Food Consumption and Growth in a Two Sector Economy

Thorsten Wichmann

Technical University Berlin

December 1995

I discuss the dynamics of growth and structural change in a two-sector dual economy characterized by a positive relationship between nutrition and productivity. A rise in agricultural production increases food consumption as well as labor productivity. The different outcomes are discussed when the level of food consumption increases productivity either in agriculture or in industry (static relationship) or when it influences the productivity of learning by doing (dynamic relationship). Since these relationships generate non-stationary dynamics, they are studied numerically. The outcome depends on the sector in which the relationship is assumed to exist. The contribution of improved nutrition to increases in consumption over time is considerably larger in the presence of a dynamic nutrition effect than in the presence of a static effect.

Keywords: growth and development theory, nutrition, efficiency wages, numerical simulations

Address: Technical University Berlin, Department of Economics – WW 16 –, Uhlandstr. 4-5,
10623 Berlin, Germany
E-Mail: T.Wichmann@ww.TU-Berlin.DE

This work as well as earlier versions have benefitted considerably from comments by Laszlo Goerke, Jürgen Kromphardt, Beate Scheidt, and Jessica deWolff. I would also like to thank participants at the Linz meeting of the Verein für Socialpolitik for encouraging comments.

I. Introduction

This paper studies the effects of a positive relationship between nutrition and productivity on the dynamics of growth and structural change within a two sector dual economy model consisting of agriculture and industry. This kind of relationship has recently found more interest, both theoretically (*Dasgupta*, 1993) as well as empirically (*Fogel* 1994).

The idea of a technically determined relationship at low levels of income between the state of nutrition or health and labor productivity is familiar to development economists. It has been developed independently in the late 1950s by *Leibenstein* (1957) and *Mazumdar* (1959) and became soon known among development economists as "Efficiency Wage Hypothesis". It gained popularity in economics when *Stiglitz* (1976) made the first step to generalize the idea to the today under this label subsumed links between wages and efficiency in terms of incentives, morale and effort-intensity.[1]

For developing countries a number of empirical studies exist that estimate the influence of nutrition or other nutrition-related health indicators on labor productivity, mostly in agriculture. *Behrman* and *Deolalikar* (1988) review this literature and note a general positive relationship between nutrition and productivity, although they criticize that several of these studies seem to suffer from methodological problems caused by self-selection bias or health endogeneity. But also studies without major methodological problems seem to support a positive relationship between nutrition and productivity.

Recently, *Fogel* (1994) made an attempt to estimate the importance of this nutrition-productivity relationship (NPR) for the development process of Britain. Rather than confining his work to the agricultural sector, he estimated the effects for the whole economy including industry. He concludes that improvements in gross nutrition account for 30% of the increase in per-capita income between 1790 and 1980. *Fogel* assigns one third of this effect to increased labor force participation, and asserts that this rise had been caused by improved nutrition which had strengthened the population and thus brought people into the labor force who previously were too weak to work. The remaining two thirds of the growth effect are said to be due to an increased labor productivity in production.

Fogel recalls that especially the poor have been too weak for intense work at the beginning of the industrial revolution around 1790:

> [I]n France the bottom 10 percent of the labor force lacked the energy for regular work, and the next 10 percent had enough energy for less than three hours of light work daily (0.52 hours of heavy work). Although the English situation was somewhat better, the bottom 3 per-

1. For a short review of this development see *Bardhan* (1993).

> cent of its labor force lacked the energy for any work, but the balance of the bottom 20 percent had enough energy for about 6 hours of light work (1.09 hours of heavy work) each day. (*ibid.*, 373)

But also the wealthier part of the society seems to have been far less healthy than today:

> [E]ven persons in the top half of the income distribution in Britain during the 18th century were stunted and wasted, suffered far more extensively from chronic diseases at young adult and middle ages than is true today and died 30 years sooner than today. (*ibid.*, 383)

Fogel regrets that these nutritional factors are largely neglected in "new" and "old" growth theory alike, although they could easily be included as labor-enhancing technical progress brought about, for example, by improvements in agricultural production. This paper makes an attempt to do so. In addition to the influence of nutrition on labor productivity in production (henceforth called static NPR) this paper also considers the influence on the ability to increase productivity, namely on the productivity of the learning by doing process (dynamic NPR). From the point of view of the new growth theory, pioneered by the work of *Romer* (1986) and *Lucas* (1988), this extension is only consequent: First of all, the mechanisms behind productivity improvements, be it innovations, human capital accumulation, or learning by doing, make up the main topic of this literature. And secondly, this field of research is based on the insight that rather small growth effects can outweigh level effects over time.

Therefore both, the static as well as the dynamic NPR are considered here and compared to situations where either malnutrition keeps productivity permanently below its maximum or no nutrition-productivity relationship exists. This comparison should yield some insights about the influence of the different relationships on the growth process. The analysis is conducted within a dual economy model that includes an agricultural as well as an industrial sector. In addition to explicitly considering production of food, which is responsible for the NPRs, this model type allows discussion of growth effects and structural change alike. Both should a priori be regarded as possibly important. We would especially want to know whether the existence of a NPR together with a technologically stagnant agricultural sector can influence an economy's structure. Following the dual economy literature, we characterize the latter by the fraction of labor in agriculture.

A distinct feature of the NPRs discussed here is their importance only at low levels of nutrition. In fact, some empirical studies even show a negative relationship at higher levels of nutrition (see below). However, the fading NPR with rising nutrition makes a steady-state discussion of this mechanism impossible. When the economy finally grows with a constant rate, the NPR has already ceased to exist. Therefore a two-step approach is chosen. In the first baseline model, which does possess a steady-state equilibrium, is presented and its properties are discussed. In the second step this model is modified to capture different NPRs. Since differences between the modifications cannot be discussed analytically, the actual comparison of the outcomes is then conducted on the basis of numerical simulations. These simulations are numerical solutions to the respective optimal control problems which are obtained by applying a modification of *Mulligan*'s and *Sala-i-Martin*'s (1991, 1993) time elimination method. The outcomes are complete time-paths for all variables conditional on the assumed parameter values and initial conditions.

The remainder of the paper is organized as follows: In section II the basic steady-state model of a dual economy is presented. Section III presents the model modifications to capture different nutrition effects. In section IV the simulation method is discussed, and in section V the different simulation outcomes are compared. Section VI concludes.

II. The Dual Economy Model

The simple model of the dual economy is based on the work of *Jorgenson* (1961) and a later generalization by *Zarembka* (1970). While these two assume constant saving, the model presented here is an optimal control model so that the nutrition effect can also influence the saving and capital accumulation decision. The economy considered is dual in the sense that different production functions exist for the traditional good, food, in the first sector and the industrial or manufacturing good, widgets, in the second sector. While the former is produced from land and labor, input factors for the latter are labor and capital. The economy also shows a second asymmetry in consumption. Food consumption raises productivity up to a certain level while widget consumption does not.

For simplicity the economy is assumed to consist of a constant number of identical individuals, so that the decision problem of a single individual can be analyzed.[2] Each individual can spend a fraction n of her inelastically supplied working time in the traditional sector and the other fraction $(1 - n)$ in the modern sector. The sectors are indicated by the subscripts A and M respectively. Food in the traditional sector is produced with a constant returns to scale production function from labor and land. The latter is normalized to unity. All agricultural output goes to labor and is consumed. Thus (per-capita) consumption of food can be expressed by:

2. The model can easily be extended to a growing labor force. However, since this does not change any of the crucial results and only adds complexity to the problem, the labor force is assumed to remain constant.

(1) $$c_A = An^{\alpha}, \qquad 0 < \alpha < 1.$$

A denotes the level of total factor productivity which grows with a constant rate ν.

In manufacturing widgets are produced by labor and capital with a Cobb-Douglas technology. As usual the output can be either consumed or invested. Then investment – and thus the change in the capital stock since there is no depreciation – is characterized by:

(2) $$\dot{k} = Mk^{1-\alpha}(1-n)^{\alpha} - c_M$$

where c_M denotes (per-capita) consumption of widgets. Similar to the agricultural sector the state of technology in manufacturing, M, grows with a constant exogenous rate μ.[3]

The individual maximizes her utility which is given by a two-good CRRA function where consumption of both goods enters in a Cobb-Douglas manner:

(3) $$u(c_A, c_M) = \begin{cases} \dfrac{(c_A^{\gamma} c_M^{1-\gamma})^{1-\sigma}}{1-\sigma} & \text{for} \quad \sigma \neq 1 \\ \ln(c_A^{\gamma} c_M^{1-\gamma}) & \text{for} \quad \sigma = 1 \end{cases} \qquad \text{where} \qquad 0 < \gamma < 1, \sigma > 0.$$

This function implies an elasticity of substitution between both goods of one. The intertemporal elasticity of substitution is $1/\sigma$, the inverse of the Arrow-Pratt measure of relative risk aversion. Note that c_A and c_M are not completely substitutable.

Equations (1) - (3) make up the basic functions for the optimal control problem the individual has to solve, namely choosing a time path for c_A, c_M and n (the control variables) which is optimal because it maximizes utility over the whole time period considered. Given these paths and a given stock of capital at $t = 0$, equation (2) implies a time path for the capital stock k. The time paths for M and A are given exogenously. The problem can be simplified by substituting the production function for food, (1), into the utility function and thereby reducing the control variables to c_M and n. Formalized the individual's decision problem can then be written as:[4]

(4) $$\max_{n,\, c_M} \int_0^{\infty} \frac{\left[(An^{\alpha})^{\gamma} c_M^{1-\gamma}\right]^{1-\sigma}}{1-\sigma} e^{-\rho t} dt$$

$$\text{s.t.} \qquad \dot{k} = Mk^{1-\alpha}(1-n)^{\alpha} - c_M$$

where ρ is the discount factor for future consumption.

3. Note that the output elasticity of labor is the same in both sectors. This assumption is not crucial for the results obtained below but simplifies the analysis.
4. In the following we do not explicitly state the special form of the utility function for $\sigma = 1$.

The familiar way to tackle this problem is to solve the current-value Hamiltonian where a shadow price θ is assigned to the capital accumulation constraint. The Hamiltonian is:

$$(5)\qquad H_c = \frac{\left[(An^{\alpha})^{\gamma} c_M^{1-\gamma} \right]^{1-\sigma}}{1-\sigma} + \theta\,(Mk^{1-\alpha}(1-n)^{\alpha} - c_M)\,.$$

H_c is the sum of current-period utility and capital investment; the latter valued at the shadow price θ. An optimal allocation must maximize H_c at every point in time. This is the case if the following four solution equations are satisfied:

$$(6)\qquad \frac{\partial H_c}{\partial c_M} = (1-\gamma)\left[(An^{\alpha})^{\gamma} c_M^{1-\gamma} \right]^{1-\sigma} c_M^{-1} - \theta = 0$$

$$(7)\qquad \frac{\partial H_c}{\partial n} = \alpha\gamma\left[(An^{\alpha})^{\gamma} c_M^{1-\gamma} \right]^{1-\sigma} n^{-1} - \alpha\theta Mk^{1-\alpha}(1-n)^{\alpha-1} = 0$$

$$(8)\qquad \frac{\partial H_c}{\partial \theta} = \dot{k} = Mk^{1-\alpha}(1-n)^{\alpha} - c_M$$

$$(9)\qquad \dot{\theta} = \theta\rho - \theta(1-\alpha)Mk^{-\alpha}(1-n)^{\alpha}$$

To be an optimal solution, the control variables c_M and n must be chosen in a way that satisfies the boundary conditions. These consist of an initial value for capital, k_0, as well as the transversality condition $\lim_{t \to \infty} e^{-\rho t}\theta k = 0$. The solution must satisfy the sufficiency conditions as well to ensure that it is indeed a maximum and not a minimum. We show in the appendix that Mangasarian's sufficiency conditions are always met for the assumed restrictions on the parameter values.

Within the system of equations (6) – (9), equation (6) describes the choice of c_M in every period. It must be balanced in a way such that the value of consuming a unit today must equal the value of saving it today and consuming the growth proceeds tomorrow. Equation (7) is a labor market condition. Labor will be allocated between the sectors such that the marginal utility from working in the different productions is equal. Equation (8) simply assures that the optimal path is also feasible, and (9) delivers the rate of decrease of the shadow price for capital θ.[5] Equations (6) - (9) together with the transversality condition define the family of optimal paths. We consider only one of these paths, namely the steady-state equilibrium where all variables grow with a constant though not necessarily equal growth rate.[6] In the steady-state also the growth rate of the shadow price θ has to be

5. For a general economic interpretation of the optimal control technique cf. *Dorfman* (1969).

constant and therefore from equation (9) the marginal product of capital is also a constant:

(10) $$(1-\alpha)\, M k^{-\alpha} (1-n)^{\alpha} = \rho - \frac{\dot{\theta}}{\theta} = \text{const.}$$

Comparing this result with equation (8), one can see that the growth rate of capital can only be constant if the fraction c_M / k remains unchanged. Therefore c_M and k have to grow with the same rate in the steady-state. Differentiation of equation (10) with respect to time (acknowledging that the growth rate of θ as well as n are constant in the steady-state) finally yields the growth rates of widget consumption and of capital:

(11) $$\left(\frac{\dot{c}_M}{c_M}\right)^{*} = \left(\frac{\dot{k}}{k}\right)^{*} = \frac{\mu}{\alpha}.$$

Therefore in the steady-state equilibrium the growth rate of per-capita consumption of industrial goods depends only on technical progress in this sector, not in any way on the outcome of agriculture. One can observe as a further result a somewhat similar feature of the model for the agricultural sector. The growth rate of per-capita food consumption is given by differentiating the agricultural production function (1) with respect to time as:

(12) $$\left(\frac{\dot{c}_A}{c_A}\right)^{*} = \nu.$$

Recall that the system of equations (6) – (9) must satisfy the transversality condition $\lim_{t \to \infty} e^{-\rho t} \theta k = 0$ which has not been shown so far. This boundary condition can only be met if the product of θ and k grows with a rate smaller than the discount rate ρ in the steady-state. Combining this condition with equations (6) and (11) yields:

(13) $$\rho > (1-\sigma)\left[\gamma\nu + (1-\gamma)\frac{\mu}{\alpha}\right].$$

In the remainder this condition is assumed to hold.

The economy's structure can be characterized by n, the fraction of labor in agriculture. In the steady-state n has to be constant; it cannot increase or decrease forever. Its steady-state value can be calculated from equations (6), (7), (10) and (11) as:

6. Since the fraction of labor in agricultural production, n, is a bounded control, it has to be constant in the steady-state. At first this does not seem to be in accordance with the empirically observed continuous decline of agriculture in the process of economic development. However, it is well known that taking into account Engel's law leads to a replication of this behavior. This is not done in this model since it would only complicate the analysis without producing much additional insight. However, it can be added easily by introducing subsistence consumption for food in the utility function as has been done, e.g., by *Matsuyama* (1992) or *Wichmann* (1995).

$$n^* = \frac{\gamma(\rho - (1-\sigma)(\gamma\nu + (1-\gamma)\frac{\mu}{\alpha}) + \mu)}{\rho - (1-\sigma)\gamma\nu + \sigma(1-\gamma)\frac{\mu}{\alpha} + \gamma\mu}. \quad (14)$$

The steady-state value for n is positive and smaller one by transversality condition (13).

Equations (11), (12), and (14) can be used to study the influences of economic policies on the steady-state. A rise in the rates of technical progress, ν and μ leads to a larger growth rate of consumption of the respective sector's output according to (11) and (12). By differentiation of (14) one can easily show that an increase in the rate of technical progress in industry leads to a decrease of the labor fraction in agriculture. A rise in the rate of agricultural technical progress, though, increases n as long as $\sigma > 0$.[7] We regard this as the most realistic case.[8] Thus, according to this comparative static analysis, a rise in μ would be the appropriate policy to industrialize a country while a rise in ν would lead to deindustrialization.

This completes the description of the basic model. It is shown in the appendix that the steady-state equilibrium is unique and saddle-path stable for a large range of parameter values. The model is rather simple in excluding, for example, Engel's law or labor force growth which are both considered important elements influencing economic development. While these elements could be easily added to the model, refraining from doing so will make the mechanism of the nutrition-productivity relationship clearer since fewer effects interfere.

III. Model Extensions: Nutrition-Productivity Relationships

A relation between nutrition and productivity can exist in several different ways. First of all, the effect might be of a static nature, that is, an increase in food consumption raises output productivity in production. This effect might occur in agricultural as well as in industrial production. Here these possibilities are discussed separately to keep their consequences as clear as possible. Combining such a relationship in agriculture with one in industry, while certainly more realistic, would only lead to superposition of their influences.

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7. For $\sigma \leq 1$ we get $\partial n^* / \partial \nu \leq 0$.
8. *Hall* (1988), for example, has estimated values around 10 for σ. *Giovannini* (1985) has obtained similar low values for the intertemporal elasticity of substitution, in some estimations not even significantly different from zero.

A further possibility is a dynamic relationship where an increase in food consumption raises the productivity growth rate. This could, for example, happen via learning or schooling since malnutrition not only reduces physical ability but also impairs mental capabilities. Therefore the model is extended to include the simplest possible element of endogenous productivity improvements, namely learning by doing. The productivity of this learning by doing process is assumed to depend positively on the level of food consumption. A NPR via learning by doing is of course only a very crude approximation of the true dynamic effects of malnutrition. Empirical evidence suggests that the main effects of malnutrition occur early in life. *Glewwe* and *Jacoby* (1995), for example, show that early childhood malnutrition causes delayed school enrollment. *Pollitt* (1984, 1990) reviews studies showing that children with severe malnutrition prior to school enrollment perform significantly worse on intelligence tests than better-nourished children. If, however, such effects last beyond school age, the grown-up children, who were malnourished as infants, will also perform worse in activities like technology adoption or learning.

To derive the model modifications, the NPR has to be specified first. A distinct feature of the relationship discussed here is that it exists only at low levels of nutrition. In fact, some empirical studies even show it to be negative at higher levels. *Strauss* (1986), for example, who analyzed farm households in Sierra Leone, estimated output elasticities of per-consumer equivalent calorie availability in agricultural production and found this elasticity to be 0.33 at the sample mean level of family calorie availability, 0.49 at 1500 calories per day, and 0.12 at 4500 calories per day. Above a daily consumption of 5200 calories, the estimated elasticity was negative. While this estimation describes relatively well the upper part of the functional form, its lower end is not clear. *Stiglitz* (1976) hypothesizes a logistic functional form but acknowledges that direct empirical evidence is difficult to obtain and therefore the functional form remains an open question. *Dasgupta* (1993) proposes a concave functional form. We follow *Dasgupta* in defining nutrition caused productivity in the following way:

$$\Pi(c_A) = \frac{\pi c_A}{\pi + c_A}, \qquad \pi > 0 \tag{15}$$

In this function productivity Π increases with rising food consumption c_A but is bounded from above by π. It shows diminishing returns to food consumption but does not possess a convex region like the logistic function. However, it is much easier to handle.

Consider first nutrition effects in agriculture itself. Assume that the agricultural production function takes on the form[9] $c_A = A\Pi n^{\alpha}$. As an additional restriction it is assumed that the nutrition effect has the character of an externality. The agent does not take into account in her optimization that increased food consumption raises her productivity and thus the amount of food available for consumption. Rather, she assumes that Π grows exogenously.[10] The interpretation of the resulting dynamic equilibrium, which is not anymore an optimal equilibrium, follows *Lucas* (1988): If the agent's expectations about the exogenous path of Π are met, if thus the actual development of Π coincides with the expected development, then the economy is said to be in a dynamic equilibrium.

The individual faces the following problem:[11]

$$(16) \quad \max_{n,\, c_M} \int_0^{\infty} \frac{\left[(A\Pi n^{\alpha})^{\gamma} c_M^{1-\gamma} \right]^{1-\sigma}}{1-\sigma} e^{-\rho t} dt$$

$$\text{s.t.} \quad \dot{k} = Mk^{1-\alpha}(1-n)^{\alpha} - c_M$$

This leads to the current-value Hamiltonian:

$$(17) \quad H_c = \frac{\left[(A\Pi n^{\alpha})^{\gamma} c_M^{1-\gamma} \right]^{1-\sigma}}{1-\sigma} + \theta (Mk^{1-\alpha}(1-n)^{\alpha} - c_M)$$

with the following solution equations:

$$(18) \quad \frac{\partial H_c}{\partial c_M} = (1-\gamma) \left[(A\Pi n^{\alpha})^{\gamma} c_M^{1-\gamma} \right]^{1-\sigma} c_M^{-1} - \theta = 0$$

$$(19) \quad \frac{\partial H_c}{\partial n} = \gamma\alpha \left[(A\Pi n^{\alpha})^{\gamma} c_M^{1-\gamma} \right]^{1-\sigma} n^{-1} - \alpha\theta Mk^{1-\alpha}(1-n)^{\alpha-1} = 0$$

$$(20) \quad \frac{\partial H_c}{\partial \theta} = \dot{k} = Mk^{1-\alpha}(1-n)^{\alpha} - c_M$$

$$(21) \quad \dot{\theta} = \theta\rho - \theta(1-\alpha)Mk^{-\alpha}(1-n)^{\alpha}$$

where $\Pi = \pi(1 - 1/(An^{\alpha}))$.[12]

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9. A more reasonable assumption would be a purely labor augmenting nutrition effect. However, this would make the problem intractable.
10. The justification for this assumption is mainly simplicity. The optimal solution to this problem becomes too complicate to be tractable – even numerically – especially for the third case of a dynamic nutrition effect.
    It is also intuitively clear what the differences between optimal and market solution should be. Since the latter neglects the productivity enhancing effect of nutrition and thus of food production, the market will allocate less labor than optimal to agriculture during the period where this effect is relevant.
11. Thus, the model only considers the productivity rising effect of better nutrition, not the labor force participation effect observed by *Fogel*.

Equations (18) – (21) together with the familiar boundary conditions describe the equilibrium. However, contrary to the basic model there does not exist a steady-state solution, since Π neither remains constant nor grows without bound. Rather, if A grows forever, Π asymptotically converges towards its upper limit π. Therefore the process can only be analyzed numerically. However, the properties of this asymptotic steady-state the economy eventually converges to can be analyzed analytically: If $\nu > 0$, Π will eventually be close to its upper limit π. Then equations (18) and (19) simplify to:

$$\frac{\partial H_c}{\partial c_M} = (1-\gamma)\left[(A\pi n^{\alpha})^{\gamma} c_M^{1-\gamma}\right]^{1-\sigma} c_M^{-1} - \theta = 0$$

$$\frac{\partial H_c}{\partial n} = \gamma\alpha\left[(A\pi n^{\alpha})^{\gamma} c_M^{1-\gamma}\right]^{1-\sigma} n^{-1} - \alpha\theta M k^{1-\alpha}(1-n)^{\alpha-1} = 0$$

Comparing this outcome with the result of the basic model presented above, one can see that the asymptotic steady-state of a model including a NPR equals the steady-state of a model without.[13] This implies that in the very long run there is no difference in growth rates of the two economies and even the structures (in terms of fractions of the labor force in agriculture) are identical. There might be a difference, however, in levels of consumption and capital as well as in the growth and development experience on the equilibrium path towards the (asymptotic) steady-state.

Next, consider a NPR existing only in industrial production. For this sector we can make the more reasonable assumption that Π works in a labor augmenting way since there are no analytical problems. After all, the workers are the food consumers and become more productive. Then the industrial production function changes into

(22) $$y_M = Mk^{1-\alpha}[(1-n)\Pi]^{\alpha}.$$

If the agent does not take into account the relationship, the current-value Hamiltonian for her optimal control problem becomes for this case:

(23) $$H_c = \frac{\left[(An^{\alpha})^{\gamma} c_M^{1-\gamma}\right]^{1-\sigma}}{1-\sigma} + \theta\left(Mk^{1-\alpha}[(1-n)\Pi]^{\alpha} - c_M\right)$$

This current-value Hamiltonian leads to the solution equations:

12. This is simply a transformation of: $\Pi = (\pi A \Pi n^{\alpha})/(\pi + A\Pi n^{\alpha})$.

13. The parameter π vanishes when deriving the steady-state in the same way as above.

(24) $$\frac{\partial H_c}{\partial c_M} = (1-\gamma)\left[(An^{\alpha})^{\gamma} c_M^{1-\gamma}\right]^{1-\sigma} c_M^{-1} - \theta = 0$$

(25) $$\frac{\partial H_c}{\partial n} = \gamma\alpha\left[(An^{\alpha})^{\gamma} c_M^{1-\gamma}\right]^{1-\sigma} n^{-1} - \alpha\theta M k^{1-\alpha}(1-n)^{\alpha-1}\Pi^{\alpha} = 0$$

(26) $$\frac{\partial H_c}{\partial \theta} = \dot{k} = Mk^{1-\alpha}\Pi^{\alpha}(1-n)^{\alpha} - c_M$$

(27) $$\dot{\theta} = \theta\rho - \theta(1-\alpha)Mk^{-\alpha}\Pi^{\alpha}(1-n)^{\alpha}$$

where $\Pi = \pi A n^{\alpha} / (\pi + An^{\alpha})$.

Again it is easy to see that in the asymptotic steady-state growth rates and structure of the economy are equal to those in the basic model.

Finally, consider a dynamic NPR in industry.[14] In this case nutrition does influence the growth rate of productivity rather than its level. To keep the model as simple as possible, this relationship is assumed to exist in a learning mechanism. The model follows *Arrow* (1962) in assuming that learning-effects are caused by capital accumulation. The continuous introduction of new capital goods confronts the worker continuously with new occasions to learn. Learning, in turn, increases the stock of knowledge and thus productivity. Following this idea, industrial production can be characterized as follows:

(28) $$y_M = Mk^{1-\alpha}[(1-n)h]^{a}$$
$$\text{where} \quad h = k^{\Pi} \quad \text{and} \quad \Pi = \frac{\pi A n^{\alpha}}{\pi + An^{\alpha}}.$$

The level of human capital or knowledge is denoted by h and works in a labor augmenting way. Capital accumulation increases this knowledge with an elasticity Π. This is where the NPR enters. It is assumed that workers learn better from the introduction of new capital goods if their nutrition level is higher. As before this effect is bounded from above.

Assuming again that the agent takes Π as exogenous, the current-value Hamiltonian from her optimization problem with h substituted by k^{Π} is:[15]

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14. The reason for not considering dynamic effects in agriculture is mainly technical. Since there is no capital accumulation in this sector, the effect would have to work via the stock of knowledge. Simulation of this relationship, however, is not possible with the method used here since A is still time-depended but not any more exogenous. Cf. section IV.

15. *Mangasarian*'s sufficiency conditions are still met as long as $\Pi < 1$.

(29) $$H_c = \frac{\left[(An^{\alpha})^{\gamma} c_M^{1-\gamma} \right]^{1-\sigma}}{1-\sigma} + \theta \left(Mk^{1-\alpha(1-\Pi)} (1-n)^{\alpha} - c_M \right).$$

This Hamiltonian has the solution equations:

(30) $$\frac{\partial H_c}{\partial c_M} = (1-\gamma) \left[(An^{\alpha})^{\gamma} c_M^{1-\gamma} \right]^{1-\sigma} c_M^{-1} - \theta = 0$$

(31) $$\frac{\partial H_c}{\partial n} = \gamma\alpha \left[(An^{\alpha})^{\gamma} c_M^{1-\gamma} \right]^{1-\sigma} n^{-1} - \alpha\theta M k^{1-\alpha(1-\Pi)} (1-n)^{\alpha-1} = 0$$

(32) $$\frac{\partial H_c}{\partial \theta} = \dot{k} = Mk^{1-\alpha(1-\Pi)} (1-n)^{\alpha} - c_M$$

(33) $$\dot{\theta} = \theta\rho - \theta(1-\alpha) Mk^{-\alpha(1-\Pi)} (1-n)^{\alpha}$$

with Π as in equation (28). In the same way as above the (asymptotic) steady-state values for growth rates and the fraction of labor in agriculture can be derived. These are slightly different now which is due to the higher rate of technical progress in industry:[16]

(34) $$\left(\frac{\dot{c}_M}{c_M}\right)^* = \left(\frac{\dot{k}}{k}\right)^* = \frac{\mu}{\alpha(1-\pi)}$$

(35) $$n^* = \frac{\gamma \left[\rho - (1-\sigma) \left(\gamma\nu + (1-\gamma) \frac{\mu}{\alpha(1-\pi)} \right) + \frac{\mu}{(1-\pi)} \right]}{\rho - (1-\sigma)\gamma\nu + \sigma(1-\gamma) \frac{\mu}{\alpha(1-\pi)} + \frac{\gamma\mu}{(1-\pi)}}$$

Having derived the models' solutions, the different growth and development paths described by the baseline model and the three variants from this section can now be compared by numerical simulation.

IV. Simulation Method

The three numerical problems to be solved are two-point boundary value problems. This name refers to the fact that one constraint is given at time zero (the starting value for k) and another at infinity (the transversality condition). If the equilibrium path is unique, there exists only one possible set of control values in each period that leads the economy

16. Therefore also the transversality condition changes into $\rho > (1-\sigma) \left(\gamma\nu + (1-\gamma) \frac{\mu}{\alpha(1-\pi)} \right)$. Note that a positive growth rate requires $\pi < 1$. For $\pi = 1$ the model becomes an endogenous growth model in the '*AK*'-tradition, as has been pointed out to me by Uwe *Walz*.

towards the (asymptotic) steady-state. The latter can be used as second boundary value since it satisfies the transversality condition. A common numerical routine to solve such problems is called “shooting”: choose starting values for the controls, numerically integrate forward and see if you have hit the second boundary condition. If not, aim higher or lower.[17] While simple, this method involves considerable amounts of computations.

For this reason a different method is chosen, namely the time elimination method proposed by *Mulligan* and *Sala-i-Martin* (1991, 1993). This is an algorithm which is based on the transformation of the original dynamic system from a time dependent two-point boundary value problem into a state-variable dependent initial value problem where the initial value is the model’s steady-state. The resulting equations are solved to yield those values for the control variables for each value of the state variable that keep the model on the equilibrium path. Although the models discussed here only have an asymptotic steady-state, the time elimination method can still be used with only a slight modification.

The original method is applied as follows (neglect for a moment the nutrition effect): In a first step the system with a constant growth path solution is transformed into one where all variables are stationary in the steady-state. This is done by defining new variables which are constant in the steady-state (e.g., c_M / k). Those can be called state-like if they contain only state variables and control-like if they are made up of state and control variables. Next the original system is transformed into a system of differential equations in the new variables. Using the chain rule of calculus, the resulting differential equations can be used to obtain the slope of policy functions which yield the value of the controls to be chosen for each value of the state variables in order to stay on the equilibrium path.[18] Taking the steady-state as initial condition, the policy functions can be derived by standard numerical routines. Here the routine NDSolve in *Mathematica* is used.[19]

Including the nutrition effect into the model changes the method in the simplest case only slightly: As long as there exists any technical progress in agriculture, Π will eventually approach its upper bound π, although this might be in the very distant future. We can

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17. For an overview of these problems and numerical solution techniques see *Goffe* (1993), *Press et al.* (1992), or *Dixon et al.* (1992).
18. This slope is derived in the following way: Suppose that the differential equations for a state-like variable z and a control-like variable n are given by $\dot{z} = \xi_1(n, z)$ and $\dot{n} = \xi_2(n, z)$. Then the time path for the control can be stated as $n(t) = n(z(t))$ and therefore

$$n'(z) = \frac{\dot{n}}{\dot{z}} = \frac{\xi_1(n, z)}{\xi_2(n, z)}.$$

19. For details of this routine see *Wolfram* (1991).

therefore derive as a first step the differential equations in the new variables which depend on Π. If Π were a constant, these equations would be exactly of the type needed for the time-elimination method.

Consider first the situation, where the nutrition effect occurs in the agricultural sector itself. The dynamic system in terms of c_M, n, k, and θ is given by equations (18) – (21). It can be transformed into differential equations in the control-like variables $z_1 = c_M / k$ and n as well as the state-like variable $z_2 = M / k^{\alpha}$. The resulting system of three differential equations can be reduced by one equation since, due to equations (18) and (19), z_1 is always given as $z_1 = \frac{(1-\gamma)}{\gamma} z_2 n (1-n)^{\alpha-1}$. This leaves:

(36) $$\dot{z}_2 = z_2 \left(\mu - \alpha z_2 (1-n)^{\alpha-1} \frac{(\gamma - n)}{\gamma}\right)$$

(37) $$\dot{n} = \frac{(1-\sigma)\left(\mu(1-\gamma) + \frac{\gamma\nu\pi}{\Pi}\right) - \mu - \rho + (1-\alpha)(1-\gamma) z_2 (1-n)^{\alpha} \frac{(\sigma n + \gamma(1-\sigma))}{\gamma(1-n)}}{(n(1-n))^{-1}(\sigma(1-\alpha n) + (1-\sigma)\gamma(1-\alpha) - (1-\sigma)\gamma\alpha\left(\frac{\pi}{\Pi} - 1\right)(1-n))}$$

where $\Pi = \pi(1 - 1/(An^{\alpha}))$.

The second problem with a nutrition effect in the industrial sector is set up in a similar way. Defining z_1 and z_2 as before, we get the following relationship between both variables: $z_1 = \frac{(1-\gamma)}{\gamma} z_2 n \Pi^{\alpha} (1-n)^{\alpha-1}$.

The time dependent model given by equations (24) – (27) can be transformed into:

(38) $$\dot{z}_2 = z_2 \left(\mu - \alpha z_2 \Pi^{\alpha} (1-n)^{\alpha-1} \frac{(\gamma - n)}{\gamma}\right).$$

(39) $$\dot{n} = \frac{(1-n) n \left[(1-\sigma)(\mu(1-\gamma) + \gamma\nu) - \mu - \rho + \alpha\nu((1-\sigma)(1-\gamma) - 1)\left(1 - \frac{\Pi}{\pi}\right) \right]}{(\sigma(1-\alpha n) + (1-\sigma)\gamma(1-\alpha)) - \alpha^2((1-\sigma)(1-\gamma) - 1)\left(1 - \frac{\Pi}{\pi}\right)} + \frac{(1-\alpha)(1-\gamma) z_2 \Pi^{\alpha} \frac{(\sigma n + \gamma(1-\sigma))}{\gamma}}{\sigma(1-\alpha n) + (1-\sigma)\gamma(1-\alpha) - \alpha^2((1-\sigma)(1-\gamma) - 1)\left(1 - \frac{\Pi}{\pi}\right)}$$

where Π is given by $\Pi = (\pi A n^{\alpha}) / (\pi + A n^{\alpha})$.

Finally, for the dynamic nutrition effect in industry the following variables can be defined: $z_1 = c_M / k$ and $z_2 = M / k^{\alpha(1-\Pi)}$. Since $z_1 = \frac{(1-\gamma)}{\gamma} z_2 n (1-n)^{\alpha-1}$, the outcome reduces to two equations:

(40) $$\dot{z}_2 = z_2 \left(\mu - \alpha (1-\Pi) z_2 (1-n)^{\alpha-1} \frac{(\gamma - n)}{\gamma}\right)$$

(41) $$\dot{n} = \frac{n(1-n)\left[(1-\sigma)(\mu(1-\gamma)+\gamma\nu)-\mu-\rho+z_2(1-\alpha)(1-\gamma)(1-\sigma)(1-n)^{\alpha-1}\right]}{(\sigma(1-\alpha n)+(1-\sigma)\gamma(1-\alpha))} + \frac{z_2 n(1-n)^{\alpha}(n\sigma(1-\alpha-\gamma+\alpha\gamma)+\alpha\Pi(n-\gamma)(\gamma(1-\sigma)+\sigma))}{\gamma(\sigma(1-\alpha n)+(1-\sigma)\gamma(1-\alpha))}$$

where $\Pi = (\pi A n^{\alpha}) / (\pi + A n^{\alpha})$.

For each of these three cases, combination of the differential equations for n and z_2 yields the derivative of a single policy function. However, if the definition for Π is substituted into the equations to eliminate Π, the differential equation still depends on A which makes the problem time-dependent. To solve this problem, we apply the time-elimination method within a two-step approach. Starting with values for A_0 and z_{20}, the trajectory towards that (fictive) equilibrium where A remains at A_0 is calculated in the first step. The outcome is a policy function $n(z_2, A_0)$, giving the optimal value for n. In the second step, A is increased exogenously and z_2 according to its differential equation. Then the first step is conducted again. Eventually agricultural productivity A will be so large that the nutrition effect disappears and the economy takes on the steady-state values from the baseline model.[20] The intuition behind this procedure is the following: in every period the stable trajectory conditional on A is calculated. For each A exists a different trajectory. The combination of movements on this trajectory (for n and z_2) and movements between trajectories (due to changes in A) describes how the economy's variables evolve over time.

V. Simulation Results

To conduct the simulations, we choose the following parameter values: $\rho = 0.05$, $\alpha = 0.7$, and $\sigma = 5$ which are conventional values. The upper limit of the nutrition caused productivity, π, is set to unity in the static case and to $\pi = 0.05$ in the dynamic scenario. For the

20. Strictly speaking, this occurs only in infinite time. But for a numerical solution it is sufficient to require that the difference be less than the precision used in solving the problem. For example, with a rate of technical progress in agriculture of 3% per year and a starting value of $A_0 = 1$ productivity Π is very close to π after approximately 200 years. While this is longer than one would sensibly expect, equation (15) could be easily modified to converge towards its limit more quickly, for example by using c_A^2 instead of c_A. However, since this would only complicate the analytical parts of the solution while not yielding much new insight, equation (15) is left as above.

former, $\pi = 1$ corresponds to a situation without an NPR. Since in the latter $\pi = 1$ is not feasible (in the steady-state $\dot{c}_M / c_M = \mu / \alpha(1 - \pi)$ would go to infinity), we choose π as well as μ half as large as in the static case implying the same steady-state growth rates of widget consumption as well as the same steady-state fractions of labor in agriculture.

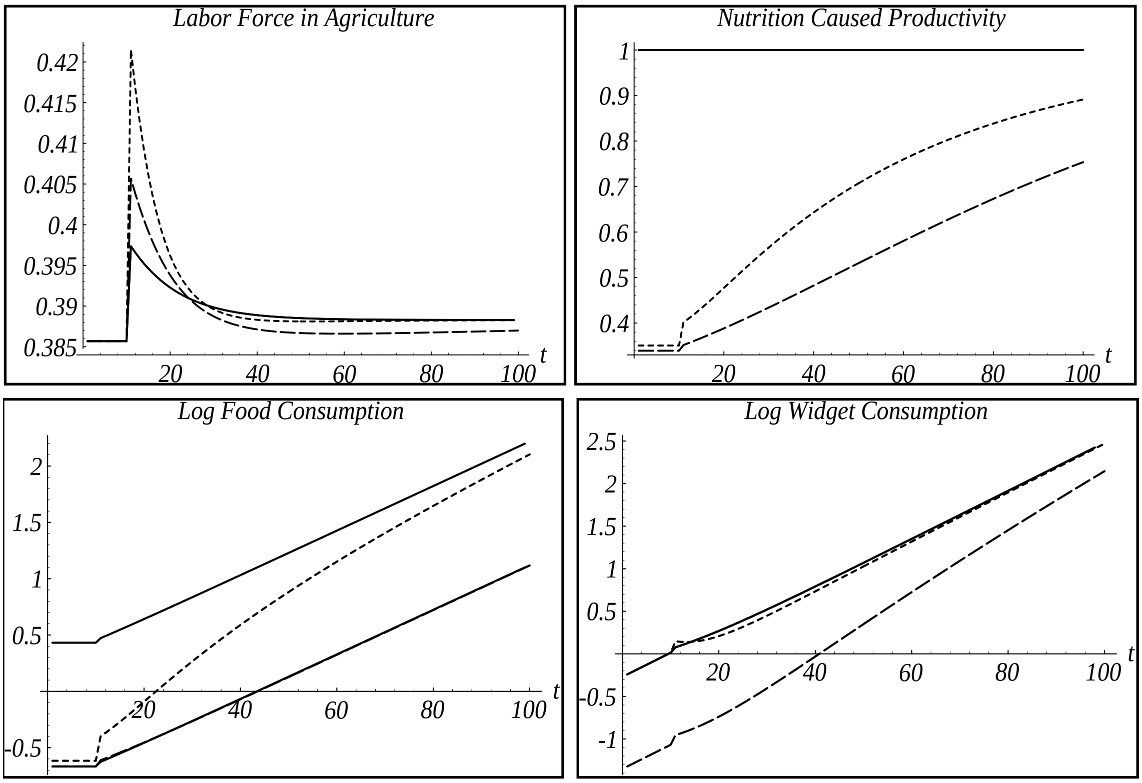
With these parameter values we conduct a simulation where in the beginning the agricultural sector is technologically stagnant. At some point in time the rate of agricultural technical progress becomes positive. We study the economy's behavior after this shock for two cases: normal agricultural technical progress ($\nu = 0.02$) and fast technical progress ($\nu = 0.04$). These two cases are simulated for each of the three possible NPRs. In all cases does the rate of industrial technical progress remain constant at $\mu = 0.02$. Increasing its value does not influence nutrition caused productivity but only leads to a new steady-state division of labor and a higher growth rate of widget consumption.

The outcome of these simulations is depicted in figures 1-4. Figures 1 and 2 contain the two static NPRs in agriculture and industry while figures 3 and 4 show the dynamics of the two possible relationships in industry.[21] The first ten periods show the pre-shock case where $\nu = 0$ and nutrition caused productivity remains permanently below its maximum value. The solid line in each picture describes the benchmark case, namely the path an economy would take when the nutrition caused productivity is always at its maximum π. This baseline economy from section II is exposed to the same shock in ν. A comparison of both paths shows the influence of the NPR on the pre-shock steady-state as well as on the transitional dynamics towards the new steady-state equilibrium. A further benchmark case also contained in the figures is the simple no-change scenario. Extending the time-paths from the first ten periods into the future yields the development of an economy where agriculture remains technologically stagnant.

Consider the static relationships first. Figure 1 shows the slow progress case and figure 2 the time paths with fast agricultural technical progress. Although agricultural and industrial NPR are depicted together, they can only be compared very carefully. First of all, the nutrition caused productivity Π is labor augmenting in industry but labor and land augmenting in agriculture. And secondly, different starting values for A had to be chosen due to computational problems: $A_0 = 1$ for the industrial relationship and $A_0 = 3$ for the agricultural one.

21. Note that the number of simulation periods has been chosen differently between the pairs to make the dynamics clearer.

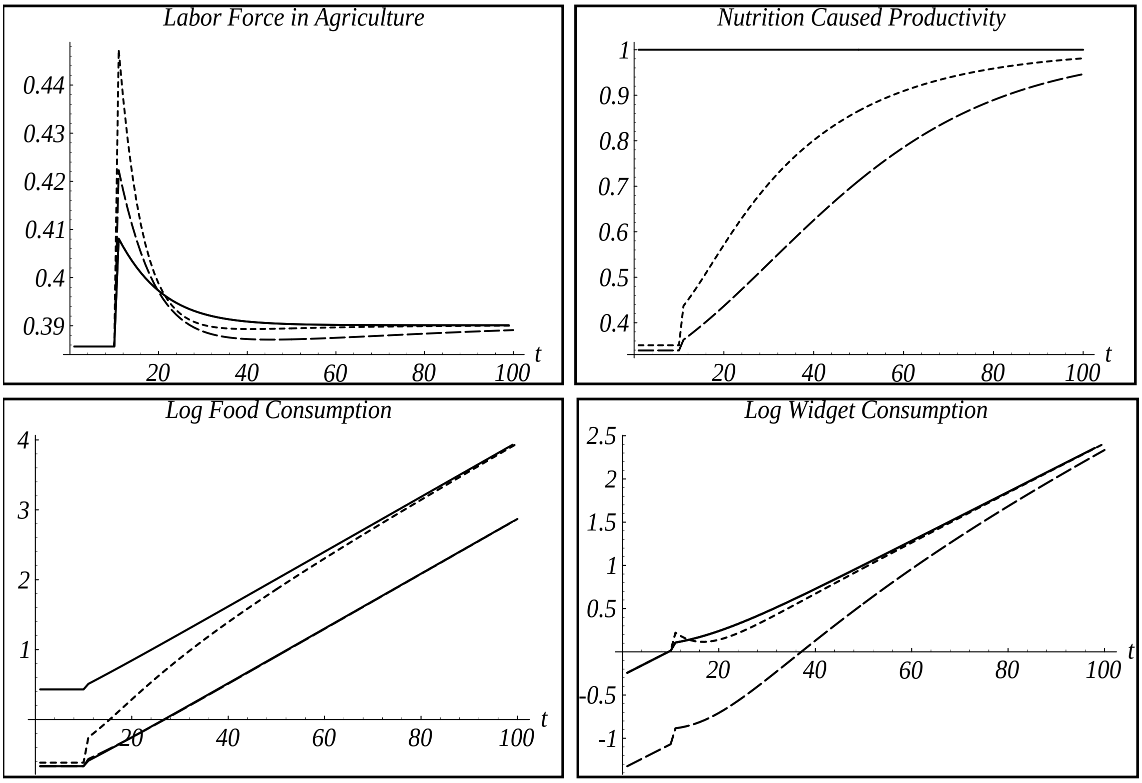
Figure 1: Static NPRs, Slow Technical Progress



Legend: solid line: baseline model(s); dotted line: agriculture effect; dashed line: industry effect.
Note: dashed and lower solid lines for food consumption coincide.

Nevertheless, several observations can be made: First of all, the simulations show that agricultural stagnation together with the existence of static NPRs does not have a large effect on the economy's structure. There is no observable difference between the values for the baseline economy and those with NPR. We can only observe the effect derived in section II, namely a rise of n after ν has been increased. Secondly, the fraction of labor in agriculture increases sharply right after the shock and subsequently decreases again slowly towards its new steady-state (with some overshooting). These peaks are larger with NPR than without. This is due to the suddenly increased marginal utility of using labor in agricultural production. For both NPRs it takes about 30 years until the economy is close to its new steady-state value for n. This is rather short compared to the time it takes the economy to reach its upper limit of nutrition caused productivity. The duration of the latter, however, is probably unrealistically long due to the functional form chosen. Thirdly, the simulations show that the consequences from nutrition-productivity relationships occur mainly in the sector where it exists. This is a consequence of the small effect on the division of labor between sectors.

Figure 2: Static NPRs, Fast Technical Progress

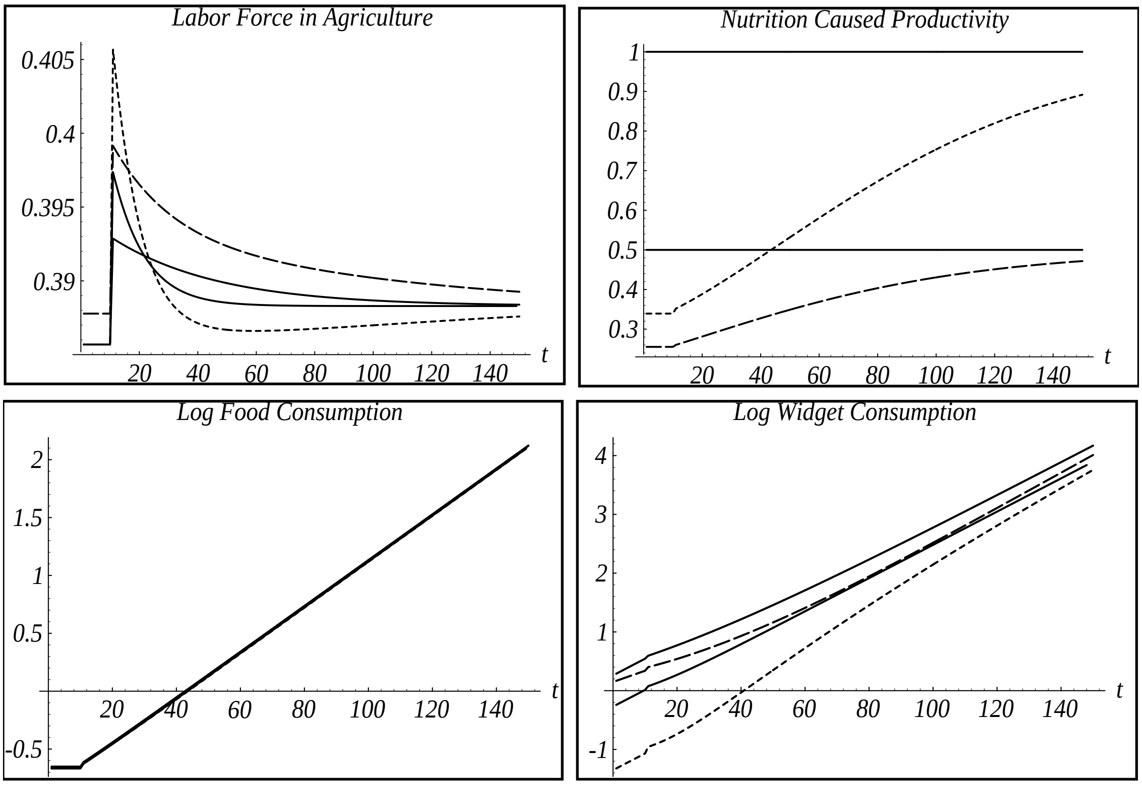


Legend: solid line: baseline model(s); dotted line: agriculture effect; dashed line: industry effect. Note: dashed and lower solid lines for food consumption coincide.

Although the structural effects are small, the output effects of both nutrition-productivity relationships are rather large. The gap between dashed or dotted lines and the solid line(s) is the forgone consumption which is lost due to the fact that productivity was not always at its highest possible level. This gap can be used to calculate the contribution of the NPR to the increase of consumption. For the static relationships this contribution can be calculated as fraction of the initial gap between solid and dashed or dotted line to the total increase of consumption over the time period considered.[22] For industry this rough calculation yields a contribution of about 6% ($\nu = 0.02$) and 5% ($\nu = 0.04$) and for agriculture of 13% ($\nu = 0.02$) and 2% ($\nu = 0.04$). Most of the values are far below those obtained by *Fogel*, even more so since he has considered a time-period twice as long as ours and the contribution of the NPR decreases as the level of (overall) productivity rises.

22. This assumes that nutrition caused productivity is at its maximum in the final period as is approximately the case in figure 2.

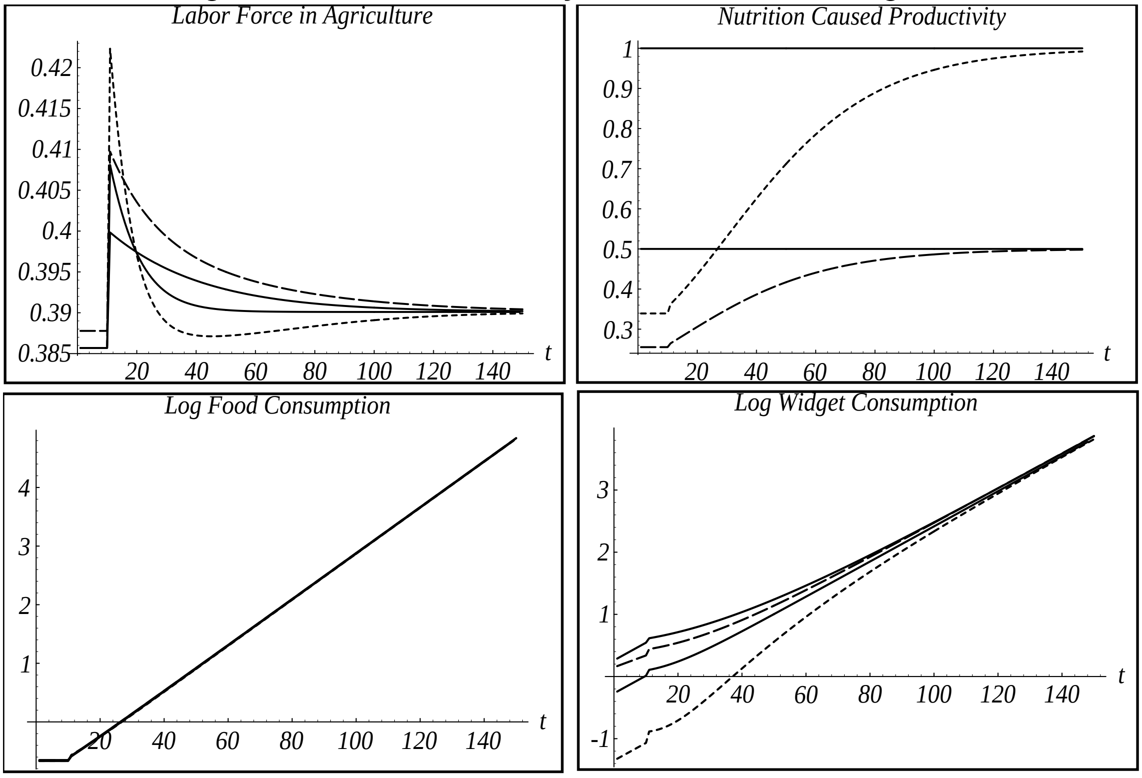
Figure 3: NPRs in Industry, Slow Technical Progress



Legend: solid line: baseline model(s); dotted line: static effect; dashed line: dynamic effect.
Note: All lines for food consumption coincide.

Figures 3 and 4 show the outcome for the static and dynamic nutrition-productivity relationships in industry together. According to these plots the static relationship seems to have larger consequences since upper solid and dashed line in the plots for widget consumption are closer together than lower solid and dotted lines. This reflects the influence of the static NPR on the *level* of widget production which does not exist in this extend for the dynamic relationship. Since the two levels of production deviate less in the latter scenario, the time-paths are closer together. However, in the long-run the dynamic effect is more important. With static nutrition caused productivity permanently below its maximum, the growth path of widget consumption would run below but parallel to the growth path characterizing an economy where $\Pi = \pi$. With a dynamic relationship in such a scenario, though, the path would be lower and flatter. In figures 3 and 4 it would be given by extending the slope of the dashed path from the pre-shock period into the future. The gap between the two paths would thus be widening over time. In addition, the rate of technical progress as well as the highest possible value for nutrition caused productivity are only half as large in the dynamic than in the static case. Using the same maximum value in the latter scenario would shift the growth path of widget consumption down. In addition cutting the rate of technical progress in half would also make the path flatter. Taking this into account, the dynamic nutrition-productivity relationship becomes even more important.

Figure 4: NPRs in Industry, Fast Technical Progress



Legend: solid line: baseline model(s); dotted line: static effect; dashed line: dynamic effect
Note: All lines for food consumption coincide.

The contribution of the increases in nutrition caused productivity to the raise in widget consumption can be calculated in the following way for the dynamic case: extend the first ten years of the dashed line into the future. This line then describes the growth path of an economy with Π fixed below its maximum. Compare the level of food consumption under this scenario with that from the simulation with dynamic NPR. After 100 years the contribution of better nutrition to the increase in widget consumption would be 55% ($\nu = 0.02$) or 50% ($\nu = 0.04$), respectively. 50 years later these numbers would have increased to 165% ($\nu = 0.02$) and 130% ($\nu = 0.04$), respectively. These values are considerably larger than those obtained for the static relationship and also larger than those obtained by *Fogel*.

VI. Conclusions

This paper has shown that a relationship between nutrition and the level of labor productivity can have considerable effects on the growth dynamics of an economy. While the simulations have shown influences on the growth rates of consumption of both goods, the influences on the economy's structure have remained negligible. The consequences of such a relationship depend very much on the sector in which it exists. An effect in agriculture primarily influences the level of food consumption while an effect in industry influences mainly output of this sector. If the nutrition productivity relationship is static, these effects are only important for a certain length of time and become negligible as total factor productivity becomes large compared to nutrition caused productivity. In addition, malnutrition has only level effects.

This is different for a dynamic nutrition-productivity relationship where better nutrition increases the productivity of the learning by doing process. Such a relationship has not only level effects but also growth effects. In a malnourished economy the growth rates of consumption remain permanently below those possible with better nutrition. The contribution of better nutrition to consumption growth is several times larger for a dynamic relationship than for a static one.

The contribution of nutrition-productivity relationships to the total increase in consumption of food or manufacturing goods implied by our model is considerably lower for the static relationships than stated by *Fogel* (1994). Under presence of dynamic effects, the model implies much larger contributions. These calculations have to be take with care, however. While *Fogel*'s calculations are based on real data, the results of the model can at best form the basis for a calibration exercise. The results are subject to the length of the time period considered, the parameter values, as well as the specific functional forms assumed.

Overall, the model has shown the usefulness of numerical simulations for the analysis of transitional dynamics. Since the long-run behavior of economies with or without NPRs is identical, means for analyzing the transitional dynamics are crucial to understand the different behavior. Analytical solutions alone are not sufficient. Since these transitional dynamics of even standard models of growth and development as well as their implications are not really well understood, improved numerical methods and decreasing computing costs will probably raise interest in these issues in the future.

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Appendix

1. Sufficiency of basic problem: *Mangasarian*'s sufficiency conditions are the following: For a problem

$$\max \int_0^\infty F(x, u)\, e^{-\rho t} dt$$
$$\text{s.t.} \quad \dot{x} = f(x, u)$$

where x denotes the vector of state variables and u the vector of control variables, the necessary conditions are also sufficient if $F(x, u)$ and $f(x, u)$ are both jointly concave in x and u and $\lambda \geq 0 \ \forall\ t$. A function $f(\mathbf{x})$ is concave on an open convex subset S in $\mathbf{R}^n$ if and only if for all $\mathbf{x} \in S$ and for all Δ_r, $(-1)^r\ \Delta_r(\mathbf{x}) \geq 0$ for $r = 1,..., n$, where the principal minors $\Delta_r(\mathbf{x})$ of order r in the Hessian matrix $\boldsymbol{f}''(\mathbf{x})$ are the determinants of the sub-matrices obtained by deleting $n - r$ arbitrary rows and then deleting the $n - r$ columns having the same numbers (*Berck* and *Sydsæter* 1991)

For the problem (4) we have

$$F(k, n, c_M) = \frac{\left[(An^{\alpha})^{\gamma} c_M^{1-\gamma} \right]^{1-\sigma}}{1-\sigma}$$

$$f(k, n, c_M) = Mk^{1-\alpha}(1-n)^{\alpha} - c_M$$

For a function to be concave the Hessian determinant must be negative semidefinite, which is the case if the principal minors change signs. Consider first $f(k, n, c_M)$:

$$H_f = \begin{bmatrix} -\alpha(1-\alpha)Mk^{-1-\alpha}(1-n)^{\alpha} & \alpha(1-\alpha)Mk^{-\alpha}(1-n)^{-1+\alpha} & 0 \\ \alpha(1-\alpha)Mk^{-\alpha}(1-n)^{-1+\alpha} & -(1-\alpha)\alpha Mk^{1-\alpha}(1-n)^{-2+\alpha} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

It can be seen from H_f that two principal minors of order one are negative since $1 - \alpha > 0$ and the third one is zero. All other principal minors are zero, too. Thus $f(k, n, c_M)$ is concave in k, n, and c_M.

Next consider $F(k, n, c_M)$. For this equation the Hessian is:

$$H_F = L\begin{bmatrix} 0 & 0 & 0 \\ 0 & \dfrac{\alpha\gamma(\alpha\gamma(1-\sigma)-1)\left(c_M^{1-\gamma}(An^{\alpha})^{\gamma}\right)^{1-\sigma}}{n^2} & \dfrac{\alpha\gamma(1-\gamma)(1-\sigma)\left(c_M^{1-\gamma}(An^{\alpha})^{\gamma}\right)^{1-\sigma}}{nc_M} \\ 0 & \dfrac{\alpha\gamma(1-\gamma)(1-\sigma)\left(c_M^{1-\gamma}(An^{\alpha})^{\gamma}\right)^{1-\sigma}}{nc_M} & \dfrac{-(1-(1-\gamma)(1-\sigma))(1-\gamma)\left(c_M^{1-\gamma}(An^{\alpha})^{\gamma}\right)^{1-\sigma}}{c_M^2} \end{bmatrix}$$

The terms on the diagonal (the principal minors of order one) are all less than or equal to zero which are the required signs. Since one of the diagonal elements is zero, only one minor of order two remains, namely:

$$\Delta_2^3 = \frac{\alpha\gamma(1-(1-\sigma)(1-\gamma+\alpha\gamma))(1-\gamma)\left(c_M^{1-\gamma}(An^{\alpha})^{\gamma}\right)^{2(1-\sigma)}}{n^2c_M^2} > 0.$$

The principal minor of order three (the Hessian's determinant) is zero. Therefore the conclusion is that *Mangasarian*'s sufficiency conditions are met by the assumptions about parameter values.

2. Stability of basic model: Stability of the steady-state equilibrium can be checked by transforming the model (6) – (9) into a system of differential equations in variables that remain constant in the steady-state, just like in section IV (cf. *Benhabib* and *Perli* (1994)). With variables $z_1 = c_M / k$, $z_2 = M / k^{\alpha}$, and n these equations are:

(A.1) $$\dot{z}_2 = z_2\left(\mu - \alpha z_2(1-n)^{\alpha-1}\left(\frac{\gamma-n}{\gamma}\right)\right).$$

(A.2) $$\dot{n} = \frac{n(1-n)\left[-\mu-\rho+(1-\sigma)(\mu(1-\gamma)+\gamma\nu)\right]}{\sigma(1-\alpha n)+(1-\sigma)\gamma(1-\alpha)} + \frac{(1-\alpha)(1-\gamma)z_2 n(1-n)^{\alpha}\dfrac{(\sigma n+\gamma(1-\sigma))}{\gamma}}{\sigma(1-\alpha n)+(1-\sigma)\gamma(1-\alpha)}$$

where $z_1 = \dfrac{(1-\gamma)}{\gamma} z_2 n(1-n)^{\alpha-1}$.

These equations have the steady-state values

(A.3) $$n^* = \frac{\gamma\left(\rho-(1-\sigma)\left(\gamma\nu+(1-\gamma)\dfrac{\mu}{\alpha}\right)+\mu\right)}{\rho-(1-\sigma)\gamma\nu+\sigma(1-\gamma)\dfrac{\mu}{\alpha}+\gamma\mu}$$

(A.4) $$z_1^* = \frac{\alpha\rho + \mu - (1-\sigma)(\alpha\gamma\nu + (1-\gamma)\mu)}{\alpha(1-\alpha)\left[\frac{(1-\gamma)(\alpha\rho + \mu - (1-\sigma)(\alpha\gamma\nu + (1-\gamma)\mu))}{\alpha\rho + \mu - (1-\sigma)(\alpha\gamma\nu + (1-\gamma)\mu) - \gamma(1-\alpha)\mu}\right]^{\alpha}}$$

The equilibrium described by (A.3) - (A.4) is unique and locally stable (or the steady-state growth path is determinate as *Benhabib* and *Perli* call it) if the system's Jacobian evaluated at the steady-state has one eigenvalue with positive and one with a negative real part. The Jacobian can be obtained from (A.1) and (A.2) and evaluated at the steady-state given by (A.3) and (A.4). However, the expression is rather complicate and the eigenvalues cannot be obtained analytically. We therefore give the analytical solution for the special case that $\sigma = 1$ and calculate eigenvalues for a large range of plausible parameter values for non-logarithmic utility numerically.

The eigenvalues of J^* are given by the solution to its characteristic equation

$$r^2 - \mathrm{Tr}\,J^* + \mathrm{Det}\,J^* = 0$$

where TrJ^* is the trace of the evaluated Jacobian J^* and DetJ^* its determinant. Instead of calculating the eigenvalues by solving the characteristic equation – which would result in a huge mess – the Routh-Hurwitz conditions may be used. According to their theorem the number of roots with positive real parts is equal to the number of variations of sign in the following scheme (see also *Benhabib* and *Perli* (1994, Theorem 1):

$$1, \quad -\mathrm{Tr}\,J^*, \quad \mathrm{Det}\,J^*$$

Determinant and trace of the above Jacobian for the special case $\sigma = 1$can be obtained as:

(A.5) $$\mathrm{Det}\,J^* = -\frac{\alpha\left[\rho + \frac{\mu}{\alpha}\right][\rho + \mu]\left[\rho + (1-\gamma+\alpha\gamma)\frac{\mu}{\alpha}\right]}{(1-\alpha)\left[(1-\gamma+\alpha\gamma(1-\alpha))\frac{\mu}{\alpha} + \rho(1-\alpha\gamma)\right]}$$

(A.6) $$\mathrm{Tr}\,J^* = \rho$$

It is easy to see that the determinant is always negative. Also the trace is strictly positive. Therefore the scheme has the order (+, -, +) implying two sign changes, and thus two eigenvalues with positive real parts. Hence, the equilibrium is locally saddle-path stable and unique.

For the more general case of non-logarithmic utility no simple analytical solutions for the Routh-Hurwitz conditions can be found. For the numerical calculations *Mathematica*'s Eigenvalue routine[23] has been employed. We have chosen $\alpha = 0.7$ and $\rho = 0.05$ as above.

For μ, ν, and γ a low and a high value are chosen to obtain results for a broad range. The variable σ is varied over the range from 0.1 to 10. The results are given in table 1.

Table 1: Saddle Path Stability for Baseline Model

γ	μ	ν	σ	γ	μ	ν	σ
0.8	0	(1-α)λ	0.1 - 10	0.4	0	(1-α)λ	0.1 - 10
		0.02	0.1 - 10			0.02	0.1 - 10
		0.04λ	0.1 - 10			0.04	0.1 - 10
	0.02	(1-α)λ	0.1 - 10		0.02	(1-α)λ	0.1 - 10
		0.02	0.1 - 10			0.02	0.1 - 10
		0.04λ	0.1 - 10			0.04	0.1 - 10

Table 1 shows that the basic dual economy model is saddle path stable for a broad range of parameter values, not only for the special case of σ = 1. We can therefore quite safely rule out the possibility of multiple equilibria or instability for reasonable parameters. Note that this does not mean that multiple equilibria or instability are impossible.

23. See *Wolfram* (1991).