CONSISTENT TESTING FOR STOCHASTIC DOMINANCE: A SUBSAMPLING APPROACH

by

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Abstract

We study a very general setting, and propose a procedure for estimating the critical values of the extended Kolmogorov-Smirnov tests of First and Second Order Stochastic Dominance due to McFadden (1989) in the general k-prospect case. We allow for the observations to be generally serially dependent and, for the first time, we can accommodate general dependence amongst the *prospects* which are to be ranked. Also, the prospects may be the residuals from certain conditional models, opening the way for *conditional* ranking. We also propose a test of Prospect Stochastic Dominance. Our method is based on subsampling and we show that the resulting data tests are consistent.

Keywords: Prospect theory; stochastic dominance; stochastic equicontinuity; subsampling.
JEL Nos.: C13, C14, C15.

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1 Introduction

There is considerable interest in uniform weak ordering of investment strategies, welfare outcomes (income distributions, poverty levels), and in program evaluation exercises. Partial *strong* orders are commonly used on the basis of specific utility (loss) functions. This is the predominant form of evaluation and is done when one employs indices of inequality or poverty in welfare, mean-variance (return-volatility) analysis in finance, or performance indices in program evaluation. By their very nature, strong orders do not command consensus. The most popular *uniform* order relations are the Stochastic Dominance (SD) relations of various orders, based on the expected utility paradigm and its mathematical regularity conditions. These relations are defined over relatively large classes of utility functions and represent "majority" preferences.

In this paper we propose an alternative procedure for estimating the critical values of a suitably extended Kolmogorov-Smirnov test due to McFadden (1989), and Klecan, McFadden, and McFadden (1991) for first and second order stochastic dominance in the general k-prospect case. Our method is based on subsampling. We prove that the resulting test is consistent. Our sampling scheme is quite general: for the first time in this literature, we allow for general dependence amongst the prospects, and for the observations to be autocorrelated over time. This is especially necessary in substantive empirical settings where income distributions, say, are compared before and after taxes (or some other policy decision), or returns on different funds are compared in the same or interconnected markets.

We also allow the prospects themselves to be residuals from some estimated model. This latter generality can be important if one wishes to control for certain characteristics before comparing outcomes. Our method offers tests of Conditional Stochastic Dominance (CSD) when the residuals are ranked. Finally, we propose a 'new' test of Prospect Stochastic Dominance and propose consistent critical values using subsampling.

Finite sample performance of our method is investigated on simulated data and found to be quite good provided the sample sizes are appropriately large for distributional rankings. An empirical application to Dow Jones and S&P daily returns demonstrates the potential of these tests and concludes the paper. The following brief definitions will be useful:

Let X_1 and X_2 be two variables (incomes, returns/prospects) at either two different points in time, or for different regions or countries, or with or without a program (treatment). Let X_{ki}, $i = 1, \ldots, N$; $k = 1, \ldots, K$ denote the not necessarily i.i.d. observations. Let $\mathcal{U}_1$ denote the class of all von Neumann-Morgenstern type utility functions, u, such that $u' \geq 0$, (increasing). Also, let $\mathcal{U}_2$ denote the class of all utility functions in $\mathcal{U}_1$ for which $u'' \leq 0$ (strict concavity), and $\mathcal{U}_3$ denote a subset of $\mathcal{U}_2$ for which $u''' \geq 0$. Let $X_{(1p)}$ and $X_{(2p)}$ denote the p-th quantiles, and $F_1(x)$ and $F_2(x)$ denote the cumulative distribution functions, respectively.

Definition 1 *X_1 First Order Stochastic Dominates X_2, denoted $X_1 \succeq_f X_2$, if and only if:*

(1) $E[u(X_1)] \geq E[u(X_2)]$ for all $u \in \mathcal{U}_1$, with strict inequality for some u; Or

(2) $F_1(x) \leq F_2(x)$ for all $x \in \mathcal{X}$, the support of X_k, with strict inequality for some x; Or

(3) $X_{(1p)} \geq X_{(2p)}$ for all $0 \leq p \leq 1$, with strict inequality for some p.

Definition 2 *X_1 Second Order Stochastic Dominates X_2, denoted $X_1 \succeq_s X_2$, if and only if one of the following equivalent conditions holds:*

(1) $E[u(X_1)] \geq E[u(X_2)]$ for all $u \in \mathcal{U}_2$, with strict inequality for some u; Or:

(2) $\int_{-\infty}^{x} F_1(t)dt \leq \int_{-\infty}^{x} F_2(t)dt$ for all $x \in \mathcal{X}$, with strict inequality for some x;Or:

(3) $\Phi_1(p) = \int_0^p X_{(1t)}dt \geq \Phi_2(p) = \int_0^p X_{(2t)}dt$ for all $0 \leq p \leq 1$, with strict inequality for some value(s) p.

Weak orders of SD obtain by eliminating the requirement of strict inequality at some point. When these conditions are not met, as when either Lorenz or Generalized Lorenz Curves of two distributions cross, unambiguous First and Second order SD is not possible. Any partial ordering by specific *indices* that correspond to the utility functions in $\mathcal{U}_1$ and $\mathcal{U}_2$ classes, will not enjoy general consensus. Whitmore introduced the concept of third order stochastic dominance (TSD) in finance, see (e.g.) Whitmore and Findley (1978). Shorrocks and Foster (1987) showed that the addition of a "transfer sensitivity" requirement leads to TSD ranking of income distributions. This requirement is stronger than the Pigou-Dalton principle of transfers since it makes regressive transfers less desirable at lower income levels. Higher order SD relations correspond to increasingly smaller subsets of $\mathcal{U}_2$. Davidson and Duclos (2000) offer a very useful characterization of these relations and their tests.

In this paper we shall also consider the concept of prospect stochastic dominance. Kahneman and Tversky (1979) mounted a critique of expected utility theory and introduced an alternative theory, called prospect theory. They argued that their model provided a better rationalization of the many observations of actual individual behavior taken in laboratory experiments. Specifically, they proposed an alternative model of decision making under uncertainty in which: (a) gains and losses are treated differently; (b) individuals act as if they had applied monotonic transformations to the underlying probabilities before making payoff comparisons.[1] Taking only part (a), individuals would rank prospects according to the expected value of S-shaped utility functions $u \in \mathcal{U}_P \subseteq \mathcal{U}_1$ for which $u''(x) \leq 0$ for all $x > 0$ but $u''(x) \geq 0$ for all $x < 0$. These properties represent risk seeking for losses but risk aversion for gains. This leads naturally to the concept of Prospect Stochastic Dominance.

Definition 3 *X_1 Prospect Stochastic Dominates X_2, denoted $X_1 \succeq_{PSD} X_2$, if and only if one of the following equivalent conditions holds:*

(1) $E[u(X_1)] \geq E[u(X_2)]$ for all $u \in \mathcal{U}_P$, with strict inequality for some u; Or:

(2) $\int_y^x F_1(t)dt \leq \int_y^x F_2(t)dt$ for all pairs (x, y) with $x > 0$ and $y < 0$ with strict inequality for some (x, y); Or:

(3) $\int_{p_1}^{p_2} X_{(1t)}dt \geq \int_{p_1}^{p_2} X_{(2t)}dt$ for all $0 \leq p_1 \leq F_1(0) \leq F_2(0) \leq p_2 \leq 1$, with strict inequality for some value(s) p.

Now consider the second component of prospect theory, (b), the transformation of probabilities. One question is whether stochastic dominance [of first, second, or higher order] is preserved under transformation, or rather what is the set of transformations under which an ordering is preserved. Levy and Wiener (1998) show that the PSD property is preserved under the class of monotonic transformations that are concave for gains and convex for losses. Therefore, if one can verify that a

[1] In Tversky and Kahneman (1992) this idea is refined to make the cumulative distribution function of payoffs the subject of the transformation. Thus, individuals would compare the distributions $F_k^* = T(F_k)$, where T is a monotonic decreasing transformation that can be interpreted as a subjective revision of probabilities that varies across investors.

prospect is dominated according to (2), this implies that it will be dominated even after transforming the probabilities for a range of such transformations.

Econometric tests for the existence of SD orders involve composite hypotheses on inequality restrictions. These restrictions may be equivalently formulated in terms of distribution functions, their quantiles, or other conditional moments. Different test procedures may also differ in terms of their accommodation of the inequality nature (information) of the SD hypotheses. A recent survey is given in Maasoumi (2001).

McFadden (1989) proposed a generalization of the Kolmogorov-Smirnov test of First and Second order SD among k prospects (distributions) based on i.i.d. observations *and independent* prospects. Klecan, Mcfadden, and Mcfadden (1991) extended these tests allowing for dependence in observations, and replacing independence with a general exchangeability amongst the competing prospects. Since the asymptotic null distribution of these tests depends on the unknown distributions, they proposed a Monte Carlo permutation procedure for the computation of critical values that relies on the exchangeability hypothesis. Maasoumi and Heshmati (2000) and Maasoumi et al. (1997) proposed simple bootstrap versions of the same tests which they employed in empirical applications. Barrett and Donald (1999) propose an alternative simulation method based on an idea of Hansen (1996b) for deriving critical values in the case where the prospects are mutually independent, and the data are i.i.d.

Alternative approaches for testing SD are discussed in Anderson (1996), Davidson and Duclos (2000), Kaur et al. (1994), Dardanoni and Forcina (2000), Bishop et al. (1998), and Xu, Fisher, and Wilson (1995), Crawford (1999), and Abadie (2001), to name but a few recent contributions. The Xu et al. (1995) paper is an example of the use of $\overline{\chi}^2$ distribution for testing the joint inequality amongst the quantiles (conditions (2) in our definitions). The Davidson and Duclos (2000) is the most general account of the tests for any order SD, *based on conditional moments* of distributions and, as with most of these alternative approaches, requires control of its size by Studentized maximum modulus or similar techniques. Maasoumi (2001) contains an extensive discussion of these alternatives. Tse and Zhang (2000) provide some Monte Carlo evidence on the power of some of these *alternative* tests.

2 The Test Statistics

We shall suppose that there are K prospects $X_1, \ldots, X_k$ and let $\mathcal{A} = \{X_k : k = 1, \ldots, K\}$. Let $\{X_{ki} : i = 1, \ldots, N\}$ be realizations of X_k for $k = 1, \ldots, K$. To subsume the empirically important case of "conditional" dominance, suppose that $\{X_{ki} : i = 1, \ldots, N\}$ are unobserved errors in the linear regression model:

$$Y_{ki} = Z_{ki}'\theta_{k0} + X_{ki},$$

for $i = 1, \ldots, N$ and $k = 1, \ldots, K$, where, $Y_{ki} \in \mathbb{R}$, $Z_{ki} \in \mathbb{R}^L$ and $\theta_{k0} \in \Theta_k \subset \mathbb{R}^L$. We shall suppose that $E(X_{ki}|Z_{ki}) = 0$ a.s. as well as other conditions on the random variables X_k, Y_k. We allow for serial dependence of the realizations and for mutual correlation across prospects. Let $X_{ki}(\theta) = Y_{ki} - Z_{ki}'\theta$, $X_{ki} = X_{ki}(\theta_{k0})$, and $\widehat{X}_{ki} = X_{ki}(\widehat{\theta}_k)$, where $\widehat{\theta}_k$ is some sensible estimator of θ_{k0} whose properties we detail below, i.e., the prospects can be estimated from the data. Since we have a linear model, there are many possible ways of obtaining consistent estimates of the unknown parameters. The motivation for considering estimated prospects is that when data is limited one may want to use a model to adjust for systematic differences. Common practice is to group the data into subsets, say of families with different sizes, or by educational attainment, or subgroups of funds by investment goals, and then make comparisons across homogenous populations. When data are limited this can be difficult. In addition, the preliminary regressions may identify "causes" of different outcomes which may be of substantive interest and useful to control for.[2]

For $k = 1, \ldots, K$, define

$$\begin{aligned} F_k(x, \theta) &= P\left(X_{ki}(\theta) \leq x\right) \text{ and} \\ F_{kN}(x, \theta) &= \frac{1}{N}\sum_{i=1}^{N} 1\left(X_{ki}(\theta) \leq x\right). \end{aligned}$$

We denote $F_k(x) = F_k(x, \theta_{k0})$ and $F_{kN}(x) = F_{kN}(x, \theta_{k0})$, and let $F(x)$ be the joint c.d.f. of

[2] Another way of controlling for systematic differences is to test a hypothesis about the conditional c.d.f.'s of Y_k given Z_k. Similar results can be established in this case.

$(X_1, \ldots, X_k)'$. Now define the following functionals of the joint distribution

$$
\begin{aligned}
d^* &= \min_{k \neq l} \sup_{x \in \mathcal{X}} \left[F_k(x) - F_l(x)\right] &(1)\\
s^* &= \min_{k \neq l} \sup_{x \in \mathcal{X}} \int_{-\infty}^{x} \left[F_k(t) - F_l(t)\right] dt &(2)\\
p^* &= \min_{k \neq l} \sup_{x, -y \in \mathcal{X}_+} \int_{y}^{x} \left[F_k(t) - F_l(t)\right] dt, &(3)
\end{aligned}
$$

where $\mathcal{X}$ denotes the support of X_{ki} and $\mathcal{X}_+ = \{x \in \mathcal{X}, x > 0\}$. Without loss of generality we assume that $\mathcal{X}$ is a bounded set, as do Klecan et al. (1991). The hypotheses of interest can now be stated as:

$$H_0^d : d^* \leq 0 \ vs. \ H_1^d : d^* > 0 \tag{4}$$

$$H_0^s : s^* \leq 0 \ vs. \ H_1^s : s^* > 0 \tag{5}$$

$$H_0^p : p^* \leq 0 \ vs. \ H_1^p : p^* > 0. \tag{6}$$

The null hypothesis H_0^d implies that the prospects in $\mathcal{A}$ are not first-degree stochastically maximal, i.e., there exists at least one prospect in $\mathcal{A}$ which first-degree dominates the others. Likewise for the second order and prospect stochastic dominance test.

The test statistics we consider are based on the empirical analogues of (1)-(3). They are defined to be:

$$
\begin{aligned}
D_N &= \min_{k \neq l} \sup_{x \in \mathcal{X}} \sqrt{N} \left[F_{kN}(x, \widehat{\theta}_k) - F_{lN}(x, \widehat{\theta}_l)\right]\\
S_N &= \min_{k \neq l} \sup_{x \in \mathcal{X}} \sqrt{N} \int_{-\infty}^{x} \left[F_{kN}(t, \widehat{\theta}_k) - F_{lN}(t, \widehat{\theta}_l)\right] dt\\
P_N &= \min_{k \neq l} \sup_{x, -y \in \mathcal{X}_+} \sqrt{N} \int_{y}^{x} \left[F_{kN}(t, \widehat{\theta}_k) - F_{lN}(t, \widehat{\theta}_l)\right] dt.
\end{aligned}
$$

These are precisely the Klecan et al. (1991) test statistics except that we have allowed the prospects to have been estimated from the data.

We next discuss the issue of how to compute the supremum in D_N, S_N and P_N, and the integrals in S_N and P_N. There have been a number of suggestions in the literature that exploit the step-function nature of $F_{kN}(t,\theta)$. The supremum in D_N can be (exactly) replaced by a maximum taken over all the distinct points in the combined sample. Regarding the computation of S_N, Klecan et al. (1991) propose a recursive algorithm for exact computation of S_N, see also Barratt and Donald (1999) for an extension to third order dominance statistics. Integrating by parts we have

$$\int_{-\infty}^{x} F_k(t)dt = E[\max\{0, x - X_k\}]$$

provided $E[|X_k|] < \infty$.[3] Motivated by this, Davidson and Duclos (1999) have proposed computing the empirical analogue

$$\frac{1}{N}\sum_{i=1}^{N}(x - X_{ki}(\theta))1\left(X_{ki}(\theta) \le x\right).$$

The computation of P_N can be based on the fact that

$$\int_{y}^{x} F_{kN}(t,\theta)dt = \int_{-\infty}^{x} F_{kN}(t,\theta)dt - \int_{-\infty}^{y} F_{kN}(t,\theta)dt$$

for all $x, -y > 0$.

To reduce the computation time, it may be preferable to compute approximations to the suprema in D_N, S_N, P_N based on taking maxima over some smaller grid of points $\mathcal{X}_J = \{x_1, \ldots, x_J\}$, where $J < n$. This is especially true of P_N, which requires a grid on $\mathbb{R}_+ \times \mathbb{R}_-$. Thus, we might compute

$$P_N^J = \min_{k \neq l} \max_{0<x,0>y\in\mathcal{X}_J} \frac{1}{\sqrt{N}}\sum_{i=1}^{N}(x - X_{ki}(\widehat{\theta}))1\left(X_{ki}(\widehat{\theta}) \le x\right) - (y - X_{li}(\widehat{\theta}))1\left(X_{li}(\widehat{\theta}) \le y\right).$$

Theoretically, provided the set of evaluation points becomes dense in the joint support, the distribution theory is unaffected by using this approximation.

[3] A similar relation holds for higher order integrated c.d.f.s. In fact, one can define 'fractional dominance' relations based on the quantity

$$\frac{1}{\Gamma(\alpha+1)}E[|\max\{0, x - X\}|^{\alpha}],$$

which is defined for all $\alpha > 0$; here, Γ is the gamma function. See Ogryczak and Ruszcynski (1997).

3 Asymptotic Null Distributions

3.1 Regularity Conditions

We need the following assumptions to analyze the asymptotic behavior of D_N:

Assumption 1: (i) $\{(X_{ki}\ ,\ Z_{ki}) : i = 1, \ldots, n\}$ is a strictly stationary and α- mixing sequence with $\alpha(m) = O(m^{-A})$ for some $A > \max\{(q-1)(q+1),\ 1 + 2/\delta\}$ $\forall 1 \leq k \leq K$, where q is an even integer that satisfies $q > 3(L+1)/2$ and δ is a positive constant that also appears in Assumption 2(ii) below. (ii) $E\left\|Z_{ki}\right\|^2 < \infty$ for all $1 \leq k \leq K$, for all $i \geq 1$. (iii) The conditional distribution $H_k(\cdot|Z_{ki})$ of X_{ki} given Z_{ki} has bounded density with respect to Lebesgue measure a.s. $\forall 1 \leq k \leq K$, $\forall\ i \geq 1$.

Assumption 2: (i) The parameter estimator satisfies $\sqrt{N}(\widehat{\theta}_k - \theta_{k0}) = (1/\sqrt{N}) \sum_{i=1}^{N} \Gamma_{k0} \psi_k(X_{ki}, Z_{ki}, \theta_{k0}) + o_p(1)$, where Γ_{k0} is a non-stochastic matrix for all $1 \leq k \leq K$; (ii) The function $\psi_k(y, z, \theta) : \mathbb{R} \times \mathbb{R}^L \times \Theta \to \mathbb{R}^L$ is measurable and satisfies (a) $E\psi_k(Y_{ki}, Z_{ki}, \theta_{k0}) = 0$ and (b) $E\left\|\psi_k(Y_{ki}, Z_{ki}, \theta_{k0})\right\|^{2+\delta} < \infty$ for some $\delta > 0$ for all $1 \leq k \leq K$, for all $i \geq 1$.

Assumption 3: (i) The function $F_k(x, \theta)$ is differentiable in θ on a neighborhood Θ_0 of θ_0 for all $1 \leq k \leq K$; (ii) For all sequence of positive constants $\{\xi_N : N \geq 1\}$ such that $\xi_N \to 0$, $\sup_{x \in \mathcal{X}} \sup_{\theta : \|\theta - \theta_0\| \leq \xi_N} \left\|\partial F_k(x, \theta)/\partial\theta - \Delta_{k0}(x)\right\| \to 0$ for all $1 \leq k \leq K$, where $\Delta_{k0}(x) = \partial F_k(x, \theta_{k0})/\partial\theta$; (iii) $\sup_{x \in \mathcal{X}} \left\|\Delta_{k0}(x)\right\| < \infty$ for all $1 \leq k \leq K$.

For the tests S_N and P_N we need the following modification of Assumptions 1 and 3:

Assumption 1*: (i) $\{(X_{ki}\ ,\ Z_{ki}) : i = 1, \ldots, n\}$ is a strictly stationary and α- mixing sequence with $\alpha(m) = O(m^{-A})$ for some $A > \max\{rq/(r-q),\ 1 + 2/\delta\}$ $\forall 1 \leq k \leq K$ for some $r > q \geq 2$, where q satisfies $q > L$ and δ is a positive constant that also appears in Assumption 2(ii). (ii) $E\left\|Z_{ki}\right\|^r < \infty$ $\forall 1 \leq k \leq K$, $\forall\ i \geq 1$.

Assumption 3* : (i) Assumption 3(i) holds; (ii) For all $1 \leq k \leq K$ for all sequence of positive constants $\{\xi_N : N \geq 1\}$ such that $\xi_N \to 0$, $\sup_{x \in \mathcal{X}} \sup_{\theta : \|\theta - \theta_0\| \leq \xi_N} \left\|(\partial/\partial\theta) \int_{-\infty}^{x} F_k(t, \theta)dt - \Lambda_{k0}(x)\right\| \to 0$, where $\Lambda_{k0}(x) = (\partial/\partial\theta) \int F_k(y, \theta_{k0})dy$; (iii) $\sup_{x \in \mathcal{X}} \left\|\Lambda_{k0}(x)\right\| < \infty$ for all $1 \leq k \leq K$.

Assumption 3** : (i) Assumption 3(i) holds; (ii) For all $1 \leq k \leq K$ for all sequence of positive constants $\{\xi_N : N \geq 1\}$ such that $\xi_N \to 0$, $\sup_{x, -y \in \mathcal{X}_+} \sup_{\theta : \|\theta - \theta_0\| \leq \xi_N} \left\|(\partial/\partial\theta) \int_{y}^{x} F_k(t, \theta)dt - \Xi_{k0}(x, y)\right\| \to 0$, where $\Xi_{k0}(x, y) = (\partial/\partial\theta) \int_{y}^{x} F_k(t, \theta_{k0})dt$; (iii) $\sup_{x, -y \in \mathcal{X}_+} \left\|\Xi_{k0}(x, y)\right\| < \infty$ for all $1 \leq k \leq K$.

REMARKS.

1. The mixing condition in Assumption 1 is stronger than the condition used in Klecan et. al. (1991). This assumption, however, is needed to verify the stochastic equicontinuity of the empirical process (for a class of bounded functions) indexed by estimated parameters, see proof of Lemma 1(a). Assumption 1^* introduces a trade-off between mixing and moment conditions. This assumption is used to verify the stochastic equicontinuity result for the (possibly) unbounded functions that appear in the test S_N (or P_N), see proof of Lemma 1(b)(or (c)). Without the estimated parameters, weaker conditions on the dependence can be assumed: indeed there are some results available for the weak convergence of the empirical process of long memory time series [e.g., Giraitis, Leipus, and Surgailis (1996)].

2. Assumptions 3 and 3^* (or 3^{**}) differ in the amount of smoothness required. For second order (or prospect) stochastic dominance, less smoothness is required.

3. When there are no estimated parameters: we do not need any moment conditions for D_N and only a first moment for S_N, P_N, and the smoothness conditions on F are redundant.

3.2 The Null Distributions

Define the empirical processes in x, θ

$$
\begin{aligned}
\nu_{kN}^{d}(x,\theta) &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[1\left(X_{ki}(\theta)\leq x\right)-F_k(x,\theta)\right] \\
\nu_{kN}^{s}(x,\theta) &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\int_{-\infty}^{x}1\left(X_{ki}(\theta)\leq t\right)dt-\int_{-\infty}^{x}F_k(t,\theta)dt\right] \\
\nu_{kN}^{p}(x,y,\theta) &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\int_{y}^{x}1\left(X_{ki}(\theta)\leq t\right)dt-\int_{y}^{x}F_k(t,\theta)dt\right] \qquad (7)
\end{aligned}
$$

Let $(\ \widetilde{d}_{kl}(\cdot)\quad \nu_{k0}'\quad \nu_{l0}'\ )'$ be a mean zero Gaussian process with covariance functions given by

$$
C^{d}(x_1,x_2)=\lim_{N\to\infty}E\begin{pmatrix} v_{kN}^{d}(x_1,\theta_{k0})-v_{lN}^{d}(x_1,\theta_{l0}) \\ \sqrt{N}\psi_{kN}(\theta_{k0}) \\ \sqrt{N}\psi_{lN}(\theta_{l0}) \end{pmatrix}\begin{pmatrix} v_{kN}^{d}(x_2,\theta_{k0})-v_{lN}^{d}(x_2,\theta_{l0}) \\ \sqrt{N}\psi_{kN}(\theta_{k0}) \\ \sqrt{N}\psi_{lN}(\theta_{l0}) \end{pmatrix}'. \qquad (8)
$$

We analogously define $\begin{pmatrix} \widetilde{s}_{kl}(\cdot) & \nu_{k0}' & \nu_{l0}' \end{pmatrix}'$ and $\begin{pmatrix} \widetilde{p}_{kl}(\cdot,\cdot) & \nu_{k0}' & \nu_{l0}' \end{pmatrix}'$ to be mean zero Gaussian processes with covariance functions given by $C^s(x_1, x_2)$ and $C^p(x_1, y_1, x_2, y_2)$ respectively.

The limiting null distributions of our test statistics are given in the following theorem.

Theorem 1. (a) *Suppose Assumptions 1-3 hold. Then, under the null H_0^d, we have*

$$D_N \Rightarrow \begin{cases} \min_{k\neq l} \sup_{x\in\mathcal{B}_{kl}^d} \left[\widetilde{d}_{kl}(x) + \Delta_{k0}(x)'\Gamma_{k0}\nu_{k0} - \Delta_{l0}(x)'\Gamma_{l0}\nu_{l0}\right] & \text{if } d = 0 \\ -\infty & \text{if } d < 0, \end{cases}$$

where $\mathcal{B}_{kl}^d = \{x \in \mathcal{X} : F_k(x) = F_l(x)\}$.

(b) Suppose Assumptions 1, 2 and 3* hold. Then, under the null H_0^s, we have*

$$S_N \Rightarrow \begin{cases} \min_{k\neq l} \sup_{x\in\mathcal{B}_{kl}^s} \left[\widetilde{s}_{kl}(x) + \Lambda_{k0}(x)'\Gamma_{k0}\nu_{k0} - \Lambda_{l0}(x)'\Gamma_{l0}\nu_{l0}\right] & \text{if } s = 0 \\ -\infty & \text{if } s < 0, \end{cases}$$

where $\mathcal{B}_{kl}^s = \{x \in \mathcal{X} : \int_{-\infty}^x F_k(t)dt = \int_{-\infty}^x F_l(t)dt\}$.

(c) Suppose Assumptions 1, 2 and 3** hold. Then, under the null H_0^p, we have*

$$P_N \Rightarrow \begin{cases} \min_{k\neq l} \sup_{(x,y)\in\mathcal{B}_{kl}^p} \left[\widetilde{p}_{kl}(x,y) + \Xi_{k0}(x)'\Gamma_{k0}\nu_{k0} - \Xi_{l0}(x)'\Gamma_{l0}\nu_{l0}\right] & \text{if } p = 0 \\ -\infty & \text{if } p < 0, \end{cases}$$

where $\mathcal{B}_{kl}^p = \{(x, y) : x \in \mathcal{X}_+, -y \in \mathcal{X}_+$ *and* $\int_y^x F_k(t)dt = \int_y^x F_l(t)dt\}$.

The asymptotic null distributions of D_N, S_N and P_N depend on the "true" parameters $\{\theta_{k0} : k = 1, \ldots, K\}$ and distribution functions $\{F_k(\cdot) : k = 1, \ldots, K\}$. This implies that the asymptotic critical values for D_N, S_N, P_N can not be tabulated once and for all. However, a subsampling procedure can be used to approximate the null distributions.

4 Critical Values based on Subsample Bootstrap

In this section, we consider the use of subsampling to approximate the null distributions of our test statistics. As was pointed out by Klecan et. al. (1991), even when the data are i.i.d. the standard bootstrap resample does not work because one needs to impose the null hypothesis in that case, which is very difficult given the complicated system of inequalities that define it. The mutual dependence of the prospects and the time series dependence in the data also complicate the issue considerably.

Horowitz (2000) gives an overview of many of the problems of using bootstrap with dependent data. The subsampling method is very simple to define and yet provides consistent critical values in a very general setting. In contrast to the simulation approach of Klecan et. al. (1991), our procedure does not require the assumption of generalized exchangeability of the underlying random variables. Indeed, we require no additional assumptions beyond those that have already been made.

We now discuss the asymptotic validity of the subsampling procedure for the test D_N (The argument for the tests S_N and P_N is similar and hence is omitted). Let $W_i = \{(Y_{ki}, Z_{ki}) : k = 1, \ldots, K\}$ for $i = 1, \ldots, N$. With some abuse of notation, the test statistic D_N can be re-written as a function of the data $\{W_i : i = 1, \ldots, N\}$:

$$D_N = \sqrt{N} d_N(W_1, \ldots, W_N),$$

where

$$d_N(W_1, \ldots, W_N) = \min_{k \neq l} \sup_{x \in \mathcal{X}} \left[F_{kN}(x, \widehat{\theta}_k) - F_{lN}(x, \widehat{\theta}_l) \right]. \tag{9}$$

Let

$$G_N(w) = \Pr\left(\sqrt{N} d_N(W_1, \ldots, W_N) \leq w \right) \tag{10}$$

denote the distribution function of D_N. Let $d_{N,b,i}$ be equal to the statistic d_b evaluated at the subsample $\{W_i, \ldots, W_{i+b-1}\}$ of size b, i.e.,

$$d_{N,b,i} = d_b(W_i, W_{i+1}, \ldots, W_{i+b-1}) \text{ for } i = 1, \ldots, N - b + 1.$$

This means that we have to recompute $\widehat{\theta}_l(W_i, W_{i+1}, \ldots, W_{i+b-1})$ using just the subsample as well. We note that each subsample of size b (taken without replacement from the original data) is indeed a sample of size b from the true sampling distribution of the original data. Hence, it is clear that one can approximate the sampling distribution of D_N using the distribution of the values of $d_{N,b,i}$ computed over $N - b + 1$ different subsamples of size b. That is, we approximate the sampling distribution G_N of D_N by

$$\widehat{G}_{N,b}(w) = \frac{1}{N - b + 1} \sum_{i=1}^{N-b+1} 1\left(\sqrt{b} d_{N,b,i} \leq w \right).$$

Let $g_{N,b}(1-\alpha)$ denote the $(1-\alpha)$-th sample quantile of $\widehat{G}_{N,b}(\cdot)$, i.e.,

$$g_{N,b}(1-\alpha) = \inf\{w : \widehat{G}_{N,b}(w) \geq 1-\alpha\}.$$

We call it the *subsample critical value* of significance level α. Thus, we reject the null hypothesis at the significance level α if $D_N > g_{N,b}(1-\alpha)$. The computation of this critical value is not particularly onerous, although it depends on how big b is. The subsampling method has been proposed in Politis and Romano (1994) and is thoroughly reviewed in Politis, Romano, and Wolf (1999). It is well known to be a universal method that can 'solve' almost any problem. In particular, it works in heavy tailed distributions, in unit root cases, in non-standard asymptotics, etc.

We now justify the above subsampling procedure. Let $g(1-\alpha)$ denote the $(1-\alpha)$-th quantile of the asymptotic null distribution of D_N (given in Theorem 1(a)).

Theorem 2. *Suppose Assumptions 1-3 hold. Assume $b/N \to 0$ and $b \to 0$ as $N \to \infty$. Then, under the null hypothesis H_0^d, we have when $d = 0$ that*

$$(a)\ g_{N,b}(1-\alpha) \xrightarrow{p} g(1-\alpha)$$

$$(b)\ \Pr[D_N > g_{N,b}(1-\alpha)] \to \alpha$$

as $N \to \infty$.

Since $d = 0$ is the least favorable case, we have that

$$\sup_{d \in H_0^d} \Pr[D_N > g_{N,b}(1-\alpha)] \leq \alpha + o(1).$$

The following theorem establishes the consistency of our test:

Theorem 3. *Suppose Assumptions 1-3 hold. Assume $b/N \to 0$ and $b \to 0$ as $N \to \infty$. Then, under the alternative hypothesis H_1^d, we have*

$$\Pr\left[D_N > g_{N,b}(1-\alpha)\right] \to 1 \text{ as } N \to \infty.$$

Remark. Results analogous to Theorems 2 and 3 hold for the test S_N (P_N) under Assumptions 1^*, 2 and $3^*(3^{**})$. The proof is similar to those of the latter theorems.

In practice, the choice of b is important and rather difficult. It is rather akin to choosing bandwidth in tests of parametric against nonparametric hypotheses. Delgado, Rodriguez-Poo, and Wolf (2001) propose a method for selecting b to minimize size distortion in the context of hypothesis testing within the maximum score estimator, although no optimality properties of this method were proven. The main problem here is that usually b that is good for size distortion is not good for power and vice a versa.

5 Numerical Results

In this section we report some numerical results on the performance of the test statistics and the subsample critical values.

5.1 Simulations

We examined three sets of designs, the Burr distributions most recently examined by Tse and Zhang (2000), the lognormal distributions most recently studied by Barrett and Donald (1999), and the exchangeable normal processes of Klecan et al. (1991). These cases allow an assessment of the power properties of the tests, and to a limited extent, the question of suitable subsample sizes.

In computing the suprema in D_N, S_N, we took a maximum over an equally spaced grid of size n on the 98% range of the pooled empirical distribution - that is, we took the 1% and 99% quantiles of this empirical distribution and then formed an equally spaced grid between these two extremes. We chose a total of nine different subsamples for each sample size $n \in \{50, 500, 1000\}$. In earlier work we tried fixed rules of the form $b(n) = c_j n^{a_j}$, but found it did not work so well. Instead, we took an equally spaced grid of subsample sizes on the range $2 \times n^{0.3} < b < 3 \times n^{0.7}$. In each case we did $1,000$ replications.

5.1.1 Tse and Zhang (2000)

In the context of independent prospects and i.i.d. observations, Tse and Zhang (2000) have provided some Monte Carlo evidence on the power of the alternative tests proposed by Davidson and Duclos

(2000), the "DD test", and Anderson (1996). They also shed light on the convergence to the Gaussian limiting distribution of these tests. The evidence on the latter issue is not very encouraging except for very large sample sizes, and they conclude that the DD test has better power than the Anderson test for the cases they considered.

In the income distribution field, an often empirically plausible candidate is the Burr Type XII distribution, $B(\alpha, \beta)$. This is a two parameter family defined by:

$$F(x) = 1 - (1 + x^{\alpha})^{-\beta}, \quad x \geq 0$$

where $E(X) < \infty$ if $\beta > 1/\alpha > 0$. This distribution has a convenient inverse: $F^{-1}(v) = [(1-v)^{-\frac{1}{\beta}} - 1]^{\frac{1}{\alpha}}, \quad 0 \leq v < 1$. We investigated the five different Burr designs of Tse and Zhang (2000), which are given below along with the population values of d^*, s^* :

X_1	X_2	d^*	s^*
$B(4.7, 0.55)$	$B(4.7, 0.55)$	0.000	0.0000
$B(2.0, 0.65)$	$B(2.0, 0.65)$	0.0000	0.0000
$B(4.7, 0.55)$	$B(2.0, 0.65)$	0.1395	0.0784
$B(4.6, 0.55)$	$B(2.0, 0.65)$	0.1368	0.0773
$B(4.5, 0.55)$	$B(2.0, 0.65)$	0.1340	0.0761

The first two designs are in the null hypothesis, while the remaining three are in our alternative. Note that Tse and Zhang (2000) actually report results for different hypotheses, so that only their first two tables are comparable. We report our results in Tables 1a-e below.

The first two designs are useful for an evaluation of size characteristics of our tests, but in the demanding context of the "least favorable" case of equality of the two distributions. The estimated CDFs "kiss" at many more points than do the integrated CDFs. As a result, large sample sizes will be needed for accurate size of FSD, as well as relatively large subsamples. For SSD, however, the accuracy is quite good for moderate sample sizes and in all but the smallest of subsample cases. Given the nature of the testing problem, sample sizes less than 100 are very small indeed. In such cases the tests will overreject at conventional levels, indicating an inability to distinguish between the "unrankable" and "equal" cases. Even in this demanding case, however, one is led to the correct decision that the two (equal) prospects here do not dominate each other. The accuracy of size estimation for SSD is rather impressive.

In the last three designs (Tables 1c-1e), the power of our tests are forcefully demonstrated. This is so even at relatively small samples sizes. Even with a sample of size 50 there is appreciable power. We note a certain phenomenon with very small samples: the power declines as the number of subsamples declines (the subsample size increases). This seems to indicate that larger *number of subsamples* are needed for more accurate estimation especially when small samples are available. The performance of the tests in these cases is quite satisfactory.

5.1.2 The lognormal distributions

The lognormal distribution is a long celebrated case in both finance and income and wealth distribution fields. It was most recently investigated in Barrett and Donald (1999) in an examination of the McFadden tests. Let,

$$X_j = \exp(\mu_j + \sigma_j Z_j),$$

where Z_j are standard normal and mutually independent.

X_1	X_2	d^*	s^*
$LN(0.85, 0.6^2)$	$LN(0.85, 0.6^2)$	0.0000	0.0000
$LN(0.85, 0.6^2)$	$LN(0.7, 0.5^2)$	0.0000	0.0000
$LN(0.85, 0.6^2)$	$LN(1.2, 0.2^2)$	0.0834	0.0000
$LN(0.85, 0.6^2)$	$LN(0.2, 0.1^2)$	0.0609	0.0122

The results are shown in Tables 2a-d.

The first two designs are in the null and the next two (2c-2d) are in the alternative for FSD, borderline null for SSD in design 2c, and in the alternative for SSD in design 2d. The first design is a "least favorable" case and, at least for the FSD test, it demonstrates the demand for higher sample sizes as well as subsample sizes. The tendency is toward moderate overrejection for very small samples. Accuracy improves quite rapidly with sample size for Second order SD tests and is impressive for most subsample sizes and moderate sample sizes.

The second design is quite instructive. While the overall results are similar to the previous case, the differences reflect the fact that there is no FSD ranking, (or equality) and only a mild degree of Second Order Dominance. For moderate to reasonable sample sizes the tendency is to sligtly underreject FSD. This tendency is reduced by increasing the size of the subsamples. The results for SSD, confirm the theoretical consistency properties of our tests.

Results for design 2c are quite conclusive. For moderate to large sample sizes, FSD is powerfully rejected, while SSD is not. Very small samples are seen to be dangerous in cases where CDFs cross (no FSD) and the degree of SSD is moderate. A comparison with the last design (case 2d) is quite instructive. Here there is no FSD or SSD and the test is quite capable of producing the correct inference. Accuracy is again improved with increasing number of subsamples.

5.1.3 Klecan, McFadden, and McFadden (1991)

The previous designs had independent prospects and i.i.d observations. In this section we investigate the three different exchangeable multinormal processes of Klecan et al. (1991),

$$X_{jt} = (1-\lambda)\left[\alpha_j + \beta_j\left(\sqrt{\rho}Z_{0t} + \sqrt{1-\rho}Z_{jt}\right)\right] + \lambda X_{j,t-1},$$

where (Z_{0t}, Z_{1t}, Z_{2t}) are i.i.d. standard normal random variables, mutually independent. The parameters $\lambda = \rho = 0.1$ determine the mutual correlation of X_{1t} and X_{2t} and their autocorrelation. The parameters α_j, β_j are actually the mean and standard deviation of the marginal distributions of X_{1t} and X_{2t}. This scheme produces autocorrelated and mutually dependent prospects. The marginals and the true values of the statistics are:

X_1	X_2	d^*	s^*
$N(0,1)$	$N(-1,16)$	0.1981	0.0000
$N(0,16)$	$N(1,16)$	-0.0126	0.0000
$N(0,1)$	$N(1,16)$	0.1981	0.5967

The results are given in Tables 3a-c. Design 3a is in the alternative for FSD, and in the null for SSD. Again we note that we need large samples and subsample sizes to infer this low degree of SSD, but have very good power in rejecting FSD (especially for large number of subsamples even in very small samples of 50). Design 3b is rather strongly in the null (note FSD implies SSD). Inappropriately small sample sizes lead to over estimation of size but, again, the larger number of subsamples do better in these situations. Interestingly, the number and size of subsamples do not appear consequential for moderate to large samples. Otherwise the theoretical power and consistency properties are strongly confirmed. The final design 3c is clearly in the alternative for both FSD and SSD. Our procedures show their expected power in rejecting dominance. For very small samples (50), again we note that larger number of subsamples do uniformly much better than otherwise (the subsample size seems not as important).

While we have looked at other designs and subsample/sample combinations and found the qualitative results here to be robust, we think the issue of optimal subsample size and numbers deserves further independent investigation in many contexts.

5.2 Daily Stock Index Returns

Finally, we applied our tests to a dataset of daily returns on the Dow Jones Industrials and the S&P500 stock returns from 8/24/88 to 8/22/00, a total of 3131 observations. The means are 0.00055 and 0.00068 respectively, while the standard deviations are 0.00908 and 0.0223 respectively; the series are certainly mutually dependent and dependent over time. Figure 1 plots the c.d.f.'s and integrated c.d.f. [denoted s.d.f.] of the two series. This shows that the two c.d.f.'s cross near zero, but the integrated c.d.f. of the Dow Jones index dominates that of the S&P500 index over this time period.

In Figure 2 we plot the surface $\int_y^x [F_{1N}(t) - F_{2N}(t)]\, dt$ against x, y on a grid of $x > 0, y < 0$. This surface is also everywhere positive, consistent with the hypothesis that the Dow Jones index prospect dominates the S&P500 index.

In Figure 3 we plot the p-value of our tests of the null hypotheses $d^* \leq 0$, $s^* \leq 0$, and $p^* \leq 0$ against subsample size. The results suggest strongly that the evidence is against $d^* \leq 0$ but in favour of $s^* \leq 0$ and $p^* \leq 0$.[4]

This is a rather striking result and implies the following. These excess daily returns on these indices (for this period) cannot be uniformly ranked on the basis of the returns alone. Any indexed-

[4] In the test of prospect dominance we subtracted off the risk free rate measured by one month t-bill rates.

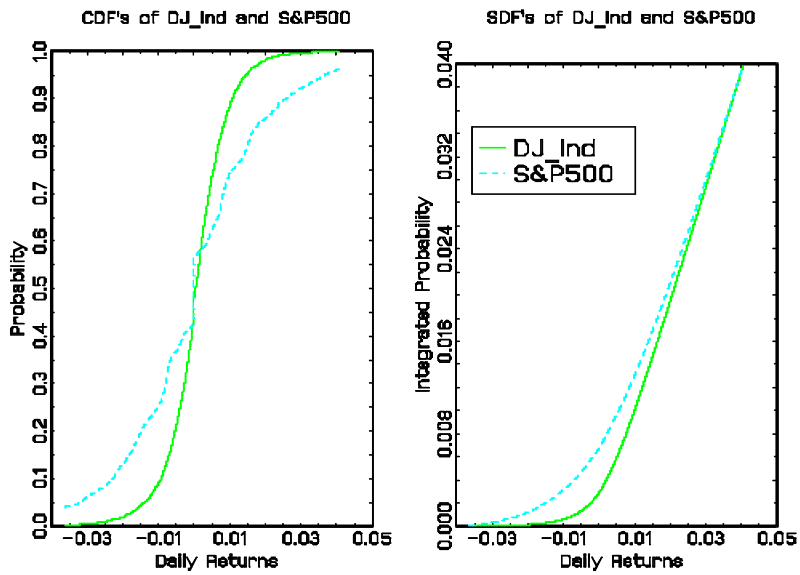

Figure 1:

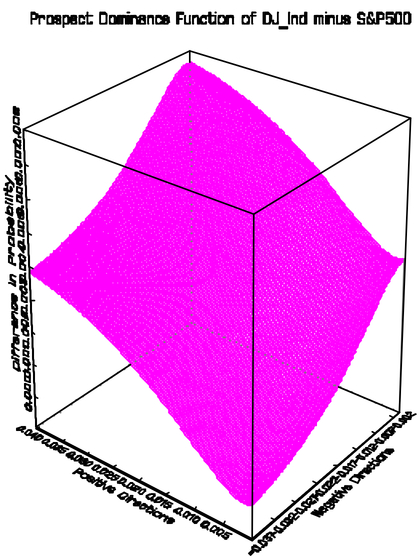

Figure 2:

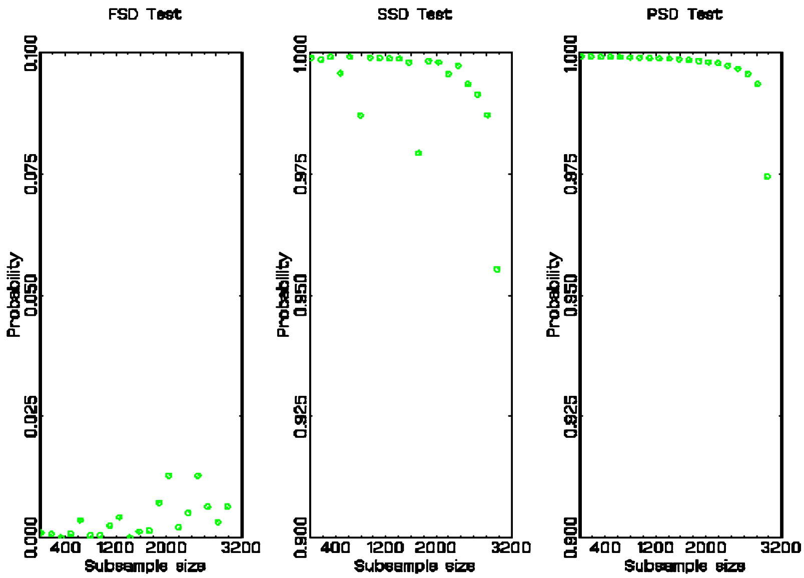

Figure 3:

based (strong) rankings at this level must necessarily depend on preferences that must be, (i) clearly revealed and declared, and (ii) defended vigorously in context. And when issues of risk and volatility are added in, most index based rankings must agree, to a statistical degree of confidence, with the uniform SSD ranking observed here. In particular, dominated index driven strategies require a revelation of their preference bases.

6 Concluding Remarks

Based on subsampling estimation of the critical values, we have obtained the asymptotic distribution of well known tests for FSD and SSD and demostrated their consistency in a very general setting that allows generic dependence of prospects and non i.i.d observations. The availability of this technique for empirical situations in which ranking is done conditional on desirable controls is of consequence for widespread use of uniform ranking in empirical finance and welfare. We have not pursued this aspect of our work here.

It is sometimes argued that the subsample bootstrap only works when the sample sizes are astronomically large, if $b=\sqrt{N}$ the argument goes, we will need N^2 observations to achieve usual accuracy. We find this argument to be somewhat misguided - the issues here are the same as in nonparametric estimation where sample sizes of 200 are routinely analyzed by these methods. If the underlying process is simple enough a subsample of size $b=30$ with $N=200$ will be quite accurate. If the underlying process is very complicated, subsample bootstrap will not work so well, but neither will any of the alternatives in general. In the designs we analyzed we found that the subsample bootstrap appears to be an effective way of computing critical values in this test of stochastic dominance, delivering good performance for sample sizes as low as 250. Some methodology for choosing b is desirable, although difficult.

A Appendix

We let C_j for some integer $j \geq 1$ denote a generic constant. (It is not meant to be equal in any two places it appears.) Let $\|Z\|_q$ denote the L^q norm $(E|Z|^q)^{1/q}$ for a random variable Z.

Lemma 1 *(a) Suppose Assumption 1 holds. Then, for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\overline{\lim_{N\to\infty}} \left\| \sup_{\rho_d^*((x_1,\theta_1),(x_2,\theta_2))<\delta} \left|\nu_{kN}^d(x_1,\theta_1) - \nu_{kN}^d(x_2,\theta_2)\right| \right\|_q < \varepsilon, \tag{A.1}$$

where

$$\rho_d^*\left((x_1,\theta_1),(x_2,\theta_2)\right) = \left\{E\left[1(X_{ki}(\theta_1) \leq x_1) - 1(X_{ki}(\theta_2) \leq x_2)\right]^2\right\}^{1/2}. \tag{A.2}$$

(b) Suppose Assumptions 1 hold. Then, for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\overline{\lim_{N\to\infty}} \left\| \sup_{\rho_s^*((x_1,\theta_1),(x_2,\theta_2))<\delta} \left|\nu_{kN}^s(x_1,\theta_1) - \nu_{kN}^s(x_2,\theta_2)\right| \right\|_q < \varepsilon, \tag{A.3}$$

where

$$\rho_s^*\left((x_1,\theta_1),(x_2,\theta_2)\right) = \left\{E\left|\int_{-\infty}^{x_1} 1(X_{ki}(\theta_1) \leq t)dt - \int_{-\infty}^{x_2} 1(X_{ki}(\theta_2) \leq t)dt\right|^r\right\}^{1/r}. \tag{A.4}$$

(c) Suppose Assumptions 1 hold. Then, for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\overline{\lim_{N\to\infty}} \left\| \sup_{\rho_p^*((x_1,y_1,\theta_1),(x_2,y_2,\theta_2))<\delta} \left|\nu_{kN}^p(x_1,\theta_1) - \nu_{kN}^p(x_2,\theta_2)\right| \right\|_q < \varepsilon, \tag{A.5}$$

where

$$\rho_p^*\left((x_1,y_1,\theta_1),(x_2,y_2,\theta_2)\right) = \left\{ E\left| \int_{y_1}^{x_1} 1(X_{ki}(\theta_1) \le t)dt - \int_{y_2}^{x_2} 1(X_{ki}(\theta_2) \le t)dt \right|^r \right\}^{1/r}. \tag{A.6}$$

Proof of Lemma 1. We first verify the conditions for part (a) of Lemma 1. The result follows from Theorem 2.2 of Andrews and Pollard (1994) with $Q = q$ and $\gamma = 1$ if we verify the mixing and bracketing conditions in the theorem. The mixing condition is implied by Assumption 1(i). The bracketing condition also holds by the following argument: Let

$$\mathcal{F}_d = \left\{1\left(X_{ki}(\theta) \le x\right) : (x,\theta) \in \mathcal{X} \times \Theta\right\}. \tag{A.7}$$

Then, $\mathcal{F}_d$ is a class of uniformly bounded functions satisfying the L^2-continuity condition, because we have

$$\begin{aligned}
&\sup_{i\ge1} E \sup_{\substack{(x',\theta')\in\mathcal{X}\times\Theta: \\ |x'-x|\le r_1, \|\theta'-\theta\|\le r_2, \sqrt{r_1^2+r_2^2}\le r}} \left|1\left(X_{ki}(\theta') \le x'\right) - 1\left(X_{ki}(\theta) \le x\right)\right|^2 \\
= \;& E \sup_{\substack{(x',\theta')\in\mathcal{X}\times\Theta: \\ |x'-x|\le r_1, \|\theta'-\theta\|\le r_2, \sqrt{r_1^2+r_2^2}\le r}} \left|1\left(X_{ki} \le Z_{ki}'(\theta'-\theta_0)+x'\right) - 1\left(X_{ki} \le Z_{ki}'(\theta-\theta_0)+x\right)\right|^2 \\
\le \;& E1\left(\left|X_{ki} - Z_{ki}'(\theta-\theta_0) - x\right| \le \left\|Z_{ki}\right\| r_1 + r_2\right) \\
\le \;& C_1\left(E\left\|Z_{ki}\right\| r_1 + r_2\right) \\
\le \;& C_2 r,
\end{aligned}$$

where the second inequality holds by Assumption 1(iii) and $C_2 = \sqrt{2}C_1\left(E\left\|Z_{ki}\right\| \vee 1\right)$ is finite by Assumption 1(ii). Now the desired bracketing condition holds because the L^2-continuity condition implies that the bracketing number satisfies

$$N(\varepsilon, \mathcal{F}_d) \le C_3 \left(\frac{1}{\varepsilon}\right)^{L+1}, \tag{A.8}$$

see Andrews and Pollard (1994, p.121).

We next verify part (b). The result follows from Theorem 3 of Hansen (1996a) with $a = L$, $\lambda = 1$, $q = q$ and $r = r$. To see this, let

$$\mathcal{F}_s = \left\{ \int_{-\infty}^{x} 1\left(X_{ki}(\theta) \le t\right) dt : (x, \theta) \in \mathcal{X} \times \Theta \right\}. \tag{A.9}$$

Then, the functions in $\mathcal{F}_s$ satisfy the Lipschitz condition:

$$\begin{aligned} & \left| \int_{-\infty}^{x'} 1\left(X_{ki}(\theta') \le t\right) dt - \int_{-\infty}^{x} 1\left(X_{ki}(\theta) \le t\right) dt \right| \\ = \; & \left| \max\left\{x' + Z_{ki}'(\theta' - \theta_{k0}) - X_{ki}, 0\right\} - \max\left\{x + Z_{ki}'(\theta - \theta_{k0}) - X_{ki}, 0\right\} \right| \\ \le \; & \sqrt{2}\left(\|Z_{ki}\| \vee 1\right)\left((x' - x)^2 + \|\theta' - \theta\|^2\right)^{1/2}, \end{aligned} \tag{A.10}$$

where the third line follows from the inequality $|\max\{a, 0\} - \max\{b, 0\}| \le |a - b|$ and Cauchy-Schwarz inequality. We also have $\sup_{k,i} E \|Z_{ki}\|^r < \infty$ by Assumption 1^*(ii) which yields the condition (12) and (13) of Hansen (1996a). Finally, the mixing condition (11) in Hansen (1996a, p.351) holds by Assumption 1^*(i), as desired.

The proof of part (c) is similar to that of part (b) except that we now take

$$\mathcal{F}_p = \left\{ \int_{y}^{x} 1\left(X_{ki}(\theta) \le t\right) dt : (x, -y, \theta) \in \mathcal{X}_+ \times \mathcal{X}_+ \times \Theta \right\} \tag{A.11}$$

and verify the Lipschitz condition using (A.10) and triangle inequality. ■

Lemma 2 *(a) Suppose Assumptions 1-3 hold. Then, we have* $\forall k = 1, \ldots, K$,

$$\sup_{x \in \mathcal{X}} \left| \nu_{kN}^{d}(x, \widehat{\theta}_k) - \nu_{kN}^{d}(x, \theta_{k0}) \right| \xrightarrow{p} 0. \tag{A.12}$$

(b) Suppose Assumptions 1^*, 2 and 3^* hold. Then, we have $\forall k = 1, \ldots, K$,

$$\sup_{x \in \mathcal{X}} \left| \nu_{kN}^{s}(x, \widehat{\theta}_k) - \nu_{kN}^{s}(x, \theta_{k0}) \right| \xrightarrow{p} 0. \tag{A.13}$$

(c) Suppose Assumptions 1^*, 2 and 3^{**} hold. Then, we have $\forall k = 1, \ldots, K$,

$$\sup_{x, -y \in \mathcal{X}_+} \left| \nu_{kN}^{p}(x, y, \widehat{\theta}_k) - \nu_{kN}^{p}(x, y, \theta_{k0}) \right| \xrightarrow{p} 0. \tag{A.14}$$

Proof of Lemma 2. We first verify part (a). Consider the pseudometric (A.2). We have

$$
\begin{aligned}
& \sup_{x\in\mathcal{X}} \rho_d^*\left(\left(x,\widehat{\theta}_k\right),(x,\theta_{k0})\right)^2 \\
= \; & \sup_{x\in\mathcal{X}} E\left[1(X_{ki}(\theta)\leq x)-1(X_{ki}(\theta_{k0})\leq x)\right]^2\Big|_{\theta=\widehat{\theta}_k} \\
= \; & \sup_{x\in\mathcal{X}} \iint \left[1\left(\widetilde{x}\leq x+z'(\widehat{\theta}_k-\theta_{k0})\right)-1\left(\widetilde{x}\leq x\right)\right]^2 dH_k(\widetilde{x}|z)dP_k(z) \qquad \text{(A.15)} \\
\leq \; & \sup_{x\in\mathcal{X}} \iint 1\left(x-\left\|z'(\widehat{\theta}_k-\theta_{k0})\right\|\leq\widetilde{x}\leq x+\left\|z'(\widehat{\theta}_k-\theta_{k0})\right\|\right) dH_k(\widetilde{x}|z)dP_k(z) \\
\leq \; & C_1\int\left\|z'(\widehat{\theta}_k-\theta_{k0})\right\|dP_k(z) \\
\leq \; & C_1\left\|\widehat{\theta}_k-\theta_{k0}\right\|E\left\|Z_{ki}\right\|\xrightarrow{p}0,
\end{aligned}
$$

where $P_k(\cdot)$ denotes the distribution function of Z_{ki} and the inequality in the 5th line holds by Assumption 1(iii) and a one-term Taylor expansion, and the last convergence to zero holds by Assumptions 1(ii) and 2. Now, (A.12) holds since we have: $\forall\varepsilon>0$, $\forall\eta>0$, $\exists\delta>0$ such that

$$
\begin{aligned}
& \overline{\lim_{N\to\infty}} P\left(\sup_{x\in\mathcal{X}}\left|\nu_{kN}^d(x,\widehat{\theta}_k)-\nu_{kN}^d(x,\theta_{k0})\right|>\eta\right) \\
\leq \; & \overline{\lim_{N\to\infty}} P\left(\sup_{x\in\mathcal{X}}\left|\nu_{kN}^d(x,\widehat{\theta}_k)-\nu_{kN}(x,\theta_{k0})\right|>\eta,\ \sup_{x\in\mathcal{X}}\rho_d^*\left(\left(x,\widehat{\theta}_k\right),(x,\theta_{k0})\right)<\delta\right) \\
& +\overline{\lim_{N\to\infty}} P\left(\sup_{x\in\mathcal{X}}\rho_d^*\left(\left(x,\widehat{\theta}_k\right),(x,\theta_{k0})\right)\geq\delta\right) \qquad \text{(A.16)} \\
\leq \; & \overline{\lim_{N\to\infty}} P^*\left(\sup_{\rho_d^*((x_1,\theta_1),(x_2,\theta_2))<\delta}\left|\nu_{kN}^d(x_1,\theta_1)-\nu_{kN}^d(x_2,\theta_2)\right|>\eta\right) \\
< \; & \frac{\varepsilon}{\eta},
\end{aligned}
$$

where the last term on the right hand side of the first inequality is zero by (A.15) and the last inequality holds by the stochastic equicontinuity result (A.1) Since $\varepsilon/\eta>0$ is arbitrary, (A.12) follows.

We next establish part (b). We have

$$
\begin{aligned}
& \sup_{x\in\mathcal{X}} \rho_s^*\left(\left(x,\widehat{\theta}_k\right),(x,\theta_{k0})\right)^r \\
= \; & \sup_{x\in\mathcal{X}} E\left|\int_{-\infty}^{x}\left(1(X_{ki}(\theta)\leq t)-1(X_{ki}(\theta_{k0})\leq t)\right)dt\right|^r\Big|_{\theta=\widehat{\theta}_k} \\
\leq \; & \left\|\widehat{\theta}_k-\theta_{k0}\right\|^r E\left\|Z_{ki}\right\|^r\xrightarrow{p}0
\end{aligned}
$$

by Assumptions 1^*(ii) and 2. Now part (b) holds using an argument similar to the one used to verify part (a). The proof of part (c) is similar. $\blacksquare$

Lemma 3 *(a) Suppose Assumptions 1-3 hold. Then, we have* $\forall k = 1, \ldots, K$,

$$\sqrt{N} \sup_{x \in \mathcal{X}} \left\| F_k(x, \widehat{\theta}_k) - F_k(x, \theta_{k0}) - \Delta'_{k0}(x) \Gamma_{k0} \overline{\psi}_{kN}(\theta_{k0}) \right\| = o_p(1).$$

(b) Suppose Assumptions 1^**, 2 and* 3^* *hold. Then, we have* $\forall k = 1, \ldots, K$,

$$\sqrt{N} \sup_{x \in \mathcal{X}} \left\| \int_{-\infty}^{x} F_k(t, \widehat{\theta}_k) dt - \int_{-\infty}^{x} F_k(t, \theta_{k0}) dt - \Lambda'_{k0}(x) \Gamma_{k0} \overline{\psi}_{kN}(\theta_{k0}) \right\| = o_p(1).$$

(c) Suppose Assumptions 1^**, 2 and* 3^{**} *hold. Then, we have* $\forall k = 1, \ldots, K$,

$$\sqrt{N} \sup_{x, -y \in \mathcal{X}_+} \left\| \int_{y}^{x} F_k(t, \widehat{\theta}_k) dt - \int_{y}^{x} F_k(t, \theta_{k0}) dt - \Xi'_{k0}(x, y) \Gamma_{k0} \overline{\psi}_{kN}(\theta_{k0}) \right\| = o_p(1).$$

Proof of Lemma 3. We verify part (a). Proof of parts (b) and (c) is similar. A mean value expansion gives

$$F_k(x, \widehat{\theta}_k) = F_k(x, \theta_{k0}) + \frac{\partial F_k(x, \theta_k^*(x))}{\partial \theta'} (\widehat{\theta}_k - \theta_{k0}),$$

where $\theta_k^*(x)$ lies between $\widehat{\theta}_k$ and θ_{k0}. By Assumption 2, we have $\sqrt{N}(\widehat{\theta}_k - \theta_{k0}) = O_p(1)$. This implies that there exists a sequence of constants $\{\xi_N : N \geq 1\}$ such that $\xi_N \to 0$ and $P\left(\left\|\widehat{\theta}_k - \theta_{k0}\right\| \leq \xi_N\right) \to 1$. The latter implies that $P\left(\sup_{x \in \mathcal{X}} \|\theta_k^*(x) - \theta_{k0}\| \leq \xi_N\right) \to 1$. Let

$$\begin{aligned} A_N &= \sup_{x \in \mathcal{X}} \left\| \frac{\partial F_k(x, \theta_k^*(x))}{\partial \theta} - \Delta_{k0}(x) \right\| \text{ and} \\ B_N &= \sup_{x \in \mathcal{X}} \sup_{\theta : \|\theta - \theta_{k0}\| \leq \xi_N} \left\| \frac{\partial F_k(x, \theta)}{\partial \theta} - \Delta_{k0}(x) \right\|. \end{aligned}$$

Then, we have $A_N = o_p(1)$ since $P(A_N \leq B_N) \to 1$ by construction and $B_N = o(1)$ by Assumption 3(ii). Now we have the desired result:

$$\begin{aligned} & \sqrt{N} \sup_{x \in \mathcal{X}} \left\| F_k(x, \widehat{\theta}_k) - F_k(x, \theta_{k0}) - \Delta'_{k0}(x) \Gamma_{k0} \overline{\psi}_{kN}(\theta_{k0}) \right\| \\ = \; & \sqrt{N} \sup_{x \in \mathcal{X}} \left\| \frac{\partial F_k(x, \theta_k^*(x))}{\partial \theta'} (\widehat{\theta}_k - \theta_{k0}) - \Delta'_{k0}(x) \Gamma_{k0} \overline{\psi}_{kN}(\theta_{k0}) \right\| \\ \leq \; & A_N \sqrt{N} \left\| \widehat{\theta}_k - \theta_{k0} \right\| + \sup_{x \in \mathcal{X}} \|\Delta_{k0}(x)\| \left\| \sqrt{N}(\widehat{\theta}_k - \theta_{k0}) - \Gamma_{k0} \sqrt{N} \overline{\psi}_{kN}(\theta_{k0}) \right\| \\ = \; & o_p(1), \end{aligned}$$

where the inequality holds by the triangle inequality and the last equality holds by Assumptions 2 and 3(iii). ■

Lemma 4 *(a) Suppose Assumptions 1-3 hold. Then, we have*

$$\begin{pmatrix} v^d_{kN}(\cdot,\theta_{k0}) - v^d_{lN}(\cdot,\theta_{l0}) \\ \sqrt{N}\psi_{kN}(\theta_{k0}) \\ \sqrt{N}\psi_{lN}(\theta_{l0}) \end{pmatrix} \Rightarrow \begin{pmatrix} \widetilde{d}_{kl}(\cdot) \\ \nu_{k0} \\ \nu_{l0} \end{pmatrix}$$

$\forall\, k,l = 1,\ldots,K$ *and the sample paths of* $\widetilde{d}_{kl}(\cdot)$ *are uniformly continuous with respect to pseudometric* ρ_d *on* $\mathcal{X}$ *with probability one, where*

$$\rho_d\left(x_1,x_2\right) = \left\{E\left[\left(1(X_{ki}\le x_1) - 1(X_{li}\le x_1)\right) - \left(1(X_{ki}\le x_2) - 1(X_{li}\le x_2)\right)\right]^2\right\}^{1/2}.$$

(b) Suppose Assumptions 1, 2 and 3* hold. Then, we have*

$$\begin{pmatrix} v^s_{kN}(\cdot,\theta_{k0}) - v^s_{lN}(\cdot,\theta_{l0}) \\ \sqrt{N}\psi_{kN}(\theta_{k0}) \\ \sqrt{N}\psi_{lN}(\theta_{l0}) \end{pmatrix} \Rightarrow \begin{pmatrix} \widetilde{s}_{kl}(\cdot) \\ \nu_{k0} \\ \nu_{l0} \end{pmatrix}$$

$\forall\, k,l = 1,\ldots,K$ *and the sample paths of* $\widetilde{s}_{kl}(\cdot)$ *are uniformly continuous with respect to pseudometric* ρ_s *on* $\mathcal{X}$ *with probability one, where*

$$\rho_s\left(x_1,x_2\right) = \left\{E\left|\int_{-\infty}^{x_1}\left(1(X_{ki}\le t) - 1(X_{li}\le t)\right)dt - \int_{-\infty}^{x_2}\left(1(X_{ki}\le t) - 1(X_{li}\le t)\right)dt\right|^r\right\}^{1/r}.$$

(c) Suppose Assumptions 1, 2 and 3** hold. Then, we have*

$$\begin{pmatrix} v^p_{kN}(\cdot,\cdot,\theta_{k0}) - v^p_{lN}(\cdot,\cdot,\theta_{l0}) \\ \sqrt{N}\psi_{kN}(\theta_{k0}) \\ \sqrt{N}\psi_{lN}(\theta_{l0}) \end{pmatrix} \Rightarrow \begin{pmatrix} \widetilde{p}_{kl}(\cdot,\cdot) \\ \nu_{k0} \\ \nu_{l0} \end{pmatrix}$$

$\forall\, k,l = 1,\ldots,K$ *and the sample paths of* $\widetilde{p}_{kl}(\cdot,\cdot)$ *are uniformly continuous with respect to pseudometric* ρ_p *on* $\mathcal{X}_+\times\mathcal{X}_-$ *with probability one, where*

$$\rho_s\left(\left(x_1,y_1\right),\left(x_2,y_2\right)\right) = \left\{E\left|\int_{y_1}^{x_1}\left(1(X_{ki}\le t) - 1(X_{li}\le t)\right)dt - \int_{y_2}^{x_2}\left(1(X_{ki}\le t) - 1(X_{li}\le t)\right)dt\right|^r\right\}^{1/r}.$$

Proof of Lemma 4. Consider part (a) first. By Theorem 10.2 of Pollard (1990), the result of Lemma 4 holds if we have (i) total boundedness of pseudometric space $(\mathcal{X}, \rho_d)$ (ii) stochastic equicontinuity of $\{v_{kN}^d(\cdot, \theta_{k0}) - v_{lN}^d(\cdot, \theta_{l0}) : N \geq 1\}$ and (iii) finite dimensional (fidi) convergence. Conditions (i) and (ii) follow from Lemma 1. We now verify condition (iii). We need to show that $v_{kN}^d(x_1, \theta_{k0}) - v_{lN}^d(x_1, \theta_{l0}), \ldots, v_{kN}^d(x_J, \theta_{k0}) - v_{lN}^d(x_J, \theta_{l0}), \sqrt{N}\overline{\psi}_{kN}(\theta_{k0})', \ \sqrt{N}\overline{\psi}_{lN}(\theta_{l0})'$ converges in distribution to $\left(\widetilde{d}_{kl}(x_1), \ldots, \widetilde{d}_{kl}(x_J), \nu_{k0}', \ \nu_{l0}'\right)'$ $\forall x_j \in \mathcal{X}$, $\forall j \leq J, \forall J \geq 1$. This result holds by the Cramer-Wold device and a CLT for bounded random variables (e.g., Hall and Heyde (1980, Corollary 5.1, p.132)) because the underlying random sequence$\{X_{ki} : i = 1, \ldots, n\}$ is strictly stationary and α- mixing with the mixing coefficients satisfying $\sum_{m=1}^{\infty} \alpha(m) \ < \infty$ by Assumption 1 and we have $|1(X_{ki} \leq x) - 1(X_{li} \leq x)| \leq 2 < \infty$. This establishes part (a).

Next, for part (b), we need to verify the fidi convergence (ii) again. Note that the moment condition of Hall and Heyde (1980, Corollary 5.1) holds since we have

$$E\left|\int_{-\infty}^{x} \left(1(X_{ki} \leq t) - 1(X_{li} \leq t)\right) dt\right|^{2+\delta} \leq E\left|X_{ki} - X_{li}\right|^{2+\delta} < \infty.$$

The mixing condition also holds since we have $\sum \alpha(m)^{-A} \leq C \sum m^{-A\delta/(2+\delta)} < \infty$ by Assumption 1*(i) as desired. Proof of part (c) is similar. ■

Proof of Theorem 1. We only verify part (a). Proof of parts (b) and (c) is analogous. Consider first the case when $d = 0$. In this case, we verify that, if $F_k(x) \leq F_l(x)$ with equality holding for $x \in \mathcal{B}_{kl}^d$, then

$$\begin{aligned} \widehat{D}_{kl} &\equiv \sup_{x \in \mathcal{X}} \sqrt{N}\left[F_{kN}(x, \widehat{\theta}_k) - F_{lN}(x, \widehat{\theta}_l)\right] \\ &\Rightarrow \sup_{x \in \mathcal{B}_{kl}^d} \left[\widetilde{d}_{kl}(\cdot) + \Delta_{k0}(\cdot)'\Gamma_{k0}\nu_{k0} - \Delta_{l0}(\cdot)'\Gamma_{l0}\nu_{l0}\right]. \end{aligned} \tag{A.17}$$

Then, the result of Theorem 1 (a) follows immediately from continuous mapping theorem.

We now establish (A.17). Lemmas 2 and 3 imply

$$\begin{aligned} \widehat{D}_{kl}(x) &\equiv \sqrt{N}\left[F_{kN}(x, \widehat{\theta}_k) - F_{lN}(x, \widehat{\theta}_l)\right] \\ &= \nu_{kN}^d(x, \widehat{\theta}_k) - \nu_{lN}^d(x, \widehat{\theta}_l) + \sqrt{N}\left[F_k(x, \widehat{\theta}_k) - F_l(x, \widehat{\theta}_l)\right] \\ &= \overline{D}_{kl}(x) + o_p(1) \text{ uniformly in } x \in \mathcal{X}, \end{aligned}$$

where

$$\begin{aligned}
\overline{D}_{kl}(x) &= D_{kl}^0(x) + D_{kl}^1(x) && \text{(A.18)} \\
D_{kl}^0(x) &= \nu_{kN}^d(x, \theta_{k0}) - \nu_{lN}^d(x, \theta_{l0}) \\
&\quad + \Delta_{k0}(x)\Gamma_{k0}\sqrt{N}\overline{\psi}_{kN}(\theta_{k0}) - \Delta_{l0}(x)\Gamma_{l0}\sqrt{N}\overline{\psi}_{lN}(\theta_{l0}) \\
D_{kl}^1(x) &= \sqrt{N}\left[F_k(x) - F_l(x)\right]. && \text{(A.19)}
\end{aligned}$$

We need to verify

$$\sup_{x \in \mathcal{X}} \overline{D}_{kl}(x) \Rightarrow \sup_{x \in \mathcal{B}_{kl}^d} d_{kl}(x). \tag{A.20}$$

Note that

$$\sup_{x \in \mathcal{B}_{kl}^d} D_{kl}^0(x) \Rightarrow \sup_{x \in \mathcal{B}_{kl}^d} d_{kl}(x) \tag{A.21}$$

by Lemma 4 and continuous mapping theorem. Note also that $\overline{D}_{kl}(x) = D_{kl}^0(x)$ for $x \in \mathcal{B}_{kl}^d$. Given $\varepsilon > 0$, this implies that

$$P\left(\sup_{x \in \mathcal{X}} \overline{D}_{kl}(x) \le \varepsilon\right) \le P\left(\sup_{x \in \mathcal{B}_{kl}^d} D_{kl}^0(x) \le \varepsilon\right). \tag{A.22}$$

On the other hand, Lemma 4 and Assumptions 1(i), 2(ii) and 3(iii) imply that given λ and $\gamma > 0$, there exists $\delta > 0$ such that

$$P\left(\sup_{\substack{\rho(x,y)<\delta \\ y \in \mathcal{B}_{kl}^d}} \left|D_{kl}^0(x) - D_{kl}^0(y)\right| > \lambda\right) < \gamma \tag{A.23}$$

and

$$\sup_{x \in \mathcal{X}} \left|D_{kl}^0(x)\right| = O_p(1). \tag{A.24}$$

The results (A.23) and (A.24) imply that we have

$$P\left(\sup_{x \in \mathcal{B}_{kl}^d} D_{kl}^0(x) \le \varepsilon\right) \le P\left(\sup_{x \in \mathcal{X}} \overline{D}_{kl}(x) \le \varepsilon + \lambda\right) + 2\gamma \tag{A.25}$$

for N sufficiently large, which follows from arguments similar to those in the proof of Theorem 6 of Klecan et. al. (1990, p.15). Taking λ and γ small and using (A.21), (A.22) and (A.25) now establish the desired result (A.20).

Next suppose $d < 0$. In this case, the set $\mathcal{B}_{kl}^d$ is an empty set and hence $F_k(x) < F_l(x)\ \forall x \in \mathcal{X}$ for some k, l. Then, $\sup_{x\in\mathcal{X}} \overline{D}_{kl}(x)$ defined in (A.18) will be dominated by the term $D_{kl}^1(x)$ which diverges to minus infinity for any $x \in \mathcal{X}$ as required. ■

Proof of Theorem 2. Let

$$d_\infty^* = \min_{k\neq l} \sup_{x\in\mathcal{B}_{kl}^d} \left[\widetilde{d}_{kl}(x) + \Delta_{k0}(x)'\Gamma_{k0}\nu_{k0} - \Delta_{l0}(x)'\Gamma_{l0}\nu_{l0}\right].$$

Let the asymptotic null distribution of D_N be given by $G(w) \equiv P\left(d_\infty^* \leq w\right)$. This distribution is absolutely continuous because it is a functional of a Gaussian process whose covariance function is nonsingular, see Lifshits (1982). Therefore, part (a) of Theorem 2 holds if we establish

$$\widehat{G}_{N,b}(w) \xrightarrow{p} G(w) \quad \forall w \in \mathbb{R}. \tag{A.26}$$

Let

$$\begin{aligned} G_b(w) &= P\left(\sqrt{b}d_{N,b,i} \leq w\right) \\ &= P\left(\sqrt{b}d_b(W_i, \ldots, W_{i+b-1}) \leq w\right) \\ &= P\left(\sqrt{b}d_b(W_1, \ldots, W_b) \leq w\right). \end{aligned}$$

By Theorem 1(a), we have $\lim_{b\to\infty} G_b(w) = G(w)$, where w is a continuity point of $G(\cdot)$. Therefore, to establish (A.26), it suffices to verify

$$\widehat{G}_{N,b}(w) - G_b(w) \xrightarrow{p} 0 \quad \forall w \in \mathbb{R}. \tag{A.27}$$

We now verify (A.27). Note first that

$$E\widehat{G}_{N,b}(w) = G_b(w). \tag{A.28}$$

Let

$$I_i = 1\left(\sqrt{b}d_b(W_i, \ldots, W_{i+b-1}) \leq w\right)$$

for $i = 1, \ldots, N$. We have

$$\begin{aligned}
\operatorname{var}\left(\widehat{G}_{N,b}(w)\right) &= \operatorname{var}\left(\frac{1}{N-b+1}\sum_{i=1}^{N-b+1} I_i\right) \\
&= \frac{1}{N-b+1}\left[S_{N-b+1,0} + 2\sum_{m=1}^{b-1} S_{N-b+1,m} + 2\sum_{m=b}^{N-b} S_{N-b+1,m}\right] \\
&\equiv A_1 + A_2 + A_3\ ,\ \ say,
\end{aligned}$$

where

$$S_{N-b+1,m} = \frac{1}{N-b+1}\sum_{i=1}^{N-b+1-m} Cov\left(I_i, I_{i+m}\right).$$

Note that

$$|A_1 + A_2| \leq O(\frac{b}{N}) = o(1). \tag{A.29}$$

Also, we have

$$\begin{aligned}
|A_3| &= \left|\frac{2}{N-b+1}\sum_{m=b}^{N-b}\left\{\frac{1}{N-b+1}\sum_{i=1}^{N-b+1-m} Cov\left(I_i, I_{i+m}\right)\right\}\right| \\
&\leq \frac{8}{\left(N-b+1\right)^2}\sum_{m=b}^{N-b}\sum_{i=1}^{N-b+1-m} \alpha_X(m-b+1) \\
&\leq \frac{8}{N-b+1}\sum_{m=1}^{N-2b+1} \alpha_X(m) \\
&\to 0 \text{ as } N \to \infty,
\end{aligned} \tag{A.30}$$

where the first inequality holds by Theorem A.5 of Hall and Heyde (1980) and the last convergence to zero holds by Assumption 1(i). Now the desired result (A.27) follows immediately from (A.28)-(A.30). This establishes part (a) of Theorem 2. Given this result, part (b) of Theorem 2 holds since we have

$$P\left(D_N > g_{N,b}(1-\alpha)\right) = P\left(D_N > g(1-\alpha) + o_p(1)\right) \to \alpha \text{ as } n \to \infty.$$

■

Proof of Theorem 3. By lemmas 2-4, we have

$$d_N(W_1, \dots, W_N) \xrightarrow{p} d^*,$$

where d^* is as defined in (1). Note that under H_1^d, we have $d^* > 0$. Now consider the empirical distribution of $d_{N,b,i} = d_b(W_i, \dots, W_{i+b-1})$:

$$\widehat{G}^0_{N,b}(w) = \frac{1}{N-b+1} \sum_{i=1}^{N-b+1} 1(d_{N,b,i} \le w) = \widehat{G}_{N,b}\left(\sqrt{b}w\right).$$

Let

$$G^0_b(w) = P\left(d_b(W_1, \dots, W_b) \le w\right).$$

By an argument analogous to those used to verify (A.27), we have

$$\widehat{G}^0_{N,b}(w) - G^0_b(w) \xrightarrow{p} 0 .$$

Since $d_b(W_1, \dots, W_b) \xrightarrow{p} d^*$, $\widehat{G}^0_{N,b}(\cdot)$ converges in distribution to a point mass at d^*. It also follows that

$$g^0_{N,b}(1-\alpha) = \inf\left\{w : \widehat{G}^0_{N,b}(w) \ge 1-\alpha\right\} \xrightarrow{p} d^*.$$

Therefore, we have

$$\begin{aligned}
P\left(D_N > g_{N,b}(1-\alpha)\right) &= P\left(\sqrt{N} d_N(W_1, \dots, W_N) > \sqrt{b} g^0_{N,b}(1-\alpha)\right) \\
&= P\left(\sqrt{\frac{N}{b}} d_N(W_1, \dots, W_N) > g^0_{N,b}(1-\alpha)\right) \\
&= P\left(\sqrt{\frac{N}{b}} d_N(W_1, \dots, W_N) > d^* + o_p(1)\right) \\
&= P\left(\sqrt{\frac{N}{b}} d_N(W_1, \dots, W_N) > d^*\right) + o(1) \\
&\to 1,
\end{aligned}$$

where the last convergence holds since $\underline{\lim}_{N\to\infty}\left(\frac{N}{b}\right) > 1$ and $d_N(W_1, \dots, W_N) \xrightarrow{p} d^* > 0$ as desired.
■

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n	b(n)	FSD95	FSD90	FSD80	SSD95	SSD90	SSD80
	6	0.4140	0.5470	0.5730	0.1170	0.1840	0.3080
	11	0.1580	0.2220	0.3730	0.1010	0.1480	0.2690
	16	0.1820	0.2260	0.3720	0.1040	0.1370	0.2740
	21	0.1500	0.1780	0.2840	0.1110	0.1490	0.2640
50	26	0.1370	0.2180	0.2990	0.1240	0.2030	0.2890
	31	0.1710	0.2380	0.3420	0.1440	0.1940	0.2880
	36	0.2210	0.2210	0.3230	0.1480	0.1480	0.2650
	41	0.1320	0.2140	0.2740	0.2320	0.3250	0.4030
	46	0.1310	0.2330	0.2330	0.3240	0.4680	0.4680
	13	0.2370	0.3320	0.3330	0.0850	0.1670	0.2760
	37	0.1180	0.2050	0.3250	0.0610	0.1320	0.2250
	61	0.0860	0.1490	0.2880	0.0590	0.1120	0.2070
	85	0.0850	0.1470	0.2490	0.0590	0.1150	0.2180
500	109	0.0740	0.1360	0.2490	0.0560	0.1180	0.2070
	133	0.0800	0.1210	0.2280	0.0580	0.1130	0.2080
	157	0.0750	0.1170	0.2140	0.0660	0.1120	0.2080
	181	0.0880	0.1280	0.2270	0.0630	0.1240	0.2070
	205	0.0680	0.1170	0.2080	0.0740	0.1200	0.2080
	16	0.1370	0.3820	0.4660	0.0610	0.1200	0.2110
	56	0.0880	0.1480	0.2910	0.0530	0.0970	0.1940
	96	0.0790	0.1350	0.2460	0.0400	0.0830	0.1750
	136	0.0650	0.1230	0.2330	0.0540	0.0810	0.1740
1000	176	0.0570	0.1060	0.2120	0.0430	0.0830	0.1810
	216	0.0580	0.0980	0.2080	0.0460	0.0770	0.1730
	256	0.0540	0.0900	0.2000	0.0460	0.0870	0.1830
	296	0.0530	0.0890	0.1900	0.0480	0.0930	0.1790
	336	0.0560	0.0890	0.1830	0.0630	0.1060	0.1770

Table 1a

n	b(n)	FSD95	FSD90	FSD80	SSD95	SSD90	SSD80
	6	0.3960	0.4720	0.4840	0.1360	0.2190	0.3260
	11	0.1620	0.2500	0.3850	0.1310	0.1830	0.2760
	16	0.1720	0.2180	0.3380	0.1290	0.1540	0.2830
	21	0.1660	0.1940	0.2810	0.1340	0.1670	0.2760
50	26	0.1390	0.2500	0.3490	0.1260	0.2160	0.3020
	31	0.1930	0.2370	0.3280	0.1620	0.2120	0.3030
	36	0.2140	0.2140	0.3120	0.1920	0.1920	0.3040
	41	0.1410	0.2170	0.2740	0.2070	0.2910	0.3700
	46	0.1460	0.2380	0.2380	0.2860	0.4250	0.4250
	13	0.3060	0.3200	0.3220	0.1230	0.2120	0.3320
	37	0.1580	0.2410	0.3660	0.0690	0.1450	0.2690
	61	0.1160	0.1950	0.3270	0.0660	0.1300	0.2590
	85	0.1110	0.1800	0.2940	0.0550	0.1220	0.2530
500	109	0.1000	0.1670	0.2680	0.0600	0.1170	0.2350
	133	0.0910	0.1460	0.2490	0.0580	0.1200	0.2320
	157	0.0840	0.1340	0.2380	0.0690	0.1120	0.2250
	181	0.0870	0.1430	0.2360	0.0740	0.1170	0.2260
	205	0.0910	0.1320	0.2180	0.0750	0.1150	0.2110
	16	0.2740	0.4480	0.4480	0.0980	0.1720	0.2930
	56	0.1140	0.1950	0.3800	0.0740	0.1410	0.2410
	96	0.1060	0.1910	0.3130	0.0540	0.1130	0.2160
	136	0.0780	0.1650	0.2960	0.0650	0.1120	0.2100
1000	176	0.0810	0.1380	0.2560	0.0600	0.1120	0.2130
	216	0.0840	0.1400	0.2430	0.0560	0.1090	0.2020
	256	0.0840	0.1530	0.2380	0.0530	0.1030	0.2140
	296	0.0810	0.1370	0.2260	0.0590	0.1040	0.2000
	336	0.0810	0.1280	0.2300	0.0660	0.1130	0.1980

Table 1b

n	b(n)	FSD95	FSD90	FSD80	SSD95	SSD90	SSD80
	6	0.8560	0.9410	0.9670	0.5550	0.7370	0.8560
	11	0.7320	0.7880	0.8780	0.4590	0.5750	0.7510
	16	0.6540	0.7000	0.8270	0.3260	0.3860	0.6040
	21	0.6200	0.6570	0.7780	0.3060	0.3730	0.5390
50	26	0.5000	0.6060	0.7000	0.2910	0.4210	0.5320
	31	0.4410	0.5080	0.6150	0.3030	0.3690	0.4850
	36	0.4380	0.4380	0.5870	0.3070	0.3070	0.4600
	41	0.3590	0.4770	0.5660	0.2260	0.3440	0.4430
	46	0.2920	0.4220	0.4220	0.2420	0.4000	0.4000
	13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	37	1.0000	1.0000	1.0000	0.9980	1.0000	1.0000
	61	1.0000	1.0000	1.0000	0.9870	0.9980	1.0000
	85	1.0000	1.0000	1.0000	0.9760	0.9920	1.0000
500	109	1.0000	1.0000	1.0000	0.9570	0.9880	1.0000
	133	0.9990	1.0000	1.0000	0.9430	0.9750	0.9960
	157	0.9970	0.9990	1.0000	0.9350	0.9610	0.9920
	181	0.9940	0.9970	0.9990	0.9240	0.9500	0.9840
	205	0.9860	0.9950	0.9980	0.9160	0.9500	0.9760
	16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	56	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	96	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	136	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1000	176	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	216	1.0000	1.0000	1.0000	0.9990	1.0000	1.0000
	256	1.0000	1.0000	1.0000	0.9980	1.0000	1.0000
	296	1.0000	1.0000	1.0000	0.9970	0.9990	1.0000
	336	1.0000	1.0000	1.0000	0.9940	0.9970	1.0000

Table 1c

n	b(n)	FSD95	FSD90	FSD80	SSD95	SSD90	SSD80
	6	0.8540	0.9400	0.9660	0.5410	0.7050	0.8460
	11	0.7020	0.7790	0.8830	0.4310	0.5640	0.7280
	16	0.6400	0.6880	0.8190	0.3280	0.3950	0.5930
	21	0.6200	0.6640	0.7620	0.3090	0.3650	0.5350
50	26	0.5050	0.6360	0.7270	0.3090	0.4450	0.5500
	31	0.4570	0.5240	0.6280	0.2760	0.3570	0.4920
	36	0.4320	0.4320	0.5810	0.3030	0.3030	0.4630
	41	0.3800	0.4820	0.5670	0.2380	0.3430	0.4540
	46	0.2740	0.4270	0.4270	0.2430	0.4200	0.4200
	13	1.0000	1.0000	1.0000	0.9990	1.0000	1.0000
	37	1.0000	1.0000	1.0000	0.9950	1.0000	1.0000
	61	1.0000	1.0000	1.0000	0.9800	0.9990	1.0000
	85	1.0000	1.0000	1.0000	0.9740	0.9980	0.9990
500	109	1.0000	1.0000	1.0000	0.9570	0.9850	0.9990
	133	0.9990	1.0000	1.0000	0.9340	0.9740	0.9970
	157	0.9940	0.9980	1.0000	0.9270	0.9660	0.9910
	181	0.9920	0.9970	1.0000	0.9150	0.9500	0.9870
	205	0.9890	0.9960	0.9990	0.8920	0.9390	0.9750
	16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	56	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	96	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	136	1.0000	1.0000	1.0000	0.9990	1.0000	1.0000
1000	176	1.0000	1.0000	1.0000	0.9990	1.0000	1.0000
	216	1.0000	1.0000	1.0000	0.9960	0.9990	1.0000
	256	1.0000	1.0000	1.0000	0.9970	0.9990	1.0000
	296	1.0000	1.0000	1.0000	0.9960	0.9980	1.0000
	336	1.0000	1.0000	1.0000	0.9940	0.9980	1.0000

Table 1d

n	b(n)	FSD95	FSD90	FSD80	SSD95	SSD90	SSD80
	6	0.8330	0.9260	0.9580	0.5340	0.7060	0.8200
	11	0.6830	0.7630	0.8800	0.4250	0.5580	0.7090
	16	0.6350	0.6950	0.8020	0.3150	0.3760	0.5990
	21	0.5880	0.6350	0.7510	0.2770	0.3380	0.5260
50	26	0.4920	0.5960	0.6830	0.3010	0.4320	0.5390
	31	0.4440	0.5070	0.6050	0.3010	0.3610	0.4860
	36	0.4360	0.4360	0.5660	0.2930	0.2930	0.4310
	41	0.3730	0.4690	0.5350	0.2480	0.3340	0.4330
	46	0.2810	0.4130	0.4130	0.2550	0.4000	0.4000
	13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	37	1.0000	1.0000	1.0000	0.9950	0.9990	1.0000
	61	1.0000	1.0000	1.0000	0.9850	0.9980	1.0000
	85	1.0000	1.0000	1.0000	0.9660	0.9940	1.0000
500	109	0.9980	1.0000	1.0000	0.9480	0.9820	0.9990
	133	0.9960	0.9990	1.0000	0.9290	0.9660	0.9930
	157	0.9950	0.9960	0.9990	0.9190	0.9530	0.9860
	181	0.9910	0.9950	0.9980	0.8970	0.9450	0.9790
	205	0.9840	0.9920	0.9970	0.8740	0.9290	0.9640
	16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	56	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	96	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	136	1.0000	1.0000	1.0000	0.9990	1.0000	1.0000
1000	176	1.0000	1.0000	1.0000	0.9980	1.0000	1.0000
	216	1.0000	1.0000	1.0000	0.9990	0.9990	1.0000
	256	1.0000	1.0000	1.0000	0.9970	0.9990	1.0000
	296	1.0000	1.0000	1.0000	0.9960	0.9990	1.0000
	336	1.0000	1.0000	1.0000	0.9950	0.9980	0.9990

Table 1e

n	b(n)	FSD95	FSD90	FSD80	SSD95	SSD90	SSD80
	6	0.4100	0.5360	0.5640	0.0900	0.1570	0.2720
	11	0.1550	0.2190	0.3590	0.0930	0.1380	0.2460
	16	0.1750	0.2050	0.3200	0.0920	0.1240	0.2490
	21	0.1490	0.1780	0.2610	0.1090	0.1410	0.2390
50	26	0.1250	0.2200	0.3050	0.1240	0.2080	0.2950
	31	0.1700	0.2270	0.3250	0.1700	0.2140	0.3180
	36	0.2120	0.2120	0.3090	0.1810	0.1810	0.2840
	41	0.1180	0.1970	0.2580	0.2560	0.3290	0.4040
	46	0.1070	0.2040	0.2040	0.3430	0.4700	0.4700
	13	0.2390	0.3620	0.3660	0.0710	0.1420	0.2480
	37	0.1160	0.2220	0.3400	0.0540	0.1120	0.2140
	61	0.0840	0.1550	0.2900	0.0500	0.1050	0.2170
	85	0.0930	0.1510	0.2610	0.0590	0.1030	0.2120
500	109	0.0870	0.1320	0.2630	0.0530	0.1060	0.2060
	133	0.0800	0.1290	0.2350	0.0660	0.1190	0.2140
	157	0.0750	0.1240	0.2230	0.0660	0.1170	0.2070
	181	0.0810	0.1360	0.2410	0.0690	0.1110	0.2100
	205	0.0800	0.1230	0.2160	0.0780	0.1100	0.2070
	16	0.1360	0.3510	0.4840	0.0640	0.1120	0.2170
	56	0.0720	0.1500	0.2880	0.0520	0.1070	0.1930
	96	0.0760	0.1380	0.2500	0.0540	0.0940	0.1940
	136	0.0580	0.1190	0.2310	0.0520	0.0910	0.1940
1000	176	0.0620	0.1160	0.2130	0.0500	0.0970	0.1870
	216	0.0620	0.1150	0.2050	0.0530	0.0990	0.1800
	256	0.0650	0.1240	0.2160	0.0590	0.0990	0.1990
	296	0.0650	0.1060	0.1970	0.0580	0.0980	0.1830
	336	0.0660	0.1080	0.1900	0.0650	0.1070	0.1910

Table 2a

n	b(n)	FSD95	FSD90	FSD80	SSD95	SSD90	SSD80
	6	0.2950	0.4040	0.4260	0.0820	0.1480	0.2710
	11	0.1200	0.1680	0.2670	0.0760	0.1290	0.2260
	16	0.1110	0.1390	0.2400	0.0940	0.1160	0.2330
	21	0.1160	0.1440	0.2090	0.1030	0.1290	0.2380
50	26	0.1270	0.1940	0.2560	0.1120	0.1860	0.2660
	31	0.1440	0.1820	0.2620	0.1330	0.1690	0.2710
	36	0.1740	0.1740	0.2630	0.1610	0.1610	0.2680
	41	0.1280	0.1880	0.2490	0.2720	0.3470	0.4320
	46	0.1280	0.2310	0.2310	0.3190	0.4560	0.4560
	13	0.0860	0.1650	0.1730	0.0050	0.0270	0.1130
	37	0.0360	0.0720	0.1380	0.0010	0.0050	0.0570
	61	0.0170	0.0360	0.1190	0.0020	0.0020	0.0360
	85	0.0200	0.0470	0.1160	0.0010	0.0070	0.0370
500	109	0.0230	0.0430	0.1200	0.0020	0.0100	0.0450
	133	0.0250	0.0520	0.1400	0.0030	0.0130	0.0630
	157	0.0350	0.0640	0.1310	0.0090	0.0220	0.0800
	181	0.0350	0.0710	0.1540	0.0120	0.0380	0.1100
	205	0.0490	0.0870	0.1730	0.0220	0.0510	0.1380
	16	0.0540	0.1500	0.2390	0.0030	0.0120	0.0590
	56	0.0150	0.0510	0.1430	0.0000	0.0020	0.0280
	96	0.0120	0.0490	0.1250	0.0000	0.0010	0.0320
	136	0.0160	0.0480	0.1380	0.0000	0.0020	0.0370
1000	176	0.0200	0.0530	0.1350	0.0000	0.0060	0.0590
	216	0.0260	0.0600	0.1350	0.0010	0.0130	0.0680
	256	0.0370	0.0640	0.1390	0.0060	0.0170	0.0810
	296	0.0380	0.0700	0.1490	0.0090	0.0290	0.1020
	336	0.0470	0.0790	0.1530	0.0160	0.0360	0.1170

Table 2b

n	b(n)	FSD95	FSD90	FSD80	SSD95	SSD90	SSD80
	6	0.5940	0.7600	0.8780	0.3040	0.5730	0.8510
	11	0.3430	0.4360	0.6110	0.5450	0.6740	0.8310
	16	0.3150	0.3700	0.5660	0.6900	0.7420	0.8660
	21	0.3370	0.3760	0.4980	0.8110	0.8330	0.8980
50	26	0.3290	0.4460	0.5320	0.8460	0.8900	0.9100
	31	0.3220	0.3740	0.4840	0.6880	0.7230	0.7910
	36	0.3670	0.3670	0.5040	0.1770	0.1770	0.2730
	41	0.2600	0.3480	0.4560	0.9130	0.9400	0.9640
	46	0.1460	0.2900	0.2900	0.9180	0.9520	0.9520
	13	1.0000	1.0000	1.0000	0.1140	0.6220	0.9950
	37	0.9960	0.9990	1.0000	0.8560	0.9860	1.0000
	61	0.9930	0.9960	1.0000	0.4890	0.6990	0.9160
	85	0.9820	0.9960	1.0000	0.0040	0.0110	0.0610
500	109	0.9690	0.9880	1.0000	0.0010	0.0010	0.0010
	133	0.9560	0.9780	1.0000	0.0010	0.0010	0.0010
	157	0.9500	0.9730	0.9950	0.0010	0.0010	0.0010
	181	0.9350	0.9620	0.9890	0.0010	0.0010	0.0010
	205	0.9190	0.9490	0.9810	0.0010	0.0010	0.0010
	16	1.0000	1.0000	1.0000	0.1720	0.8230	1.0000
	56	1.0000	1.0000	1.0000	0.9860	1.0000	1.0000
	96	1.0000	1.0000	1.0000	0.8740	0.9740	0.9990
	136	1.0000	1.0000	1.0000	0.0010	0.0010	0.0010
1000	176	0.9990	1.0000	1.0000	0.0010	0.0010	0.0010
	216	0.9990	0.9990	1.0000	0.0010	0.0010	0.0010
	256	0.9980	0.9990	1.0000	0.0010	0.0010	0.0010
	296	0.9970	0.9980	1.0000	0.0010	0.0010	0.0010
	336	0.9960	0.9970	1.0000	0.0010	0.0010	0.0010

Table 2c

n	b(n)	FSD95	FSD90	FSD80	SSD95	SSD90	SSD80
	6	0.5220	0.6970	0.7830	0.0240	0.0690	0.2060
	11	0.3770	0.4860	0.6590	0.0390	0.0750	0.2000
	16	0.3300	0.3780	0.5760	0.1390	0.1910	0.3900
	21	0.3300	0.3780	0.4840	0.1630	0.2180	0.3520
50	26	0.3670	0.4760	0.5630	0.1570	0.2600	0.3430
	31	0.3780	0.4330	0.5330	0.1750	0.2310	0.3350
	36	0.3880	0.3880	0.5020	0.1920	0.1920	0.3080
	41	0.2230	0.2900	0.3710	0.2240	0.2960	0.3790
	46	0.1420	0.2300	0.2300	0.1930	0.3090	0.3090
	13	1.0000	1.0000	1.0000	0.4850	0.7950	0.9800
	37	0.9980	1.0000	1.0000	0.4770	0.7210	0.9420
	61	0.9890	1.0000	1.0000	0.4790	0.6720	0.9170
	85	0.9860	0.9920	1.0000	0.4760	0.6440	0.8750
500	109	0.9770	0.9910	1.0000	0.4920	0.6190	0.8430
	133	0.9480	0.9710	0.9960	0.4820	0.6120	0.8050
	157	0.9350	0.9750	0.9950	0.4710	0.6130	0.7750
	181	0.9240	0.9570	0.9880	0.4910	0.6140	0.7680
	205	0.8980	0.9350	0.9750	0.4880	0.6120	0.7500
	16	1.0000	1.0000	1.0000	0.9270	0.9950	1.0000
	56	1.0000	1.0000	1.0000	0.9150	0.9870	1.0000
	96	1.0000	1.0000	1.0000	0.9120	0.9820	1.0000
	136	1.0000	1.0000	1.0000	0.8620	0.9570	0.9980
1000	176	1.0000	1.0000	1.0000	0.8660	0.9540	0.9940
	216	1.0000	1.0000	1.0000	0.8630	0.9420	0.9900
	256	0.9980	1.0000	1.0000	0.8280	0.8970	0.9780
	296	0.9980	1.0000	1.0000	0.8360	0.9070	0.9700
	336	0.9940	1.0000	1.0000	0.7980	0.8760	0.9450

Table 2d

n	b(n)	FSD95	FSD90	FSD80	SSD95	SSD90	SSD80
	6	0.9660	0.9830	0.9990	0.0970	0.2870	0.6210
	11	0.8810	0.9210	0.9700	0.3140	0.4370	0.6720
	16	0.7930	0.8270	0.9100	0.4510	0.5150	0.7170
	21	0.7340	0.7710	0.8620	0.6310	0.6780	0.7900
50	26	0.6220	0.7370	0.8130	0.7100	0.7860	0.8480
	31	0.5780	0.6440	0.7440	0.6460	0.6970	0.7750
	36	0.5380	0.5380	0.6560	0.1980	0.1980	0.2820
	41	0.3770	0.4870	0.5680	0.8340	0.8720	0.9010
	46	0.2970	0.5100	0.5100	0.7740	0.8300	0.8300
	13	1.0000	1.0000	1.0000	0.0020	0.0460	0.6540
	37	1.0000	1.0000	1.0000	0.5150	0.8070	0.9860
	61	1.0000	1.0000	1.0000	0.7140	0.9000	0.9940
	85	1.0000	1.0000	1.0000	0.0280	0.0980	0.3590
500	109	1.0000	1.0000	1.0000	0.0010	0.0010	0.0010
	133	1.0000	1.0000	1.0000	0.0010	0.0010	0.0010
	157	1.0000	1.0000	1.0000	0.0010	0.0010	0.0010
	181	1.0000	1.0000	1.0000	0.0010	0.0010	0.0010
	205	1.0000	1.0000	1.0000	0.0010	0.0010	0.0010
	16	1.0000	1.0000	1.0000	0.0010	0.0440	0.8580
	56	1.0000	1.0000	1.0000	0.8080	0.9840	1.0000
	96	1.0000	1.0000	1.0000	0.9560	0.9950	1.0000
	136	1.0000	1.0000	1.0000	0.0010	0.0010	0.0010
1000	176	1.0000	1.0000	1.0000	0.0010	0.0010	0.0010
	216	1.0000	1.0000	1.0000	0.0010	0.0010	0.0010
	256	1.0000	1.0000	1.0000	0.0010	0.0010	0.0010
	296	1.0000	1.0000	1.0000	0.0010	0.0010	0.0010
	336	1.0000	1.0000	1.0000	0.0010	0.0010	0.0010

Table 3a

n	b(n)	FSD95	FSD90	FSD80	SSD95	SSD90	SSD80
50	6	0.2180	0.3060	0.3350	0.0540	0.1500	0.3900
	11	0.0850	0.1120	0.1830	0.0830	0.1410	0.3480
	16	0.0850	0.1040	0.1800	0.0850	0.1090	0.2650
	21	0.0930	0.1060	0.1530	0.1370	0.1820	0.3000
	26	0.0820	0.1300	0.2010	0.1710	0.2650	0.3540
	31	0.1060	0.1440	0.2130	0.2460	0.3020	0.3990
	36	0.1290	0.1290	0.2180	0.1990	0.1990	0.3190
	41	0.1050	0.1660	0.2190	0.4520	0.5330	0.5860
	46	0.1250	0.2380	0.2380	0.5300	0.6170	0.6170
500	13	0.0010	0.0010	0.0010	0.0000	0.0010	0.0090
	37	0.0010	0.0010	0.0020	0.0000	0.0010	0.0010
	61	0.0000	0.0000	0.0010	0.0000	0.0000	0.0230
	85	0.0000	0.0000	0.0030	0.0000	0.0010	0.0190
	109	0.0000	0.0000	0.0010	0.0020	0.0050	0.0170
	133	0.0010	0.0010	0.0030	0.0020	0.0070	0.0270
	157	0.0000	0.0010	0.0020	0.0010	0.0040	0.0180
	181	0.0030	0.0040	0.0100	0.0020	0.0070	0.0260
	205	0.0050	0.0060	0.0140	0.0040	0.0100	0.0230
1000	16	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	56	0.0000	0.0000	0.0000	0.0000	0.0000	0.0150
	96	0.0000	0.0000	0.0000	0.0000	0.0000	0.0050
	136	0.0000	0.0000	0.0000	0.0000	0.0000	0.0030
	176	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	216	0.0000	0.0000	0.0000	0.0000	0.0010	0.0030
	256	0.0000	0.0000	0.0000	0.0000	0.0000	0.0030
	296	0.0000	0.0000	0.0010	0.0000	0.0010	0.0020
	336	0.0000	0.0000	0.0010	0.0000	0.0000	0.0040

Table 3b

n	b(n)	FSD95	FSD90	FSD80	SSD95	SSD90	SSD80
	6	0.9700	0.9860	0.9980	0.8520	0.8880	0.9250
	11	0.8860	0.9280	0.9750	0.7610	0.8200	0.8790
	16	0.8240	0.8440	0.9120	0.7110	0.7520	0.8480
	21	0.7690	0.8110	0.8830	0.6490	0.6930	0.7840
50	26	0.6590	0.7460	0.8140	0.5630	0.6740	0.7440
	31	0.6040	0.6610	0.7620	0.4770	0.5460	0.6650
	36	0.5710	0.5710	0.6920	0.4180	0.4180	0.5630
	41	0.3860	0.4880	0.5960	0.3360	0.4610	0.5540
	46	0.3340	0.5170	0.5170	0.2670	0.4710	0.4710
	13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	37	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	61	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	85	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
500	109	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	133	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	157	1.0000	1.0000	1.0000	0.9990	1.0000	1.0000
	181	1.0000	1.0000	1.0000	0.9990	0.9990	1.0000
	205	1.0000	1.0000	1.0000	0.9990	0.9990	1.0000
	16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	56	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	96	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	136	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1000	176	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	216	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	256	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	296	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	336	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 3c