WEAK CONVERGENCE OF MULTIVARIATE FRACTIONAL PROCESSES*

by

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Contents:

Discussion Paper
No.EM/98/352
July 1998

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* Research supported by ESRC grant R000235892.

Abstract

Weak convergence to a form of fractional Brownian motion is established for a wide class of nonstationary fractionally integrated multivariate processes. Instrumental for the main argument is a result of some independent interest on approximations for partial sums of stationary linear vector sequences. A functional central limit theorem for smoothed processes is analyzed under more general assumptions.

Keywords: Nonstationary fractional integration; functional central limit theorem.

JEL No.: C32

1. INTRODUCTION

Let $\{\varepsilon_t\}$ be an independent and identically distributed ($i.i.d.$) sequence of random variables such that $E\varepsilon_1 = 0$, $E\varepsilon_1^2 = \sigma^2 < \infty$; for $0 < d_u < \frac{1}{2}$ consider the class of processes $\{u_t\}$ satisfying

$$u_t = \sum_{j=-\infty}^{t} \phi_{t-j}\varepsilon_j \ , \ t = 1, 2, \dots \ , \ \phi_0 = 1 \ , \ \phi_j \sim \ell(j) j^{d_u - 1} \ , \ j = 1, 2\dots \ , \tag{1}$$

where $\ell(.)$, (like $\ell_{ab}(.)$, $\ell'_{ab}(.)$, $\ell''_{ab}(.)$ to be introduced later) denotes a Lebesgue-measurable, real-valued function varying slowly at infinity, defined on $[0, \infty)$, bounded on every compact subset therein, and positive on $[X, \infty)$, for some $X > 0$, while " $\sim$ " indicates that the ratio of left- and right-hand sides tends to one; we term a sequence satisfying (1) long range dependent, or fractionally integrated of order d_u. Parametric, semiparametric and nonparametric statistical inference under long range dependence has been extensively investigated in recent years, for instance by Giraitis and Koul (1997), Robinson (1995) and Csőrgo and Mielniczuk (1995), respectively; models that combine long range dependence and infinite variance innovations have also been considered, for instance by Kokoszka and Taqqu (1995,1996), and Kokoszka and Mikosch (1997).

For many statistical applications partial sums of long range dependent variables are of interest. Denote by $B(r)$ standard Brownian motion, i.e. a zero-mean Gaussian process on R with independent increments and such that

$$B(0) = 0 \ , \ \text{a.s.} \ , \tag{2}$$

$$EB(r_1)B(r_2) = \min(r_1, r_2) \ , \ r_1, r_2 \geq 0 \ . \tag{3}$$

For $\frac{1}{2} < d < \frac{3}{2}$, denote by $B(r; d)$ fractional Brownian motion (cf. Samorodnitsky and Taqqu (1994)), given by

$$\begin{aligned} B(r, d) &= 0 \ , \ \text{a.s.} \ , \ r = 0 \ , \\ B(r; d) &= C_1(d)\sigma^2 \int_R \left[\{(r-s)_+\}^{d-1} - \{(-s)_+\}^{d-1} \right] dB(s) \ , \ r \in R^+ \ , \\ C_1(d) &= \left\{ \frac{1}{2d-1} + \int_0^\infty \left[(1+s)^{d-1} - s^{d-12} \right]^2 ds \right\}^{-1/2} , \end{aligned}$$

where $(t)_+ = \max(t, 0)$. Under (1) and higher moment conditions on ε_t, we have the invariance principle

$$(\ell(n)n^{d-1/2})^{-1}\left(\sum_{t=1}^{[nr]} u_t\right) \Rightarrow \sigma^2 B(r; d) , \quad 0 \le r \le 1 \ , \text{as } n \to \infty , \tag{4}$$

where $d = d_u + 1$, $\Rightarrow$ signifies weak convergence in a suitable metric space (Billingsley (1968)), and [.] is integer part. The convergence (4) was established by Davydov (1970), Taqqu (1975) and Gorodetskii (1977), and extended by Chan and Terrin (1995), Csőrgo and Mielniczuk (1995), and others.

Partial sum processes can be restrictive for applications and more general forms of dependence may be considered, for instance nonstationary fractional integration. For $t \ge 1$ let

$$z_t = \sum_{j=1}^{t} \psi_{t-j}\eta_j \ , \quad \psi_0 = 1 \ , \ \psi_j \sim \ell(j)j^{d-1} \ , \ j = 1, 2, ..., \ t \ge 1 \ , \tag{5}$$

where $d > \frac{1}{2}$ and

$$\eta_t = \sum_{j=-\infty}^{\infty} a_{t-j}\varepsilon_j \ , \ 0 < \sum_{j=-\infty}^{\infty} |a_j| < \infty \ , \ t = 1, 2, ... \ ,$$

the latter inequality implying η_t has "short range dependence", by contrast with (1). For $d = 1$ and $\ell(.) \equiv 1$ we have partial sums of short range dependent innovations $z_t = \sum_{j=1}^{t} \eta_j$. The sum in (5) has to be finite, because the ψ_j are not square-summable for $d > \frac{1}{2}$; it can be verified that $Var(\sum_{t=1}^{m} z_t) \sim cm^{2d+1}$ and $\sum_{t=1}^{m} Ez_t^2 \sim c'm^{2d}$ as $m \to \infty$, $0 < c, c' < \infty$, and hence z_t is nonstationary long range dependent in the extended sense of Heyde and Yang (1997).

The purpose of this paper is to study weak convergence for vector processes generalizing (5). Vector stationary long range dependent processes have been studied by Robinson (1995); a preliminary investigation for a class of nonstationary univariate processes related to those we analyze here was considered by Akonom and Gourieroux (1987) and Silveira (1991). Our main result, established in Section 2, specializes in the univariate case to

$$(\ell(n)n^{d-1/2})^{-1} z_{[nr]} \Rightarrow \sigma^2 a(1)^2 W(r; d) \ , \quad 0 \le r \le 1 \ , \text{as } n \to \infty \ , \tag{6}$$

with $a(1) = \sum_{j=-\infty}^{\infty} a_j$ and where the process $W(r;d)$, itself denoted fractional Brownian motion by Akonom and Gourieroux (1987), is defined formally for $d > \frac{1}{2}$ as a Holmgren-Riemann-Liouville fractional integral

$$\begin{aligned} W(0;d) &= 0 \text{ , a.s. } , \\ W(r;d) &= \int_0^r (r-s)^{d-1} dB(s) \ , \ r > 0 \ . \end{aligned}$$

Like $B(r;d)$, $W(r;d)$ is Gaussian with almost surely continuous sample paths, and $W(r,1) = B(r)$, but $W(r;d)$ and $B(r;d)$ are different processes, in particular their autocovariances differ, as discussed by Marinucci and Robinson (1998), where $W(r;d)$ is labelled "type II" fractional Brownian motion.

The main ingredient for the proof of (6) is the representation

$$\begin{aligned} (\ell(n)n^{d-1/2})^{-1} z_{[nr]} &= q_n(r) + r_n(r) \ , \\ \sup_{r\in[0,1]} |r_n(r)| &= o_p(1) \ , \end{aligned} \qquad (7)$$

where $q_n(r) = \sigma^2 a(1)^2 \sum_{i=1}^{[nr]-1} (r - \frac{i}{n})^{d-1} w_i$ for an $i.i.d.$ sequence $w_i \equiv N(0,1)$, " $\equiv$ " signifying equality in distribution. Weak convergence of $q_n(r)$ to $W(r;d)$ entails convergence of the finite dimensional distributions and tightness of $q_n(r)$. For (7) we extend results for partial sums of $i.i.d.$ vectors due to Einmahl (1989) to short range dependent vectors, by employing a decomposition used in another context by Phillips and Solo (1992).

In the sequel, C denotes a generic, positive constant, I_p the p-rowed identity matrix, and $\|.\|$ the Euclidean norm. Section 2 presents the main result of the paper, while technical lemmas are collected in Section 3.

2. MAIN RESULT

Let us introduce the following assumptions.

ASSUMPTION A1 The $p \times 1$ vector sequence $\{z_t\}$ satisfies

$$z_t = \sum_{j=1}^{t} \Psi_{t-j} \eta_j \ , \ t = 1, 2, \ldots \ ,$$

where $\Psi_0 = I_p$ and for $j \geq 1$ Ψ_j has (a, b)-th element

$$\psi_{ab,j} = g_{ab} \ell_{ab}(j) j^{d_a - 1} \ ,$$

where $d_a > \frac{1}{2}$, $a, b = 1, \ldots, p$, and

$$\max_{0 \leq \theta \leq 1} |\ell_{aa}(j+\theta) - \ell_{aa}(j+1)| \leq C \frac{\ell'_{aa}(j)}{j} \ , \ \ell'_{aa}(j) > 0 \ , \tag{8}$$

$$|\ell_{ab}(j) - \ell_{aa}(j)| \leq C \frac{\ell''_{ab}(j)}{j} \ , \ \ell''_{ab}(j) > 0 \ . \tag{9}$$

ASSUMPTION A2 The η_t in Assumption A1 satisfy

$$\eta_t = \sum_{j=-\infty}^{\infty} A_{t-j} \varepsilon_j \ , \ \sum_{j=0}^{\infty} \left\{ \sum_{k=j+1}^{\infty} \| A_k \|^2 + \sum_{k=j+1}^{\infty} \| A_{-k} \|^2 \right\} < \infty \ . \tag{10}$$

ASSUMPTION A3 The ε_t in Assumption A2 are *i.i.d.* with

$$E\varepsilon_1 = 0 \ , \ E\varepsilon_1 \varepsilon_1' = \Sigma \ , \ E\|\varepsilon_1\|^q < \infty \ , \text{some } q > 2 \ . \tag{11}$$

ASSUMPTION A4

$$rank(\Sigma) = rank(G) = rank(\sum_{j=-\infty}^{\infty} A_j) = p \ , \tag{12}$$

where $G = \{g_{ab}\}_{a,b}$, $a, b = 1, \ldots, p$.

We refer collectively to Assumptions A1-A4 as Assumption A. Condition (8) is a mild smoothness restriction on $\ell_{ab}(.)$, holding, for instance, for $\ell_{ab}(.) = (\log(.))^\xi$, any real ξ. More generally, $\ell_{ab}(.)$ can be a *normalised* slowly varying function, (Bingham et al. (1989), p.15),

$$\ell_{ab}(x) = c \exp \left\{ \int_0^x \frac{e(u)}{u} du \right\} \ , \ c > 0 \ , \ e(u) \to 0 \text{ as } u \to \infty \ ,$$

with $d\ell_{ab}(x)/dx = e(x)\ell_{ab}(x)/x = o(x^{-1})$ a.e. as x goes to infinity. This class coincides with the *Zygmund class l(.)*, such that $x^{\xi} l(x)$ is eventually increasing and $x^{-\xi} l(x)$ is eventually decreasing as $x \to \infty$ for any $\xi > 0$ (Bingham et al. (1989), p.24). (9) is a homogeneity condition for the coefficients on the a-th row of Ψ_j and it is satisfied by parametric models of fractional integration, cf. Corollary 1. The stationary linear specification for η_t in (10)/(11) entails a mild form of short range dependence condition, which is for instance implied by $\sum_{j=-\infty}^{\infty} j^{1/2} \|A_j\| < \infty$ (cf. Phillips and Solo (1992)). Condition (12) ensures that the asymptotic limit process will have nondegenerate finite dimensional distributions.

The following lemma follows from Theorems 1, 2 and 4 in Einmahl (1989).

Lemma 1 *(Einmahl (1989))* Let $\{\varepsilon_t\} : S \to R^p$ be a sequence of *i.i.d.* vectors such that Assumptions A3 holds. Then we can construct a probability space $(S_0, \Im_0, P_0)$ and two sequences of *i.i.d.* vectors $\{\widehat{\varepsilon}_t\}$, $\{w_t\}$ with $\widehat{\varepsilon}_t \equiv \varepsilon_t$, $w_t \equiv N(0, \Sigma)$, $n = 1, 2, ...$, such that, as $n \to \infty$,

$$\| \sum_{t=1}^{n} \widehat{\varepsilon}_t - \sum_{t=1}^{n} w_t \| = o(n^{1/q}) \text{ , a.s. ,} \qquad (13)$$

and

$$\max_{1 \le k \le n} \| \sum_{t=1}^{k} \widehat{\varepsilon}_t - \sum_{t=1}^{k} w_t \| = o_p(n^{1/q}) \text{ .} \qquad (14)$$

We can now establish approximations for partial sums of multivariate linear sequences as follows.

Lemma 2 Under Assumptions A2 and A3 we can construct a probability space $(S_0, \Im_0, P_0)$ and two sequences of *i.i.d.* vectors $\{\widehat{\eta}_t\}$, $\{w_t\}$ with $\widehat{\eta}_t \equiv \eta_t$, $w_t \equiv N(0, \Sigma)$, $n = 1, 2, ...$ such that as $n \to \infty$, for $2 < s < q$

$$S_n - V_n = o(n^{1/s}) \text{ , a.s. ,} \qquad (15)$$

$$\sup_{j \le n} \frac{\|S_j - V_j\|}{n^{1/s}} \quad = \quad o_p(1) \ , \tag{16}$$

where $S_j = \sum_{t=1}^{j} \widehat{\eta}_t$ and $V_j = A(1) \sum_{t=1}^{j} w_t$, $A(x) = \sum_{j=-\infty}^{\infty} A_j x^j$.

Proof For $|x| \le 1$ we have

$$A(x) = A(1) + (x-1) \left\{ A^+(x) - A^-(x) \right\} \ ,$$

where

$$\begin{aligned} A^+(x) &= \sum_{j=0}^{\infty} A_j^+ x^j \ , \ A^-(x) = \sum_{j=0}^{\infty} A_j^- x^j \ , \\ A_j^+ &= \sum_{k=j+1}^{\infty} A_k \ , \ A_j^- = \sum_{k=j+1}^{\infty} A_{-k} \ . \end{aligned}$$

Then

$$\sum_{t=1}^{n} \eta_t = A(1) \sum_{t=1}^{n} \varepsilon_t + \overline{\varepsilon}_{0n}$$

where $\overline{\varepsilon}_{0n} = \varepsilon_0^+ - \varepsilon_0^- - \varepsilon_n^+ + \varepsilon_n^-$, $\varepsilon_t^+ = A^+(L)\varepsilon_t$, $\varepsilon_t^- = A^-(L)\varepsilon_t$. Also

$$E\|\overline{\varepsilon}_{0n}\|^q \le C \left\{ E\|\varepsilon_0^+\|^q + E\|\varepsilon_0^-\|^q \right\} \ ,$$

where

$$\begin{aligned} E\|\varepsilon_0^+\|^q &\le CE \left\{ \sum_{j=0}^{\infty} \|A_j^+\|^2 \|\varepsilon_{-j}\|^2 \right\}^{q/2} \\ &\le C \left\{ \sum_{j=0}^{\infty} \left(E\|A_j^+\|^q \|\varepsilon_{-j}\|^q \right)^{2/q} \right\}^{q/2} \\ &\le C \left\{ \sum_{j=0}^{\infty} \|A_j^+\|^2 \right\}^{q/2} E\|\varepsilon_0\|^q < \infty \end{aligned}$$

using Burkholder's (1973) and Minkowski's inequalities. In the same way $E\|\varepsilon_0^-\|^q < \infty$. Thus $\overline{\varepsilon}_{0n} = o(n^{1/s})$ a.s. by Markov's inequality and the Borel-Cantelli lemma. The proof of

(15) is completed by application of (13) to $A(1)\varepsilon_t$ and the identification $\widehat{\eta}_t = \sum_{j=-\infty}^{\infty} A_j \widehat{\varepsilon}_{t-j}$ for $\widehat{\varepsilon}_t$. To establish (16) write

$$S_j - V_j = A(1) \left\{ \sum_{t=1}^{j} \widehat{\varepsilon}_t - \sum_{t=1}^{j} w_t \right\} + \bar{\varepsilon}_{0j} \ ,$$

and hence $P(\sup_{j \le n} \|S_j - V_j\| > \lambda n^{1/s})$ is bounded by

$$P(\sup_{j \le n} \|A(1)(\sum_{t=1}^{j} \widehat{\varepsilon}_t - \sum_{t=1}^{j} w_t)\| > \frac{\lambda n^{1/s}}{2}) + P(\sup_{j \le n} \|\bar{\varepsilon}_{0j}\| > \frac{\lambda n^{1/s}}{2}) \ . \tag{17}$$

Also, the first component of (17) is bounded by

$$P(\sup_{j \le n} \|(\sum_{t=1}^{j} \widehat{\varepsilon}_t - \sum_{t=1}^{j} w_t)\| > c\lambda n^{1/s})$$

for some $0 < c < \infty$. From (14) it follows that the w_t can be chosen as *i.i.d.* $N(0, \Sigma)$ such that

$$\sup_{j \le n} \|(\sum_{t=1}^{j} \widehat{\varepsilon}_t - \sum_{t=1}^{j} w_t)\| = o_p(n^{1/s})$$

because λ is arbitrary. The second component of (17) is bounded by

$$\begin{aligned} P(\sup_{j \le n} \|\varepsilon_{0j}\| \quad &\ge \quad \frac{\lambda n^{1/s}}{2}) \le C \frac{E \sup_{j \le n} \|\varepsilon_{0j}\|^q}{(\lambda n^{1/s})^q} \\ &\le \quad C \frac{nE(\|\varepsilon_0^+\|^q + \|\varepsilon_0^-\|^q)}{(\lambda n^{1/s})^q} \to 0 \end{aligned}$$

as $n \to \infty$, in view of the previous evaluation. □

Results from Silveira (1991) suggest that for $q < 3$ and under moment conditions stronger than Assumption A3, (15) can be extended to cover also forms of dependence that are neither strictly stronger nor weaker than our linearity, such as absolute regularity (cf. Pham and Tran, (1985)). The only part of Lemma 2 which is used in the sequel is (16), but we have included (15) to mirror the derivation of analogous results by Einmahl (1989) and others.

ASSUMPTION B For q defined by Assumption A3,

$$q > \max(2, \frac{2}{2d_* - 1}) \ , \ d_* = \min_{1 \le a \le p} d_a \ . \tag{18}$$

Herrndorf (1984) considered normalized partial sums of covariance stationary mixing sequences u_t, the argument to establish weak convergence requiring tighter moment conditions on u_t the slower the mixing rate. On the other hand in (18) a larger amount of "persistence", i.e. a larger d_*, entails weaker moment conditions, at least for $d_* < 1$. A heuristic explanation is as follows: while the mixing rate in classical central limit theorems does not affect the $\sqrt{n}$-normalization, in Theorem 1 a lower value of d_* entails a smaller normalization, and hence tighter bounds on the remainder terms are needed (cf. Davydov (1970) and Gorodetskii (1977)). In view of Assumption A3, Assumption B is vacuous for $d_* \geq 1$.

Define the normalizing matrix function $D(n; d_z)$, for $d_z = (d_1, ..., d_p)$, as

$$D(n; d_z) = diag\left\{ \left(\ell_{11}(n)n^{d_1-1/2}\right)^{-1}, ..., \left(\ell_{pp}(n)n^{d_p-1/2}\right)^{-1} \right\} ,$$

and "type II" multivariate fractional Brownian motion for $r \geq 0$ as

$$W(r; d_z, \Omega) = (0, ...0)' \text{ , a.s. , } r = 0 \text{ ,} \tag{19}$$

$$W(r; d_z, \Omega) = \int_0^r G(r, s) dB(r; \Omega) \text{ , } r > 0 \text{ ,} \tag{20}$$

where $\Omega = A(1)\Sigma A(1)'$ is a $p \times p$ full rank matrix (by Assumption A4), $B(.; \Omega)$ is $p \times 1$ scaled Brownian motion such that

$$B(r; \Omega) = (0, ...0)' \text{ , a.s. , } r = 0 \text{ ,}$$

$$EB(r_1; \Omega)B(r_2; \Omega)' = \Omega \min(r_1, r_2) \text{ ,}$$

and $G(r, s)$ has (a, b)-th element $g_{ab}(r - s)^{d_a - 1}$, $a, b = 1, ..., p$, for $0 \leq s < r$, and zero otherwise.

Define $z_n(r) = D(n; d_z)\widehat{z}_{[nr]}$, for $0 \leq r \leq 1$, and note that $z_n(r) \in D[0, 1]^p$, the space of R^p-valued vector functions on $[0, 1]$ whose components are continuous on the right and with finite left limit, endowed with the product σ-algebra $\mathcal{D}^p$. The latter is generated by the open sets with respect to the metric that induces the Skorohod J_1 topology on the component spaces; this makes $D[0, 1]^p$ complete and separable, like $D[0, 1]$. The proof of weak convergence in $D[0, 1]^p$ involves the same steps as for the univariate case (see e.g.

Csörgö and Mielniczuk (1995)), namely convergence of the finite dimensional distributions and tightness of the components of $z_n(r)$.

Theorem 1 Under Assumptions A and B, for $0 \leq r \leq 1$

$$z_n(r) \Rightarrow W(r; d_z, \Omega) \text{ as } n \to \infty ,$$

where $\Rightarrow$ signifies convergence in the Skorohod J_1 topology of $D[0,1]^p$.

Proof For S_j, V_j defined in Lemma 2 we can write

$$\begin{aligned}
\widehat{z}_n(r) &= D(n; d_z) \sum_{k=1}^{[nr]} \Psi_{[nr]-k} \left(S_k - S_{k-1}\right) , \\
&= Q_{1n}(r) + Q_{2n}(r) + Q_{3n}(r) + Q_{4n}(r) + Q_{5n}(r) + Q_{6n}(r) ,
\end{aligned}$$

with

$$\begin{aligned}
Q_{1n}(r) &= \sum_{k=1}^{[nr]-1} \mathcal{G}(r,k,n) n^{-1/2} \left[V_k - V_{k-1}\right] 1_{[nr]>2}(r) , \\
Q_{2n}(r) &= \sum_{k=1}^{[nr]-1} D(n; d_z) \Psi_{[nr]-k} \left[(S_k - S_{k-1}) - (V_k - V_{k-1})\right] 1_{[nr]>2}(r) , \\
Q_{3n}(r) &= \sum_{k=1}^{[nr]-1} \left(D(n; d_z)\Psi_{[nr]-k} - \mathcal{G}(r,k,n) n^{-1/2}\right) \left[V_k - V_{k-1}\right] 1_{[nr]>2}(r) , \\
Q_{4n}(r) &= D(n; d_z) \left[S_{[nr]} - S_{[nr]-1}\right] 1_{[nr]>2}(r) , \\
Q_{5n}(r) &= D(n; d_z) \widehat{z}_{[nr]} 1_{[nr]\leq 2}(r) , \\
Q_{6n}(r) &= \sum_{k=1}^{[nr]-1} \mathcal{H}(r,k,n) n^{-1/2} \left[S_k - S_{k-1}\right] 1_{[nr]>2}(r) ,
\end{aligned}$$

where $1_A(.)$ is the indicator function of the set A, and $\mathcal{G}(r,k,n)$, $\mathcal{H}(r,k,n)$ have (a,b)-th element

$$\begin{aligned}
&g_{ab} \frac{\ell_{aa}(nr-k)}{\ell_{aa}(n)} (r - \frac{k}{n})^{d_a - 1} , \\
&n^{1-d_a} g_{ab} \frac{\ell_{ab}([nr]-k) - \ell_{aa}([nr]-k)}{\ell_{aa}(n)} ([nr]-k)^{d_a - 1} n^{-1/2} ,
\end{aligned}$$

respectively, for $a, b = 1, ..., p$. The Theorem will follow if, as $n \to \infty$,

$$Q_{1n}(r) \Rightarrow W(r; d_z, \Omega),\ r \in [0,1]\ , \tag{21}$$

$$\sup_{0 \le r \le 1} \|Q_{in}(r)\| = o_p(1)\ ,\ i = 2, ..., 6\ . \tag{22}$$

Now (21) follows from Lemma 4, while (22) with $i = 2$ follows from Lemma 5, with $i = 3$ from Lemma 6, with $i = 4, 5$ from Lemma 7 and with $i = 6$ from Lemma 8. □

We now focus on more specific models of fractional integration.

ASSUMPTION A' For $d_a > \frac{1}{2}$, $a = 1, ..., p$, let Assumptions A2, A3, A4 hold, and let

$$z_t = \Delta(L)\Lambda(L)\left\{\eta_t 1_{t>0}(t)\right\}\ ,\ t = 0, \pm 1, ..., \tag{23}$$

for

$$\Delta(L) = diag\left\{(1-L)^{-d_1}, ..., (1-L)^{-d_p}\right\}\ ,\ \Lambda(L) = \sum_{j=0}^{\infty} \Lambda_j L^j\ ,$$

where L is the lag operator and the coefficient Δ_{ak} of L^k in the expansion of $(1-L)^{-d_a}$ is defined by

$$\Delta_{ak} = \frac{\Gamma(k+d_a)}{\Gamma(d_a)\Gamma(k+1)}\ ,\ \Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1} dx\ ,\ k = 1, 2, ...\ .$$

ASSUMPTION C For $\Lambda(L)$ defined by Assumption A', $\Lambda_0 = I_p$ and $\lambda_{ab,j}$ the a, b-th element of Λ_j

$$|\lambda_{ab,j}| \le C j^{-\gamma_a}\ ,\ a, b = 1, ..., p\ ,\ j = 1, 2, ...\ ,$$

where $\gamma_a = \max(4 - d_a, d_a)$.

Assumption A' allows more generality than the class of vector autoregressive fractionally integrated moving average processes, which are defined by the equations

$$z_t = \Delta(L)\left\{\eta_t 1_{t>0}(t)\right\}\ ,\ \eta_t = \Phi^{-1}(L)\Theta(L)\varepsilon_t\ ,\ t = 1, 2, ...\ , \tag{24}$$

with $\Phi(s)$ and $\Theta(s)$ $p \times p$ matrix polynomials with real coefficients which satisfy

$$\Phi(s) = I_p - \Phi_1 s - ... - \Phi_{q_1} s^{q_1} \ , \ \Theta(s) = I_p - \Theta_1 s - ... - \Theta_{q_2} s^{q_2} \ ,$$

where q_1 and q_2 are positive integers and we assume that $\Phi(s)$ has no roots in the closed disk $\{s : |s| \le 1\}$. The coefficients in the series expansion of $\Phi^{-1}(s)\Theta(s)$ tend exponentially to zero and therefore Assumption A2 is trivially satisfied after the identification $A(s) = \Phi^{-1}(s)\Theta(s)$, whence (24) follows from (23) on picking $\Lambda(L) \equiv I_p$. Also, (24) provides a natural generalization and more modelling flexibility than non-fractional vector autoregressive integrated moving averages, which are highly popular among time series analysts and correspond to (24) in the special case where $d_1, ..., d_p$ are positive integers. For $p = 1$, the class considered by Akonom and Gourieroux (1987) is given by (23) with *i.i.d.* η_t (i.e. $A(L) \equiv I_p$) and $\Lambda(s) = \Phi^{-1}(s)\Theta(s)$; this class does not cover (24), however.

From Assumption B we may write

$$z_t = \sum_{j=1}^{t} \Pi_{t-j} \eta_j \ , \ t = 1, 2, ... \ ,$$

where Π_j has (a, b)-th element

$$\pi_{ab,j} = \sum_{i=0}^{j} \frac{\Gamma(i + d_a)}{\Gamma(d_a)\Gamma(i + 1)} \lambda_{ab,j-i} \ , \ j = 1, 2, ... \ . \tag{25}$$

We need to approximate the $\pi_{ab,j}$ by means of the following lemma, which extends Lemma 3.2 in Kokoszka and Taqqu (1995) allowing for $d > \frac{1}{2}$ and for algebraically (rather than exponentially) decaying coefficients λ_j (cf. Silveira (1991)).

Lemma 3 Let $d > -\frac{1}{2}$ and set

$$\gamma = \begin{cases} 4, & d < 0 \\ 4 - d, & 0 < d < 2 \\ 2, & 2 < d \end{cases} .$$

Let

$$\pi_0 = 1 \ , \ \pi_j = \sum_{i=0}^{j} \lambda_{j-i} \frac{\Gamma(i + d)}{\Gamma(d)\Gamma(i + 1)} \ , \ j > 0 \ ,$$

where

$$\lambda_0 = 1 \ , \ |\lambda_j| \le C j^{-\gamma} \ . \tag{26}$$

Then as $j \to \infty$

$$|\pi_j - \frac{\lambda(1)}{\Gamma(d)} j^{d-1}| \le C j^{d-2} \ , \tag{27}$$

where $\lambda(1) = \sum_{j=0}^{\infty} \lambda_j$.

Proof Note first that from Abramowitz and Stegun (1970), formula 6.1.47,

$$|\frac{\Gamma(j+d)}{\Gamma(j+1)} - j^{d-1}| \le C j^{d-2} \ , \ j = 1, 2, \dots \ . \tag{28}$$

The left side of (27) is thus bounded by $\Gamma(d)^{-1}\{I + II + III + IV\}$, where

$$\begin{aligned} (I) &= |\lambda_j|\Gamma(d) \ , \ (III) = \sum_{i=1}^{j} |\lambda_{j-i}||i^{d-1} - j^{d-1}| \\ (II) &= \sum_{i=1}^{j} |\lambda_{j-i}||\frac{\Gamma(i+d)}{\Gamma(i+1)} - i^{d-1}| \ , \ (IV) = |\sum_{i=j+1}^{\infty} \lambda_i| j^{d-1} \ . \end{aligned}$$

By (26), $I \le C j^{-\gamma} \le C j^{d-2}$, and $IV \le C j^{d-\gamma} \le C j^{d-2}$. By (28) $II \le C \sum_{i=1}^{j} |\lambda_{j-i}| i^{d-2}$. For $d \ge 2$ this is $O(j^{d-2})$ by summability of $\{\lambda_i\}$ implied by (26). For $d < 2$ it is bounded by

$$\sum_{i=1}^{[j/2]} |\lambda_{j-i}| i^{d-2} + \sum_{i=[j/2]}^{j} |\lambda_{j-i}| i^{d-2} = \sum_{i=[j/2]}^{\infty} |\lambda_i| + C j^{d-2} \sum_{i=0}^{\infty} |\lambda_i| \ ,$$

and this is $O(j^{d-3} + j^{d-2}) = O(j^{d-2})$ by (26). For $d \ge 2$, by the mean value theorem $III < C j^{d-2} \sum_{i=1}^{j} |\lambda_{j-i}|(j-i) \le C j^{d-2}$ by (26). For $0 \le d < 2$

$$\begin{aligned} III &\le C \sum_{i=1}^{[j/2]} |\lambda_{j-i}|(j-i) + C j^{d-2} \sum_{i=[j/2]}^{j} |\lambda_{j-i}|(j-i) \\ &\le C(j^{-2} + j^{d-2}) \le 2C j^{d-2} \ . \end{aligned}$$

For $-\frac{1}{2} < d < 0$

$$III \le \sum_{i=1}^{[j/2]} |\lambda_{j-i}|(j^{1-d} - i^{1-d})(ji)^{d-1} + C j^{d-2} \sum_{i=[j/2]}^{j} |\lambda_{j-i}|(j-i) \ .$$

The second term is $O(j^{d-2})$, whereas the first is, by the mean value theorem, less than $Cj^{d-1}j^{-d}\sum_{i\geq[j/2]} i|\lambda_i| \leq Cj^{-3}$. $\square$

From Lemma 3 and Theorem 1 we derive the following result.

Corollary 1 Under Assumptions A', B and C, for $0 \leq r \leq 1$

$$z_n(r) \Rightarrow W(r; d_z, \Omega) \text{ as } n \to \infty ,$$

where $W(r; d_z, \Omega)$ is as defined in (19)/(20) with $g_{ab} = \left\{\Gamma(d_a)^{-1}\lambda_{ab}(1)\right\}$, $\lambda_{ab}(1) = \sum_{j=0}^{\infty} \lambda_{ab,j}$, $a, b = 1, ..., p$.

Proof Under Assumption C, from a straightforward application of Lemma 3 to (25) we have

$$\pi_{ab,j} = \frac{\lambda_{ab}(1)}{\Gamma(d_a)} j^{d_a-1} + O(j^{d_a-2}) \text{ , as } j \to \infty .$$

Hence Assumption A1 is satisfied after the identification

$$\ell_{ab}(j) = 1 + \left\{ \frac{\Gamma(d_a)}{\lambda_{ab}(1)} \frac{\pi_{ab,j}}{j^{d-1}} - 1 \right\} = 1 + O(j^{-1}) \text{ , as } j \to \infty ,$$

and the result follows by appealing to Theorem 1. $\square$

For $d_* > 1$ a much simpler proof of Corollary 1 follows from Abel summation by parts and the continuous mapping theorem (cf. Akonom and Gourieroux (1987)). Applications to asymptotic inference on nonstationary time series are presented in Robinson and Marinucci (1998).

The conditions on the moments of the innovation sequence $\{\varepsilon_t\}$ can be relaxed (cf. Silveira (1991)) if we focus on the smoothed multivariate series $\widetilde{z}_{[nr]} = \sum_{t=1}^{[nr]} z_t$, $0 \leq r \leq 1$, which represents fluctuations of partial sums of $\{z_t\}$.

Corollary 2 Let Assumptions A hold. Then as $n \to \infty$, for $0 \leq r \leq 1$

$$\widetilde{D}(n; d_z^+)\widetilde{z}_{[nr]} \Rightarrow W(r; d_z^+, \Omega) , \tag{29}$$

where

$$\widetilde{D}(n;d_z^+) = diag\left\{\left(\frac{\ell_{11}(n)n^{d_1}}{d_1}\right)^{-1},...,\left(\frac{\ell_{pp}(n)n^{d_p}}{d_p}\right)^{-1}\right\} ,$$

and $d_z^+ = (d_1+1,....,d_p+1)$.

Proof For $0 \le r \le 1$, write

$$\begin{aligned}\widetilde{D}(n;d_z^+)\sum_{t=1}^{[nr]} z_t &= \widetilde{D}(n;d_z^+)\sum_{t=1}^{[nr]}\sum_{j=1}^{t}\Psi_{t-j}\eta_j \\ &= \widetilde{D}(n;d_z^+)\sum_{j=1}^{[nr]}\widetilde{\Psi}_{[nr]-j}\eta_j ,\end{aligned}$$

where $\widetilde{\Psi}_0 = I_p$ and for $j \ge 1$, $\widetilde{\Psi}_j = \sum_{i=0}^{j}\Psi_i$ has (a,b)-th element

$$\widetilde{\psi}_{ab,j} = 1 + g_{ab}\sum_{i=1}^{j}\ell_{ab}(i)i^{d_a-1} \sim \frac{g_{ab}}{d_a}\ell_{ab}(j)j^{d_a} \text{ , as } j \to \infty ,$$

the approximation following from the direct half of Karamata's Theorem (Bingham et al. (1989), p.26). Hence Assumption A is satisfied if $\widetilde{z}_t$ is viewed as a nonstationary fractionally integrated process of order d_z^+; because $\min_{1\le a\le p}(d_a+1) > \frac{3}{2}$, Assumption B holds and (29) follows from Theorem 1. □

3. TECHNICAL LEMMAS

Lemma 4 Under Assumptions A and B, as $n \to \infty$

$$Q_{1n}(r) \Rightarrow W(r;d_z,\Omega) \text{ , } r \in [0,1] .$$

Proof Since $Q_{1n}(r)$ and $W(r;d,\Omega)$ are Gaussian, convergence of the finite dimensional distributions follows if we establish asymptotic equivalence of their first two moments. The fact that

$$\lim_{n\to\infty} EQ_{1n}(r) = (0,...,0)' = EW(r;d_z,\Omega) \ ,$$

is obvious. Fix w.l.o.g. $r_2 \geq r_1$ and recall that, from Assumption A1 and Potter's theorem (Bingham et al., (1989), p.25) we have

$$|\frac{\ell_{aa}(nr-k)}{\ell_{aa}(n)}| \leq C(r-\frac{k}{n})^{-\delta} \ , \ a=1,...,p \ , \ k=1,...,[nr]-1 \ , \tag{30}$$

where δ is any positive constant, which we shall hereafter choose such that $d_a - \delta > \frac{1}{2}$. Hence the a,b-th component of $\mathcal{G}(r,k,n)$ can be bounded by a constant if $d_a > 1$, and by $g_{ab}(r-s)^{d_a-1-\delta}$ if $d_a \leq 1$; by dominated convergence

$$\begin{aligned}
\lim_{n\to\infty} EQ_{1n}(r_1)Q'_{1n}(r_2) &= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{[nr_1]-1} \mathcal{G}(r_1,k,n)\Omega\mathcal{G}'(r_2,k,n) \\
&= \lim_{n\to\infty} \sum_{k=1}^{[nr_1]-1} \int_{k/n}^{(k+1)/n} \mathcal{G}(r_1,k,n)\Omega\mathcal{G}'(r_2,k,n)ds \\
&= \int_0^{r_1} G(r_1,s)\Omega G'(r_2,s)ds \\
&= EW(r_1;d_z,\Omega)W'(r_2;d_z,\Omega) \, ,
\end{aligned}$$

so that convergence of the finite-dimensional distributions is established. Also, Akonom and Gourieroux (1987), p.13 show that a tightness criterion for Gaussian series is given by, for $a=1,...,p$, $0 \leq r_1 < r < r_2 \leq 1$

$$E\left\{Q_{1n}^{(a)}(r) - Q_{1n}^{(a)}(r_1)\right\}^2 E\left\{Q_{1n}^{(a)}(r_2) - Q_{1n}^{(a)}(r)\right\}^2 \leq C|r_2 - r_1|^{\gamma} \ , \tag{31}$$

for constants $C, \gamma > 0$. To prove (31), define for $0 \leq r \leq 1$, $a=1,...,p$,

$$R_{an}(r) = \sum_{k=1}^{[nr]-1} (r-\frac{k}{n})^{d_a-1-\delta}\left(B(\frac{k}{n}) - B(\frac{k-1}{n})\right) \ , \tag{32}$$

where $B(.)$ is univariate standard Brownian motion as introduced in(2)/(3); to simplify notation, we shall use $d_a^- = d_a - \delta$.

Consider first the case where $r_1 > 0$. The inequality (31) is trivial for any fixed n (n_0, say); we can take $n_0 = [\frac{2}{r_1}]$ and focus without loss of generality on $n > n_0$, so that $r_1 > \frac{2}{n}$ always holds. Denote by g_a' the a-th row of G; we have

$$\begin{aligned} Q_{1n}^{(a)}(r) &= \sum_{k=1}^{[nr]-1} (r - \frac{k}{n})^{d_a - 1} \frac{\ell_{aa}(nr-k)}{\ell_{aa}(n)} g_a' \left(B(\frac{k}{n};\Omega) - B(\frac{k-1}{n};\Omega) \right) \\ &\equiv (g_a'\Omega g_a)^{1/2} \sum_{k=1}^{[nr]-1} (r - \frac{k}{n})^{d_a - 1} \frac{\ell_{aa}(nr-k)}{\ell_{aa}(n)} \left(B(\frac{k}{n}) - B(\frac{k-1}{n}) \right) , \end{aligned}$$

and therefore in view of (30)

$$\begin{aligned} & E\left\{ Q_{1n}^{(a)}(r) - Q_{1n}^{(a)}(r_1) \right\}^2 E\left\{ Q_{1n}^{(a)}(r_2) - Q_{1n}^{(a)}(r) \right\}^2 \\ & \leq CE\left\{ R_{an}(r) - R_{an}(r_1) \right\}^2 E\left\{ R_{an}(r_2) - R_{an}(r) \right\}^2 \ , \end{aligned} \tag{33}$$

From (32) we obtain easily, for $0 < \rho_1 < \rho_2 \leq 1$,

$$\begin{aligned} E\left\{ R_{an}(\rho_2) - R_{an}(\rho_1) \right\}^2 &= \frac{1}{n} \sum_{k=1}^{[n\rho_1]-1} \left[(\rho_2 - \frac{k}{n})^{d_a^- - 1} - (\rho_1 - \frac{k}{n})^{d_a^- - 1} \right]^2 \\ &\quad + \frac{1}{n} \sum_{k=1+[n\rho_1]}^{[n\rho_2]-1} (\rho_2 - \frac{k}{n})^{2d_a^- - 2} \\ &= M_1(\rho_1, \rho_2) + M_2(\rho_1, \rho_2) \ . \end{aligned}$$

Now if $d_a^- > 2$, $M_1(\rho_1, \rho_2) \leq (d_a^- - 1)^2 (\rho_2 - \rho_1)^2$ by the mean value theorem and easy manipulations; if $1 < d_a^- \leq 2$, $M_1(\rho_1, \rho_2) \leq (\rho_2 - \rho_1)^{2d_a^- - 2}$ from the inequality $|u + v|^\theta \leq |u|^\theta + |v|^\theta$, $1 \leq \theta \leq 2$; if $d_a^- = 1$, $M_1 = 0$. Finally, if $\frac{1}{2} < d_a^- < 1$, we note that for $\rho_2 > \rho_1$, $s \in (-\infty, \rho_1)$

$$f(s) = (\rho_1 - s)^{d_a^- - 1} - (\rho_2 - s)^{d_a^- - 1}$$

is non-decreasing, having derivative

$$(1 - d_a^-) \left\{ (\rho_1 - s)^{d_a^- - 2} - (\rho_2 - s)^{d_a^- - 2} \right\} > 0 \ .$$

Therefore

$$M_1(\rho_1, \rho_2) \ \leq \ \int_0^{\rho_1} f(s)^2 ds$$

$$
\begin{aligned}
&= (\rho_2-\rho_1)^{2d_a^- -2}\int_0^{\rho_1}\left[(1+\frac{s}{\rho_2-\rho_1})^{d_a^- -1}-(\frac{s}{\rho_2-\rho_1})^{d_a^- -1}\right]^2 ds \\
&\le (\rho_2-\rho_1)^{2d_a^- -1}\int_0^{\infty}\left[(1+v)^{d_a^- -1}-v^{d_a^- -1}\right]^2 dv \\
&\le C(\rho_2-\rho_1)^{2d_a^- -1} ,
\end{aligned}
$$

because for $\frac{1}{2} < d_a^- < \frac{3}{2}$

$$
\int_0^{\infty}\left[(1+v)^{d_a^- -1}-v^{d_a^- -1}\right]^2 dv < \infty ,
$$

as discussed in Samorodnitsky and Taqqu (1994, p.321). It follows that $M_1(\rho_1,\rho_2) \le C(\rho_2-\rho_1)^{\gamma}$ for $\gamma > 0$. Let us now consider $M_2(\rho_1,\rho_2)$; we assume without loss of generality $[n\rho_2] > [n\rho_1]$. If $d_a^- \ge 1$, we have

$$
\begin{aligned}
M_2(\rho_1,\rho_2) &\le \rho_2 - \frac{[n\rho_1]+1}{n} \\
&\le (\rho_2-\rho_1)+\frac{1}{n} . \qquad (34)
\end{aligned}
$$

If instead $\frac{1}{2} < d_a^- < 1$, we have that $(r-s)^{2d_a^- -2}$ is non-decreasing in s, for $s < r$. Hence

$$
\begin{aligned}
M_2(\rho_1,\rho_2) &\le \int_{(1+[n\rho_1])/n}^{[n\rho_2]/n}(\rho_2-s)^{2d_{ia}^- -2}ds \\
&\le \int_{\rho_1}^{\rho_2}(\rho_2-s)^{2d_a^- -2}ds \\
&= \frac{1}{2d_a-1}(\rho_2-\rho_1)^{2d_a^- -1} , \qquad (35)
\end{aligned}
$$

which, together with (34), gives

$$
M_2(\rho_1,\rho_2) \le C(\rho_2-\rho_1)^{\xi}+\frac{1}{n} , \text{ some } \xi > 0 ,
$$

for all $d_a^- > \frac{1}{2}$. Now we identify $\rho_2 = r$, $\rho_1 = r_1$ to bound the first element on the right hand side of (33), and $\rho_2 = r_2$, $\rho_1 = r$ to bound the second element on the right hand side of (33), so we consider together $M_1(r_1,r), M_2(r_1,r)$ and $M_1(r,r_2), M_2(r,r_2)$. For $r_2 - r_1 < \frac{1}{n}$ implies either $M_2(r_1,r) = 0$ or $M_2(r,r_2) = 0$; we assume $M_2(r_1,r) = 0$. Hence for $r_2 - r_1 > \frac{1}{n}$ we deduce from (34) and (35) that

$$
E\left\{R_{an}(r)-R_{an}(r_1)\right\}^2 E\left\{R_{an}(r_2)-R_{an}(r)\right\}^2
$$

$$= CM_1(r_1, r)\left[M_1(r, r_2) + M_2(r, r_2)\right] \leq C(r_2 - r_1)^{\gamma} \; ,$$

some $\gamma > 0$. Otherwise, when $r_2 - r_1 > \frac{1}{n}$, we have

$$\begin{aligned}
& E\left\{R_{an}(r) - R_{an}(r_1)\right\}^2 E\left\{R_{an}(r_2) - R_{an}(r)\right\}^2 \\
\leq\; & C\max((r - r_1)^{\gamma}, (r_2 - r)^{\gamma}) + \frac{1}{n^2} \\
\leq\; & C\max((r - r_1)^{\gamma}, (r_2 - r)^{\gamma}) + (r_2 - r_1)^2 \\
\leq\; & C(r_2 - r_1)^{\xi} \; , \text{ some } \xi > 0 \; .
\end{aligned}$$

The same bounds hold for $r_1 = 0$, and the result then follows from (33). □

Lemma 5 Under Assumptions A and B, as $n \to \infty$

$$\sup_{0 \leq r \leq 1} \|Q_{2n}(r)\| = o_p(1) \; .$$

Proof By Abel summation by parts

$$\begin{aligned}
Q_{2n}(r) &= D(n; d_z) \sum_{k=1}^{[nr]-1} \Psi_{[nr]-k}\left[(S_k - S_{k-1}) - V_k - V_{k-1}\right] \\
&= D(n; d_z) \sum_{k=1}^{[nr]-2} \left[\Psi_{[nr]-k} - \Psi_{[nr]-k-1}\right]\left[S_k - V_k\right] \\
&\quad + D(n; d_z)\Psi_1\left[S_{[nr]-1} - V_{[nr]-1}\right] \; .
\end{aligned}$$

Define $\widehat{\Psi}_j = \Psi_j - \Psi_{j-1}$, for $j \geq 1$; from (8) we have

$$\begin{aligned}
& |\ell_{aa}(j) j^{d_a - 1} - \ell_{aa}(j-1)(j-1)^{d_a - 1}| \\
=\; & |\left\{\ell_{aa}(j) j^{d_a-1} - \ell_{aa}(j)(j-1)^{d_a-1}\right\} + \left\{\ell_{aa}(j)(j-1)^{d_a-1} - \ell_{aa}(j-1)(j-1)^{d_a-1}\right\}| \\
\leq\; & C j^{d_a - 2}\left\{|\ell_{aa}(j)| + \ell'_{aa}(j)\right\} \; ,
\end{aligned}$$

by the mean value theorem and (8). Hence we obtain

$$\|D(n; d_z)\widehat{\Psi}_j\| \leq C n^{1/2 - d_a} \frac{|\ell_{aa}(j)| + \ell'_{aa}(j)}{|\ell_{aa}(n)|} j^{d_a - 2} \; .$$

Thus $\|Q_{2n}(r)\|$ is bounded by

$$C\sum_{k=1}^{[nr]-2}\left[\sum_{a=1}^{p} n^{1/2-d_a}\frac{|\ell_{aa}([nr]-k)|+\ell'_{aa}([nr]-k)}{|\ell_{aa}(n)|}([nr]-k)^{d_a-2}\|S_k-V_k\|\right]$$
$$+C\|\Psi_1\|\sum_{a=1}^{p} n^{1/2-d_a}\|S_{[nr]-1}-V_{[nr]-1}\| ,$$

and hence by Lemma 2, for some $s>\max(2,\frac{2}{2d_*-1})$,

$$\begin{aligned}\sup_{0\le r\le 1}\|Q_{2n}(r)\| &= o_p(n^{1/s}\sum_{a=1}^{p}\sum_{k=1}^{n-2}\frac{|\ell_{aa}(n-k)|+\ell'_{aa}(n-k)}{|\ell_{aa}(n)|}(n-k)^{d_a-2}n^{1/2-d_a}+1)\\ &= o_p(1+n^{1/s}\sum_{a=1}^{p}\sum_{k=1}^{n-2}(n-k)^{d_a-2+\delta}n^{1/2-d_a+\delta})\\ &= o_p(1+n^{1/s}n^{2\delta-\min(1/2,d_*-1/2)})=o_p(1) ,\end{aligned}$$

because $\ell'_{aa}(t),\ell_{aa}(t)<Ct^{\delta}$, $\delta>0$, and on picking δ such that $0<\delta<\frac{1}{2}\left\{\min(\frac{1}{2},d_*-\frac{1}{2})-\frac{1}{s}\right\}$.

□

Lemma 6 Under Assumptions A and B, as $n\to\infty$

$$\sup_{0\le r\le 1}\|Q_{3n}(r)\|=o_p(1) .$$

Proof We have

$$\begin{aligned}\|Q_{3n}(r)\| &\le \max_{k\le n}\|A(1)w_k\|\sup_{0\le r\le 1}\sum_{k=1}^{[nr]-1}\|D(n;d_z)\Psi_{[nr]-k}-\mathcal{G}(r,k,n)n^{-1/2})\|\\ &= \max_{k\le n}\|A(1)w_k\|\sup_{0\le r\le 1}\sum_{k=1}^{[nr]-1}\|G\|n^{-1/2}\|\mathcal{L}(r,k,n)+\mathcal{R}(r,k,n)\| .\end{aligned}$$

where $\mathcal{L}(.,.,.)$ and $\mathcal{R}(.,.,.)$ are diagonal matrices with a-th diagonal elements

$$([nr]-k)^{d_a-1}\frac{\ell_{aa}([nr]-k)-\ell_{aa}(nr-k)}{n^{d_a-1}\ell_{aa}(n)} ,$$
$$\frac{\ell_{aa}(nr-k)}{\ell_{aa}(n)}\left(\frac{([nr]-k)^{d_a-1}-(nr-k)^{d_a-1}}{n^{d_a-1}}\right) ,$$

respectively, for $a = 1, ..., p$. Now

$$\|\mathcal{L}(r,k,n)\| = O\left(\sum_{a=1}^{p} |\frac{\ell'_{aa}([nr]-k)}{\ell_{aa}(n)}| \frac{([nr]-k)^{d_a-2}}{n^{d_a-1}}\right) , \tag{36}$$

$$\|\mathcal{R}(r,k,n)\| = O\left(\sum_{a=1}^{p} |\frac{\ell_{aa}(nr-k)}{\ell_{aa}(n)}| \frac{([nr]-k)^{d_a-2}}{n^{d_a-1}}\right) , \tag{37}$$

where (36) follows from (8), and (37) from the mean value theorem. By (30), both (36) and (37) are bounded by $C\sum_{a=1}^{p} n^{d_a-1+\delta}([nr]-k)^{d_a-2}$, any $\delta > 0$, and these bounds are uniform over r; hence we obtain

$$\begin{aligned}\sup_{0\le r\le 1} \|Q_{3n}(r)\| &\le C \max_{k\le n} \|w_k\| n^{\delta} \sum_{a=1}^{p} n^{-\min(1/2, d_a-1/2)} \\ &\le Cn^{-\xi} \max_{k\le n} \|w_k\|, \text{ some } \xi > 0 \ ,\end{aligned}$$

on picking $\delta < \min(\frac{1}{2} - q, d_a - \frac{1}{2} - q)$. Now denote by w_{ak} the a-th component of the vector process w_k, where $w_{ak} \equiv N(0, \sigma_a^2)$, for σ_a^2 the a-th element on the main diagonal of Σ; for any $\lambda > 0$,

$$\begin{aligned}P\left\{n^{-\xi} \max_{k\le n} \|w_k\| > \lambda\right\} &= O(n \sum_{a=1}^{p} P\left\{|w_{ak}| > \lambda n^{\xi}\right\}) \\ &= o(n \sum_{a=1}^{p} e^{-(\lambda n^{\xi})/2\sigma_a^2}) = o(1) \ ,\end{aligned}$$

where the second step follows from the inequality $\int_{\theta}^{\infty} e^{-u^2/2} du < \theta^{-1} e^{-\theta^2/2}$, which holds for $\theta > 0$. □

Lemma 7 Under Assumptions A and B, as $n \to \infty$

$$\sup_{0\le r\le 1} \|Q_{in}(r)\| = o_p(1) \ , \ i = 4, 5 \ .$$

Proof For any $\lambda > 0$

$$P(\sup_{0\le r\le 1} \|Q_{4n}(r)\| > \lambda) \le CP(\max_{k\le n} \|\widehat{\eta}_k\| > \lambda n^{d_*-1/2})$$

$$
\begin{aligned}
&\leq CnP(\|\widehat{\eta}_1\| \geq \lambda n^{d_* -1/2}) \\
&= O(n \times n^{q(1/2-d_*)}) = o(1) \ ;
\end{aligned}
$$

likewise

$$
\begin{aligned}
\sup_{0\leq r\leq 1} \|Q_{5n}(r)\| &= \max\left\{\|D(n;d_z)\widehat{\eta}_1\|, \|D(n;d_z)(\Psi_1\widehat{\eta}_1 + \widehat{\eta}_2)\|\right\} \\
&= o_p(1) \ ,
\end{aligned}
$$

because $E\|\Psi_1\widehat{\eta}_1 + \widehat{\eta}_2\| = E\|\Psi_1\eta_1 + \eta_2\| \leq \{\|\Psi_1\| + p\} \|E\|\eta_1\|$ and under Assumption A

$$
E\|\eta_1\| < \sum_{j=-\infty}^{\infty} \|A_j\| E\|\varepsilon_{1-j}\| < \infty \ .
$$

□

Lemma 8 Under Assumptions A and B, as $n \to \infty$

$$
\sup_{0\leq r\leq 1} \|Q_{6n}(r)\| = o_p(1) \ .
$$

Proof In view of Assumption A1, for $d_* \geq 1$ we write

$$
\begin{aligned}
&\sup \| \sum_{k=1}^{[nr]-1} \mathcal{H}(r,k,n) n^{-1/2} \left[S_k - S_{k-1}\right] \| \\
&\leq C \sum_{a=1}^{p} \left\{ \sum_{k=1}^{n} \ell''_{ab}(k) n^{-1/2+\theta} k^{-1} \right\} \frac{n^{-\theta}}{|\ell_{aa}(n)|} \max_{k\leq n} \|\eta_k\| \ ,
\end{aligned}
$$

where we pick $\frac{1}{q} < \theta < \frac{1}{2}$; now as $n \to \infty$, for $a, b = 1, ..., p$

$$
\left\{ \sum_{k=1}^{n} \ell''_{ab}(k) n^{-1/2+\theta} k^{-1} \right\} = O(1) \ , \ \frac{n^{-\theta}}{|\ell_{aa}(n)|} \max_{k\leq n} \|\eta_k\| = o_p(1) \ ,
$$

in view of previous calculations. For $d_* < 1$, assume w.l.o.g. that $d_a < 1$ for $a = 1, ..., p^*$, $p^* < p$ and $d_a \geq 1$ otherwise; $\sup_{0\leq r\leq 1} \|Q_{6n}(r)\|$ is then bounded by

$$
\begin{aligned}
&C \sum_{a=1}^{p^*} n^{1-d_a} \left\{ \sum_{k=1}^{n} \ell''_{ab}(k) k^{d_a - 2} \right\} n^{-1/2} \max_{k\leq n} \|\eta_k\| + C \sum_{a=p^*+1}^{p} \frac{n^{-\theta}}{|\ell_{aa}(n)|} \max_{k\leq n} \|\eta_k\| \\
&\leq Cn^{1/2-d_*} \max_{k\leq n} \|\eta_k\| = o_p(1) \text{ as } n \to \infty \ .
\end{aligned}
$$

□

REFERENCES

Abramowitz, M. and Stegun, I. (1970) *Handbook of Mathematical Functions,* Dover, New York

Akonom, J. and Gourieroux, C. (1987) "A Functional Central Limit Theorem for Fractional Processes", *Discussion Paper #8801*, CEPREMAP, Paris

Billingsley, P. (1968) *Convergence of Probability Measures,* Wiley, New York

Bingham, N.H., Goldie, C.M. and Teugels, J.L. (1989) *Regular Variation,* 2nd edition, Cambridge University Press, Cambridge

Burkholder, D.L. (1973) "Distribution Function Inequalities for Martingales", *Ann.Prob.* 1, 19-42

Chan, N.H. and Terrin, N. (1995) "Inference for Unstable Long-Memory Processes with Applications to Fractional Unit Root Autoregressions", *Ann.Statist.,* 23, 1662-1683

Csőrgo, S. and Mielniczuk, J. (1995) "Distant Long-Range Dependent Sums and Regression Estimation", *Stoch.Proc.Applic.,* 59, 143-155

Davydov, Y. (1970) "The Invariance Principle for Stationary Processes", *Theory Prob.Applic.,* 15, 487-498

Einmahl, U. (1989) "Extensions of Results of Komlos, Major and Tusnady to the Multivariate Case", *J.Mult.Analysis* 28, 20-68

Giraitis, L. and Koul, H. (1997) "Estimation of the Dependence Parameter in Linear Regression with Long Range Dependent Errors", *Stoch.Proc.Applic.,* 71, 207-224

Gorodetskii, V.V. (1977) "On Convergence to Semi-Stable Gaussian Processes", *Theory Prob.Applic.,* 22, 498-508

Herrndorf, N. (1984) "A Functional Central Limit Theorem for Weakly Dependent Sequences of Random Variables", *Ann.Prob.,* 12, 141-153

Heyde, C.C. and Yang, Y. (1997) "On Defining Long-Range Dependence", *J.Appl.Prob.,* 34, 939-944

Kokoszka, P. and Mikosch, T. (1997) "The Integrated Periodogram for Long-

Memory Processes with Finite or Infinite Variance", *Stoch.Proc.Applic.,* 66, 55-78

Kokoszka, P. and Taqqu, M.S. (1995) "Fractional ARIMA with Stable Innovations", *Stoch.Proc.Applic.,* 60, 19-47

Kokoszka, P. and Taqqu, M.S. (1996) "Parameter Estimation for Infinite Variance Fractional ARIMA", *Ann.Statist.,* 24, 1880-1913

Marinucci, D. and Robinson, P.M. (1998) "Alternative Forms of Fractional Brownian Motion", revision with *J.Statist.Plann.Inf.*.

Pham, T.D. and Tran, L.T. (1985) "Some Mixing Properties of Time Series Models", *Stoch.Proc.Applic.,* 19, 297-303

Phillips, P.C.B. and Solo, V. (1992) "Asymptotics for Linear Processes", *Ann.Statist.,* 20, 971-1001

Robinson, P.M. (1995) "Log-Periodogram Regression of Time Series with Long Range Dependence", *Ann.Statist.,* 23, 1048-1072

Robinson, P.M. and Marinucci, D. (1998) "Semiparametric Frequency Domain Analysis of Fractional Cointegration", STICERD Discussion Paper #348, London School of Economics.

Samorodnitsky, G. and Taqqu, M.S. (1994) *Stable Non-Gaussian Random Processes*, Chapman and Hall, New York.

Silveira, G. (1991) *Contributions to Strong Approximations in Time Series with Applications in Nonparametric Statistics and Functional Central Limit Theorems*, Ph.D. thesis, University of London

Taqqu, M.S. (1975) "Weak Convergence to Fractional Brownian Motion and to the Rosenblatt Process", *Z. Wahrscheinlichkeitstheorie Verw.Geb.,* 50, 53-83