{\eta}_{2t}, \tag{2.15}$$

where inty^1_{2t} are exports of goods from country 1 and inty^2_{1t} imports from country 2.

Final goods are an aggregate of the goods produced by intermediate firms of the two countries. They are assembled with a constant returns-to-scale technology $\text{GDP}_{it} = [a_i(\text{inty}^1_{it})^{1-a_3} + (1-a_i)(\text{inty}^2_{it})^{1-a_3}]^{1/(1-a_3)}$, where $a_3 \geqslant -1$ while a_1 and $(1-a_2)$ measure the domestic content of domestic spending. The resource constraint in the final goods market is $\text{GDP}_{it} = c_{it} + \text{inv}_{it}$. The two countries differ in the realizations of technology shocks. We assume $\ln(\zeta_{it})$ is an AR(1) with persistence $|\rho_\zeta| < 1$ and variance σ^2_ζ.

To map this setup into a Bellman equation assume that there is a social planner who attributes the weights $\mathbb{W}_1$ and $\mathbb{W}_2$ to the utilities of the households of the two countries. Let the planner's objective function be $u^{\text{SP}}(c_{1t}, c_{2t}, N_{1t}, N_{2t}) = \sum_{i=1}^{2} \mathbb{W}_i E_0 \sum_{t=0}^{\infty} \beta^t [c^{\vartheta}_{it}(1-N_{it})^{1-\vartheta}]^{1-\varphi}/(1-\varphi)$; let $y_{2t} = [K_{1t}, K_{2t}, B_{1t}]$, $y_{3t} = [\zeta_{1t}, \zeta_{2t}]$, and $y_{1t} = [\text{inty}^1_{it}, \text{inty}^2_{it}, c_{it}, N_{it}, K_{it+1}, B_{1t+1},\ i = 1, 2]$. Then the Bellman equation is given by $\mathbb{V}(y_2, y_3) = \max_{\{y_1\}} u^{\text{SP}}(c_1, c_2, N_1, N_2) + E\beta\mathbb{V}(y^+_2, y^+_3 \mid y_2, y_3)$ and the constraints are given by (2.14) and (2.15), the law of motion of the shocks and the resource constraint $c_{1t} + c_{2t} + K_{1t+1} + K_{2t+1} = \text{GDP}_{1t} + \text{GDP}_{2t} - \frac{1}{2}b(K_{1t+1}/K_{1t} - 1)^2 K_{1t} - \frac{1}{2}b(K_{2t+1}/K_{2t} - 1)^2 K_{2t}$.

Clearly, the value function has the same format as in (2.5). Since the functional form for utility is the same in both countries, the utility function of the social planner will also have the same functional form. Some information about the properties of the model can be obtained by examining the first-order conditions and the properties of the final good production function. In fact, we have

$$c_{it} + \text{inv}_{it} = p_{1t}\text{inty}^1_{it} + p_{2t}\text{inty}^2_{it}, \tag{2.16}$$

$$\text{ToT}_t = \frac{p_{2t}}{p_{1t}}, \tag{2.17}$$

$$\text{nx}_t = \text{inty}^1_{2t} - \text{ToT}_t\text{inty}^2_{1t}. \tag{2.18}$$

Equation (2.16) implies that output of the final good is allocated to the inputs according to their prices, $p_{2t} = \partial\,\text{GDP}_{1t}/\partial\,\text{inty}^2_{1t}$, $p_{1t} = \partial\,\text{GDP}_{1t}/\partial\,\text{inty}^1_{1t}$; (2.17) gives an expression for the terms of trade and (2.18) defines the trade balance.

Exercise 2.11. (i) Show that the demand functions for the two goods in country 1 are

$$\begin{aligned}
\text{inty}^1_{1t} &= a_1^{1/a_3}[a_1^{1/a_3} + (1-a_1)^{1/a_3}\,\text{ToT}_t^{-(1-a_3)/a_3}]^{-a_3/(1-a_3)}\,\text{GDP}_{1t},\\
\text{inty}^2_{1t} &= (1-a_1)^{1/a_3}\,\text{ToT}_t^{-1/a_3}\\
&\quad\times[a_1^{1/a_3} + (1-a_1)^{1/a_3}\,\text{ToT}_t^{-(1-a_3)/a_3}]^{-a_3/(1-a_3)}\,\text{GDP}_{1t}.
\end{aligned}$$

(ii) Describe how the terms of trade relate to the variability of final goods demands.

(iii) Noting that $\text{ToT}_t = (1-a_1)(\text{inty}^2_{1t})^{-a_3}/(a_1(\text{inty}^1_{1t})^{-a_3})$, show that when the elasticity of substitution between domestic and foreign good $1/a_3$ is high, any excess of demand in either of the two goods induces small changes in the terms of trade and large changes in the quantities used.

Exercise 2.12. Consider the same two-country model of example 2.6 but now assume that financial markets are incomplete. That is, households are forced to trade only a one-period bond which is assumed to be in zero net supply (i.e., $B_{1t} + B_{2t} = 0$). How would you solve this problem? What does the assumption of incompleteness imply? Would it make a difference if the household of country 1 has limited borrowing capabilities, e.g., $B_{1t} \leqslant K_{1t}$?

Interesting insights can be added to a basic RBC model when some agents are not optimizers.

Example 2.7. Suppose that the economy is populated by standard RBC households (their fraction in the total population is Ψ) which maximize $E_0 \sum_t \beta^t u(c_t, c_{t-1}, N_t)$ subject to the constraint $c_t + \text{inv}_t + B_{t+1} = w_t N_t + r_t K_t + (1 + r_t^{\text{B}})B_t + \text{prf}_t + T_t$, where prf_t are the firm's profits, T_t are government transfers, and B_t are real bonds. Suppose that capital accumulates according to $K_{t+1} = (1-\delta)K_t + \text{inv}_t$. The remaining $1-\Psi$ households are myopic and consume all their income every period, that is, $c_t^{\text{RT}} = w_t N_t + T_t^{\text{RT}}$ and supply all their work time inelastically at each t.

Rule-of-thumb households play the role of an insensitive buffer in this economy. Therefore, total hours, aggregate output, and aggregate consumption will be much less sensitive to shocks than in an economy where all households are optimizers. For example, government expenditure shocks crowd out consumption less and under some efficiency wage specification, they can even make it increase.

Exercise 2.13 (Kiyotaki and Moore). Consider a model with two goods, land La, which is in fixed supply, and fruit which is not storable, and a continuum of two types of agent: farmers of measure 1 and gatherers of measure Ψ. Utilities are of the form $E_t \sum_t \beta_j^t c_{j,t}$, where $c_{j,t}$ is the consumption of fruit of type j, j = farmers, gatherers, and where $\beta_{\text{farmers}} < \beta_{\text{gatherers}}$. Let p_t^{L} be the price of land in terms of fruit and r_t the rate of exchange of a unit of fruit today for tomorrow. There are technologies to produce fruit from land. Farmers use $f(\text{La}_t)_{\text{farmer}} = (b_1 + b_2)\text{La}_{t-1}$, where b_1 is the tradable part and b_2 the bruised one (nontradable); gatherers use

$f(\mathrm{La}_t)_{\text{gatherer}}$, where f_{gatherer} displays decreasing returns-to-scale and all output is tradable. The budget constraint for the two agents is $p_t^{\mathrm{L}}(\mathrm{La}_{jt} - \mathrm{La}_{jt-1}) + r_t B_{jt-1} + c_{jt}^{\dagger} = f(\mathrm{La}_t)_j + B_{jt}$, where $c_{jt}^{\dagger} = c_{jt} + b_2 \mathrm{La}_{t-1}$ for farmers and $c_{jt}^{\dagger} = c_{jt}$ for gatherers, B_{jt} are loans, and $p_t^{\mathrm{L}}(\mathrm{La}_{jt} - \mathrm{La}_{jt-1})$ is the value of new land acquisitions. The farmers' technology is idiosyncratic so that only farmer i has the skill to produce fruit from it. The gatherers' technology does not require specific skills. Note that, if no labor is used, fruit output is zero.

(i) Show that in equilibrium $r_t = r = 1/\beta_{\text{gatherers}}$ and that for farmers to be able to borrow a collateral is required. Show that the maximum amount of borrowing is $B_t \leqslant p_{t+1}^{\mathrm{L}} \mathrm{La}_t / r$.

(ii) Show that, if there is no aggregate uncertainty, farmers borrow from gatherers up to their maximum, invest in land, and consume $b_2 \mathrm{La}_{t-1}$. That is, for farmers $\mathrm{La}_t = (1/(p_t^{\mathrm{L}} - r^{-1} p_{t+1}^{\mathrm{L}}))(b_1 + p_t^{\mathrm{L}})\mathrm{La}_{t-1} - rB_{t-1}$, where $p_t^{\mathrm{L}} - r^{-1} p_{t+1}^{\mathrm{L}}$ is the user cost of land (the down payment needed to purchase land) and $B_t = r^{-1} \times p_{t+1}^{\mathrm{L}} \mathrm{La}_t$. Argue that, if p_t^{L} increases, La_t and B_t will increase provided $b_1 + p_t^{\mathrm{L}} > rB_{t-1}/\mathrm{La}_{t-1}$. Hence, the higher the land price, the higher the net worth of farmers and the more they will borrow.

2.1.3 Monetary Models

The next set of models explicitly includes monetary factors. Finding a role for money in a general equilibrium model is difficult: with a full set of Arrow–Debreu claims, money is a redundant asset. Therefore, frictions of some sort need to be introduced for money to play some role. This means that the allocations produced by the competitive equilibrium are no longer optimal and that the Bellman equation formulation needs to be modified to take this into account (see, for example, Cooley 1995, pp. 50–60). We focus attention on two popular specifications — a competitive model with transactional frictions and a monopolistic competitive framework where either sticky prices or sticky wages or both are exogenously imposed — and examine what they have to say about two questions of interest to macroeconomists: do monetary shocks generate liquidity effects? That is, do monetary shocks imply negative comovements between short-term interest rates and (a narrow measure of) money? Do expansionary monetary shocks imply expansionary and persistent output effects?

Example 2.8 (Cooley and Hansen). The representative household maximizes $E_0 \times \sum_t \beta^t u(c_{1t}, c_{2t}, N_t)$, where c_{1t} is consumption of a cash good, c_{2t} is consumption of a credit good, and N_t is the number of hours worked. The budget constraint is $c_{1t} + c_{2t} + \mathrm{inv}_t + M_{t+1}/p_t \leqslant w_t N_t + r_t k_t + M_t/p_t + T_t/p_t$, where $T_t = M_{t+1} - M_t$ and p_t is the price level. There is a cash-in-advance constraint that forces households to buy c_{1t} with cash. We require $p_t c_{1t} \leqslant M_t + T_t$ and assume that the monetary authority sets $\ln M_{t+1}^s = \ln M_t^s + \ln M_t^{\mathrm{g}}$, where $\ln M_t^{\mathrm{g}}$ is an AR(1) process with mean $\bar{M}$, persistence ρ_M, and variance σ_M^2. The household chooses sequences for the two consumption goods, for hours, for investment, and for real balances to satisfy the budget constraint. We assume that shocks are realized at the beginning of each t so that the household knows the value of the shocks when taking decisions. The resource constraint is $c_{1t} + c_{2t} + \text{inv}_t = f(K_t, N_t, \zeta_t)$, where $\ln \zeta_t$ is an AR(1) process with persistence ρ_ζ and variance σ_ζ^2. Since the expected rate of return on money is lower than the expected return on capital, the cash-in-advance constraint will be binding and agents hold just the exact amount of money needed to purchase c_{1t}.

When $\bar{M} > 0$, money (and prices) grow over time. To map this setup into a stationary problem define $M_t^* = M_t/M_t^s$ and $p_t^* = p_t/M_{t+1}^s$. The value function is

$$\begin{aligned} &V(K, k, M^*, \zeta, M^{\text{g}}) \\ &\quad = \max \left[U\left(\frac{M^* + M^{\text{g}} - 1}{p^* M^{\text{g}}}; wN + [r + (1-\delta)]k - k^+ - \frac{(M^*)^+}{p^*}; N \right) \right] \\ &\qquad + \beta E V[K^+, k^+, (M^*)^+, \zeta^+, (M^{\text{g}})^+], \end{aligned} \tag{2.19}$$

where $K^+ = (1-\delta)K + \text{INV}$, $k^+ = (1-\delta)k + \text{inv}$, $c_1 = (M^* + M^{\text{g}} - 1)/(M^{\text{g}} p^*)$, and K represents the aggregate capital stock. The problem is completed by the consistency conditions $k^+ = h_1(K, \zeta, M^{\text{g}})$, $N = h_2(K, \zeta, M^{\text{g}})$, $p^* = h_3(K, \zeta, M^{\text{g}})$, where h_j are functions mapping aggregate shocks and states into optimal per capita decision variables and the aggregate price level.

Not much can be done with this model without taking some approximation. However, we can show that monetary disturbances have perverse output effects and produce expected inflation but not liquidity effects. Suppose $c_{2t} = 0, \forall t$. Then an unexpected increase in M_t^{g} makes agents substitute away from c_{1t} (which is now more expensive) toward credit goods — leisure and investment — which are cheaper. Hence, consumption and hours fall while investment increases. With a standard Cobb–Douglas production function output then declines. Also, since positive monetary shocks increase inflation, the nominal interest rate will increase, because both the real rate and expected inflation have temporarily increased. Hence, a surprise increase in M_t^{g} does not produce a liquidity effect or output expansions.

There are several ways to correct for the lack of positive correlation between money and output. For example, introducing one-period labor contracts (as we have done in exercise 2.8) does change the response of output to monetary shocks. The next exercise provides a way to generate the right output and interest rate effects by introducing a loan market, forcing the household to take decisions before shocks are realized and the firm to borrow to finance its wage bill.

Exercise 2.14 (working capital). Consider the same economy of example 2.8 with $c_{2t} = 0, \forall t$, but assume that the household deposits part of its money balances at the beginning of each t in banks. Assume that deposit decisions are taken before shocks occur and that the representative firm faces a working capital constraint, i.e., it has to pay for the factors of production before the receipts from the sale of the goods are received. The representative household maximizes utility by choice of consumption, labor, capital, and deposits, i.e., $\max_{\{c_t, N_t, K_{t+1}, \text{dep}_t\}} E_0 \sum_t \beta^t \times [c_t^{\vartheta}(1-N_t)^{1-\vartheta}]^{1-\varphi}/(1-\varphi)$. There are three constraints. First, goods must be purchased with money, i.e., $c_t p_t \leqslant M_t - \text{dep}_t + w_t N_t$. Second, there is a budget constraint $M_{t+1} = \text{prf}_{1t} + \text{prf}_{2t} + r_t p_t K_t + M_t - \text{dep}_t + w_t N_t - c_t p_t - \text{inv}_t p_t$, where $\text{prf}_{1t}(\text{prf}_{2t})$ represent the share of firm's (bank's) profits and r_t is the real return to capital. Third, capital accumulation is subject to an adjustment cost $b \geqslant 0$, i.e., $\text{inv}_t = K_{t+1} - (1-\delta)K_t - \frac{1}{2}b(K_{t+1}/K_t - 1)^2 K_t$. The representative firm rents capital and labor and borrows cash from the representative banks to pay for the wage bill. The problem is $\max_{\{K_t, N_t\}} \text{prf}_{1t} = p_t \zeta_t K_t^{1-\eta} N_t^{\eta} - p_t r_t K_t - (1+i_t) w_t N_t$, where i_t is the nominal interest rate. The representative bank takes deposits and lends them together with new money to firms. Profits, prf_{2t}, are distributed pro rata to the household. The monetary authority sets its instrument according to

$$M_t^{a_0} = i_t^{a_1} \text{GDP}_t^{a_2} \pi_t^{a_3} M_t^{g}, \tag{2.20}$$

where a_i are parameters and $\text{GDP}_t = \zeta_t K_t^{1-\eta} N_t^{\eta}$. For example, if $a_0 = 0$, $a_1 = 1$, the monetary authority sets the nominal interest rate as a function of output and inflation and stands ready to provide money when the economy needs it. Let $(\ln \zeta_t, \ln M_t^{g})$ be AR(1) processes with persistence ρ_ζ, ρ_M and variances σ_ζ^2, σ_M^2.

(i) Set $b = 0$. Show that the labor demand and the labor supply are $-U_{N,t} = (w_t/p_t) E_t \beta U_{c,t+1} p_t/p_{t+1}$ and $w_t i_t/p_t = f_{N,t}$. Argue that labor supply changes in anticipation of inflation while labor demand is directly affected by interest rate changes so that output will be positively related to money shocks.

(ii) Show that the optimal saving decision satisfies $E_{t-1} U_{c,t}/p_t = E_{t-1} i_t \beta \times U_{c,t+1}/p_{t+1}$. How does this compare with the saving decisions of the basic cash-in-advance (CIA) model of example 2.8?

(iii) Show that the money demand can be written as $p_t \text{GDP}_t/M_t = 1/(1+\eta/i_t)$. Conclude that velocity $p_t \text{GDP}_t/M_t$ and the nominal rate are positively related and that a liquidity effect is generated in response to monetary disturbances.

Exercise 2.15 (Dunlop–Tarshis puzzle). Suppose the representative household maximizes $E_0 \sum_{t=0}^{\infty} \beta^t [\ln c_t + \vartheta_{\text{m}} \ln M_{t+1}/p_t + \vartheta_N \ln(1-N_t)]$ subject to $c_t + M_{t+1}/p_t + K_{t+1} = w_t N_t + r_t K_t + (M_t + T_t)/p_t$. Let $\pi_{t+1} = p_{t+1}/p_t$ be the inflation rate. The representative firm rents capital from the household and produces using $\text{GDP}_t = \zeta_t K_t^{1-\eta} N_t^{\eta}$, where $\ln \zeta_t$ is a technological disturbance and capital depreciates in one period. Let the quantity of money evolve according to $\ln M_{t+1}^{s} = \ln M_t^{s} + \ln M_t^{g}$ and assume that at each t the government takes away G_t units of output.

(i) Assume $G_t = G$, $\forall t$. Write down the first-order conditions for the optimization problem of the household and the firm and find the competitive equilibrium for $(c_t, K_{t+1}, N_t, w_t, r_t, M_{t+1}/p_t)$.

(ii) Show that, in equilibrium, hours worked are independent of the shocks, that output and hours are uncorrelated, and that real wages are perfectly correlated with output.

(iii) Show that monetary disturbances are neutral. Are they also superneutral, i.e., do changes in the growth rate of money have real effects?

(iv) Suppose there are labor contracts where the nominal wage rate is fixed one period in advance according to $w_t = E_{t-1}M_t + \ln(\eta) - \ln(\vartheta_{\mathrm{m}}(\eta\beta)/(1-\beta)) - E_{t-1}\ln N_t$. Show that monetary disturbances produce a contemporaneous negative correlation between real wages and output.

(v) Now assume that G_t is stochastic and set $\ln M_t^{\mathrm{g}} = 0, \forall t$. What is the effect of government expenditure shocks on the correlation between real wages and output? Give some intuition for why adding labor contracts or government expenditure could reduce the correlation between real wages and output found in (ii).

The final type of model we consider adds nominal rigidities to a structure where monopolistic competitive firms produce intermediate goods which they sell to competitive final goods producers.

Example 2.9 (sticky prices). Suppose the representative household maximizes $E_0 \sum_t \beta^t [c_t^{\vartheta}(1-N_t)^{1-\vartheta}]^{1-\varphi}/(1-\varphi) + (1/(1-\varphi_{\mathrm{m}}))(M_{t+1}/p_t)^{1-\varphi_{\mathrm{m}}}$ by choices of c_t, N_t, K_{t+1}, M_{t+1} subject to the budget constraint $p_t(c_t + \mathrm{inv}_t) + B_{t+1} + M_{t+1} \leqslant r_t p_t K_t + M_t + (1+i_t)B_t + w_t N_t + \mathrm{prf}_t$ and the capital accumulation equation $\mathrm{inv}_t = K_{t+1} - (1-\delta)K_t - \frac{1}{2}b(K_{t+1}/K_t - 1)^2 K_t$, where b is an adjustment cost parameter. Here $\mathrm{prf}_t = \int \mathrm{prf}_{it}\,\mathrm{d}i$ are profits obtained from owning intermediate firms. There are two types of firm: monopolistic competitive, intermediate-good-producing firms and perfectly competitive, final-good-producing firms. Final goods firms take the continuum of intermediate goods and bundle it up for final consumption. The production function for final goods is $\mathrm{GDP}_t = (\int_0^1 \mathrm{inty}_{it}^{1/(1+\varsigma_p)}\,\mathrm{d}i)^{1+\varsigma_p}$, where $\varsigma_p > 0$. Profit maximization implies that the demand for each input i is $\mathrm{inty}_{it}/\mathrm{GDP}_t = (p_{it}/p_t)^{-(1+\varsigma_p)/\varsigma_p}$, where p_{it} is the price of intermediate good i and p_t the price of the final good, $p_t = (\int_0^1 p_{it}^{-1/\varsigma_p}\,\mathrm{d}i)^{-\varsigma_p}$.

Intermediate firms minimize costs and choose prices to maximize profits. Price decisions cannot be taken every period: only $(1-\zeta_{\mathrm{p}})$ of the firms are allowed to change prices at t. Their costs-minimization problem is $\min_{\{K_{it},N_{it}\}}(r_t K_{it} + w_t N_{it})$ subject to $\mathrm{inty}_{it} = \zeta_t K_{it}^{1-\eta} N_{it}^{\eta}$ and their profit-maximization problem is $\max_{\{p_{it+j}\}} E_t \sum_j \beta^j (U_{c,t+j}/p_{t+j})\zeta_{\mathrm{p}}^j \mathrm{prf}_{it+j}$, where $\beta^j U_{c,t+1}/p_{t+1}$ is the value of a unit of profit, prf_{it}, to shareholders next period, subject to the demand function from final goods firms. Here $\mathrm{prf}_{t+j} = (p_{it+j} - \mathrm{mc}_{it+j})\mathrm{inty}_{it+j}$ and mc_{it} are nominal marginal costs.

We assume that the monetary authority uses a rule of the form (2.20). Since only a fraction of the firms can change prices at each t, aggregate prices evolve according to $p_t = (\zeta_{\mathrm{p}} p_{t-1}^{-1/\varsigma_p} + (1-\zeta_{\mathrm{p}})\tilde{p}_t^{-1/\varsigma_p})^{-\varsigma_p}$, where $\tilde{p}_t$ is the common solution (all firms allowed to change prices are identical) to the following optimality condition (dropping the subscript i):

$$0 = E_t \sum_j \beta^j \zeta_{\mathrm{p}}^j \frac{U_{c,t+j}}{p_{t+j}} \left(\frac{\pi^j p_t}{1+\varsigma_p} - \mathrm{mc}_{t+j} \right) \mathrm{inty}_{t+j}, \tag{2.21}$$

where π is the steady-state inflation rate. Hence, intermediate firms choose prices so that the discounted marginal revenues equals the discounted marginal costs in expected terms. Note that, if $\zeta_{\mathrm{p}} \to 0$ and no capital is present, (2.21) reduces to the standard condition that the real wage equals the marginal product of labor. Expression (2.21) is the basis for the so-called New Keynesian Phillips curve (see, for example, Woodford 2003, chapter 3), an expression relating current inflation to expected future inflation and to current marginal costs. To explicitly obtain such a relationship, (2.21) needs to be log-linearized around the steady state.

To see what expression (2.21) involves, consider the case in which utility is logarithmic in consumption, linear in leisure, and the marginal utility of real balances is negligible, i.e., $U(c_t, N_t, M_{t+1}/p_t) = \ln c_t + (1 - N_t)$, output is produced with labor, prices are set every two periods, and, in each period, half of the firms change their price. Optimal price setting is

$$\frac{\tilde{p}_t}{p_t} = (1+\varsigma_p) E_t \left(\frac{U_{c,t} c_t w_t + \beta U_{c,t+1} c_{t+1} w_{t+1} \pi_{t+1}^{(1+\varsigma_p)/\varsigma_p}}{U_{c,t} c_t + \beta U_{c,t+1} c_{t+1} \pi_{t+1}^{1/\varsigma_p}} \right), \tag{2.22}$$

where $\tilde{p}_t$ is the optimal price, p_t the aggregate price level, w_t the wage rate, and $\pi_t = p_{t+1}/p_t$ the inflation rate. Ideally, firms would like to charge a price which is a constant markup $(1+\varsigma_p)$ over marginal (labor) costs. However, because individual prices are set for two periods, firms cannot do this and when prices are allowed to be changed, they are set as a constant markup over current and expected future marginal costs. Note that, if there are no shocks, $\pi_{t+1} = 1$, $w_{t+1} = w_t$, $c_{t+1} = c_t$, and $\tilde{p}_t / p_t = (1+\varsigma_p) w_t$.

Exercise 2.16. (i) Cast the household problem of example 2.9 into a Bellman equation format. Define states, controls, and the value function.

(ii) Show that, if prices are set one period in advance, the solution to (2.21) is

$$p_{it} = (1+\varsigma_p) E_{t-1} \frac{E_t (U_{c,t+j}/p_{t+j}) p_t^{(1+\varsigma_p)/\varsigma_p} \mathrm{inty}_{it}}{E_{t-1} (U_{c,t+j}/p_{t+j}) p_t^{(1+\varsigma_p)/\varsigma_p} \mathrm{inty}_{it}} \mathrm{mc}_{it}.$$

Give conditions that ensure that intermediate firms set prices as a constant markup over marginal costs.

(iii) Intuitively explain why monetary expansions are likely to produce positive output effects. What conditions need to be satisfied for monetary expansions to produce a liquidity effect?

Extensions of the model that also allow for sticky wages are straightforward. We ask the reader to study a model with both sticky prices and sticky wages in the next exercise.

Exercise 2.17 (sticky wages). Assume that households are monopolistic competitive in the labor market so that they can choose the wage at which to work. Suppose capital is in fixed supply and that the period utility function is $u_1(c_t)+u_2(1-N_t)+(M_{t+1}/p_t)^{1-\varphi_{\rm m}}/(1-\varphi_{\rm m})$. Suppose that households set nominal wages in a staggered way and that a fraction $1-\zeta_{\rm w}$ can do this every period. When the household is allowed to reset the wage, she maximizes the discounted sum of utilities subject to the budget constraint.

(i) Show that utility maximization leads to

$$E_t \sum_{j=0}^{\infty} \beta^j \zeta_{\rm w}^j \left(\frac{\pi^j w_t}{(1+\varsigma_{\rm w}) p_{t+j}} U_{1,t+j} + U_{2,t+j} \right) N_{t+j} = 0, \tag{2.23}$$

where β is the discount factor and $\varsigma_{\rm w} > 0$ is a parameter in the labor aggregator $N_t = [\int N_t(i)^{1/(1+\varsigma_{\rm w})}\,{\rm d}i]^{1+\varsigma_{\rm w}}$, $i \in [0,1]$. (Note: whenever the wage rate cannot be changed $w_{t+j} = \pi^j w_t$, where π is the steady-state inflation.)

(ii) Show that, if $\zeta_{\rm w} = 0$, (2.23) reduces to $w_t/p_t = -U_{2,t}/U_{1,t}$.

(iii) Calculate the equilibrium output, the real rate, and the real wage when prices and wages are flexible.

Exercise 2.18 (Taylor contracts). Consider a sticky wage model with no capital. Here labor demand is $N_t = {\rm GDP}_t$, real marginal costs are ${\rm mc}_t = w_t = 1$, where w_t is the real wage and ${\rm GDP}_t = c_t$. Suppose consumption and real balances are not substitutable in utility so that the money demand function is $M_{t+1}/p_t = c_t$. Suppose $\ln M^s_{t+1} = \ln M^s_t + \ln M^{\rm g}_t$, where $\ln M^{\rm g}$ is i.i.d. with mean $\bar{M} > 0$ and assume two-period staggered labor contracts.

(i) Show that $w_t = [0.5(\tilde{w}_t/p_t)^{-1/\varsigma_{\rm w}} + (w_{t-1}/p_t)^{-1/\varsigma_{\rm w}}]^{-\varsigma_{\rm w}}$, where $\tilde{w}_t$ is the nominal wage reset at t.

(ii) Show that $\pi_t \equiv p_t/p_{t-1} = [(\tilde{w}_{t-1}/p_{t-1})^{-1/\varsigma_{\rm w}}/(2-\tilde{w}_t/p_t^{-1/\varsigma_{\rm w}})]^{-\varsigma_{\rm w}}$ and that $N_{it} = N_t[(\tilde{w}_t/p_t)/w_t]^{-(1+\varsigma_{\rm w})/\varsigma_{\rm w}}$ if the wage was set at t and $N_{it} = N_t[(\tilde{w}_{t-1}/p_{t-1})/(w_t\pi_t)]^{-(1+\varsigma_{\rm w})/\varsigma_{\rm w}}$ if the wage was set at $t-1$.

(iii) Show that if utility is linear in N_t, monetary shocks have no persistence.

While expansionary monetary shocks in models with nominal rigidities produce expansionary output effects, their size is typically small and their persistence minimal, unless nominal rigidities are extreme. The next example shows a way to make output effects of monetary shocks sizeable.

Example 2.10 (Benhabib and Farmer). Consider an economy where utility is $E_0\sum_t \beta^t [c^{1-\varphi_c}/(1-\varphi_c) - (1/(1-\varphi_n))(n_t^{1-\varphi_n}/N_t^{\varphi_N-\varphi_n})]$, where n_t is individual employment, N_t is aggregate employment, and ϕ_c, ϕ_n, ϕ_N are parameters. Suppose output is produced with labor and real balances, i.e., ${\rm GDP}_t = (a_1 N_t^{\eta} + a_2(M_t/p_t)^{\eta})^{1/\eta}$, where η is a parameter. The consumers' budget constraint is $M_t/p_t = M_{t-1}/p_t + f[N_t, (M_{t-1}+M^{\rm g}_t)/p_t] - c_t$ and assume that $M^{\rm g}_t$ is i.i.d. with mean $\bar{M} \geqslant 0$. Equilibrium in the labor market implies $-U_N/U_c = f_N(N_t, M_t/p_t)$

and the demand for money is $E_t(f_{M,t+1}U_{c,t+1}/\pi_{t+1}) = E_t(i_{t+1}U_{c,t+1}/\pi_{t+1})$, where $1 + i_t$ is the gross nominal rate on a one-period bond, π_t the inflation rate, and $f_M = \partial f/\partial(M/p)$. These two standard conditions are somewhat special in this model. Decentralizing in a competitive equilibrium and log-linearizing the labor market condition, we have $\varphi_c \ln c_t + \varphi_n n_t - (\varphi_N + \varphi_n) \ln N_t = \ln w_t - \ln p_t$. Since agents are all identical, the aggregate labor supply will be a downward-sloping function of the real wage and given by $\varphi_c \ln c_t - \varphi_N \ln N_t = \ln w_t - \ln p_t$. Hence, a small shift in labor demand increases consumption (which is equal to output in equilibrium) and makes real wages fall and employment increase. As a consequence, a demand shock can generate procyclical consumption and employment paths. Note also that, since money enters the production function, an increase in money could shift labor demand as in the working-capital model. However, contrary to that case, labor market effects can be large because of the slope of the aggregate labor supply curve, and this occurs even when money is relatively unimportant as a productive factor.

We will see in exercise 2.34 that there are other more conventional ways to increase output persistence following monetary shocks while maintaining low price stickiness.

Sticky price models applied to an international context produce two interesting implications for exchange rate determination and for international risk sharing.

Example 2.11 (Obstfeld and Rogoff). Consider a structure like the one of example 2.9 where prices are chosen one period in advance, there are two countries, purchasing power parity holds, and international financial markets are incomplete, in the sense that only a real bond, denominated in the composite consumption good, is traded. In this economy the domestic nominal interest rate is priced by arbitrage and satisfies $1 + i_{1t} = E_t(p_{1t+1}/p_{1t})(1 + r_t^{\mathrm{B}})$, where r_t^{B} is the real rate on internationally traded bonds and uncovered interest parity holds, i.e., $1 + i_{1t} = E_t(\mathrm{ner}_{t+1}/\mathrm{ner}_t)(1 + i_{2t})$, where $\mathrm{ner}_t = p_{1t}/p_{2t}$ and p_{jt} is the consumption-based money price index in country j, $j = 1, 2$. Furthermore, the Euler equations imply the international risk-sharing condition $E_t[(c_{1t+1}/c_{1t})^{-\varphi} - (c_{2t+1}/c_{2t})^{-\varphi}] = 0$. Hence, while consumption growth need not be a random walk, the difference in scaled consumption growth is a martingale difference.

The money demand in country j is $M_{jt+1}/p_{jt} = \vartheta_{\mathrm{m}} c_{jt}[(1 + i_{jt})/i_{jt}]^{1/\varphi_{\mathrm{m}}}$, $j = 1, 2$. Using uncovered interest parity and log-linearizing, $\hat{M}_{1t} - \hat{M}_{2t} \propto (1/\varphi_{\mathrm{m}})(\hat{c}_{2t} - \hat{c}_{1t}) + [\beta/(1-\beta)\varphi_{\mathrm{m}}]\,\widehat{\mathrm{ner}}_t$, where the hat indicates deviations from the steady state. Hence, whenever $\hat{M}_{1t} - \hat{M}_{2t} \neq 0$ or $\hat{c}_{2t} - \hat{c}_{1t} \neq 0$, the nominal exchange rate jumps to a new equilibrium.

Variations or refinements of the price (wage) technology exist in the literature (see Rotemberg 1984; Dotsey et al. 1999). Since these refinements are tangential to the scope of this chapter, we invite the interested reader to consult the original sources for details and extensions.

2.2 Approximation Methods

As mentioned, finding a solution to the Bellman equation is, in general, complicated. The Bellman equation is a functional relationship and a fixed point needs to be found in the space of functions. When the regularity conditions for existence and uniqueness are satisfied, calculation of this fixed point requires iterations which involve the computation of expectations and the maximization of the value function.

We have also seen in example 2.2 and exercise 2.3 that, when the utility function is quadratic (logarithmic) and time separable and the constraints are linear, the form of the value function and of the decision rules is known. In these two situations, if the solution is known to be unique, the method of undetermined coefficients can be used to find the unknown parameters. Quadratic utility functions are not very appealing, however, as they imply implausible behavior for consumption and asset returns. Log-utility functions are easy to manipulate but they are also restrictive regarding the attitude of agents toward risk. Based on a large body of empirical research, the macroeconomic literature typically uses a general power specification for preferences. With this choice one has either to iterate on the Bellman equation or resort to approximations to find a solution.

We have also mentioned that solving general nonlinear expectational equations, such as those emerging from the first-order conditions of a stochastic Lagrangian multiplier problem, is complicated. Therefore, approximations also need to be employed in this case.

This section considers a few approximation methods currently used in the literature. The first approximates the objective function quadratically around the steady state. In the second, the approximation is calculated forcing the states and the exogenous variables to take only a finite number of possible values. This method can be applied to both the value function and to the first-order conditions. The other two approaches directly approximate the optimal conditions of the problem. In one case a log-linear (or a second-order) approximation around the steady state is calculated. In the other, the expectational equations are approximated by nonlinear functions and a solution is obtained by finding the parameters of these functions.

2.2.1 Quadratic Approximations

Quadratic approximations are easy to compute but work under two restrictive conditions. The first is that there exists a point — typically, the steady state — around which the approximation can be taken. Although this requirement may appear innocuous, it should be noted that some models do not possess a steady or a stationary state and in others the steady state may be multiple. The second is that local dynamics are well-approximated by linear difference equations. Consequently, such approximations are inappropriate when problems involve large perturbations away from the approximation point (e.g., policy shifts), dynamic paths are nonlinear, or transitional issues are considered. Moreover, they are likely to give incorrect answers for problems with inequality (e.g., borrowing or irreversibility) constraints, since the nonstochastic steady state ignores them.

Quadratic approximations of the objective function are used in situations where the social planner decisions generate competitive equilibrium allocations. When this is not the case the method requires some adaptation to take into account the fact that aggregate variables are distinct from individual ones (see, for example, Hansen and Sargent 2005; Cooley 1995, chapter 2), but the same principle works in both cases.

Quadratic approximations can be applied to both value function and Lagrangian multiplier problems. We will discuss applications to the first type of problem only since the extension to the second type of problem is straightforward. Let the Bellman equation be

$$\mathbb{V}(y_2, y_3) = \max_{\{y_1\}} \tilde{u}(y_1, y_2, y_3) + \beta E \mathbb{V}(y_2^+, y_3^+ \mid y_2, y_3), \tag{2.24}$$

where y_2 is an $m_2 \times 1$ vector of the states, y_3 is an $m_3 \times 1$ vector of exogenous variables, and y_1 is an $m_1 \times 1$ vector of the controls. Suppose that the constrains are $y_2^+ = h(y_3, y_1, y_2)$ and the law of motion of the exogenous variables is $y_3^+ = \rho_3 y_3 + \epsilon^+$, where h is continuous and ϵ a vector of martingale difference disturbances. Using the constraints into (2.24) we have

$$\mathbb{V}(y_2, y_3) = \max_{\{y_2^+\}} u(y_2, y_3, y_2^+) + \beta E \mathbb{V}(y_2^+, y_3^+ \mid y_2, y_3). \tag{2.25}$$

Let $\bar{u}(y_2, y_3, y_2^+)$ be the quadratic approximation of $u(y_2, y_3, y_2^+)$ around $(\bar{y}_2, \bar{y}_3, \bar{y}_2)$. If $\mathbb{V}^0$ is quadratic, then (2.25) maps quadratic functions into quadratic functions and the limit value of $V(y_2, y_3)$ will also be quadratic. Hence, under some regularity conditions, the solution to the functional equation is quadratic and the decision rule for y_2^+ linear. When the solution to (2.25) is known to be unique, an approximation to it can be found either by iterating on (2.25) starting from a quadratic $\mathbb{V}^0$ or by guessing that $\mathbb{V}(y_2, y_3) = \mathbb{V}_0 + \mathbb{V}_1[y_2, y_3] + [y_2, y_3]\mathbb{V}_2[y_2, y_3]'$, and finding $\mathbb{V}_0, \mathbb{V}_1, \mathbb{V}_2$.

It is important to stress that certainty equivalence is required when computing the solution to a quadratic approximation. This principle allows us to eliminate the expectation operator from (2.25) and reinsert it in front of all future unknown variables once a solution is found. This operation is possible because the covariance matrix of the shocks does not enter the decision rule. That is, certainty equivalence implies that we can set the covariance matrix of the shocks to zero and replace random variables with their unconditional mean.

Exercise 2.19. Consider the basic RBC model with no habit persistence in consumption and utility given by $u(c_t, c_{t-1}, N_t) = c_t^{1-\varphi}/(1-\varphi)$, no government sector, and no taxes and consider the recursive formulation provided by the Bellman equation.

(i) Compute the steady states and a quadratic approximation to the utility function.

(ii) Compute the value function assuming that the initial V^0 is quadratic and calculate the optimal decision rule for capital, labor, and consumption.

While exercise 2.19 takes a brute force approach to iterations, one should remember that approximate quadratic value function problems fit into the class of optimal linear regulator problems. Therefore, an approximate solution to the functional equation (2.25) can also be found by using methods developed in the control literature. One example of an optimal linear regulator problem was encountered in exercise 2.5. Recall that, in that case, we want to maximize $E_t \sum_t \beta^t([y_{2t}, y_{3t}]'\mathcal{Q}_2[y_{2t}, y_{3t}] + y_{1t}'\mathcal{Q}_1 y_{1t} + 2[y_{2t}, y_{3t}]'\mathcal{Q}_3' y_1)$ with respect to y_{1t}, y_{20} given, subject to $y_{2t+1} = \mathcal{Q}_4' y_{2t} + \mathcal{Q}_5' y_{1t} + \mathcal{Q}_6' y_{3t+1}$. The Bellman equation is

$$\begin{aligned}\mathbb{V}(y_2, y_3) = \max_{\{y_1\}} [y_2, y_3]'\mathcal{Q}_2[y_2, y_3] + y_1'\mathcal{Q}_1 y_1 + 2[y_2, y_3]'\mathcal{Q}_3' y_1 \\ + \beta E \mathbb{V}(y_2^+, y_3^+ \mid y_2, y_3). \quad (2.26)\end{aligned}$$

Hansen and Sargent (2005) show that, starting from arbitrary initial conditions, iterations on (2.26) yield at the jth step the quadratic value function $\mathbb{V}^j = y_2'\mathbb{V}_2^j y_2 + \mathbb{V}_0^j$, where

$$\begin{aligned}\mathbb{V}_2^{j+1} &= \mathcal{Q}_2 + \beta\mathcal{Q}_4\mathbb{V}_2^j\mathcal{Q}_4' \\ &\quad - (\beta\mathcal{Q}_4\mathbb{V}_2^j\mathcal{Q}_5' + \mathcal{Q}_3')(\mathcal{Q}_1 + \beta\mathcal{Q}_5\mathbb{V}_2^j\mathcal{Q}_5')^{-1}(\beta\mathcal{Q}_5\mathbb{V}_2^j\mathcal{Q}_4' + \mathcal{Q}_3) \quad (2.27)\end{aligned}$$

and $\mathbb{V}_0^{j+1} = \beta\mathbb{V}_0^j + \beta\,\mathrm{tr}(\mathbb{V}_2^j\mathcal{Q}_6'\mathcal{Q}_6)$. Equation (2.27) is the so-called matrix Riccati equation which depends on the parameters of the model (i.e., the matrices $\mathcal{Q}_i$), but it does not involve $\mathbb{V}_0^j$. Equation (2.27) can be used to find the limit value $\mathbb{V}_2$ which, in turn, allows us to compute the limit of $\mathbb{V}_0$ and of the value function. The decision rule which attains the maximum at iteration j is $y_{1t}^j = -(\mathcal{Q}_1 + \beta\mathcal{Q}_5\mathbb{V}_2^j\mathcal{Q}_5')^{-1} \times (\beta\mathcal{Q}_5\mathbb{V}_2^j\mathcal{Q}_4' + \mathcal{Q}_3)y_{2t}$ and can be calculated given $\mathbb{V}_2^j$, y_{2t}, and the parameters of the model.

While it is common to iterate on (2.27) to find the limits of $\mathbb{V}_0^j, \mathbb{V}_2^j$, the reader should be aware that algorithms which produce this limit in one step are available (see, for example, Hansen et al. 1996).

Exercise 2.20. Consider the two-country model analyzed in example 2.6.

(i) Take a quadratic approximation to the objective function of the social planner around the steady state and map the problem into a linear regulator framework.

(ii) Use the matrix Riccati equation to find a solution to the maximization problem.

Example 2.12. Consider the setup of exercise 2.7, where the utility function is $u(c_t, G_t, N_t) = \ln(c_t + \vartheta_{\mathrm{G}} G_t) + \vartheta_N(1 - N_t)$ and where G_t is an AR(1) process with persistence ρ_{G} and variance σ_{G}^2 and is financed with lump sum taxes. The resource constraint is $c_t + K_{t+1} + G_t = K_t^{1-\eta}N_t^{\eta}\zeta_t + (1-\delta)K_t$, where $\ln\zeta_t$ is an AR(1) disturbance with persistence ρ_ζ and variance σ_ζ^2. Setting $\vartheta_{\mathrm{G}} = 0.7$, $\eta = 0.64$, $\delta = 0.025$, $\beta = 0.99$, $\vartheta_N = 2.8$, we have that $(K/\mathrm{GDP})^{\mathrm{ss}} = 10.25$, $(c/\mathrm{GDP})^{\mathrm{ss}} = 0.745$, $(\mathrm{inv}/\mathrm{GDP})^{\mathrm{ss}} = 0.225$, $(G/\mathrm{GDP})^{\mathrm{ss}} = 0.03$, and $N^{\mathrm{ss}} = 0.235$. Approximating the utility function quadratically and the constraint linearly, we can use the matrix Riccati equation to find a solution. Convergence was achieved at iteration 243 and the increment in the value function at the last iteration was 9.41×10^{-6}. The value function is proportional to $[y_2, y_3]\mathbb{V}_2[y_2, y_3]'$, where $y_2 = K$, $y_3 = (G, \zeta)$, and

$$\mathbb{V}_2 = \begin{bmatrix} 1.76\times 10^{-9} & 3.08\times 10^{-7} & 7.38\times 10^{-9} \\ -1.54\times 10^{-8} & -0.081 & -9.38\times 10^{-8} \\ -2.14\times 10^{-6} & -3.75\times 10^{-4} & -8.98\times 10^{-6} \end{bmatrix}.$$

The decision rule for $y_1 = (c, N)'$ is

$$y_{1t} = \begin{bmatrix} -9.06\times 10^{-10} & -0.70 & -2.87\times 10^{-9} \\ -9.32\times 10^{-10} & -1.56\times 10^{-7} & -2.95\times 10^{-9} \end{bmatrix} y_{2t}.$$

The alternative to brute force or Riccati iterations is the method of undetermined coefficients. Although the approach is easy conceptually, it may be mechanically cumbersome, even for small problems. If we knew the functional form of the value function (and/or of the decision rule), we could posit a specific parametric representation and use the first-order conditions to solve for the unknown parameters, as we did in exercise 2.3. We highlight a few steps of the approach in the next example and let the reader fill in the details.

Example 2.13. Suppose that the representative household chooses sequences for $(c_t, M_{t+1}/p_t)$ to maximize $E_0 \sum_t \beta^t [c_t^{\vartheta} + (M_{t+1}/p_t)^{1-\vartheta}]$, where c_t is consumption and $M_{t+1}^{\dagger} = M_{t+1}/p_t$ are real balances. The budget constraint is $c_t + M_{t+1}/p_t = (1 - T^y)w_t + M_t/p_t$, where T^y is an income tax. We assume that w_t and M_t are exogenous and stochastic. The government budget constraint is $G_t = T^y w_t + (M_{t+1} - M_t)/p_t$, which, together with the consumer budget constraint, implies $c_t + G_t = w_t$. Substituting the constraints in the utility function we have $E_0\sum_t \beta^t \{[(1 - T^y)w_t + M_t^{\dagger}/\pi_t + M_{t+1}^{\dagger}]^{\vartheta} + (M_{t+1}^{\dagger})^{1-\vartheta}\}$, where π_t is the inflation rate. The states of the problem are $y_{2t} = (M_t^{\dagger}, \pi_t)$ and the shocks are $y_{3t} = (w_t, M_t^{g})$. The Bellman equation is $\mathbb{V}(y_2, y_3) = \max_{\{c, M^{\dagger}\}}[u(c, M^{\dagger}) + \beta E\mathbb{V}(y_2^{+}, y_3^{+} \mid y_2, y_3)]$. Let $(c^{ss}, M^{\dagger ss}, w^{ss}, \pi^{ss})$ be the steady-state value of consumption, real balances, income, and inflation. For $\pi^{ss} = 1, w^{ss} = 1$, consumption and real balances in the steady state are $c^{ss} = (1 - T^y)$ and $(M^{\dagger})^{ss} = \{[(1-\beta)\vartheta(1 - T^y)^{\vartheta-1}]/(1-\vartheta)\}^{-1/\vartheta}$. A quadratic approximation to the utility function is $\mathfrak{B}_0 + \mathfrak{B}_1 x_t + x_t'\mathfrak{B}_2 x_t$, where $x_t = (w_t, M_t^{\dagger}, \pi_t, M_{t+1}^{\dagger})$,

$$\mathfrak{B}_0 = (c^{ss})^{\vartheta} + [(M^{\dagger})^{ss}]^{1-\vartheta},$$

$$\mathfrak{B}_1 = \Bigg[\vartheta(c^{ss})^{\vartheta-1}(1 - T^y); \frac{\vartheta(c^{ss})^{\vartheta-1}}{\pi^{ss}}; \vartheta(c^{ss})^{\vartheta-1}\left(-\frac{(M^{\dagger})^{ss}}{(\pi^{ss})^2}\right); \\ -\vartheta(c^{ss})^{\vartheta-1} + (1+\vartheta)((M^{\dagger})^{ss})^{-\vartheta}\Bigg],$$

and the matrix $\mathfrak{B}_2$ is

$$\begin{bmatrix} \kappa(1-T^y)^2 & \kappa(1-T^y)/\pi^{\text{ss}} & \kappa(1-T^y)[-(M^\dagger)^{\text{ss}}/(\pi^{\text{ss}})^2] & -\kappa(1-T^y) \\ \kappa(1-T^y)/\pi^{\text{ss}} & \kappa/(\pi^{\text{ss}})^2 & [-(M^\dagger)^{\text{ss}}/(\pi^{\text{ss}})^2][\kappa/\pi^{\text{ss}}+\vartheta(c^{\text{ss}})^{\vartheta-1}] & -\kappa/\pi^{\text{ss}} \\ \kappa(1-T^y)[-(M^\dagger)^{\text{ss}}/(\pi^{\text{ss}})^2] & [-(M^\dagger)^{\text{ss}}/(\pi^{\text{ss}})^2][\kappa/\pi^{\text{ss}}+\vartheta(c^{\text{ss}})^{\vartheta-1}] & \kappa\left(-\dfrac{(M^\dagger)^{\text{ss}}}{(\pi^{\text{ss}})^2}\right)\left[-\dfrac{2c^{\text{ss}}}{(\vartheta-1)\pi^{\text{ss}}}-\left(-\dfrac{(M^\dagger)^{\text{ss}}}{(\pi^{\text{ss}})^2}\right)\right] & -\kappa\left[-\dfrac{(M^\dagger)^{\text{ss}}}{(\pi^{\text{ss}})^2}\right] \\ -\kappa(1-T^y) & -\kappa/\pi^{\text{ss}} & -\kappa[-(M^\dagger)^{\text{ss}}/(\pi^{\text{ss}})^2] & \kappa+\vartheta(1+\vartheta)[(M^\dagger)^{\text{ss}}]^{-\vartheta-1} \end{bmatrix},$$

where $\kappa = \vartheta(\vartheta-1)(c^{\text{ss}})^{\vartheta-2}$. One could then guess a quadratic form for the value function and solve for the unknown coefficients. Alternatively, if only the decision rule is needed, one could directly guess a linear policy function (in deviation from steady states) of the form $M^\dagger_{t+1} = \mathcal{Q}_0 + \mathcal{Q}_1 M^\dagger_t + \mathcal{Q}_2\pi_t + \mathcal{Q}_3 w_t + \mathcal{Q}_4 M^{\text{g}}_t$ and solve for $\mathcal{Q}_i$ by using the linear version of the first-order conditions.

Exercise 2.21. Find the approximate first-order conditions of the problem of example 2.13. Show the form of $\mathcal{Q}_j$, $j = 0, 1, 2, 3$. (Hint: use the certainty equivalence principle.)

When the number of states is large, analytic calculation of first- and second-order derivatives of the utility function may take quite some time. As an alternative, numerical derivatives, which are much faster to calculate and only require the solution of the model at a pivotal point, could be used. Hence, in example 2.13, to approximate $\partial u/\partial c$, one could use $\{[(1-T^y)w^{\text{ss}}+\iota]^\vartheta - [(1-T^y)w^{\text{ss}}-\iota]^\vartheta\}/2\iota$, for ι small.

Exercise 2.22 (Ramsey). Suppose that the representative household maximizes $E_0\sum_t \beta^t[\upsilon_t c_t^{1-\varphi_c}/(1-\varphi_c) - N_t^{1-\varphi_n}/(1-\varphi_n)]$, where υ_t is a preference shock and φ_c, φ_n are parameters. The consumer budget constraint is $E_0\sum_t \beta^t p^0_t \times [(1-T^y_t)\,\text{GDP}_t + s^{0\text{b}}_t - c_t] = 0$, where $s^{0\text{b}}_t$ is a stream of coupon payments promised by the government at time 0 and p^0_t is the Arrow–Debreu price. The resource constraint is $c_t + G_t = \text{GDP}_t = \zeta_t N_t^\eta$. The government budget constraint is $E_0\sum_t \beta^t p^0_t[(G_t + s^{0\text{b}}_t) - T^y_t\text{GDP}_t] = 0$. Given a process for G_t and the present value of coupon payments $E_0\sum_t \beta^t p^0_t s^{0\text{b}}_t$, a feasible tax process must satisfy the government budget constraint. Assume that $(\upsilon_t, \zeta_t, s^{0\text{b}}_t, G_t)$ are random variables with AR(1) representation. The representative household chooses sequences for consumption and hours and the government selects the tax process preferred by the household. The government commits at time 0 to follow the optimal tax system, once and for all.

(i) Take a quadratic approximation to the problem, calculate the first-order conditions of the household problem, and show how to calculate p_t^0.

(ii) Show the allocations for c_t, N_t and the optimal tax policy T_t^y. Is it true that the optimal tax rate implies tax smoothing (random walk taxes), regardless of the process for G_t?

2.2.2 Discretization

As an alternative to quadratic approximations, one could solve the value function problem by discretizing the state space and the space over which the exogenous processes take values. This is the method popularized, for example, by Merha and Prescott (1985). The idea is that the states are forced to lie in the set $Y_2 = \{y_{21}, \dots, y_{2n_1}\}$ and the exogenous processes in the set $Y_3 = \{y_{31}, \dots, y_{3n_2}\}$. Then the space of possible (y_{2t}, y_{3t}) combination has $n_1 \times n_2$ points. For simplicity, assume that the process for the exogenous variables is first-order Markov with transition $P(y_{3t+1} = y_{3j'} \mid y_{3t} = y_{3j}) = p_{j'j}$. The value function associated with each pair of states and exogenous processes is $\mathbb{V}(y_{2i}, y_{3j})$, which is of dimension $n_1 \times n_2$. Because of the Markov structure of the shocks, and the assumptions made, we have transformed an infinite-dimensional problem into the problem of mapping $n_1 \times n_2$ matrices into $n_1 \times n_2$ matrices. Therefore, iterations on the Bellman equation are easier to compute. The value function can be written as $(\mathcal{T}\mathbb{V}_{ij})(y_2, y_3) = \max_n u(y_1, y_{2i}, y_{3j}) + \beta \sum_{l=1}^{n_2} \mathbb{V}_{n,l}\, p_{l,j}$, where y_{1n} is such that $h(y_{1n}, y_{2i}, y_{3j}) = y_{2n}, n = 1, \dots, n_1$. An illustration of the approach is given in the next example.

Example 2.14. Consider an RBC model where a random stream of government expenditure is financed by distorting income taxes, labor supply is inelastic, and production uses only capital. The social planner chooses $\{c_t, K_{t+1}\}$ to maximize $E_0 \sum_t \beta^t c_t^{1-\varphi}/(1-\varphi)$, given G_t and K_t, subject to $c_t + K_{t+1} - (1-\delta)K_t + G_t = (1 - T^y)K_t^{1-\eta}$, where G_t is an AR(1) with persistence ρ_{G}, variance σ_{G}^2, and $(\varphi, \beta, T^y, \eta, \delta)$ are parameters. Given K_0, the Bellman equation is

$$\mathbb{V}(K, G) = \max_{\{K^+\}} [(1 - T^y)K^{1-\eta} + (1-\delta)K - G - K^+]^{1-\varphi}/(1-\varphi) + \beta E[\mathbb{V}(K^+, G^+ \mid K, G)].$$

Suppose that the capital stock and government expenditure can take only two values, and let the transition for G_t be $p_{j',j}$. Then the discretization algorithm works as follows.

Algorithm 2.2.

(1) Choose values for $(\delta, \eta, \varphi, T^y, \beta)$ and specify the elements of $p_{j',j}$.

(2) Choose an initial 2×2 matrix $\mathbb{V}(K, G)$, e.g., $\mathbb{V}^0 = 0$.

(3) For each $i, j = 1, 2$, calculate

$$(\mathcal{T}\mathbb{V}_{i,j})(K,G) = \max \left\{ \frac{[(1-T^y)K_i^{1-\eta} + (1-\delta)K_i - K_i - G_j]^{1-\varphi}}{1-\varphi} + \beta[\mathbb{V}_{i,j}p_{j,j} + \mathbb{V}_{i,j'}p_{j,j'}], \frac{[(1-T^y)K_i^{1-\eta} + (1-\delta)K_i - K_{i'} - G_j]^{1-\varphi}}{1-\varphi} + \beta[\mathbb{V}_{i',j}p_{j,j} + \mathbb{V}_{i',j'}p_{j,j'}] \right\}.$$

(4) Iterate on (3) until, for example, $\max_{i,i'} |\mathcal{T}^l\mathbb{V}_{i,j} - \mathcal{T}^{l-1}\mathbb{V}_{i,j}| \leqslant \iota$, ι small, $l = 2, 3, \dots$.

Suppose $T^y = 0.1$, $\delta = 0.1$, $\beta = 0.9$, $\varphi = 2$, $\eta = 0.66$; choose $G_1 = 1.1$, $G_2 = 0.9$, $K_1 = 5.3$, $K_2 = 6.4$, $p_{11} = 0.8$, $p_{22} = 0.7$, $\mathbb{V}^0 = 0$. Then

$$(\mathcal{T}\mathbb{V}_{11}) = \max_{1,2} \left\{ \frac{[(1-T^y)K_1^{1-\eta} + (1-\delta)K_1 - K_1 - G_1]^{1-\varphi}}{1-\varphi}, \frac{[(1-T^y)K_1^{1-\eta} + (1-\delta)K_1 - K_2 - G_1]^{1-\varphi}}{1-\varphi} \right\}$$
$$= \max_{1,2}\{14.38, 0.85\} = 14.38.$$

Repeating for the other entries,

$$\mathcal{T}\mathbb{V} = \begin{bmatrix} 14.38 & 1.03 \\ 12.60 & -0.81 \end{bmatrix}, \quad \mathcal{T}^2\mathbb{V} = \begin{bmatrix} 24.92 & 3.91 \\ 21.53 & 1.10 \end{bmatrix}, \quad \lim_{l\to\infty} \mathcal{T}^l\mathbb{V} = \begin{bmatrix} 71.63 & 31.54 \\ 56.27 & 1.10 \end{bmatrix}.$$

Implicitly the solution defines the decision rule; for example, from $(\mathcal{T}\mathbb{V}_{11})$ we have that $K_t = K_1$.

Clearly, the quality of the approximation depends on the fineness of the grid. It is therefore a good idea to start from coarse grids and after convergence is achieved check whether finer grids produce different results.

The discretization approach is well-suited for problems of modest dimension (i.e., when the size of the state variables and of the exogenous processes is small) since constructing a grid which systematically and effectively covers high-dimensional spaces is difficult. For example, when we have one state, two shocks, and 100 grid points, 1 000 000 evaluations are required in each step. Nevertheless, even with this large number of evaluations, it is easy to leave large portions of the space unexplored. Therefore, one has to be careful when using such an approach.

Exercise 2.23 (search). Suppose a worker has the choice of accepting or rejecting a wage offer. If she has worked at $t-1$, the offer is $w_t = b_0 + b_1 w_{t-1} + e_t$, where e_t is an i.i.d. shock; if she was not working at $t-1$, the offer w_t^* is drawn from some stationary distribution. Having observed w_t, the worker decides whether to work or not (i.e., whether $N_t = 0$ or $N_t = 1$). The worker cannot save so $c_t = w_t$

if $N_t = 1$ and $c_t = \bar{c}$ if $N_t = 0$, where $\bar{c}$ measures unemployment compensations. The worker maximizes discounted utility, where $u(c) = c_t^{1-\varphi}/(1-\varphi)$ and φ is a parameter.

(i) Write down the maximization problem and the first-order conditions.

(ii) Define states and controls and the Bellman equation. Suppose $e_t = 0, b_0 = 0$, $b_1 = 1$, $\beta = 0.96$, and $w_t^* \sim \mathbb{U}(0,1)$. Calculate the optimal value function and the decision rules.

(iii) Assume that the worker now also has the option of retiring so that $x_t = 0$ or $x_t = 1$. Suppose $x_t = x_{t-1}$ if $x_{t-1} = 0$ and that $c_t = w_t$ if $N_t = 1, x_t = 1$; $c_t = \bar{c}$ if $N_t = 0, x_t = 1$ and $c_t = \bar{\bar{c}}$ if $N_t = 0, x_t = 0$, where $\bar{\bar{c}}$ is the retirement pay. Write down the Bellman equation and calculate the optimal decision rules.

(iv) Suppose that the worker now has the option to migrate. For each location $i = 1,2$ the wage is $w_t^i = b_0 + b_1 w_{t-1}^l + e_t^i$ if she has worked at $t-1$ in location i, and $w_t^i \sim \mathbb{U}(0,i)$ otherwise. Consumption is $c_t = w_t$ if $i_t = i_{t-1}$ and $c_t = \bar{c} - \varrho$ if $i_t \neq i_{t-1}$, where $\varrho = 0.1$ is a migration cost. Write down the Bellman equation and calculate the optimal decision rules.

Exercise 2.24 (Lucas tree model). Consider an economy where an infinitely lived representative household has a random stream of perishable endowments sd_t and decides how much to consume and save, where savings can take the form of either stocks or bonds, and let $u(c_t, c_{t-1}, N_t) = \ln c_t$.

(i) Write down the maximization problem and the first-order conditions. Write down the Bellman equation specifying the states and the controls.

(ii) Assume that the endowment process can take only two values $\text{sd}_1 = 6$, $\text{sd}_2 = 1$ with transition $\left[\begin{smallmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{smallmatrix}\right]$. Find the 2×1 vector of value functions.

(iii) Find the policy function for consumption, stock, and bond holdings and the pricing functions for stocks and bonds.

One can also employ a discretization approach to directly solve the optimality conditions of the problem. Hence, the methodology is applicable to problems where the value function may not exist.

Example 2.15. For general preferences, the Euler equation of exercise 2.24 is

$$p_t^{\text{s}}(\text{sd}_t) U_{c,t} = \beta E[U_{c,t+1}(p_{t+1}^s(\text{sd}_{t+1}) + \text{sd}_{t+1})], \tag{2.28}$$

where we have made explicit the dependence of p_t^{s} on sd_t. If we assume that $\text{sd}_t = [\text{sd}_h, \text{sd}_L]$, use the equilibrium condition $c_t = \text{sd}_t$, and let $U_i^1 \equiv p^s(\text{sd}_i) U_{\text{sd}_i}$ and $U_i^2 = \beta \sum_{i'=1}^{2} p_{ii'} U_{\text{sd}_i} \text{sd}_i$, (2.28) can be written as $U_i^1 = U_i^2 + \beta \sum_{i'} p_{ii'} U_{i'}^1$ or $U^1 = (1 - \beta P)^{-1} U^2$, where P is the matrix with typical element $\{p_{ii'}\}$. Therefore, share prices satisfy $p^s(\text{sd}_i) = \sum_{i'} (I + \beta P + \beta^2 P^2 + \cdots)_{ii'} U_{i'}^2 / U_{\text{sd}_i}$, where the sum is over the (i, i') elements of the matrix.

Exercise 2.25. Consider the intertemporal condition (2.11), the intratemporal condition (2.12) of a standard RBC economy. Assume $T^y = 0$ and that (K_t, ζ_t) can take two values. Describe how to find the optimal consumption/leisure choice when $U(c_t, c_{t-1}, N_t) = \ln c_t + \vartheta_N(1 - N_t)$.

2.2.3 Loglinear Approximations

Log-linearizations have been extensively used in recent years following the work of Blanchard and Kahn (1980), King et al. (1988a,b), and Campbell (1994). Uhlig (1999) has systematized the methodology and provided software useful for solving a variety of problems. King and Watson (1998) and Klein (2000) provided algorithms for singular systems and Sims (2001) a method for solving linear systems where the distinction between states and controls is unclear.

Loglinear approximations are similar, in spirit, to quadratic approximations and the solutions are computed by using similar methodologies. The former may work better when the problem displays some mild nonlinearities. The major difference between the two approaches is that quadratic approximations are typically performed on the objective function while log-linear approximations are calculated by using the optimality conditions of the problem. Therefore, the latter can be used in situations where, because of distortions, the competitive equilibrium is suboptimal.

The basic principles of log-linearization are simple. We need a point around which the log-linearization takes place. This could be the steady state or, in models with friction, the frictionless solution. Let $y = (y_1, y_2, y_3)$. The optimality conditions of the problem can be divided into two blocks, the first containing expectational equations and the second nonexpectational equations:

$$1 = E_t[h(y_{t+1}, y_t)], \tag{2.29}$$

$$1 = f(y_t, y_{t-1}), \tag{2.30}$$

where $f(0, 0) = 1$ and $h(0, 0) = 1$. Taking a first-order Taylor expansion around $(\bar{y}, \bar{y}) = (0, 0)$, we have

$$0 \approx E_t[h_{t+1} y_{t+1} + h_t y_t], \tag{2.31}$$

$$0 \approx f_t y_t + f_{t-1} y_{t-1}, \tag{2.32}$$

where $f_j = \partial \ln f / \partial y'_j$ and $h_j = \partial \ln h / \partial y'_j$. Equations (2.31) and (2.32) form a system of linear expectational equations.

Although log-linearization only requires the first derivatives of f and h, Uhlig (1999) suggests a set of approximations to calculate (2.31), (2.32) directly without differentiation. The tricks involve replacing Y_t with $\bar{Y} e^{\hat{y}_t}$, where $\hat{y}_t$ is small, and using the following three rules (here a_0 is a constant and b_{1t}, b_{2t} small numbers).

(i) $e^{b_{1t} + a_0 b_{2t}} \approx 1 + b_{1t} + a_0 b_{2t}$.

(ii) $b_{1t} b_{2t} \approx 0$.

(iii) $E_t[a_0 e^{b_{1t+1}}] \propto E_t[a_0 b_{1t+1}]$.

Example 2.16. To illustrate these rules, consider the resource constraint $C_t + G_t + \text{Inv}_t = \text{GDP}_t$. Set $\bar{C}e^{\hat{c}_t} + \bar{G}e^{\hat{g}_t} + \overline{\text{Inv}}\,e^{\widehat{\text{inv}}_t} = \overline{\text{GDP}}\,e^{\widehat{\text{gdp}}_t}$ and use rule (i) to get $\bar{C}(1+\hat{c}_t) + \bar{G}(1+\hat{g}_t) + \overline{\text{Inv}}\,(1+\widehat{\text{inv}}_t) - \overline{\text{GDP}}\,(1+\widehat{\text{gdp}}_t) = 0$. Then, using $\bar{C} + \bar{G} + \overline{\text{Inv}} = \overline{\text{GDP}}$, we get $\bar{C}\hat{c}_t + \bar{G}\hat{g}_t + \overline{\text{Inv}}\,\widehat{\text{inv}}_t - \overline{\text{GDP}}\,\widehat{\text{gdp}}_t = 0$ or $(\bar{C}/\overline{\text{GDP}})\hat{c}_t + (\bar{G}/\overline{\text{GDP}})\hat{g}_t + (\overline{\text{Inv}}/\overline{\text{GDP}})\,\widehat{\text{inv}}_t - \widehat{\text{gdp}}_t = 0$.

Exercise 2.26. Suppose y_t and y_{t+1} are conditionally jointly lognormal and homoskedastic. Replace (2.29) with $0 = \ln\{E_t[e^{\bar{h}(y_{t+1}, y_t)}]\}$, where $\bar{h} = \ln(h)$. Using $\ln h(0,0) \approx 0.5\,\text{var}_t[\bar{h}_{t+1}y_{t+1} + \bar{h}_t y_t]$, show that the log-linear approximation is $0 \approx E_t[\bar{h}_{t+1}y_{t+1} + \bar{h}_t y_t]$. What is the difference between this approximation and the one in (2.31)?

Exercise 2.27. Suppose that the private production is $\text{GDP}_t = (K_t/\text{Pop}_t)^{\aleph_1/(1-\eta)} \times (N_t/\text{Pop}_t)^{\aleph_2/\eta} K_t^{1-\eta} N_t^{\eta} \zeta_t$, where (K/Pop_t) and (N_t/Pop_t) are the average endowment of capital and hours in the economy. Suppose the utility function is $E_t \times \sum_t \beta^t[\ln(c_t/\text{Pop}_t) - (1/(1-\varphi_N))(N_t/\text{Pop}_t)^{1-\varphi_N}]$. Assume that $(\ln \zeta_t, \ln \text{Pop}_t)$ are AR(1) processes with persistence equal to ρ_ζ and 1.

(i) Show that the optimality conditions of the problem are

$$\frac{c_t}{\text{Pop}_t}\left(\frac{N_t}{\text{Pop}_t}\right)^{-\varphi_N} = \eta \frac{\text{GDP}_t}{\text{Pop}_t}, \tag{2.33}$$

$$\frac{\text{Pop}_t}{c_t} = E_t \beta \frac{\text{Pop}_{t+1}}{c_{t+1}}\left[(1-\delta) + (1-\eta)\frac{\text{GDP}_{t+1}}{K_{t+1}}\right]. \tag{2.34}$$

(ii) Find expressions for the log-linearized production function, the labor market equilibrium, the Euler equation, and the budget constraint.

(iii) Write the log-linearized expectational equation in terms of an Euler equation error. Find conditions under which there are more stable roots than state variables (in which case sunspot equilibria may be obtained).

There are several economic models which do not fit the setup of (2.29), (2.30). For example, Rotemberg and Woodford (1997) describe a model where consumption at time t depends on the expectation of variables dated at $t+2$ and on. This model can be accommodated in the setup of (2.29), (2.30) by using dummy variables, as the next example shows. In general, restructuring of the timing convention of the variables, or enlarging the vector of states, suffices to fit these problems into (2.29), (2.30).

Example 2.17. Suppose that (2.29) is $1 = E_t[h(y_{2t+2}, y_{2t})]$. We can transform this second-order expectational equation into a 2×1 vector of first-order expectational equations by using a dummy variable y^*_{2t}. In fact, the above is equivalent to $1 = E_t[h(y^*_{2t+1}, y_{2t})]$ and $y_{2t+1} = y^*_{2t}$ as long as $[y_{2t}, y^*_{2t}]$ are used as state variables for the problem.

Exercise 2.28. Consider a model with optimizers and rule-of-thumb households like the one of example 2.7 and assume that optimizing households display habit in consumption. In particular, assume that their utility function is $(c_t - \gamma c_{t-1})^{\vartheta} \times (1 - N_t)^{1-\vartheta}$. Derive the first-order conditions of the model and map them into (2.29), (2.30).

Example 2.18. Log-linearizing around the steady state the equilibrium conditions of the model of exercise 2.13, and assuming an unexpected change in the productivity of farmers' technology (represented by Δ) lasting one period, we have $(1 + 1/\varrho)\widehat{\text{La}}_t = \Delta + (r/(r-1))\hat{p}_t^{\text{L}}$ for $\tau = 0$ and $(1 + 1/\varrho)\widehat{\text{La}}_{t+\tau} = \widehat{\text{La}}_{t+\tau-1}$ for $\tau \geqslant 1$, where ϱ is the elasticity of the supply of land with respect to the user costs in the steady state and $\hat{p}_t^{\text{L}} = ((r-1)/(r\varrho))\{1/[1 - \varrho/(r(1+\varrho))]\}\widehat{\text{La}}_t$, where the hat indicates percentage deviations from the steady state. Solving these two expressions we have $\hat{p}_t^{\text{L}} = \Delta/\varrho$ and $\widehat{\text{La}}_t = [1/(1+1/\varrho)][1 + r/((r-1)\varrho)]\Delta$. Three interesting conclusions follow. First, if $\varrho = 0$, temporary shocks have permanent effects on farmers' land and on its price. Second, since $[1/(1 + 1/\varrho)][1 + r/((r-1)\varrho)] > 1$, the effect on land ownership is larger than the shock. Finally, in the static case $(\widehat{\text{La}}_t)^* = \Delta < \widehat{\text{La}}_t$ and $(\hat{p}_t^{\text{L}})^* = [(r-1)/(r\varrho)]\Delta < \hat{p}_t^{\text{L}}$. This is because Δ affects the net worth of farmers: a positive Δ reduces the value of the obligations and implies a larger use of capital by the farmers, therefore magnifying the effect of the shock on land ownership.

Exercise 2.29. Show that the log-linearized first-order conditions of the sticky price model of example 2.9 when $K_t = 1, \forall t$, and when monopolistic firms use $\beta u_{c,t+1}/u_{c,t}$ as discount facter are

$$\left.\begin{aligned}
0 &= \hat{w}_t + \frac{N^{\text{ss}}}{1-N^{\text{ss}}}\hat{N}_t - \hat{c}_t,\\
\left(\frac{1}{1+i^{\text{ss}}}\right)\hat{\imath}_{t+1} &= [1 - \vartheta(1-\varphi)](\hat{c}_{t+1} - \hat{c}_t)\\
&\quad - (1-\vartheta)(1-\varphi)(\hat{N}_{t+1} - \hat{N}_t)\frac{N^{\text{ss}}}{1-N^{\text{ss}}} - \hat{\pi}_{t+1},\\
\left(\widehat{\frac{M_{t+1}}{p_t}}\right) &= \frac{\vartheta(1-\varphi)-1}{\varphi_{\text{m}}}\hat{c}_t + \frac{N^{\text{ss}}}{1-N^{\text{ss}}}\frac{(1-\vartheta)(1-\varphi)}{\varphi_{\text{m}}}\hat{N}_t\\
&\quad - \frac{1}{\varphi_{\text{m}}(1+i^{\text{ss}})}\hat{\imath}_t,\\
\beta E_t\hat{\pi}_{t+1} &= \hat{\pi}_t - \frac{(1-\zeta_{\text{p}})(1-\zeta_{\text{p}}\beta)}{\zeta_{\text{p}}}\widehat{\text{mc}}_t,
\end{aligned}\right\} \tag{2.35}$$

where mc_t are the real marginal costs, ζ_{p} is the probability of not changing prices, w_t is the real wage, φ is the risk-aversion parameter, ϑ is the share of consumption in utility, φ_{m} is the exponent on real balances in utility, the superscript "ss" refers to the steady state, and a hat denotes percentage deviation from the steady state.

As with quadratic approximations, the solution of the system of equations (2.31), (2.32) can be obtained in two ways when the solution is known to exist and to be unique: using the method of the undetermined coefficients or finding the saddle-point solution (Vaughan's method). The method of undetermined coefficients is analogous to the one described in exercise 2.19. Vaughan's method works with the state-space representation of the system. Both methods require the computation of eigenvalues and eigenvectors. For a thorough discussion of the methods, the reader should consult, for example, the chapter of Uhlig in Marimon and Scott (1999) or Klein (2000). Here we briefly describe the building blocks of the procedure and highlight the important steps with some examples.

Rather than using (2.31) and (2.32), we employ a slightly more general setup which directly allows for structures like those considered in example 2.17 and exercise 2.28, without any need to enlarge the state space.

Let y_{1t} be of dimension $m_1 \times 1$, y_{2t} of dimension $m_2 \times 1$, and y_{3t} of dimension $m_3 \times 1$, and suppose the log-linearized optimality conditions and the law of motion of the exogenous variables can be written as

$$0 = \mathcal{Q}_1 y_{2t} + \mathcal{Q}_2 y_{2t-1} + \mathcal{Q}_3 y_{1t} + \mathcal{Q}_4 y_{3t}, \tag{2.36}$$

$$0 = E_t(\mathcal{Q}_5 y_{2t+1} + \mathcal{Q}_6 y_{2t} + \mathcal{Q}_7 y_{2t-1} + \mathcal{Q}_8 y_{1t+1} + \mathcal{Q}_9 y_{1t} + \mathcal{Q}_{10} y_{3t+1} + \mathcal{Q}_{11} y_{3t}), \tag{2.37}$$

$$0 = y_{3t+1} - \rho y_{3t} - \epsilon_t, \tag{2.38}$$

where $\mathcal{Q}_3$ is an $m_4 \times m_1$ matrix of rank $m_1 \leqslant m_4$, and ρ has only stable eigenvalues. Assume that a solution is given by

$$y_{2t} = \mathcal{A}_{22} y_{2t-1} + \mathcal{A}_{23} y_{3t}, \tag{2.39}$$

$$y_{1t} = \mathcal{A}_{12} y_{2t-1} + \mathcal{A}_{13} y_{3t}. \tag{2.40}$$

Letting $\mathcal{Z}_1 = \mathcal{Q}_8 \mathcal{Q}_3^+ \mathcal{Q}_2 - \mathcal{Q}_6 + \mathcal{Q}_9 \mathcal{Q}_3^+ \mathcal{Q}_1$, Uhlig (1999) shows the following.

(a) $\mathcal{A}_{22}$ satisfies the (matrix) quadratic equations:

$$\left.\begin{aligned} 0 &= \mathcal{Q}_3^0 \mathcal{Q}_1 \mathcal{A}_{22} + \mathcal{Q}_3^0 \mathcal{Q}_2, \\ 0 &= (\mathcal{Q}_5 - \mathcal{Q}_8 \mathcal{Q}_3^+ \mathcal{Q}_1)\mathcal{A}_{22}^2 - \mathcal{Z}_1 \mathcal{A}_{22} - \mathcal{Q}_9 \mathcal{Q}_3^+ \mathcal{Q}_2 + \mathcal{Q}_7. \end{aligned}\right\} \tag{2.41}$$

The equilibrium is stable if all eigenvalues of $\mathcal{A}_{22}$ are less than 1 in absolute value.

(b) $\mathcal{A}_{12}$ is given by $\mathcal{A}_{12} = -\mathcal{Q}_3^+(\mathcal{Q}_1 \mathcal{A}_{22} + \mathcal{Q}_2)$.

(c) Given $\mathcal{Z}_2 = (\mathcal{Q}_5 \mathcal{A}_{22} + \mathcal{Q}_8 \mathcal{A}_{12})$ and $\mathcal{Z}_3 = \mathcal{Q}_{10}\rho + \mathcal{Q}_{11}$, $\mathcal{A}_{13}$ and $\mathcal{A}_{23}$ satisfy

$$\begin{bmatrix} I_{m_3} \otimes \mathcal{Q}_1 & I_{m_3} \otimes \mathcal{Q}_3 \\ \rho' \otimes \mathcal{Q}_5 + I_{m_3} \otimes (\mathcal{Z}_2 + \mathcal{Q}_6) & \rho' \otimes \mathcal{Q}_8 + I_{m_3} \otimes \mathcal{Q}_9 \end{bmatrix} \begin{bmatrix} \operatorname{vec}(\mathcal{A}_{23}) \\ \operatorname{vec}(\mathcal{A}_{13}) \end{bmatrix} = - \begin{bmatrix} \operatorname{vec}(\mathcal{Q}_4) \\ \operatorname{vec}(\mathcal{Z}_3) \end{bmatrix},$$

where $\operatorname{vec}(\cdot)$ is columnwise vectorization, $\mathcal{Q}_3^{\mathrm{G}}$ is a pseudo-inverse of $\mathcal{Q}_3$ and satisfies $\mathcal{Q}_3^{\mathrm{G}} \mathcal{Q}_3 \mathcal{Q}_3^{\mathrm{G}} = \mathcal{Q}_3^{\mathrm{G}}$ and $\mathcal{Q}_3 \mathcal{Q}_3^{\mathrm{G}} \mathcal{Q}_3 = \mathcal{Q}_3$. $\mathcal{Q}_3^0$ is an $(m_4 - m_1) \times m_4$ matrix whose rows are a basis for the space of $\mathcal{Q}_3'$ and I_{m_3} is the identity matrix of dimension m_3.

Example 2.19. Consider an RBC model with an intermediate monopolistic competitive sector. Let the profits in firm i be $\text{prf}_{it} = (p_{it} - \text{mc}_{it})\text{inty}_t$ and let $\text{mk}_{it} = (p_{it} - \text{mc}_{it})$ be the markup. If the utility function is of the form $u(c_t, c_{t-1}, N_t) = c_t^{1-\varphi}/(1-\varphi) + \vartheta_N(1-N_t)$, the dynamics depend on the markup only via the steady states. For this model the log-linearized conditions are

$$0 = -\text{Inv}^{\text{ss}}\,\widehat{\text{inv}}_t - C^{\text{ss}}\hat{c}_t + \text{GDP}^{\text{ss}}\,\widehat{\text{gdp}}_t, \tag{2.42}$$

$$0 = -\text{Inv}^{\text{ss}}\,\widehat{\text{inv}}_t - K^{\text{ss}}\hat{k}_{t+1} + (1-\delta)K^{\text{ss}}\hat{k}_t, \tag{2.43}$$

$$0 = (1-\eta)\hat{k}_t - \widehat{\text{gdp}}_t + \eta\hat{N}_t + \zeta_t, \tag{2.44}$$

$$0 = -\varphi\hat{c}_t + \widehat{\text{gdp}}_t - \hat{N}_t, \tag{2.45}$$

$$0 = \text{mk}^{\text{ss}}(1-\eta)(\text{GDP}^{\text{ss}}/K^{\text{ss}})[\hat{k}_t + \widehat{\text{gdp}}_t] - r^{\text{ss}}\hat{r}_t, \tag{2.46}$$

$$0 = E_t[-\varphi\hat{c}_{t+1} + \hat{r}_{t+1} + \varphi\hat{c}_t], \tag{2.47}$$

$$0 = \hat{\zeta}_{t+1} - \rho_\zeta\hat{\zeta}_t - \hat{\epsilon}_{1t+1}, \tag{2.48}$$

where $(\text{Inv}^{\text{ss}}/\text{GDP}^{\text{ss}})$ and $(C^{\text{ss}}/\text{GDP}^{\text{ss}})$ are the steady-state investment and consumption to output ratios, r^{ss} is the steady-state real rate, and mk^{ss} the steady-state markup. Letting $y_{1t} = (\hat{c}_t, \widehat{\text{gdp}}_t, \hat{N}_t, \hat{r}_t, \widehat{\text{inv}}_t)$, $y_{2t} = \hat{k}_t$, $y_{3t} = \hat{\zeta}_t$, we have $\mathcal{Q}_5 = \mathcal{Q}_6 = \mathcal{Q}_7 = \mathcal{Q}_{10} = \mathcal{Q}_{11} = [0]$,

$$\mathcal{Q}_2 = \begin{bmatrix} 0 \\ (1-\delta)K^{\text{ss}} \\ 1-\eta \\ 0 \\ -D^{\text{ss}} \end{bmatrix}, \quad \mathcal{Q}_3 = \begin{bmatrix} -C^{\text{ss}} & \text{GDP}^{\text{ss}} & 0 & 0 & -\text{Inv}^{\text{ss}} \\ 0 & 0 & 0 & 0 & \text{Inv}^{\text{ss}} \\ 0 & -1 & \eta & 0 & 0 \\ -\varphi & 1 & -1 & 0 & 0 \\ 0 & D^{\text{ss}} & 0 & -r^{\text{ss}} & 0 \end{bmatrix},$$

$$\mathcal{Q}_1 = \begin{bmatrix} 0 \\ -K^{\text{ss}} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{Q}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\mathcal{Q}_8 = [-\varphi, 0, 0, 1, 0], \quad \mathcal{Q}_9 = [\varphi, 0, 0, 0, 0], \quad \rho = [\rho_\zeta],$$

where $D^{\text{ss}} = \text{mk}^{\text{ss}}(1-\eta)(\text{GDP}^{\text{ss}}/K^{\text{ss}})$.

It is important to stress that the method of undetermined coefficients properly works only when the state space is chosen to be of minimal size; that is, no redundant state variables are included. If this is not the case, $\mathcal{A}_{22}$ may have zero eigenvalues and this will produce "bubble" solutions.

Computationally, the major difficulty is to find a solution to the matrix equation (2.41). The toolkit of Uhlig (1999) recasts the problem into a generalized eigenvalue–eigenvector problem. Klein (2000) and Sims (2001) calculate a solution by using the generalized Shur decomposition. When applied to some of the problems of this chapter, the two approaches yield similar solutions. In general, the Shur (QZ) decomposition is useful when generalized eigenvalues may not be distinct. However, the QZ decomposition is not necessarily unique.

Exercise 2.30. Suppose that the representative household maximizes $E_0 \sum_t \beta^t \times (c_t^{1-\varphi_c}/(1-\varphi_c) + (M_{t+1}/p_t)^{1-\varphi_{\mathrm{m}}}/(1-\varphi_{\mathrm{m}}))$, where φ_c and φ_{m} are parameters, subject to the resource constraint $c_t + K_{t+1} + M_{t+1}/p_t = \zeta_t K_t^{1-\eta} N_t^{\eta} + (1-\delta) \times K_t + M_t/p_t$, where $\ln \zeta_t$ is an AR(1) process with persistence ρ_ζ and standard error σ_ζ. Let $M_{t+1}^{\dagger} = M_{t+1}/p_t$ be real balances, π_t the inflation rate, r_t the rental rate of capital, and assume $\ln M_{t+1}^s = \ln M_t^s + \ln M_t^{\mathrm{g}}$, where $\ln M_t^{\mathrm{g}}$ has mean $\bar{M} \geqslant 0$ and standard error σ_M.

(i) Verify that the first-order conditions of the problem are

$$\left.\begin{aligned} r_t &= (1-\eta)\zeta_t K_t^{-\eta} N_t^{\eta} + (1-\delta), \\ 1 &= E_t[\beta(c_{t+1}/c_t)^{-\varphi_c} r_{t+1}], \\ (M_{t+1}^{\dagger})^{-\vartheta_{\mathrm{m}}} c_t^{-\varphi_c} &= 1 + E_t[\beta(c_{t+1}/c_t)^{-\varphi_c} \pi_{t+1}]. \end{aligned}\right\} \tag{2.49}$$

(ii) Log-linearize (2.49), the resource constraint, and the law of motion of the shocks and cast these equations into the form of equations (2.36)–(2.38).

(iii) Guess that a solution for $[K_{t+1}, c_t, r_t, M_{t+1}^{\dagger}]$ is linear in $(K_t, M_t^{\dagger}, \zeta_t, M_t^{\mathrm{g}})$. Determine the coefficients of the relationship. Is the selected state space minimal?

Exercise 2.31. Suppose that the representative household maximizes $E_0 \sum_{t=0}^{\infty} \beta^t \times u(c_t, 1-N_t)$ subject to $c_t + M_{t+1}/p_t + K_{t+1} \leqslant (1-\delta)K_t + (\mathrm{GDP}_t - G_t) + M_t/p_t + T_t$, $M_t/p_t \geqslant c_t$, where $\mathrm{GDP}_t = \zeta_t K_t^{1-\eta} N_t^{\eta}$ and assume that the monetary authority sets $\Delta \ln M_{t+1}^s = \ln M_t^{\mathrm{g}} + a i_t$, where a is a parameter and i_t the nominal interest rate. The government budget constraint is $G_t + (M_{t+1} - M_t)/p_t = T_t$. Let $[\ln G_t, \ln \zeta_t, \ln M_t^{\mathrm{g}}]$ be a vector of random disturbances.

(i) Assume a binding CIA constraint, $c_t = M_{t+1}/p_t$. Derive the optimality conditions and the equation determining the nominal interest rate.

(ii) Compute a log-linear approximation of the first-order conditions and of the budget constraint, of the production function, of the CIA constraint, of the equilibrium pricing equation for nominal bonds, and of the government budget constraint around the steady states.

(iii) Show that the system is recursive and can be solved for $(N_t, K_t, M_{t+1}/p_t, i_t)$ first, while $(\mathrm{GDP}_t, c_t, \lambda_t, T_t)$ can be solved in a second stage as a function of $(N_t, K_t, M_{t+1}/p_t, i_t)$, where λ_t is the Lagrangian multiplier on the private budget constraint.

(iv) Write down the system of difference equations for $(N_t, K_t, M_t/p_t, i_t)$. Guess a linear solution (in deviation from steady states) in K_t and $[\ln G_t, \ln \zeta_t, \ln M_t^{\mathrm{g}}]$ and find the coefficients of the solution.

(v) Assume prices are set one period in advance as a function of the states and of past shocks, i.e., $p_t = a_0 + a_1 K_t + a_{21} \ln G_{t-1} + a_{22} \ln \zeta_{t-1} + a_{23} \ln M_{t-1}^{\mathrm{g}}$.

What is the state vector in this case? Use the method of undetermined coefficients to find a solution.

The next example shows the log-linearized decision rules of a version of the sticky price, sticky wage model described in exercise 2.17.

Example 2.20. Assume that capital is in fixed supply and the utility function is $E_0 \sum_t \beta^t [(c_t^{\vartheta}(1 - N_t)^{1-\vartheta})^{1-\varphi}/(1 - \varphi) + (\vartheta_{\mathrm{m}}/(1 - \varphi_{\mathrm{m}}))(M_{t+1}/p_t)^{1-\varphi_{\mathrm{m}}}]$. Set $N^{\mathrm{ss}} = 0.33$, $\eta = 0.66$, $\pi^{\mathrm{ss}} = 1.005$, $\beta = 0.99$, $c^{\mathrm{ss}}/\mathrm{GDP}^{\mathrm{ss}} = 0.8$, where $c^{\mathrm{ss}}/\mathrm{GDP}^{\mathrm{ss}}$ is the share of consumption in GDP, N^{ss} is the number of hours worked, and π^{ss} is the gross inflation in the steady states, η is exponent of labor in the production function, β is the discount factor. These choices imply, for example, that in the steady state the gross real interest rate is 1.01, output is 0.46, real balances 0.37, and the real (fully flexible) wage 0.88. We select the degree of price and wage rigidity to be the same and set $\zeta_{\mathrm{p}} = \zeta_{\mathrm{w}} = 0.75$. Given the quarterly frequency of the model, this choice implies that on average firms (households) change their price (wage) every three quarters. Also, we choose the elasticity of money demand $\vartheta_{\mathrm{m}} = 7$. In the monetary policy rule we set $a_2 = -1.0$, $a_1 = 0.5$, $a_3 = 0.1$, $a_0 = 0$. Finally, ζ_t and M_t^{g} are AR(1) processes with persistence 0.95. The log-linearized decision rules for the real wage, output, nominal interest rate, real balances, and inflation, in terms of lagged real wages and the two shocks, are

$$\begin{bmatrix} \hat{w}_t \\ \hat{y}_t \\ \hat{\imath}_t \\ \hat{M}_t^{\dagger} \\ \hat{\Pi}_t \end{bmatrix} = \begin{bmatrix} 0.0012 \\ 0.5571 \\ 0.0416 \\ 0.1386 \\ 0.1050 \end{bmatrix} \begin{bmatrix} \hat{w}_{t-1} \end{bmatrix} + \begin{bmatrix} 0.5823 & -0.0005 \\ 0.2756 & 0.0008 \\ 0.0128 & 0.9595 \\ 0.0427 & -0.1351 \\ -0.7812 & 0.0025 \end{bmatrix} \begin{bmatrix} \hat{\zeta}_t \\ \hat{M}_t^{\mathrm{g}} \end{bmatrix}.$$

Two features of this approximate solution are worth commenting upon. First, there is little feedback from the state to the endogenous variables, except for output. This implies that the propagation properties of the model are limited. Second, monetary disturbances have little contemporaneous impact on all variables, except interest rates and real balances. These two observations imply that monetary disturbances have negligible real effects. This is confirmed by standard statistics. For example, technology shocks explain about 99% of the variance of output at the four years' horizon and monetary shocks the rest. This model also misses the sign of a few important contemporaneous correlations. For example, using linearly detrended U.S. data, the correlation between output and inflation is 0.35. For the model, the correlation is −0.89.

Exercise 2.32 (delivery lag). Suppose that the representative household maximizes $E_0 \sum_t \beta^t [\ln c_t - \vartheta_N N_t]$ subject to $c_t + \mathrm{inv}_t \leqslant \zeta_t K_t^{1-\eta} N_t^{\eta}$ and assume one-period delivery lag, i.e., $K_{t+1} = (1 - \delta)K_t + \mathrm{inv}_{t-1}$. Show that the Euler equation is $\beta E_t[c_{t+1}^{-1}(1 - \eta)\,\mathrm{GDP}_{t+1} K_{t+1}^{-1}] + (1 - \delta)c_t^{-1} - \beta^{-1} c_{t-1}^{-1} = 0$. Log-linearize the system and find a solution by using K_t and $c_t^* = c_{t-1}$ as states.

Vaughan's method, popularized by Blanchard and Kahn (1980) and King et al. (1988a,b), takes a slightly different approach. First, using the state-space representation for the (log-)linearized version of the model, it eliminates the expectation operator either assuming certainty equivalence or substituting expectations with actual values of the variables plus an expectational error. Second, it uses the law of motion of the exogenous variables, the linearized solution for the state variables, and the costate (the Lagrangian multiplier) to create a system of first-order difference equations (if the model delivers higher-order dynamics, the dummy variable trick described in example 2.17 can be used to get the system in the required form). Third, it computes an eigenvalue–eigenvector decomposition on the matrix governing the dynamics of the system and divides the roots into explosive and stable ones. Then, the restrictions implied by the stability condition are used to derive the law of motion for the control (and the expectational error, if needed).

Suppose that the log-linearized system is $\Upsilon_t = \mathcal{A} E_t \Upsilon_{t+1}$, where $\Upsilon_t = [y_{1t}, y_{2t}, y_{3t}, y_{4t}]$, y_{2t} and y_{1t} are, as usual, the states and the controls, y_{4t} are the costates, and y_{3t} are the shocks and partition $\Upsilon_t = [\Upsilon_{1t}, \Upsilon_{2t}]$. Let $\mathcal{A} = \mathcal{P}\mathcal{V}\mathcal{P}^{-1}$ be the eigenvalue–eigenvector decomposition of $\mathcal{A}$. Since the matrix $\mathcal{A}$ is symplectic, the eigenvalues come in reciprocal pairs when distinct. Let $\mathcal{V} = \mathrm{diag}(\mathcal{V}_1, \mathcal{V}_1^{-1})$, where $\mathcal{V}_1$ is a matrix with eigenvalues greater than 1 in modulus and

$$\mathcal{P}^{-1} = \begin{bmatrix} \mathcal{P}_{11}^{-1} & \mathcal{P}_{12}^{-1} \\ \mathcal{P}_{21}^{-1} & \mathcal{P}_{22}^{-1} \end{bmatrix}.$$

Multiplying both sides by $\mathcal{A}^{-1}$, using certainty equivalence, and iterating forward, we have

$$\begin{bmatrix} \Upsilon_{1t+j} \\ \Upsilon_{2t+j} \end{bmatrix} = \mathcal{P}^{-1} \begin{bmatrix} \mathcal{V}_1^{-j} & 0 \\ 0 & \mathcal{V}_1^{j} \end{bmatrix} \begin{bmatrix} \mathcal{P}_{11}\Upsilon_{1t} + \mathcal{P}_{12}\Upsilon_{2t} \\ \mathcal{P}_{21}\Upsilon_{1t} + \mathcal{P}_{22}\Upsilon_{2t} \end{bmatrix}. \tag{2.50}$$

We want to solve (2.50) under the condition that Υ_{2t+j} goes to zero as $j \to \infty$, starting from some Υ_{20}. Since the components of $\mathcal{V}_1$ exceed unity, this is possible only if the terms multiplying $\mathcal{V}_1$ are zero. This implies $\Upsilon_{2t} = -\mathcal{P}_{22}^{-1}\mathcal{P}_{21}\Upsilon_{1t} \equiv \mathcal{Q}\Upsilon_{1t}$ so that (2.50) is

$$\begin{bmatrix} \mathcal{Q}\Upsilon_{1t+j} \\ \Upsilon_{2t+j} \end{bmatrix} = \begin{bmatrix} \mathcal{Q}\mathcal{P}_{11}^{-1}\mathcal{V}_1^{-j}(\mathcal{P}_{11}\Upsilon_{1t} + \mathcal{P}_{12}\Upsilon_{2t}) \\ \mathcal{P}_{21}^{-1}\mathcal{V}_1^{-j}(\mathcal{P}_{11}\Upsilon_{1t} + \mathcal{P}_{12}\Upsilon_{2t}) \end{bmatrix}, \tag{2.51}$$

which also implies $\mathcal{Q} = \mathcal{P}_{21}^{-1}\mathcal{P}_{11}$. Note that, for quadratic problems, the limit value of $\mathcal{Q}$ is the same as the limit of the Riccati equation (2.27).

Example 2.21. The basic RBC model with labor–leisure choice, no habit, $G_t = T_t = T^y = 0$, production function $f(K_t, N_t, \zeta_t) = \zeta_t K_t^{1-\eta} N_t^{\eta}$, and utility function $u(c_t, c_{t-1}, N_t) = \ln c_t + \vartheta_N(1 - N_t)$ when log-linearized, delivers the representation $\Upsilon_t = \mathcal{A}_0^{-1}\mathcal{A}_1 E_t \Upsilon_{t+1}$, where $\Upsilon_t = [\hat{c}_t, \hat{K}_t, \hat{N}_t, \hat{\zeta}_t]$ (since there is a one-to-one relationship between c_t, N_t, and λ_t, we can solve λ_t out of the system), where the hat indicates percentage deviations from steady states and

$$\mathcal{A}_0 = \begin{bmatrix} 1 & \eta - 1 & 1-\eta & -1 \\ -1 & 0 & 0 & 0 \\ -\left(\frac{c}{K}\right)^{\text{ss}} & (1-\eta)\left(\frac{N^{\text{ss}}}{K^{\text{ss}}}\right)^{\eta} + (1-\delta) & \eta\left(\frac{N^{\text{ss}}}{K^{\text{ss}}}\right)^{\eta} & \left(\frac{N^{\text{ss}}}{K^{\text{ss}}}\right)^{\eta} \\ 0 & 0 & 0 & \rho \end{bmatrix},$$

$$\mathcal{A}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & -\beta\eta(1-\eta)\left(\frac{N^{\text{ss}}}{K^{\text{ss}}}\right)^{\eta} & \beta\eta(1-\eta)\left(\frac{N^{\text{ss}}}{K^{\text{ss}}}\right)^{\eta} & \beta(1-\eta)\left(\frac{N^{\text{ss}}}{K^{\text{ss}}}\right)^{\eta} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let $\mathcal{A}_0^{-1}\mathcal{A}_1 = \mathcal{P}\mathcal{V}\mathcal{P}^{-1}$, where $\mathcal{P}$ is a matrix whose columns are the eigenvectors of $\mathcal{A}_0^{-1}\mathcal{A}_1$ and $\mathcal{V}$ contains, on the diagonal, the eigenvalues. Then

$$\mathcal{P}^{-1}\Upsilon_t \equiv \Upsilon_t^{\dagger} = \mathcal{V} E_t \Upsilon_{t+1}^{\dagger} \equiv \mathcal{V} E_t \mathcal{P}^{-1}\Upsilon_{t+1}. \tag{2.52}$$

Since $\mathcal{V}$ is diagonal, there are four independent equations which can be solved forward, i.e.,

$$\Upsilon_{it}^{\dagger} = v_i E_t \Upsilon_{i,t+\tau}^{\dagger}, \quad i = 1, \dots, 4. \tag{2.53}$$

Since one of the conditions describes the law of motion of the technology shocks, one of the eigenvalues is ρ_{ζ}^{-1} (the inverse of the persistence of technology shocks). One other condition describes the intratemporal efficiency condition (see equation (2.12)): since this is a static relationship, the eigenvalue corresponding to this equation is zero. The other two conditions, the Euler equation for capital accumulation (equation (2.11)) and the resource constraint (equation (2.4)), produce two eigenvalues: one above and one below 1. The stable solution is associated with the $v_i > 1$ since $\Upsilon_{it}^{\dagger} \to \infty$ for $v_i < 1$. Hence, for (2.53) to hold for each t in the stable case, it must be that $\Upsilon_{it}^{\dagger} = 0$ for all $v_i < 1$.

Assuming $\beta = 0.99$, $\eta = 0.64$, $\delta = 0.025$, $\vartheta_N = 3$, the resulting steady states are $c^{\text{ss}} = 0.79$, $K^{\text{ss}} = 10.9$, $N^{\text{ss}} = 0.29$, $\text{GDP}^{\text{ss}} = 1.06$, and

$$\Upsilon_t^{\dagger} = \begin{bmatrix} 1.062 & 0 & 0 & 0 \\ 0 & 1.05 & 0 & 0 \\ 0 & 0 & 0.93 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} E_t \begin{bmatrix} -2.18 & -0.048 & 0.048 & 24.26 \\ 0 & 0 & 0 & 23.01 \\ -2.50 & 1.36 & 0.056 & 1.10 \\ -2.62 & 0.94 & -0.94 & 2.62 \end{bmatrix} \Upsilon_{t+1}^{\dagger}.$$

The second row has $v_2 = \rho_{\zeta}^{-1}$, the last row the intertemporal condition. The remaining two rows generate a saddle path. Setting the third and fourth rows to zero ($v_3, v_4 < 1$), we have $\hat{c}_t = 0.54\hat{N}_t + 0.02\hat{K}_t + 0.44\hat{\zeta}_t$ and $\hat{N}_t = -2.78\hat{c}_t + \hat{K}_t + 2.78\hat{\zeta}_t$. The third rows of $\mathcal{A}_0$ and $\mathcal{A}_1$ provide the law of motion for capital: $\hat{K}_{t+1} = -0.07\hat{c}_t + 1.01\hat{K}_t + 0.06\hat{N}_t + 0.10\hat{\zeta}_t$.

Exercise 2.33. Suppose the representative household chooses consumption, hours, and nominal money balances to maximize $E_0 \sum_{t=0}^{\infty} u(c_t, 1 - N_t)$ subject to the following three constraints:

$$\left.\begin{aligned} \text{GDP}_t &= \zeta_t N_t^{\eta} = G_t + c_t, \\ c_t &= M_t / p_t, \\ M_{t+1} &= (M_t - p_t c_t) + p_t(y_t - G_t) + M_t(\bar{M} + M_t^{\text{g}}), \end{aligned}\right\} \tag{2.54}$$

where ζ_t is a technology shock, G_t government expenditure, c_t consumption, M_t nominal balances, and p_t prices. Here G_t, ζ_t, and M_t^{g} are exogenous. Note that the third constraint describes the accumulation of money: $\bar{M}$ is a constant and M_t^{g} is a mean zero random variable.

(i) Derive and log-linearize the first-order conditions of the problem. What are the states?

(ii) Solve the linear system assuming that the growth rate of the exogenous variables $(\zeta_t, G_t, M_t^{\text{g}})$ is an AR(1) process with common parameter ρ. Calculate the equilibrium expressions for inflation, output growth, and real balances.

(iii) Suppose you want to price the term structure of nominal bonds. Such bonds cost 1 unit of money at time t and give $1 + i_{t+\tau}$ units of money at time $t + \tau$, $\tau = 1, 2, \dots$. Write the equilibrium conditions to price these bonds. Calculate the log-linear expression of the slope for the term structure between a bond with maturity $\tau \to \infty$ and a one-period bond.

(iv) Calculate the equilibrium pricing formula and the rate of return for stocks which cost p_t^{s} units of consumption at t, and pay dividends $p_t^{\text{s}}\text{sd}_t$ which can be used for consumption only at $t + 1$. (Hint: the value of dividends at $t + 1$ is $p_t^{\text{s}}\text{sd}_t / p_{t+1}$.) Calculate a log-linear expression for the equity premium (the difference between the nominal return on stocks and the nominal return on a one-period bond).

(v) Simulate the responses of the slope of term structure and of the equity premium to a unitary shock in the technology (ζ_t), in government expenditure (G_t), and in money growth (M_t^{g}). Is the pattern of responses economically sensible?

Exercise 2.34 (Pappa). Consider the sticky price model analyzed in exercise 2.9 with the capital utilization setup but without adjustment costs to capital. Log-linearize the model and compute output responses to monetary shocks (still assume the monetary rule (2.20)). How does the specification compare in terms of persistence and amplitude of real responses to the standard one, without capacity utilization, but with capital adjustment costs?

2.2.4 Second-Order Approximations

First-order (linear) approximations are fairly easy to construct, useful for a variety of purposes, and accurate enough for fitting DSGE models to the data. However, first-order approximations are insufficient, when evaluating welfare across policies that do not affect the deterministic steady state of the model, when analyzing asset pricing problems, or when risk considerations become important. In some cases it may be enough to assume that nonlinearities, although important, are small in some sense (see, for example, Woodford 2003). In general, one may want to have methods to solve a second-order system and produce locally accurate approximations to the dynamics of the model, without having to explicitly consider global (nonlinear) approximations.

Suppose the model has the form

$$E_t[\mathfrak{J}(y_{t+1}, y_t, \sigma\epsilon_{t+1})] = 0, \tag{2.55}$$

where $\mathfrak{J}$ is an $n \times 1$ vector of functions, y_t is an $n \times 1$ vector of endogenous variables, and ϵ_t is an $n_1 \times 1$ vector of shocks. Clearly, some components of (2.55) may be deterministic and others may be static. So far we have been concerned with the first-order expansions of (2.55), i.e., with the following system of equations:

$$E_t[\mathfrak{J}_1 \,\mathrm{d}y_{t+1} + \mathfrak{J}_2 \,\mathrm{d}y_t + \mathfrak{J}_3\sigma \,\mathrm{d}\epsilon_{t+1}] = 0, \tag{2.56}$$

where $\mathrm{d}x_t$ is the deviation of x_t from some pivotal point, $x_t = (y_t, \epsilon_t)$. As we have seen, solutions to (2.56) are found positing a functional relationship $y_{t+1} = \mathfrak{J}^*(y_t, \sigma\epsilon_t, \sigma)$, linearly expanding it around the steady state $\mathfrak{J}^*(y^{\mathrm{ss}}, 0, 0)$, substituting the linear expression in (2.56), and matching coefficients.

Here we are concerned with approximations of the form

$$\begin{aligned} E_t[\mathfrak{J}_1 \,\mathrm{d}y_{t+1} &+ \mathfrak{J}_2 \,\mathrm{d}y_t + \mathfrak{J}_3\sigma \,\mathrm{d}\epsilon_{t+1} \\ &+ 0.5(\mathfrak{J}_{11} \,\mathrm{d}y_{t+1} \,\mathrm{d}y_{t+1} + \mathfrak{J}_{12} \,\mathrm{d}y_{t+1} \,\mathrm{d}y_t + \mathfrak{J}_{13} \,\mathrm{d}y_{t+1}\sigma \,\mathrm{d}\epsilon_{t+1} \\ &\quad + \mathfrak{J}_{22} \,\mathrm{d}y_t \,\mathrm{d}y_t + \mathfrak{J}_{23} \,\mathrm{d}y_t\sigma \,\mathrm{d}\epsilon_t + \mathfrak{J}_{33}\sigma^2 \,\mathrm{d}\epsilon_{t+1} \,\mathrm{d}\epsilon_{t+1})] = 0, \end{aligned} \tag{2.57}$$

which are obtained from a second-order Taylor expansion of (2.55). These differ from standard linearizations with lognormal errors since second-order terms in $\mathrm{d}y_t$, $\mathrm{d}y_{t+1}$ appear in the expression.

Since the second-order terms enter linearly in the specification, solutions to (2.57) can also be obtained with the method of undetermined coefficients, assuming there exists a solution of the form $y_{t+1} = \mathfrak{J}^*(y_t, \sigma\epsilon_t, \sigma)$, taking a second-order expansion of this guess around the steady states $\mathfrak{J}^*(y^{\mathrm{ss}}, 0, 0)$, substituting the second-order expansion for y_{t+1} into (2.57), and matching coefficients. As shown by Schmitt-Grohe and Uribe (2004), the problem can be sequentially solved, finding first the first-order terms and then the second-order ones.

Clearly, we need regularity conditions for the solution to exist and to have good properties. Kim et al. (2004) provide a set of necessary conditions. We first need the solution to imply that y_{t+1} remains in the stable manifold defined by $\mathfrak{H}(y_{t+1}, \sigma) = 0$ and satisfies $\{\mathfrak{H}(y_t, \sigma) = 0$, $\mathfrak{H}(y_{t+1}, \sigma) = 0$ a.s., and $\mathfrak{J}^1(y_{t+1}, y_t, \sigma\epsilon_{t+1}) = 0$ a.s. imply $E_t\mathfrak{J}^2(y_{t+1}, y_t, \sigma\epsilon_{t+1}) = 0\}$, where $\mathfrak{J} = (\mathfrak{J}^1, \mathfrak{J}^2)$. Second, we need $\mathfrak{H}(y_{t+1}, \sigma)$ to be continuous and twice differentiable in both its arguments. Third, we need the smallest unstable root of the first-order system to exceed the square of its largest stable root. This last condition is automatically satisfied if the dividing line is represented by a root of 1.0.

Under these conditions, Kim et al. argue that the second-order approximate solution to the dynamics of the model is accurate, in the sense that the error in the approximation converges in probability to zero at a more rapid rate than $\|\mathrm{d}y_t, \sigma\|^2$, when $\|\mathrm{d}y_t, \sigma\|^2 \to 0$. This claim does not depend on the a.s. boundedness of the process for ϵ_t, which is violated when its distribution has unbounded support, or on the stationarity of the model. However, for nonstationary systems the n-step-ahead accuracy deteriorates quicker than in the stationary case.

Example 2.22. We consider a version of the two-country model analyzed in example 2.6, where the population is the same in the two countries, the social planner equally weights the utility of the household of the two countries, there is no intermediate good sector, capital adjustment costs are zero, and output is produced with capital only. The planner's objective function is $E_0 \sum_t \beta^t (c_{1t}^{1-\varphi} + c_{2t}^{1-\varphi})/(1-\varphi)$, the resource constraint is $c_{1t} + c_{2t} + k_{1t+1} + k_{2t+1} - (1-\delta)(k_{1t} + k_{2t}) = \zeta_{1t} k_{1t}^{1-\eta} + \zeta_{2t} k_{2t}^{1-\eta}$, and $\ln \zeta_{it}$, $i = 1, 2$, is assumed to be i.i.d. with mean zero and variance σ_ζ^2. Given the symmetry of the two countries, it must be the case that in equilibrium $c_{1t} = c_{2t}$ and that the Euler equations for capital accumulations in the two countries are identical. Letting $\varphi = 2$, $\delta = 0.1$, $1 - \eta = 0.3$, $\beta = 0.95$, the steady state is $(k_i, \zeta_i, c_i) = (2.62, 1.00, 1.07)$, $i = 1, 2$, and a first-order expansion of the policy function is

$$k_{it+1} = \begin{bmatrix} 0.444 & 0.444 & 0.216 & 0.216 \end{bmatrix} \begin{bmatrix} k_{1t} \\ k_{2t} \\ \zeta_{1t} \\ \zeta_{2t} \end{bmatrix}, \quad i = 1, 2. \tag{2.58}$$

A second-order expansion of the policy function in country $i = 1, 2$ is

$$\begin{aligned} k_{it+1} = {} & \begin{bmatrix} 0.444 & 0.444 & 0.216 & 0.216 \end{bmatrix} \begin{bmatrix} k_{1t} \\ k_{2t} \\ \zeta_{1t} \\ \zeta_{2t} \end{bmatrix} - 0.83\sigma^2 \\ & + 0.5 \begin{bmatrix} k_{1t} & k_{2t} & \zeta_{1t} & \zeta_{2t} \end{bmatrix} \begin{bmatrix} 0.22 & -0.18 & -0.02 & -0.08 \\ -0.18 & 0.22 & -0.08 & -0.02 \\ -0.02 & -0.08 & 0.17 & -0.04 \\ -0.08 & -0.02 & -0.04 & 0.17 \end{bmatrix} \begin{bmatrix} k_{1t} \\ k_{2t} \\ \zeta_{1t} \\ \zeta_{2t} \end{bmatrix}. \end{aligned} \tag{2.59}$$

Hence, apart from the quadratic terms in the states, (2.58) and (2.59) differ because the variance of the technology shock enters (2.59). In particular, when technology shocks are highly volatile, more consumption and less capital will be chosen with the second-order approximation. Clearly, the variance of the shocks is irrelevant for the decision rules obtained with the first-order approximation.

Exercise 2.35. Consider the sticky price model whose log-linear approximation is described in exercise 2.29. Assuming that $\vartheta = 0.5$, $\varphi = 2$, $\varphi_{\mathrm{m}} = 0.5$, $\zeta_{\mathrm{p}} = 0.75$, $\beta = 0.99$, compare first- and second-order expansions of the solution for c_t, N_t, i_t, π_t, assuming that there are only monetary shocks, which are i.i.d. with variance σ_M^2, that monetary policy is conducted by using a rule of the form $i_t = \pi_t^{a_3} M_t^{\mathrm{g}}$, and that w_t is equal to the marginal product of labor.

2.2.5 Parametrizing Expectations

The method of parametrizing expectations was suggested by Marcet (1992) and further developed by Marcet and Lorenzoni (1999). With this approach, the approximation is globally valid as opposed to valid only around a particular point as it is the case with quadratic, log-linear, or second-order approximations. Therefore, with such a method we can undertake experiments which are, for example, far away from the steady state, unusual from the historical point of view, or involve switches of steady states. The approach has two advantages. First, it can be used when inequality constraints are present. Second, it has a built-in mechanism that allows us to check whether a candidate solution satisfies the optimality conditions of the problem. Therefore, the accuracy of the approximation can be implicitly examined.

The essence of the method is simple. First, one approximates the expectational equations of the problem with a vector of functions $\hbar$, i.e., $\hbar(\alpha, y_{2t}, y_{3t}) \approx E_t[h(y_{2t+1}, y_{2t}, y_{3t+1}, y_{3t})]$, where y_{2t} and y_{3t} are known at t and α is a vector of (nuisance) parameters. Polynomial, trigonometric, logistic, or other simple functions which are known to have good approximation properties can be used. Second, one estimates α by minimizing the distance between $E_t[h(y_{2t+1}(\alpha), y_{2t}(\alpha), y_{3t+1}, y_{3t})]$ and $\hbar(\alpha, y_{2t}(\alpha), y_{3t})$, where $\{y_{2t}(\alpha)\}_{t=1}^{T}$ are simulated time series for the states obtained with the approximate solution. Let $Q(\alpha, \alpha^*) = |E_t[h(y_{2t+1}(\alpha), y_{2t}(\alpha), y_{3t+1}, y_{3t})] - \hbar(\alpha^*, y_{2t}, y_{3t})|^q$ some $q \geqslant 1$, where α^* is the distance minimizer. The method then looks for an $\tilde{\alpha}$ such that $Q(\tilde{\alpha}, \tilde{\alpha}) = 0$.

Example 2.23. Consider a basic RBC model with inelastic labor supply, where utility is given by $u(c_t) = c_t^{1-\varphi}/(1-\varphi)$ and φ is a parameter, the budget constraint is $c_t + K_{t+1} + G_t = (1 - T^y)\zeta_t K_t^{1-\eta} + (1-\delta)K_t + T_t$, and $(\ln \zeta_t, \ln G_t)$ are AR processes with persistence $(\rho_\zeta, \rho_{\mathrm{G}})$ and unit variance. The expectational (Euler) equation is

$$c_t^{-\varphi} = \beta E_t\{c_{t+1}^{-\varphi}[(1 - T^y)\zeta_{t+1}(1-\eta)K_{t+1}^{-\eta} + (1-\delta)]\}, \qquad (2.60)$$

where β is the rate of time preferences. We wish to approximate the expression on the right-hand side of (2.60) with a function $\hbar(K_t, \zeta_t, G_t, \alpha)$, where α is a set of parameters. Then the parametrizing expectation algorithm works as follows.

Algorithm 2.3.

(1) Select $(\varphi, T^y, \delta, \rho_\zeta, \rho_{\mathrm{G}}, \eta, \beta)$. Generate (ζ_t, G_t), $t = 1, \dots, T$, choose an initial α^0.

(2) Given a choice for $\hbar$ calculate $c_t(\alpha^0)$ from (2.60) with $\hbar(\alpha^0, k_t, \zeta_t, G_t)$, in place of $\beta E_t[c_{t+1}^{-\varphi}((1-T^y)\zeta_{t+1}(1-\eta)K_{t+1}^{-\eta} + (1-\delta))]$ and $K_{t+1}(\alpha^0)$ from the resource constraint. Do this for every t. This produces a time series for $c_t(\alpha^0)$ and $K_{t+1}(\alpha^0)$.

(3) Run a nonlinear regression using simulated $c_t(\alpha^0)$, $K_{t+1}(\alpha^0)$ of $\hbar(\alpha, K_t(\alpha^0), \zeta_t, G_t)$ on $\beta c_{t+1}(\alpha^0)^{-\varphi}[(1-T^y)\zeta_{t+1}(1-\eta)K_{t+1}(\alpha^0)^{-\eta} + (1-\delta)]$. Call the resulting nonlinear estimator α^{0*} and with this α^{0*} construct $Q(\alpha^0, \alpha^{0,*})$.

(4) Set $\alpha^1 = (1-\varrho)\alpha^0 + \varrho Q(\alpha^0, \alpha^{0*})$, where $\varrho \in (0, 1]$.

(5) Repeat steps (2)–(4) until $Q(\alpha^{*L-1}, \alpha^{*L}) \leqslant \iota$ or $|\alpha^{\mathrm{L}} - \alpha^{L-1}| \leqslant \iota$, or both, ι small.

(6) Use another $\hbar$ function and repeat steps (2)–(5).

When convergence is achieved, $\hbar(\alpha^*, K_t, \zeta_t, G_t)$ is the required approximating function. Since the method does not specify how to choose $\hbar$, one typically starts with a simple function (a first-order polynomial or a trigonometric function) and then checks the robustness of the solution by using more complex functions (e.g., a higher-order polynomial).

For the model of this example, setting $\varphi = 2$, $T^y = 0.15$, $\delta = 0.1$, $\rho_{\mathrm{G}} = \rho_\zeta = 0.95$, $\eta = 0.66$, $\beta = 0.99$, $q = 2$, and choosing $\hbar = \exp(\ln \alpha_1 + \alpha_2 \ln K_t + \alpha_3 \ln \zeta_t + \alpha_4 \ln G_t)$, 100 iterations of the above algorithm led to the following optimal approximating values, $\alpha_1 = -0.0780$, $\alpha_2 = 0.0008$, $\alpha_3 = 0.0306$, $\alpha_4 = 0.007$, and with these values $Q(\alpha^{*L-1}, \alpha^{*L}) = 0.000\,008$.

Next we show how to apply the method when inequality constraints are presented.

Example 2.24. Consider a small open economy which finances current account deficits issuing one-period nominal bonds. Assume that there is a borrowing constraint $\bar{B}$ so that $B_t - \bar{B} < 0$. The Euler equation for debt accumulation is

$$c_t^{-\varphi} - \beta E_t[c_{t+1}^{-\varphi}(1 + r_t) - \lambda_{t+1}] = 0, \tag{2.61}$$

where r_t is the exogenous world real rate, λ_t the Lagrangian multiplier on the borrowing constraint, and the Kuhn–Tucker condition is $\lambda_t(B_t - \bar{B}) = 0$. To find a solution use $0 = c_t^{-\varphi} - \beta\hbar(\alpha, r_t, \lambda_t, c_t)$ and $\lambda_t(B_t - \bar{B}) = 0$ and calculate c_t and B_t, assuming $\lambda_t = 0$, for some $\alpha = \alpha^0$. If $B_t > \bar{B}$, set $B_t = \bar{B}$, find λ from the first equation and c_t from the budget constraint. Do this for every t; find α^{0*}; generate α^1 and repeat until convergence. In essence, λ_t is treated as an additional variable, to be solved for in the model.

Exercise 2.36. Suppose in the model of example 2.23 that $u(c_t, c_{t-1}, N_t) = (c_t - \gamma c_{t-1})^{1-\varphi}/(1-\varphi)$, $T_t = T^y = 0$. Provide a parametrized expectation algorithm to solve this model. (Hint: there are two state variables in the Euler equation.)

Exercise 2.37 (CIA with taxes). Consider a model where a representative household maximizes a separable utility function of the form $E_0 \sum_{t=0}^{\infty} \beta^t [\vartheta_c \ln(c_{1t}) + (1-\vartheta_c)\ln(c_{2t}) - \vartheta_N(1-N_t)]$ by choices of consumption of cash and credit goods, leisure, nominal money balances, and investments, $0 < \beta < 1$. Suppose that the household is endowed with K_0 units of capital and one unit of time. The household receives income from capital and labor which is used to finance consumption purchases, investments, and holdings of money and government bonds. c_{1t} is the cash good and needs to be purchased with money; c_{2t} is the credit good. Output is produced with capital and labor by a single competitive firm with constant returns-to-scale technology and $1-\eta$ is the share of capital. In addition, the government finances a stochastic flow of expenditure by issuing currency, taxing labor income with a marginal tax rate T_t^y, and issuing nominal bonds, which pay an interest rate i_t. Assume that money supply evolves according to $\ln M_{t+1}^s = \ln M_t^s + \ln M_t^g$. Suppose agents start at time t with holdings of money M_t and bonds B_t. Assume that all the uncertainty is resolved at the beginning of each t.

(i) Write down the optimization problem mentioning the states and the constraints and calculate the first-order conditions. (Hint: you will need to make the economy stationary.)

(ii) Solve the model by parametrizing the expectations and using a first-order polynomial.

(iii) Describe the effects of an i.i.d. shock in T_t^y on real variables, prices, and interest rates, when B_t adjusts to satisfy the government budget constraint. Would your answer change if you kept B_t fixed and instead let G_t change to satisfy the government budget constraint?

As mentioned, the method of parametrizing expectations has a built-in mechanism to check the accuracy of the approximation. In fact, whenever the approximation is appropriate, the simulated time series must satisfy the Euler equation. As we will describe in more detail in chapter 5, this implies that, if $\tilde{\alpha}$ solves $Q(\tilde{\alpha},\tilde{\alpha}) = 0$, then $Q(\tilde{\alpha},\tilde{\alpha}) \otimes \mathfrak{h}(z_t) = 0$, where z_t is any variable in the information set at time t and $\mathfrak{h}$ is a $q \times 1$ vector of continuous differential functions. Under regularity conditions, when T is large, $T \times [(1/T)\sum_t Q_t \otimes \mathfrak{h}(z_t)]' W_t [(1/T)\sum_t Q_t \otimes \mathfrak{h}(z_t)]$, where Q_t is the sample counterpart of Q, ν is equal to the dimension of the Euler conditions times the dimension of $\mathfrak{h}$, and $W_T \xrightarrow{\text{P}} W$ is a weighting matrix. For example 2.23, the first-order approximation is accurate since $\mathfrak{S}$ has a p-value of 0.36, when two lags of consumption are used as z_t.

While useful for a variety of problems, the parametrizing expectations approach has two important drawbacks. First, the iterations defined by algorithm 2.3 may lead nowhere since the fixed point problem does not define a contraction operator. In other words, there is no guarantee that the distance between the actual and approximating function will get smaller as the number of iterations grows. Second, the method relies on the sufficiency of the Euler equation. Hence, if the utility function is not strictly concave, the solution that the algorithm delivers may be inappropriate.

2.2.6 A Comparison of Methods

There exists a literature comparing various approximation approaches. For example, the special issue of the *Journal of Business and Economic Statistics* of July 1991 shows how various methods perform in approximating the decision rules of a particular version of the one-sector growth model for which an analytic solution is available. Some additional evidence is in Ruge-Murcia (2002) and Fernandez-Villaverde and Rubio-Ramirez (2003a,b). In general, little is known about the properties of various methods in specific applications. Experience suggests that even for models possessing simple structures (i.e., models without habit, adjustment costs of investment, etc.), simulated series may display somewhat different dynamics depending on the approximation used. For more complicated models no evidence is available. Therefore, caution should be employed in interpreting the results obtained by approximating models with any of the methods described in this chapter.

Exercise 2.38 (growth with corruption). Consider a representative household who maximizes $E_0 \sum_t \beta^t c_t^{1-\varphi}/(1-\varphi)$ by choices of consumption c_t, capital K_{t+1}, and bribes br_t subject to

$$c_t + K_{t+1} = (1 - T_t^y)N_t w_t + r_t K_t - \text{br}_t + (1-\delta)K_t, \tag{2.62}$$

$$T_t^y = T_t^{\text{e}}(1 - a \ln \text{br}_t) + T_0^y, \tag{2.63}$$

where w_t is the real wage, T_t^y is the income tax rate, T_t^{e} is an exogenously given tax rate, T_0^y is a constant, and (φ, a, δ) are parameters. The technology is owned by the firm and given by $f(K_t, N_t, \zeta_t, K_t^{\text{G}}) = \zeta_t K_t^{1-\eta} N_t^{\eta} (K_t^{\text{G}})^{\aleph}$, where $\aleph \geqslant 0$, K_t is the capital stock, and N_t hours worked. Government capital K_t^{G} evolves according to $K_{t+1}^{\text{G}} = (1-\delta)K_t^{\text{G}} + N_t w_t T_t^y$. The resource constraint is $c_t + K_{t+1} + K_{t+1}^{\text{G}} + \text{br}_t = f(K_t, N_t, \zeta_t) + (1-\delta)(K_t + K_t^{\text{G}})$ and $(\zeta_t, T_t^{\text{e}})$ are independent AR(1) processes, with persistence (ρ_ζ, ρ_e) and variances $(\sigma_\zeta^2, \sigma_e^2)$.

(i) Define a competitive equilibrium and compute the first-order conditions.

(ii) Assume $\varphi = 2$, $a = 0.03$, $\beta = 0.96$, $\delta = 0.10$, $\rho_e = \rho_\zeta = 0.95$, and set $\sigma_\zeta^2 = \sigma_e^2 = 1$. Take a quadratic approximation of the utility and find the decision rules for the variables of interest.

(iii) Assume that $(\zeta_t, T_t^{\text{e}})$ and the capital stock can take only two values (say, high and low). Solve the model by discretizing the state and shock spaces. (Hint: use the fact that shocks are independent and the values of the AR parameter to construct the transition matrix for the shocks.)

(iv) Solve the model by using a first-order log-linear approximation method.

(v) Use the parametrized expectations method with a first-order power function to find a global solution.

(vi) Compare the time series properties of consumption, investment and bribes in (ii)–(v).

Exercise 2.39 (transmission with borrowing constraints). Consider an economy where preferences are described by $u(c_t, c_{t-1}, N_t) = (c_t^{\vartheta}(1-N_t)^{1-\vartheta})^{1-\varphi}/(1-\varphi)$, which accumulates capital according to $K_{t+1} = (1-\delta)K_t + \text{inv}_t$, where δ is the depreciation rate. Assume that the production function is of the form $\text{GDP}_t = \zeta_t K_t^{\eta_k} N_t^{\eta_N} \text{La}_t^{\eta_L}$, where La_t is land. Suppose individual agents have the ability to borrow and trade land and that their budget constraint is $c_t + K_{t+1} + B_{t+1} + p_t^{\text{L}}\text{La}_{t+1} \leqslant \text{GDP}_t + (1-\delta)K_t + (1+r_t^{\text{B}})B_t + p_t^{\text{L}}\text{La}_t$, where B_t are bond holdings, and suppose that there is a borrowing constraint of the form $p_t^{\text{L}}\text{La}_t - B_{t+1} \geqslant 0$, where p_t^{L} is the price of land in terms of consumption goods.

(i) Show that, in the steady state, the borrowing constraint is binding if $(1 + r^{\text{B}}) < \text{GDP}^{\text{ss}}/K^{\text{ss}} + (1-\delta)$. Give conditions which ensure that the constraint is always binding.

(ii) Describe the dynamics of output following a technology shock when the borrowing constraint (a) never binds, (b) always binds, (c) binds at some t. (Hint: use an approximation method which allows the comparison across cases.)

(iii) Is it true that the presence of (collateralized) borrowing constraints amplifies and stretches over time the real effects of technology shocks?