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Abstract

This paper analyzes equilibria in hedonic economies and presents conditions for identifying structural preference and technology parameters with nonadditive marginal utility and marginal product functions. The nonadditive class is very general, allows for heterogeneity in the curvature of consumer utility, and can result in bunching. Such bunching has largely been ignored in the previous literature. The paper presents methods to identify and estimate marginal utility and marginal product functions that are nonadditive in the unobservable random terms, using observations from a single hedonic market. The new methods for nonadditive models are useful when statistical tests reject additive specifications or when prior information suggests that consumer or firm heterogeneity in the curvature of utility or production functions is likely to be important. The paper provides conditions under which nonadditive marginal utility and marginal product functions are nonparametrically identified, and proposes nonparametric estimators for them. The estimators are consistent and asymptotically normal. The paper also formalizes and extends existing results in the literatures on identifying structural parameters using multimarket data.

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1 Introduction

In hedonic models, the price of a product is a function of the attributes that characterize the product. These models are also used to estimate consumer preferences for the attributes of goods, and to determine how much consumers value attributes and the variability in consumer valuations. Hedonic models have been used to study prices for job safety, environmental quality, school quality, and automobile fuel efficiency among other applications.

In seminal papers, Tinbergen (1956) and Rosen (1974) pioneered the theoretical and empirical study of hedonic models in perfectly competitive settings. In their models, an economy is specified by a distribution of buyers and a distribution of sellers. A buyer could be a consumer buying a product or a firm buying labor services. A seller could be a firm selling a product or a worker selling labor services.

To focus the discussion, but without any loss of generality, this paper focuses on a labor market interpretation. In equilibrium, each buyer is matched with a seller. Each buyer (firm) is characterized by a profit function that depends on the attributes characterizing the product, as well as on firm characteristics (endowments; productivity and the like). Each seller (worker) is characterized by a utility function that depends on the attributes characterizing the product, as well as on some characteristics of the worker (preference parameters, endowments and the like). Given a price function for the attributes, each buyer demands the vector of attributes that maximizes profits, and each seller supplies the vector of attributes that maximizes utility. The equilibrium price function is such that the distribution of demand equals the distribution of supply for all values of the attributes. When the production and utility functions are quadratic and the heterogeneity variables are normal, the model has a closed form solution, where the equilibrium *marginal* price function is linear in the attributes. This specification was first studied by Tinbergen (1956).

Rosen (1974) suggested a two stage method to estimate preferences and technologies in hedonic models based on linear approximations to the true model. His method first estimates the marginal price function. Then he uses the first order conditions of the buyers and sellers to estimate the profit and utility functions.

Influential papers by Brown and Rosen (1982) and Brown (1983) sharply criticized the method of identification proposed by Rosen (See also Epple, 1987 and Kahn and Lang, 1988.). Using linear approximations to buyer and seller first order conditions and to the equilibrium marginal price function, Brown and Rosen argue that hedonic models are not identified using data from a single market. They claim that sorting implies that there are no natural exclusion restrictions within a single market.

Ekeland, Heckman, and Nesheim (2004) show that Brown and Rosen's nonidentification result is a consequence of their arbitrary linearization. The linear case analyzed by Brown and Rosen is nongeneric and is exactly the case suggested by Tinbergen (1956). The Tinbergen model is not identified in a single cross section.

Ekeland et al. analyze a hedonic model with additive marginal utility and additive marginal product functions. They show that these parameters are identified from single market data. They present two methods for recovering the functions. One is based on extensions of average derivative models (Powell, Stock, and Stoker, 1989) and transformation models (Horowitz, 1996, 1998). The other is based on nonparametric instrumental variables (Darolles, Florens, and Renault, 2003; Newey and Powell, 2003). The performance of those estimators is studied in Heckman, Matzkin, and Nesheim (2005).

The additivity restrictions used to establish identification in Ekeland, Heckman, and Nesheim (2004) impose strong restrictions. No heterogeneity in the curvature of production and preference functions is tolerated. Allowing for such heterogeneity in curvature is an important theoretical generalization of the additive model. In this paper, we consider identification of hedonic equilibrium models where the marginal utility and marginal product functions are nonadditive in the unobserved variables.

General nonadditive production and utility functions are not identified using data from a single market without invoking further conditions. We provide conditions under which the nonadditive marginal utility and nonadditive marginal production function are identified from the equilibrium price function, the distribution of demanded attributes conditional on the observed characteristics of the consumers, and the distribution of supplied attributes conditional on the observed characteristics of the firms. Our identification analysis proceeds as follows. First, using methods in Matzkin (1999, 2003), we show that we can identify the demand and supply functions for attributes. They are nonparametric, nonadditive functions of the observable and unobservable characteristics of, respectively, the firms and workers. This first step requires no additional assumptions beyond what she assumes. Second, we use the demand and supply functions, together with the equilibrium price function, and the restrictions imposed by the first order conditions to recover the marginal utility and marginal product functions. This second step requires an assumption on the marginal utility and marginal product functions, which reduces the number of free arguments in these functions. We provide several alternative specifications, propose nonparametric estimators for the marginal utility and marginal product functions, and show that they are consistent and asymptotically normal.

Identification of the demand and supply functions allows one to predict partial equilibrium impacts of changes in individual level observables on individual choices in a hedonic market. For example, in a market for jobs with varying levels of risk of injury, one can predict the impact of changes in education on individual choices. Conditional on education, one can also predict differences in choices for different quantiles of the distribution of unobservable heterogeneity. Such predictions hold other variables and the hedonic equilibrium price fixed.

Identification of the demand and supply functions however does not allow one to measure the welfare impacts of changes nor to predict general equilibrium effects. An upper bound on welfare impacts can be computed using hedonic prices (see Scotchmer (1985) and Kanemoto (1988)). However, identification of the structural marginal utility and marginal product functions allows one to do better. In addition, identification of these functions allows one to predict general equilibrium effects of policy and environment changes. We develop these points in more detail in Section 4.

We also show that more general nonadditive marginal production and utility functions are identified using data from multiple markets. The identification result makes use of the fact that, in general, differences in the distributions of observable variables across markets will result in price function variation across markets. This variation is an implication of equilibrium in hedonic models. The price function variation and its dependence on market level observables can be used to identify marginal utility and marginal product functions. This result formalizes and extends discussions in Rosen (1974), Brown and Rosen (1982), Epple (1987) and Kahn and Lang (1988) who discuss how to use multimarket data to identify structural parameters in hedonic models.

We also analyze equilibria in hedonic economies and study conditions that generate equilibria with bunching, i.e., in which positive masses of consumers and firms locate at a common location.[1] The conditions that lead to bunching are related to the conditions that generate bunching in non-competitive nonlinear pricing models (see for example Mussa and Rosen, 1978; Guesnerie and Laffont, 1984; Rochet and Stole, 2003) and in other competitive sorting models (Nesheim, 2001, 2004). In all cases, a Spence-Mirlees like single-crossing condition is sufficient to rule out bunching in the interior. Failure of such a condition may lead to bunching. In a competitive hedonic model, an additional consideration plays a role. Both buyers and sellers must bunch at the same point.

[1] In the cases we consider, equilibrium exists. See Gretsky, Ostroy, and Zame (1992, 1999) and Ekeland (2005).

The demand estimation techniques developed in this paper build on a long line of research on models with unobserved heterogeneity. Estimation of demand models generated by random utility functions have been studied in the past using parametric assumptions (Heckman, 1974; McFadden, 1974; Heckman and Willis, 1977), semiparametric assumptions (Manski, 1975, 1985; Cosslett, 1983; Matzkin, 1991b; Horowitz, 1992; Klein and Spady, 1993; Ichimura and Thompson, 1998, among others), and more recently, using nonparametric assumptions (Matzkin, 1992, 1993; Briesch, Chintagunta, and Matzkin, 1997; Brown and Matzkin, 1998; Horowitz, 2001; McFadden and Train, 2000; Blomquist and Newey, 2002, among others). McElroy (1981, 1987), Brown and Walker (1989, 1995) and Lewbel (1996) considered inference for random utility and random production functions in perfectly competitive, non-hedonic situations.

Work on nonadditive models also has a long lineage. Nonparametric estimation of models with nonadditive random terms has been previously studied in Matzkin (1991a), Olley and Pakes (1996), Altonji and Ichimura (1999), Altonji and Matzkin (2001, 2005), Briesch, Chintagunta, and Matzkin (1997), Brown and Matzkin (1998), Heckman and Vytlacil (1999, 2001), Matzkin (1999, 2003), Vytlacil (2002), Blundell and Powell (2004), and, more recently, by Bajari and Benkard (2001), Chesher (2001), Hong and Shum (2001), and Imbens and Newey (2002).

This paper proceeds in the following way. Section 2 describes the hedonic model for a product with a single attribute. Section 3 discusses the properties of equilibrium in hedonic models and provides several analytic examples of hedonic equilibria generated by nonadditive functions both with and without bunching. Section 4 studies the identification of nonadditive marginal utility and nonadditive marginal product functions. Section 5 discusses identification using multi-market data. Section 6 presents nonparametric estimators and their asymptotic properties for the single market case. Section 7 presents results from Monte Carlo analysis of the estimators of the model. Section 8 concludes.

2 The competitive hedonic equilibrium model

Consider a labor market setting in which jobs are characterized by their attributes. The analysis applies equally well to any spot market in which products are differentiated by their attributes, prices are set competitively and participating buyers and sellers each trade a single type of product chosen from a set of feasible products. We first present an analysis that assumes that almost all participating agents optimally choose a point (or a location) in the interior of the set of feasible job attributes. This is the framework that is most frequently assumed in empirical studies of hedonic markets. In this section, we focus on equilibria with no bunching. We focus on equilibria in which no positive measure of agents choose the same job.[2] This is the conventional starting point for a competitive hedonic model. We defer our discussion of bunching until section 3.2.

Workers (sellers) match to single worker firms (buyers). Let z denote a scalar attribute characterizing jobs, assumed to be a disamenity for the workers and an input for the firms.[3] For example, z could measure the risk of injury on the job as in Kniesner and Leeth (1995). We assume that $z \in \widetilde{Z} = [z_L, z_H] \subseteq R$ where $\widetilde{Z}$ could be the entire real line. The space $\widetilde{Z}$ is the space of technologically feasible job attributes.[4] Let $P(z)$ be a twice continuously differentiable price function. The value of $P(z)$ is the wage paid at a job with attribute z. Each worker has quasilinear utility function $P(z) - U(z, x, \varepsilon)$ where x is a vector of observable characteristics of the consumer of dimension n_x and ε is a scalar unobservable heterogeneity term.[5] We assume that ε is statistically independent of x. The population of workers is described by the pair of density functions f_x and f_ε strictly positive on $\widetilde{X} \subseteq R^{n_x}$ and $\widetilde{E} \subseteq R$ respectively. Additionally, each worker may opt out of the market (or choose not to trade) in which case they obtain reservation utility V_0.

Each firm has a production function $\Gamma(z, y, \eta)$ where y is a vector of observable characteristics of the firm of dimension n_y and η is a scalar unobservable heterogeneity term. We assume that η is statistically independent of y and that (y, η) are independent of (x, ε). The population of firms is described by the pair of density functions f_y and f_η strictly positive on $\widetilde{Y} \subseteq R^{n_y}$ and $\widetilde{H} \subseteq R$ respectively. If a firm opts out of the market, it earns reservation profits Π_0. Both U and Γ are assumed to be twice continuously differentiable with respect to all arguments.

Each consumer chooses $z \in \widetilde{Z}$, a job type or a location in the space of job attributes, to maximize

$$P(z) - U(z, x, \varepsilon).$$

[2] However, a positive measure may choose not to participate in the market, for example, when there are more workers than firms.

[3] This does not rule out that jobs may be characterized by multiple attributes. The one dimensional attribute z could be an index of job "quality" that is produced by a higher dimensional vector of attributes.

[4] In the Kniesner and Leeth analysis, $\widetilde{Z} = [0, 1]$.

[5] This is an economy with transferable utility. The econometric analysis in this paper can easily be adapted to the case where utility takes the form $U^*(P(z) + R, z, x, \varepsilon)$ where R is nonlabor income.

The first and second order conditions for an interior optimizer are

$$\begin{array}{ll} \text{FOC} & P_z(z) - U_z(z, x, \varepsilon) = 0 \\ \text{SOC} & P_{zz}(z) - U_{zz}(z, x, \varepsilon) < 0 \end{array}$$

where P_z and P_{zz} denote the first and second derivatives of P with respect to z and U_z and U_{zz} denote the first and second order partial derivatives of U with respect to z.

Assume a unique interior optimizer exists for almost all workers in equilibrium.[6] By the Implicit Function Theorem and SOC, there exists a function $z = s(x, \varepsilon)$ such that

$$P_z(s(x, \varepsilon)) - U_z(s(x, \varepsilon), x, \varepsilon) = 0. \tag{2.1}$$

Moreover,

$$\frac{\partial s(x, \varepsilon)}{\partial \varepsilon} = \frac{U_{z\varepsilon}(s(x, \varepsilon), x, \varepsilon)}{P_{zz}(s(x, \varepsilon)) - U_{zz}(s(x, \varepsilon), x, \varepsilon)}$$

so that $\frac{\partial s(x,\varepsilon)}{\partial \varepsilon} > 0$ if $U_{z\varepsilon} < 0$.

It is clarifying to substitute out for ε in terms of observables in these expressions. Let $\widetilde{s}(z, x)$ denote the inverse of s with respect to ε. This can be obtained directly from FOC assuming $U_{z\varepsilon} \neq 0$. Substituting back into FOC we obtain

$$P_z(z) - U_z(z, x, \widetilde{s}(z, x)) = 0.$$

Since $U_{z\varepsilon} \neq 0$, $\widetilde{s}(z, x)$ is a differentiable function (since we have assumed that P_z is continuously differentiable) and

$$\frac{\partial \widetilde{s}(z, x)}{\partial z} = \frac{P_{zz}(z) - U_{zz}(z, x, \widetilde{s}(z, x))}{U_{z\varepsilon}(z, x, \widetilde{s}(z, x))} \tag{2.2}$$

so that $\frac{\partial \widetilde{s}(z,x)}{\partial z} > 0$ if $U_{z\varepsilon} < 0$. In this section, we assume that $U_{z\varepsilon}(z, x, \varepsilon) < 0$ for all (z, x, ε).

A parallel analysis can be performed for the other side of the market. Each firm chooses $z \in \widetilde{Z}$ to maximize the profit function

$$\Gamma(z, y, \eta) - P(z).$$

[6] Ekeland (2005) provides sufficient conditions for this condition to be satisfied. For example, if the distributions of buyer and seller types are absolutely continuous with respect to Lesbesgue measure, $U_{z\varepsilon} < 0$, and $\Gamma_{z\eta} > 0$, then the conditions are met and each agent has at most one interior optimizer. We study some examples that relax these assumptions in the next section of this paper.

The first and second order conditions for an interior optimizer are

$$\begin{array}{ll} \text{FOC} & \Gamma_z(z,y,\eta) - P_z(z) = 0 \\ \text{SOC} & \Gamma_{zz}(z,y,\eta) - P_{zz}(z) < 0 \end{array}.$$

Assuming a unique interior optimizer exists for almost all firms, there exists a function $z = d(y,\eta)$ such that

$$\Gamma_z(d(y,\eta), y, \eta) - P_z(d(y,\eta)) = 0.$$

Moreover,

$$\frac{\partial d(y,\eta)}{\partial \eta} = \frac{\Gamma_{z\eta}(d(y,\eta), y, \eta)}{P_{zz}(d(y,\eta)) - \Gamma_{zz}(d(y,\eta), y, \eta)}$$

so that $\frac{\partial d(y,\eta)}{\partial \eta} > 0$ if $\Gamma_{z\eta} > 0$. We substitute out for η in terms of observables using $\widetilde{d}(z,y)$ for the inverse of d with respect to η. Substituting back into the firm's first order conditions we obtain

$$\Gamma_z(z, y, \widetilde{d}(z,y)) - P_z(z) = 0.$$

If $\Gamma_{z\eta}(d(y,\eta), y, \eta) \neq 0$, then $\widetilde{d}(z,y)$ is a differentiable function and

$$\frac{\partial \widetilde{d}(z,y)}{\partial z} = \frac{P_{zz}(z) - \Gamma_{zz}(z, y, \widetilde{d}(z,y))}{\Gamma_{z\eta}(z, y, \widetilde{d}(z,y))}, \tag{2.3}$$

so that $\frac{\partial \widetilde{d}(z,y)}{\partial z} > 0$ if $\Gamma_{z\eta} > 0$. In this section, we assume that $\Gamma_{z\eta} > 0$ for all (z, y, η).

In equilibrium, the density of the supplied z must equal the density of the demanded z for all values of $z \in \widetilde{Z}$. To express this condition in terms of the primitive functions, consider the transformation

$$z = s(x,\varepsilon) \;\;\&\;\; x = x$$

for all $(x,\varepsilon) \in \widetilde{X} \times \widetilde{E}$. Let

$$Z_s = \left\{ z \in \widetilde{Z} \,\middle|\, z = s(x,\varepsilon) \text{ for some } (x,\varepsilon) \in \widetilde{X} \times \widetilde{E} \right\}$$

be the range of the mapping $s(x,\varepsilon)$. For all $z \in Z_s$ and all $x \in \widetilde{X}$, the inverse of this transformation is

$$\varepsilon = \widetilde{s}(z,x) \quad \& \;\; x = x$$

and the Jacobian determinant is

$$\begin{vmatrix} \frac{\partial \widetilde{s}(z,x)}{\partial z} & \frac{\partial \widetilde{s}(z,x)}{\partial x} \\ 0 & 1 \end{vmatrix} = \frac{\partial \widetilde{s}\left(z,x\right)}{\partial z}.$$

Since $U_{z\varepsilon} < 0$, equation (2.2) implies $\frac{\partial \widetilde{s}(z,x)}{\partial z} > 0$. Using the densities of x and ε, this mapping defines the density of the supplied z. This density is

$$\int_{\widetilde{X}} f_{\varepsilon}\left(\widetilde{s}\left(z,x\right)\right) f_x(x) \frac{\partial \widetilde{s}\left(z,x\right)}{\partial z} dx \tag{2.4}$$

for $z \in Z_s$. For $z \in \widetilde{Z} \setminus Z_s$, the density of supply is zero.

The density of the demanded z is obtained by a parallel argument. Consider the transformation

$$z = d(y,\eta) \quad \& \quad y = y$$

for all $(y,\eta) \in \widetilde{Y} \times \widetilde{H}$. Let

$$Z_d = \left\{ z \in Z \middle| z = d\left(y,\eta\right) \text{ for some } \left(y,\eta\right) \in \widetilde{Y} \times \widetilde{H} \right\}$$

be the range of the mapping $d\left(y,\eta\right)$. For all $z \in Z_d$ and all $y \in \widetilde{Y}$, the inverse of this transformation is

$$\eta = \widetilde{d}\left(z,y\right) \quad \& \quad y = y$$

and the Jacobian determinant is

$$\begin{vmatrix} \frac{\partial \widetilde{d}(z,y)}{\partial z} & \frac{\partial \widetilde{d}(z,y)}{\partial y} \\ 0 & 1 \end{vmatrix} = \frac{\partial \widetilde{d}\left(z,y\right)}{\partial z}.$$

Since $\Gamma_{z\eta} > 0$, equation (2.3) implies $\frac{\partial \widetilde{d}(z,y)}{\partial z} > 0$. The density of the demanded z is

$$\int_{\widetilde{Y}} f_{\eta}\left(\widetilde{d}\left(z,y\right)\right) f_y(y) \frac{\partial \widetilde{d}\left(z,y\right)}{\partial z} dy \tag{2.5}$$

for $z \in Z_d$. For $z \in \widetilde{Z} \setminus Z_d$, the density of demand is zero.

Expressions (2.4) and (2.5) give the densities of supply and demand respectively for an arbitrary smooth price function that yields unique interior optimizers for almost all workers and firms. Among the set of smooth price functions that yield unique interior optimizers, an equilibrium price function must satisfy the equilibrium condition that the density of supply equals the density of demand, for all values of $z \in \widetilde{Z}$. This condition requires that $Z_s = Z_d$ and that

$$\int_{\widetilde{X}} f_\varepsilon\left(\widetilde{s}\left(z,x\right)\right) f_x(x) \frac{\partial \widetilde{s}\left(z,x\right)}{\partial z} dx = \int_{\widetilde{Y}} f_\eta\left(\widetilde{d}\left(z,y\right)\right) f_y(y) \frac{\partial \widetilde{d}\left(z,y\right)}{\partial z} dy \tag{2.6}$$

for all $z \in Z = Z_s = Z_d$. Equation (2.6) is a second order differential equation in P. A smooth price function defined on $\widetilde{Z}$ that yields well defined inverse supply and demand functions, that satisfies $Z_s = Z_d$, and (2.6) is an equilibrium price function. In this paper, we assume such an equilibrium price function exists and study its theoretical and empirical properties.

If some agents are indifferent between multiple locations, the equilibrium condition (2.6) must be modified. Indifferent consumers or firms must be assigned to locations in proportions sufficient to maintain equality between demand and supply distributions at all locations. For example, if all firms are identical, $\Gamma\left(z,y,\eta\right) = \Gamma\left(z\right)$ for all $\left(z,y,\eta\right)$, the equilibrium price is $P\left(z\right) = \Gamma\left(z\right)$ for $z \in \tilde{Z}$. In this case, equilibrium supply is as above, but equilibrium demand is a correspondence. The assignment of firms to locations is not unique. Equilibrium requires firms to be assigned to locations so that the distribution of demand equals the distribution of supply. For example, the differentiable function $\widetilde{d}\left(z,y\right)$ satisfying (2.6) with $\frac{\partial \widetilde{d}(z,y)}{\partial z} > 0$, is an equilibrium assignment. For an analysis of existence and uniqueness of equilibria under very general conditions, see Ekeland (2005). The next section analyzes some properties of this equilibrium. It also considers when bunching will arise.

3 Properties of equilibrium

This section discusses the curvature of the equilibrium price, bunching on the boundary and on the interior of the space of feasible job types, and demand predictions and welfare calculations

3.1 Curvature of the equilibrium price

Let P_z be the derivative of an equilibrium price function and assume that almost all consumers are interior optimizers. Let $\widetilde{d}$ and $\widetilde{s}$ be the inverse functions associated with the demand and supply functions derived under the assumption that $U_{z\varepsilon} < 0$ and $\Gamma_{z\eta} > 0$. Assume scalar heterogeneity. Under these conditions, when we substitute equations $\tilde{d}$ and $\tilde{s}$

and use equations (2.2) and (2.3) in equilibrium equation (2.6) and solve for P_{zz}, we obtain (after simplification)[7]

$$P_{zz} = \frac{\int_{\widetilde{X}} \int_{\widetilde{Y}} \left[\frac{f_\varepsilon}{-U_{z\varepsilon}} U_{zz} + \frac{f_\eta}{\Gamma_{z\eta}} \Gamma_{zz} \right] f_x f_y dx dy}{\int_{\widetilde{Y}} \left[\frac{f_\varepsilon}{-U_{z\varepsilon}} + \frac{f_\eta}{\Gamma_{z\eta}} \right] f_x f_y dx dy}. \tag{3.1}$$

We have suppressed the arguments of the functions to simplify the notation. The expression shows that the curvature of the equilibrium price function is a weighted average of $\Gamma_{zz}\left(z, y, \widetilde{d}(y, z)\right)$, the curvatures of firms' technologies and $U_{zz}\left(z, x, \widetilde{s}(z, x)\right)$, the curvatures of workers' utilities. The relevant curvatures to include in the weighted sum are the values at equilibrium. The relevant weights are positive and are determined by the distributions of worker and firm heterogeneity and by the second derivative terms $\Gamma_{z\eta}\left(z, y, \widetilde{d}(z, y)\right)$ and $U_{z\varepsilon}\left(z, x, \widetilde{s}(z, x)\right)$.

Consider, as a special case, the additive marginal return specification studied in Ekeland, Heckman, and Nesheim (2004) where, for some functions m_w, ϕ_w, m_f, and ϕ_f,

$$U_z(z, x, \varepsilon) = m_w(z) + \phi_w(x) - \varepsilon$$

$$\Gamma_z(z, y, \eta) = m_f(z) + \phi_f(y) + \eta.$$

Worker and firm heterogeneity shift the levels of the marginal utilities and marginal products but do not affect the curvatures. In this case, equation (3.1) simplifies to

$$P_{zz} = \frac{m_w'(z) \int_{\widetilde{X}} f_\varepsilon\left(\widetilde{s}(z, x)\right) f_x(x) dx + m_f'(z) \int_{\widetilde{Y}} f_\eta\left(\widetilde{d}(z, y)\right) f_y(y) dy}{\int_{\widetilde{X}} f_\varepsilon\left(\widetilde{s}(z, x)\right) f_x(x) dx + \int_{\widetilde{Y}} f_\eta\left(\widetilde{d}(z, y)\right) f_y(y) dy} \tag{3.2}$$

since $U_{zz} = m_w'(z)$, $\Gamma_{zz} = m_f'(z)$ and $-U_{z\varepsilon} = \Gamma_{z\eta} = 1$. The curvature of the hedonic price function is the weighted average of $m_w'(z)$ and $m_f'(z)$, the Hessians of worker preferences and firm technologies respectively. The weights depend on the relative magnitudes of the densities. In limiting cases, $P_{zz} = m_f'(z)$ if firms are homogenous or $P_{zz} = m_w'(z)$ if workers are homogenous.[8]

[7] This expression was also derived in Ekeland, Heckman, and Nesheim (2004).

[8] The required limit operations are not developed in this paper. They require that we collapse the distributions of (x, ε) of (y, η) to point masses. To establish the claimed result it is easier to make a direct argument as in Rosen (1974). He uses a zero profit condition for the firm with all firms being alike to trace out an isoprofit contour for different z. The gradient of the contour is the hedonic function.

In the general additive case, we have the following theorem that guarantees that all potential firms and workers participate and that (3.2) characterizes the curvature of the price function.

Theorem 3.1 *Suppose that there exists some pair* (z_0, z_1) *with* $z_L \leq z_0 < z_1 \leq z_H$ *such that*

(i)

$$m_w(z_0) + \max_{(x,\varepsilon)} \{\phi_w(x) - \varepsilon\} = m_f(z_0) + \min_{(y,\eta)} \{\phi_f(y) + \eta\},$$

$$m_w(z_1) + \min_{(x,\varepsilon)} \{\phi_w(x) - \varepsilon\} = m_f(z_1) + \max_{(y,\eta)} \{\phi_f(y) + \eta\},$$

and

(ii)

$$m'_f(z) < m'_w(z) \text{ for all } z \in \widetilde{Z}.$$

Then, the right side of (3.2) is well defined, (3.2) has a unique solution with initial condition $P_z(z_0) = m_w(z_0) + \max_{(x,\varepsilon)} \{\phi_w(x) - \varepsilon\}$, *and the solution is an equilibrium price function under which almost all agents have a unique interior optimum.*

Proof. The first condition *(i)* guarantees that $\Gamma_z(z, y, \eta) = U_z(z, x, \eta)$ has a solution for all $z \in [z_0, z_1] \subseteq \widetilde{Z}$ and for all $(y, \eta, x, \varepsilon)$. The second guarantees that all agents have a unique interior optimum. ■

The first condition guarantees that each pair of buyers and sellers can find some point z such that their indifference curves "kiss" as in Rosen (1974). The second condition guarantees that at such a point of tangency, the second order conditions for both agents are satisfied. In contrast, if $m'_f(z) > m'_w(z)$ for all $z \in \widetilde{Z}$, then (3.2) is not well defined because for any price function the SOC is violated for workers, for firms, or both.[9] In this case, given a price satisfying

$$m'_w(z) \leq P_{zz}(z) \leq m'_w(z),$$

the equilibrium condition requires that demand equals supply at z_L and z_H.

More generally, if we have an equilibrium in which all agents choose an interior optimum, then using the worker and firm SOC we have

$$\Gamma_{zz}(z, y, \widetilde{d}(z, y)) < U_{zz}(z, x, \widetilde{s}(z, x)). \tag{3.3}$$

In words, at every location z in the job attribute space, the curvature of the profit function

[9] If $m'_f(z) > m'_w(z)$ for all $z \in \widetilde{Z}$ and $\widetilde{Z} = R$, then no equilibrium exists.

of every firm choosing location z in equilibrium must be less than the curvature of the utility function of every worker choosing location z. The conditions ensure that every worker and firm who match at a point z because their indifference curves are mutually tangent are actually maximizing and not minimizing. Strong sufficient conditions to guarantee an interior maximum are

$$\Gamma_z(z_L, y, \eta) > U_z(z_L, x, \varepsilon) \text{ for almost all } (y, \eta, x, \varepsilon)$$

$$\Gamma_z(z_H, y, \eta) < U_z(z_H, x, \varepsilon) \text{ for almost all } (y, \eta, x, \varepsilon)$$

and

$$\Gamma_{zz}(z, y, \eta) < U_{zz}(z, x, \varepsilon)$$

for almost all $(z, y, \eta, z, \varepsilon)$. These conditions guarantee that for almost all pairs $\{(y, \eta), (x, \varepsilon)\}$, a point z exists such that $\Gamma_z(z, y, \eta) = U_z(z, x, \varepsilon)$ and z is an optimum for both (y, η) and (x, ε) for some price function. An example illustrates our analysis.

3.1.1 A nonadditive example

To provide a tractable example of a nonadditive economy, suppose that all heterogeneity across firms is represented by a scalar variable η and all heterogeneity across consumers is represented by a scalar variable ε. Workers and firms choose $z \in [0, \infty)$ to maximize $P(z) - \varepsilon^{-1} z^{\beta}$ and $z^{\alpha}\eta - P(z)$ respectively. Assume $0 < \alpha < \beta$ and that $F_{\varepsilon}(\varepsilon) = \frac{\varepsilon - \varepsilon_L}{\varepsilon_H - \varepsilon_L}$ and $F_{\eta}(\eta) = \frac{\eta - \eta_L}{\eta_H - \eta_L}$ so that ε and η are distributed uniformly on the respective intervals. Further assume that $\varepsilon_L = \eta_L$ and $\varepsilon_H = \eta_H$. In this economy, workers with low disutility of work (high ε) sort into high productivity (high η) firms. Exploiting the positive assortative matching, the integrated form of equilibrium condition (2.6) for this model is

$$F_{\varepsilon}\left(\frac{\beta\, z^{\beta-1}}{P_z(z)}\right) = F_{\eta}\left(\frac{P_z(z)}{\alpha z^{\alpha-1}}\right)$$

for all z such that $\varepsilon_L \leq \frac{\beta\, z^{\beta-1}}{P_z(z)} \leq \varepsilon_H$. Thus, the slope of any equilibrium price function must satisfy

$$P_z(z) = (\alpha\beta)^{\frac{1}{2}}\, z^{\frac{\alpha+\beta}{2}-1}$$

for all z such that $\varepsilon_L^{\frac{2}{\beta-\alpha}} \left(\frac{\alpha}{\beta}\right)^{\frac{1}{\beta-\alpha}} = z_0 \leq z \leq z_1 = \varepsilon_H^{\frac{2}{\beta-\alpha}} \left(\frac{\alpha}{\beta}\right)^{\frac{1}{\beta-\alpha}}$. The subset of $\widetilde{Z}$ with positive density of demand is $Z = [z_0, z_1]$ where $z_0 = \varepsilon_L^{\frac{2}{\beta-\alpha}} \left(\frac{\alpha}{\beta}\right)^{\frac{1}{\beta-\alpha}}$ and $z_1 = \varepsilon_H^{\frac{2}{\beta-\alpha}} \left(\frac{\alpha}{\beta}\right)^{\frac{1}{\beta-\alpha}}$.

These conditions pin down the slope of an equilibrium price function on the interior of the interval $[z_0, z_1]$. They imply that, on this interval, any equilibrium price function must satisfy

$$P(z) = P_0 + 2\frac{(\alpha\beta)^{\frac{1}{2}}}{(\alpha+\beta)} z^{\frac{\alpha+\beta}{2}} \tag{3.4}$$

where P_0 is a constant. Assuming all agents trade, the value of the constant must satisfy

$$P_0 + 2\frac{(\alpha\beta)^{\frac{1}{2}}}{(\alpha+\beta)} z_0^{\frac{\alpha+\beta}{2}} - \varepsilon_L^{-1} z_0^{\beta} \geq V_0$$

$$\eta z_0^{\alpha} - P_0 - 2\frac{(\alpha\beta)^{\frac{1}{2}}}{(\alpha+\beta)} z_0^{\frac{\alpha+\beta}{2}} \geq \Pi_0.$$

We assume the reservation values are low enough to ensure that all agents trade. In general, the constant P_0 is not uniquely determined. With an equilibrium price function satisfying these conditions, the supply and demand functions are

$$s(\varepsilon) = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{\beta-\alpha}} \varepsilon^{\frac{2}{\beta-\alpha}}$$

$$d(\eta) = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{\beta-\alpha}} \eta^{\frac{2}{\beta-\alpha}}.$$

These supply and demand functions are uniquely determined.

Outside the interval $[z_0, z_1]$, prices are not uniquely determined. However, we can define a set of admissible price functions any element of which supports the equilibrium in which no agents choose to trade outside the interval $[z_0, z_1]$. Let $P(z)$ be a function satisfying equation (3.4) for all $z \in [z_0, z_1]$. Then $P(z)$ is an equilibrium price function if

$$\sup_{\eta} \{\eta z^{\alpha} - \Pi(\eta)\} \leq P(z) \leq \inf_{\varepsilon} \left\{V(\varepsilon) + \varepsilon^{-1} z^{\beta}\right\} \tag{3.5}$$

for all $z \notin [z_0, z_1]$ where

$$\Pi(\eta) = \eta (d(\eta))^{\alpha} - P(d(\eta))$$

$$V(\varepsilon) = P(s(\varepsilon)) - \varepsilon^{-1} (s(\varepsilon))^{\beta}.$$

The functions $\Pi(\eta)$ and $V(\varepsilon)$ describe the equilibrium levels of profits and utility obtained by agents with different values of η and ε respectively. Any price function that satisfies the inequalities in (3.5) will support the equilibrium because almost all agents weakly prefer some $z \in [z_0, z_1]$ to all $z \notin [z_0, z_1]$. Outside the interval $[z_0, z_1]$, a price $P(z)$ can be thought of as a shadow price. It is never observed because no agents choose to trade there. However, it is necessary to support equilibrium.

Because of the positive assortative matching and the identical distributions of firms and workers, the equilibrium matching condition in this economy is $\eta = \varepsilon$. In the equilibrium, almost all agents choose a point in the interior of the space of attributes. We now consider a case with bunching on the boundary.

3.2 Bunching on the boundary

Let $\widetilde{Z} = [0, 1]$ and let $\Pi_0 = V_0 = 0$ so that reservation profits and utilities are zero. Suppose that each firm chooses z to maximize $z^\alpha \eta - P(z)$ where $\alpha = .5$ and $F_\eta(\eta) = \eta$ for $\eta \in [0, 1]$ and suppose that each worker maximizes $P(z) - z^\varepsilon$ where $F_\varepsilon(\varepsilon) = \frac{\varepsilon - 0.5\alpha}{1.5\alpha - 0.5\alpha}$ for $\varepsilon \in \left[\frac{\alpha}{2}, \frac{3\alpha}{2}\right]$. The first and second order conditions for the firm are

$$\text{FOC} \quad \alpha\, z^{\alpha-1}\eta - P_z(z) = 0 \tag{3.6a}$$

$$\text{SOC} \quad \alpha(\alpha - 1) z^{\alpha-2}\eta - P_{zz}(z) < 0 \tag{3.6b}$$

which implies that for those firms at an interior optimum

$$\eta(z) = \frac{P_z(z) z^{1-\alpha}}{\alpha}. \tag{3.7}$$

The first and second order conditions for the workers are

$$\text{FOC} \quad P_z(z) - \varepsilon\, z^{\varepsilon-1} = 0 \tag{3.8a}$$

$$\text{SOC} \quad P_{zz}(z) - \varepsilon\,(\varepsilon - 1)\, z^{\varepsilon-2} < 0. \tag{3.8b}$$

For any interior equilibrium we cannot have $\varepsilon < \alpha$. To see this, from the second order condition for the firm we obtain after substituting (3.6a) into (3.6b) and collecting terms,

$$(\alpha - 1) < \frac{z P_{zz}(z)}{P_z(z)}.$$

From the second order conditions for the workers we obtain

$$P_{zz} < \varepsilon\left(\varepsilon - 1\right) z^{\varepsilon - 2}$$

which is the same as

$$zP_{zz} < \varepsilon\left(\varepsilon - 1\right) z^{\varepsilon - 1}.$$

Using the rewritten first order condition (3.7), we can substitute $P_z\left(z\right)$ for $\left(\varepsilon - 1\right) z^{\varepsilon - 1}$ to obtain

$$\frac{zP_{zz}}{P_z\left(z\right)} < \varepsilon - 1.$$

Thus $\varepsilon > \alpha$ is required to produce an interior solution.

We conjecture the following solution for this example and show that it satisfies the equilibrium conditions. Suppose that exactly half of all workers and firms choose the corner solution $z = 0$. The rest sort positively on the heterogeneity parameters $\left(\eta, \varepsilon\right)$, and locate at an interior optima. Each of the most productive firms is at an interior optimum $\left(\text{i.e. each of those firms with } \eta > \alpha = \frac{1}{2}\right)$ and each of the high elasticity workers (the ones with low disutility of effort) participates at an interior $\left(\varepsilon > \alpha = \frac{1}{2}\right)$. Since we assume that $z \leq 1$, the high elasticity persons are the ones who have the least disutility of work.

If there is positive assortative matching

$$\eta\left(\varepsilon\right) = F_{\eta}^{-1}\left(F_{\varepsilon}\left(\varepsilon\right)\right), \ \varepsilon \in \left[\alpha, \frac{3\alpha}{2}\right].$$

Using our specific functional forms for the distributions, we obtain

$$\eta\left(\varepsilon\right) = \frac{\varepsilon - \frac{\alpha}{2}}{\alpha} = \frac{\varepsilon}{\alpha} - \frac{1}{2}, \quad \varepsilon \in \left[\alpha, \frac{3\alpha}{2}\right]. \tag{3.9}$$

Then using first order conditions (3.6a) and (3.8a) we obtain

$$\varepsilon z^{\varepsilon - 1} = \alpha\eta z^{\alpha - 1}.$$

Substituting $\eta\left(\varepsilon\right)$ in this expression, we obtain

$$z = \left(1 - \frac{\alpha}{2\varepsilon}\right)^{\frac{1}{\varepsilon - \alpha}}, \ \varepsilon \in \left[\alpha, \frac{3\alpha}{2}\right]. \tag{3.10}$$

This is the equilibrium demand function. The matching supply function can be calculated by using (3.9) to substitute out for ε in (3.10). As a consequence, the interval with positive density of demand and supply is $Z = \left[0, \left(\frac{2}{3}\right)^4\right]$. No closed form solution for the price function exists but we can characterize the marginal price function using (3.8a) and (3.10). In particular as $\varepsilon \to \alpha$, $z \to 0$ and $P_z(z)$ becomes arbitrarily large. This is an equilibrium because the supply density equals the demand density at each interior z. Consumers and firms not at the boundary are at a interior optima in this interval.

3.3 Bunching on the interior

The previous section gives conditions that produce bunching on the boundary of the space of feasible attributes. In equilibrium, a positive fraction of agents do not have an interior optimum. Bunching on the interior occurs when a positive fraction of both workers and firms have an optimum at a single point in the interior of $\widetilde{Z}$. To produce bunching at z^*, the set of workers who satisfy

$$P_z(z^*) - U_z(z^*, x, \varepsilon) = 0$$

and the set of firms that satisfy

$$\Gamma_z(z^*, y, \eta) - P_z(z^*) = 0$$

must both have positive measure. If U_z and Γ_z are differentiable and the distributions of (x, ε) and (y, η) are absolutely continuous with respect to Lebesgue measure, this can only happen at z^* if the set

$$A(z^*) = \{(y, \eta, x, \varepsilon) \,|\Gamma_z(z^*, y, \eta) = U_z(z^*, x, \varepsilon)\}$$

has dimension $n_x + n_y + 2$. The set of agents who choose z^* in equilibrium is a subset of $A(z^*)$. If $A(z^*)$ has dimension less than $n_x + n_y + 2$, then it has measure zero and the set of agents who choose z^* has measure zero.

An alternative way to see this is to note that if there is bunching at z^* in equilibrium, then

$$z^* = d(y, \eta) = s(x, \varepsilon)$$

for sets of (y, η) and (x, ε) of equal and positive measure. This means that

$$\frac{\partial d(y, \eta)}{\partial y} = \frac{\partial d(y, \eta)}{\partial \eta} = 0$$

$$\frac{\partial s(x,\varepsilon)}{\partial x} = \frac{\partial s(x,\varepsilon)}{\partial \varepsilon} = 0$$

for $\{(y,\eta,x,\varepsilon)\,|z^* = d(y,\eta) = s(x,\varepsilon)\}$.

To see how interior bunching might arise, consider the following example. Let y measure managerial skill or quality and let the distribution of manager skill be given by the distribution function F_y such that y is a continuous random variable. Let z measure hours of work on a job. A manager of type y has a production function that is quadratic in z:

$$\Gamma = \left\{ \begin{array}{ll} \Gamma_0 + \Gamma_1(y)\,z + \Gamma_4(y)\,z^2 & y \in [y_0, y_1] \\ \Gamma_0 + \Gamma_2(y)\,z + \Gamma_4(y)\,z^2 & y \in [y_1, y_2] \\ \Gamma_0 + \Gamma_3(y)\,z + \Gamma_4(y)\,z^2 & y \in [y_2, y_3] \end{array} \right\}$$

where $\Gamma_4(y) < 1$ for all y,

$$\begin{aligned} \Gamma_1(y) &= \left(-25F_y(y)^2 + 10F_y(y) + 1\right)(1 - \Gamma_4(y)) \\ \Gamma_2(y) &= 2(1 - \Gamma_4(y)) \\ \Gamma_3(y) &= \left(25F_y(y)^2 - 40F_y(y) + 18\right)(1 - \Gamma_4(y)), \end{aligned} \tag{3.11}$$

and $y_0 = F_y^{-1}(0)$, $y_1 = F_y^{-1}(0.2)$, $y_2 = F_y^{-1}(0.8)$, and $y_3 = F_y^{-1}(1)$. Assuming that F_y is twice continuously differentiable, this production function is twice continuously differentiable in all arguments and is quadratic in z. Over the relevant range of z, managers of higher quality have higher marginal productivity.

On the worker side, let x measure disutility from work and let the distribution of worker types have distribution function F_x such that x is a continuous random variable. Suppose utility for a worker with characteristic x is

$$U = \left\{ \begin{array}{ll} U_0 + U_1(x)\,z + U_4(x)\,z^2 & x \in [x_0, x_1] \\ U_0 + U_2(x)\,z + U_4(x)\,z^2 & x \in [x_1, x_2] \\ U_0 + U_3(x)\,z + U_4(x)\,z^2 & x \in [x_2, x_3] \end{array} \right\}$$

where $1 < U_4(x)$ for all x,

$$\begin{aligned} U_1(x) &= \left(-25F_x(x)^2 + 10F_x(x) + 1\right)(1 - U_4(x)) \\ U_2(x) &= 2(1 - U_4(x)) \\ U_3(x) &= \left(25F_x(x)^2 - 40F_x(x) + 18\right)(1 - U_4(x)), \end{aligned} \tag{3.12}$$

and $x_0 = F_x^{-1}(0)$, $x_1 = F_x^{-1}(0.2)$, $x_2 = F_x^{-1}(0.8)$, and $x_3 = F_x^{-1}(1)$. As with firms, this utility function is quadratic in z and twice continuously differentiable. Over the relevant range of z, workers with higher values of x have lower marginal disutility of work.

This example generalizes the seminal Tinbergen (1956) normal-quadratic hedonic model. The equilibrium price function in this economy can be shown to be

$$P(z) = p_0 + z^2.$$

When y is uniformly distributed so $F_y(y) = y$ for $y \in [0, 1]$, the demand function is shown in Figure 1. For y in the interval [.2, .8], the first order condition for the firm is

$$2(1 - \Gamma_4(y)) + 2\Gamma_4(y)z = 2z$$

or

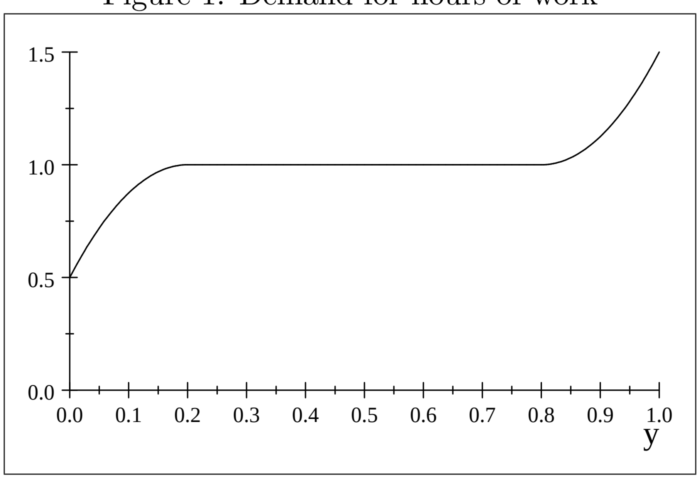
$$1 - \Gamma_4(y) = (1 - \Gamma_4(y))z.$$

So, $z = 1$ is optimal for all y in this interval $\left(\frac{\partial d(y)}{\partial y} = 0 \text{ in this interval}\right)$. Similarly, for x in the interval $[.2, .8]$, the first order condition for the worker is

$$2(1 - U_4(x)) + 2U_4(x)z = 2z$$

and again $z = 1$ is optimal for all x in the interval.

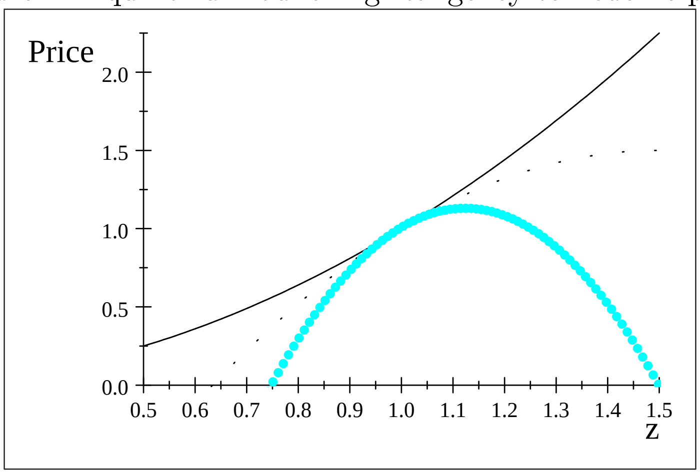
Figure 1: Demand for hours of work



In this example, 60% of the managers and 60% of the workers choose to bunch at $z = 1$.

Tangency conditions for two managers with particular values of y in the interval $[.2, .8]$ are shown in Figure 2. Both indifference curves shown in the figure are tangent to the hedonic price at $z = 1$. The two indifference curves have different curvatures at this point. In fact there is a full cluster of indifference curves with positive probability mass that are tangent to the price function at $z = 1$.

Figure 2: Equilibrium bunching: tangency to hedonic price



Over all intervals of y, the demand for z by firms of type y is

$$z(y) = \left\{ \begin{array}{cc} -\frac{25}{2}F_y(y)^2 + 5F_y(y) + \frac{1}{2} & y \in (y_0, y_1] \\ 1 & y \in [y_1, y_2] \\ \frac{25}{2}F_y(y)^2 - 20F_y(y) + 9 & y \in [y_2, y_3) \end{array} \right\}.$$

The supply function is similar. All managers with skill less than y_1 (20% of the population) employ part-time workers ($z < 1$). All managers with skill greater than y_1 and less than y_2 (60% of the population) employ full time workers ($z = 1$), and all managers with skills larger than y_2 employ workers who work overtime ($z > 1$). Similarly, sixty percent of the workforce bunch at $z = 1$ or at full time work. In this model, those choosing $z = 1$ are the mediocre managers and the mediocre workers.[10]

Such bunching is a knife-edge phenomenon. Any perturbation of the price function (so that the term in z is not quadratic or does not have a unitary coefficient) will break the

[10] In this example, the bunching point, $z = 1$ is determined for exogenous technological reasons. Such a bunching point could also emerge endogenously due to social coordination. For example, suppose that the production function were as above but utility depended on $E(z)$, the average level of z in the market. In

bunching. So will choice of a more general coefficient on the linear term of the quadratic technologies.

3.4 Demand predictions and welfare calculations

The previous sections discuss the shape of the hedonic price function and the distribution of agents across locations. The strength of the hedonic approach however lies in its ability to model the heterogeneity in individual choices and individual outcomes. The supply and demand functions, $s(x,\varepsilon)$ and $d(y,\eta)$ can be used to describe individual level choices as functions of observable (x,y) and unobservable characteristics (ε,η). The shapes of these functions depend on the the structural utility and production functions and on the shape of the equilibrium price function. From an empirical perspective, one of the goals of empirical hedonic analysis is to estimate these supply and demand functions using data on observed hedonic choices z, observed characteristics (x,y), and the observed hedonic prices. We consider identification of these supply and demand functions in the next section.

Identification of the supply function for example allows one to predict partial equilibrium impacts of changes in individual level observables on individual choices of z. Such predictions hold the individual's level of ε and the equilibrium prices fixed. Note that they do not allow one to measure the welfare impacts of such a change nor do they allow us to make predict general equilibrium impacts. These latter calculations require information about the structural utility and production functions.

Consider the partial equilibrium welfare impact on an individual of a change from x_0 to x_1. Holding everything else constant, such a change will lead the individual to move from $z_0 = s(x_0,\varepsilon)$ to $z_1 = s(x_1,\varepsilon)$. Assuming that $z_1 > z_0$ and holding ε fixed, the welfare impact

particular, in the previous example replace (3.12) with

$$\begin{aligned} U_1(x) &= \Big(25\left[1-2E(z)\right]F_x(x)^2 - 10\left[1-2E(z)\right]F_x(x)+1\Big)(1-U_4(x)) \\ U_2(x) &= 2E(z)(1-U_4(x)) \\ U_3(x) &= \Big(25\left[3-2E(z)\right]F_x(x)^2 - 40\left[3-2E(z)\right]F_x(x)+6\left[8-5E(z)\right]\Big)(1-U_4(x)). \end{aligned}$$

In this case, equilibrium bunching again emerges with 60% of the population choosing $z = E(z) = 1$. In this example $E(z) = 1$, so in a single cross-section the model in the text and this model are indistinguishable.

of this change is

$$
\begin{aligned}
& P(z_1) - U(z_1, x_1, \varepsilon) - [P(z_0) - U(z_0, x_0, \varepsilon)] \\
= \; & P(z_1) - P(z_0) - [U(z_1, x_1, \varepsilon) - U(z_0, x_0, \varepsilon)] \\
= \; & P(z_1) - P(z_0) - \int_{z_0}^{z_1} U_z(z, x_1, \varepsilon)\, dz - \int_{x_0}^{x_1} U_x(z_0, x, \varepsilon)\, dx.
\end{aligned}
\tag{3.13}
$$

The welfare changes equals the price difference between the two locations z_1 and z_0 minus the change in utility. For large changes in x that result in large changes in z, the change in hedonic prices $P(z_1) - P(z_0)$ overestimates the change in welfare if $U_x > 0$ (we have already assumed that $U_z > 0$). This is the well-known result from Scotchmer (1985) and Kanemoto (1988).

When U_z and U_x are unknown, neither of the integrals on the third line of (3.13) is known. However if U_z can be identified and estimated, then the first integral can be calculated. The hedonic model provides no information about the second integral because this integral calculates the direct impact of x on welfare holding z fixed. To estimate this last integral, either additional assumptions must be imposed (e.g. the value of $U(z, x, \varepsilon)$ is known for some value of z and for all (x, ε)) or additional information must be obtained (e.g. about how much households are willing to pay for x.)

Nevertheless, knowledge of U_z improves our knowledge about the welfare impacts of changes in x. In addition, knowledge of U_z combined with information about firms technologies allows us to compute general equilibrium impacts of changes in the hedonic market environment and improves measures of the general equilibrium welfare impacts of such changes. If the structural functions and the distributions of agents are known, then we can compute how hedonic equilibrium will change in response to changes in preferences, changes in technologies, or changes in the distribution of workers or firms. Such general equilibrium impacts and welfare measures cannot be computed without knowledge of the structural functions.

We now turn to an analysis of identification of the nonaddictive hedonic model. This analysis enables us to estimate directly U_z and U_x (and the corresponding technology parameters of firms) so that we can execute the welfare calculations in (3.13) exactly. We first consider the single market case. We develop the multimarket case in section 6.

4 Identification in a single market

This section analyzes identification of the supply function $s(x, \varepsilon)$, the marginal utility function $U_z(z, x, \varepsilon)$ and the distribution of ε in a single market. We omit discussion of the demand side of the market because the analysis is completely analagous. Our analysis assumes that the equilibrium price function $P(z)$ and the distribution of (z, x) are known where z denotes the observed hedonic location choice of an individual and x denotes the vector of observed consumer characteristics. Most of the analysis assumes that there is no bunching in hedonic equilibrium and that all agents choose to enter the market. At the end of this section we show that consideration of bunching does not substantially change our analysis.

In nonadditive hedonic models, the supply function $s(x, \varepsilon)$ is a nonseparable function of a vector of observables x and a scalar unobservable ε. By assumption ε is independent of x. Furthermore, as we show below, our theoretical structure from Section 2 implies that s is an increasing function of ε. Therefore, $s(x, \varepsilon)$ is identified using results from Matzkin (1999) and Matzkin (2003). As discussed in Matzkin (2003), identification of this function requires either a normalization of $s(x, \varepsilon)$ (fixing its value at a point) or of the distribution of the unobservable ε (assuming that the distribution is known). However, certain features of this function, such as the effect on z of changing x from x_0 to x_1 leaving the value of ε fixed, i.e. $z_1 - z_0 = s(x_1, \varepsilon) - s(x_0, \varepsilon)$, are invariant to the choice of a normalization. We next consider conditions that identify U_z. We start with a nonidentification result that illustrates the key ideas.

4.1 A nonidentification result

Given that the supply function $s(x, \varepsilon)$ is identified and the price function $P(z)$ is known, we seek to identify the marginal utility function U_z. This function must satisfy the first order condition

$$U_z(s(x, \varepsilon), x, \varepsilon) = P_z(s(x, \varepsilon)). \tag{4.1}$$

The key to understanding whether U_z is identified is this first order condition. Note that the marginal utility function is identified for those values of (z, x, ε) that lie on the surface $\{(z, x, \varepsilon) : z = s(x, \varepsilon)\}$. On this surface, the value of the marginal utility U_z is known, since it must equal the value of the marginal price function.

However, as is clear from this expression, without further restrictions it is not possible to identify the function U_z for all values of (z, x, ε) using data from a single market. For any arbitrary values of x and ε, the value $s(x,\varepsilon)$, the first argument of the function, is uniquely determined. Thus, even if we could observe ε, we could not independently vary (z,x,ε) and trace out the function on its entire n_x+2 dimensional domain.

There are three responses to this fundamental nonidentification problem: 1) Focus attention on features of U_z that are identified. 2) Impose functional restrictions on U_z that enable analysts to overcome the exact functional dependence between z, x, and ε implied by economic theory. 3) Obtain data from equilibria in different markets and make use of independent variation in hedonic equilibrium prices across markets. We consider the first two approaches in the remainder of this section and consider the third approach in section 5.

4.2 What is identified without further structure

Even though U_z is not identified using data from a single market without further structure, some features of the function U_z can be identified. For example, if x contains two variables, x_1 and x_2, then the ratio of the partial derivatives of U_z with respect to x_1 and x_2 is identified. To see this, note that we can totally differentiate equation (4.1) with respect to x_1 and x_2 to obtain

$$U_{zz}(z,x_1,x_2,\varepsilon)\frac{\partial s(x_1,x_2,\varepsilon)}{\partial x_1}+U_{zx_1}(z,x_1,x_2,\varepsilon)=P_{zz}(s(x_1,x_2,\varepsilon))\frac{\partial s(x_1,x_2,\varepsilon)}{\partial x_1}$$

and

$$U_{zz}\frac{\partial s(x_1,x_2,\varepsilon)}{\partial x_2}+U_{zx_2}=P_{zz}\left(s(x_1,x_2,\varepsilon)\right)\frac{\partial s(x_1,x_2,\varepsilon)}{\partial x_2}.$$

Hence,

$$\frac{U_{zx_1}(z,x_1,x_2,\varepsilon)}{U_{zx_2}(z,x_1,x_2,\varepsilon)}=\frac{\frac{\partial s(x_1,x_2,\varepsilon)}{\partial x_1}}{\frac{\partial s(x_1,x_2,\varepsilon)}{\partial x_2}}\;|_{s(x_1,x_2,\varepsilon)=z}\;. \tag{4.2}$$

Since $s(x,\varepsilon)$ is identified, the ratios of partial derivatives in (4.2) are identified without any further restrictions. The ratio on the left side of (4.2) measures the effect on U_z of changing x_1 relative to changing x_2. This equals the ratio of the corresponding partial derivatives of s. This identification result requires no further restrictions on the set of admissible U_z functions. Nor does it require any normalizations.

Alternatively, if x is a scalar and we assume that the distribution of ε is known, the same arguments can be used to show that the ratio of partial derivatives

$$\frac{U_{zx}(z,x,\varepsilon)}{U_{z\varepsilon}(z,x,\varepsilon)} = \frac{\frac{\partial s(x,\varepsilon)}{\partial x}}{\frac{\partial s(x,\varepsilon)}{\partial \varepsilon}} \mid_{s(x,\varepsilon)=z} \tag{4.3}$$

is identified. For example if it is known (equivalently, if we make the normalization) that ε is distributed uniformly on $[0,1]$, then the ratio in (4.3) is identified. In this case, (4.3) could be used to evaluate the relative impacts on U_z of observable x and unobservable ε for different values of x and at different quantiles of the distribution of ε. This result requires a normalization on the distribution of ε but does not require any restrictions on the set of admissible U_z.

4.3 Imposing further structure

A second way to deal with the fundamental nonidentification problem is to impose additional restrictions on the set of admissible U_z functions. Proceeding down this route, we develop three theorems that show how introducing an assumption that has the effect of reducing the number of arguments of U_z by one enables the analyst to recover U_z for values of its arguments outside the two dimensional subdomain defined by the surface $(s(x,\varepsilon), x, \varepsilon)$. The first two theorems, assume that U_z depends on two of its arguments through a known function, $q : R^2 \to R$. This separability restriction or shape restriction reduces the dimension of the domain of U_z and allows us to identify U_z. In addition to the shape restriction, these theorems require a normalization either of the distribution of ε (assuming the distribution is known as in Theorem 4.1) or of the function U_z (assuming that its value is known at a point as in Theorem 4.2). Either normalization is sufficient for identification. The third theorem presents a third alternative shape restriction on U_z that can be used when x is a vector. In this last case, assuming that U_z depends on its arguments through two known functions, $q_1 : R^2 \to R$ and $q_2 : R^2 \to R$ reduces the dimension of the domain of U_z and allows us to identify $U_{z.}$.

The first theorem shows that U_z is identified when we assume that U_z is a weakly separable function of the pair (z, x) and ε and use a normalization on either the supply function or the distribution of ε. For example, suppose that we specify the distribution of ε. Then, we can prove the following theorem:

Theorem 4.1 *Suppose that for some unknown differentiable function $m : R^2 \to R$, which is strictly increasing in its second argument, and some known differentiable function $q : R^2 \to$*

R, the marginal utility function can be written

$$U_z(z, x, \varepsilon) = m\left(q(z, x), \varepsilon\right). \tag{4.4}$$

Further, assume that F_ε is known and let $(q_l(\varepsilon), q_u(\varepsilon))$ denote the support of $q(s(x, \varepsilon), x)$ for any $\varepsilon \in \widetilde{E}$. Then, for all ε and all x such that $q(s(x, \varepsilon), x) \in (q_l(\varepsilon), q_u(\varepsilon))$, $U_z(z, x, \varepsilon)$ is identified.

Proof. See Appendix. ■

For a given ε, identification of $s\left(x, \varepsilon\right)$ allows us to find all pairs of (z, x) that are consistent with fixed ε. Then the shape restriction on U_z allows us to select from among these pairs the pair that produces a fixed value of $q\left(z, x\right)$. Combining these two points, allows us to identify U_z at an arbitrary point. A similar result can be obtained if instead of requiring that U_z be a function of $q(z, x)$, we require that U_z be a function of $q(z, \varepsilon)$. Specifically, suppose that that for some *unknown* function $m : R^2 \to R$, which is strictly increasing in its first argument, and some *known* function $q : R^2 \to R$, which is strictly increasing in its second argument

$$U_z(z, x, \varepsilon) = m\left(q(z, \varepsilon), x\right)$$

Assume that F_ε is known. For any x, let $(q_l(x), q_u(x))$ denote the support of $q(s(x, \varepsilon), \varepsilon)$. Then, for all x and all ε such that $q(s(x, \varepsilon), \varepsilon) \in (q_l(x), q_u(x))$, $U_z(z, x, \varepsilon)$ is identified. The argument follows the same lines as is used in the proof of Theorem 4.1.

These results make use of a separability restriction on U_z and a normalization of the distribution of ε. Another alternative is to impose a separability restriction and normalize the function U_z by assuming that its value is known at a point. This alternative implies a normalization on the function s instead of on the distribution of ε. Along these lines, we can obtain the following theorem:

Theorem 4.2 *Let $x \in R$. Suppose that for some unknown, differentiable function $m : R^2 \to R$, which is strictly increasing in its last argument, and some known, differentiable function $q : R^2 \to R$,*

$$U_z(z, x, \varepsilon) = m\left(q(z, x), \varepsilon\right).$$

Use the function P_z to fix the value of the unknown function U_z at one value $\overline{x}$ of x, and on the 45 degree line on the (z, ε) space, by requiring that for all t,

$$U_z(t, \overline{x}, t) = P_z(t) \tag{4.5}$$

Let ε be given. Let $q \in (q_l(\varepsilon), q_u(\varepsilon))$, the support of $q(s(x,\varepsilon),x)$. Then, for x such that $q(s(x,\varepsilon),x) \in (q_l(\varepsilon), q_u(\varepsilon))$, $U_z(z,x,\varepsilon)$ is identified.

Proof. See Appendix. ■

The result can be easily modified to apply to the case where $U_z(z,x,\varepsilon) = m\left(q(z,\varepsilon),x\right)$. Specifically, suppose that for some *unknown* function $m : R^2 \to R$, which is strictly increasing in its first coordinate and some *known* function $q : R^2 \to R$, which is strictly increasing in its second coordinate

$$U_z(z,x,\varepsilon) = m\left(q(z,\varepsilon),x\right)$$

Suppose that (4.5) is satisfied. Then, $U_z(z,x,\varepsilon)$ is identified on an appropriate set. Suppose, for example, that

$$U_z\left(z,x,\varepsilon\right) = m\left(z \cdot \varepsilon, x\right)$$

for an unknown function m. Then the normalization (4.5) is imposed by fixing the values of m when $x = \overline{x}$, by

$$m\left(t^2, \overline{x}\right) = P_z(t).$$

When x is a vector, many alternative restrictions can be used. As a prototype, we consider one alternative restriction that is sufficient for identification when x is a two dimensional vector (x_1, x_2). This restriction imposes that U_z is a weakly separable function of two known functions $q_1\left(z.x_1\right)$ and $q_2\left(z,x_2\right)$. The next theorem shows that this restriction, along with a normalization on U_z allows us to identify U_z.

Theorem 4.3 *Let $x = (x_1,x_2) \in R^2$. Suppose that for some unknown differentiable function $m : R^2 \to R$, which is strictly increasing in its second argument, and some known differentiable functions $q_1 : R^2 \to R$ and $q_2 : R^2 \to R$*

$$U_z(z,x_1,x_2,\varepsilon) = m\left(q_1(z,x_1), q_2(x_2,\varepsilon)\right) \tag{4.6}$$

where q_2 is strictly increasing in its arguments. Let $\left[t_2^l, t_2^u\right]$ denote the support of $q_2(x_2,\varepsilon)$. Assume the function m is known at one point so that for some values $\overline{z}$ of z, $\overline{x}_1$ of x_1, and $\alpha \in \left[t_2^l, t_2^u\right]$,

$$m\left(q_1(\overline{z},\overline{x}_1),\alpha\right) = P_z\left(\overline{z}\right). \tag{4.7}$$

For any $t_2 \in \left[t_2^l, t_2^u\right]$, let $[t_1^l(t_2), t_1^u(t_2)]$ denote the support of $q_1\left(s(x_1,x_2,\varepsilon),x_1\right)$ conditional on $q_2(x_2,\varepsilon) = t_2$. Then, for any (z,x_1,x_2,ε) such that $q_2(x_2,\varepsilon) \in \left[t_2^l, t_2^u\right]$ and $q_1(z,x_1) \in [t_1^l(t_2), t_1^u(t_2)]$, $U_z(z,x_1,x_2,\varepsilon)$ is identified.

Proof. See Appendix. ■

Identification of U_z is obtained in several steps. First, equation (4.6) implies that the supply function $s(x_1, x_2, \varepsilon)$ is a weakly separable function of x_1 and $q_2(x_2, \varepsilon)$. Equation (4.7) then implies that the supply function is known at one point. Further, the strict monotonicity of m and q_2 in their second arguments implies that the supply function is strictly increasing in ε. These implications guarantee that the supply function s and the distribution of ε are identified. Next, to identify the value of $m(t_1, t_2)$ at an arbitrary point (t_1, t_2) on the relevant domain, we first find values x_1^*, x_2^*, and ε^* such that when $z = s(x_1^*, x_2^*, \varepsilon^*)$, $q_1(z, x_1^*) = t_1$ and $q(x_2^*, \varepsilon^*) = t_2$. Finally, since such a z satisfies the FOC, it follows that $m(t_1, t_2) = P_z(z) = P_z\left(s(x_1^*, x_2^*, \varepsilon^*)\right)$. In short, independent variation in x_1 and x_2, the assumed dependence of U_z on only two arguments, and knowledge of the functions q_1 and q_2, allow us to trace out U_z as a function of its two arguments.

The statement and the proof of Theorem 4.3 can easily be modified to show that the function U_z is also identified when it can be expressed as a function $m(t_1, x_1)$, where $t_1 = q_1(z, t_2)$ and $t_2 = q_2(x_2, \varepsilon)$. To see this, suppose that for some unknown function $m : R^2 \to R$ and some known functions $q_1 : R^2 \to R$ and $q_2 : R^2 \to R$, such that m is strictly increasing in its first argument, q_1 is strictly increasing in its second argument, and q_2 is strictly increasing in its arguments

$$U_z(z, x_1, x_2, \varepsilon) = m\left(q_1(z, q_2(x_2, \varepsilon)), x_1\right). \tag{4.8}$$

Assuming the function m is known at one point, so that for some values $\overline{z}$ of z, $\overline{x}_1$ of x_1, and $\alpha \in R$,

$$m\left(q_1(\overline{z}, \alpha), \overline{x}_1\right) = P_z\left(\overline{z}\right). \tag{4.9}$$

Then, as in the proof of Theorem 4.3, it can be shown that, by (4.8), the supply function, $s(x_1, x_2, \varepsilon)$ is weakly separable into $q_2(x_2, \varepsilon)$, by (4.9), the value of s is fixed at one point, and by the monotonicity of m and q_1, s is strictly increasing in q_2. These properties guarantee identification of s and of the distribution of ε using the analysis of Matzkin (1999). To identify the value of $m(t_1, t_2)$ at an arbitrary vector (t_1, t_2), let $x_1^* = t_2$, and find x_2^*, and ε^* such that when $z = s(x_1^*, x_2^*, \varepsilon^*)$, $q_1(z, q_2(x_2^*, \varepsilon^*)) = t_1$. Then, as in the previous argument, $m(t_1, t_2) = P_z(z) = P_z\left(s(x_1^*, x_2^*, \varepsilon^*)\right)$.

It is interesting to consider the economic implications of some of these alternative restrictions. Focus on the types of restrictions described in Theorems 4.1 and 4.2. Consider the models

$$U_z\left(z, x, \varepsilon\right) = m_1\left(q_1\left(z, x\right), \varepsilon\right)$$

$$U_z(z,x,\varepsilon) = m_2(q_2(z,x),\varepsilon)$$

and

$$U_z(z,x,\varepsilon) = m_3(q_3(z,\varepsilon),x)$$

where (m_1, q_1), (m_2, q_2) and (m_3, q_3) are three pairs of functions satisfying the assumptions in Theorem 4.1 or its modification suggested above. Assume q_1, q_2 and q_3 are three distinct functions so the three models are distinct. In each model, the functions m_1, m_2, and m_3 are identified. Choice between the models must be based on prior information or on theory. The models do have slightly different economic interpretations. The models (m_1, q_1) and (m_2, q_2) both assume that the trade-off

$$\frac{\partial U_z / \partial z}{\partial U_z / \partial x}$$

is known and depends only on observables. These two models are distinct because they make different assumptions about this trade-off. In contrast, model (m_3, q_3) assumes that

$$\frac{\partial U_z / \partial z}{\partial U_z / \partial \varepsilon}$$

is known and only depends on unobservables.

In addition, if we consider using the results from estimation of these models to calculate welfare criteria as in equation (3.13), we will get different results depending on which model we use. For example, if we use model (m_1, q_1) to compute equation (3.13) we obtain

$$P(z_1) - P(z_0) - \int_{z_0}^{z_1} m_1(q_1(s,x),\varepsilon)\,ds - \left(\int_{z_L}^{z_0} \int_{x_0}^{x_1} m_q(q_1(s,x),\varepsilon)\, q_x(s,x)\, dx ds \right). \quad (4.10)$$

The result will be different if we use model (m_2, q_2) or model (m_3, q_3).

4.4 Identification when there is bunching

In Theorem 4.3 we make the assumption that m is strictly increasing in its second argument and that q_2 is strictly increasing in both arguments. Similarly, in Theorems 4.1 and 4.2 we assume that m is strictly increasing in its second argument. These assumptions rule out bunching because they guarantee that either $\frac{\partial s(x,\varepsilon)}{\partial x} \neq 0$ or $\frac{\partial s(x,\varepsilon)}{\partial \varepsilon} \neq 0$. However, the assumption of strict monotonicity while convenient for the proofs is stronger than is required to identify these functions. The theorems can be modified to relax the assumptions and allow

q_2 to be weakly increasing in its arguments in Theorem 4.3 and m to be weakly increasing in its second argument in each case. Moreover, weak monotonicity allows for bunching. We outline the proof for the case of Theorem 4.1.

When there is bunching, the analysis is essentially unchanged. Assume we have data from a market in which a fraction of agents bunch at a single point z^* while the rest spread themselves continuously over the domain $Z \subseteq R$. Assume that

$$U_z(z, x, \varepsilon) = m(q(z, x), \varepsilon)$$

where m is weakly increasing in its second argument, q is a known function, and F_ε is known.

Under these assumptions, the supply function $s(x, \varepsilon)$ need not be everywhere differentiable and its inverse $\widetilde{s}(z, x)$ need not be single valued. However the second order conditions and the monotonicity of m with respect to ε guarantee that $s(x, \varepsilon)$ is nondecreasing in ε. Hence, as in the proof of Theorem 4.1

$$s(x, \varepsilon) = F^{-1}_{Z|x=x}(F_\varepsilon(\varepsilon)).$$

Despite the possibility that $F_{Z|x}$ might only be right continuous and not continuous, the inverse is still well defined. Thus, even when there is bunching, the supply function is identified.

Moreover, m is identified on the support of $q(s(x, \varepsilon), x)$ and ε. The proof is identical to the last part of the proof of Theorem 4.1. So,

$$m(t_1, t_2) = P_z(s(x^*, t_2))$$

where x^* satisfies

$$t_1 = q(s(x^*, t_2), x^*).$$

When there is bunching, there may be multiple values of x^* that satisfy this last equation. However, for every value of (t_1, t_2) satisfying the support condition, by construction, there is at least one x^*.

Bunching does not change the analysis of the identification of the functions m and s using theorem 4.1 in any essential way. The only loss of information is with respect to the values of individual level unobserved heterogeneity. For those agents who bunch at z^*, the individual values of ε are not identified. For this group, all that is known is that $\varepsilon \in \widetilde{s}(z^*, x)$.

We now consider how access to multiple market choice data produces identification.

5 Identification in multiple markets

Identification of $U_z(z,x,\varepsilon)$ in a single market is limited because all consumers face the same price schedule. With multiple markets however, the marginal price function $P_z(z)$ will typically vary with underlying market conditions. For example, assuming that the marginal utility function, $U_z(z,x,\varepsilon)$, does not vary across markets[11], the marginal price function (and the supply function $s(x,\varepsilon)$) will, in general, vary across markets when the distribution of worker types or firm types varies across markets.[12] When data are available from multiple markets and cross market variation in the distributions of observables causes cross market variation in $P_z(z)$ and $s(x,\varepsilon)$, this cross market variation can be used to identify the function $U_z(z,x,\varepsilon)$ without imposing additional restrictions.

Suppose that the distributions of ε and η are the same in all markets. Further assume that the distributions of x and y with densities denoted by $(f_x, f_y) \in \mathcal{F}(X) \times \mathcal{F}(Y) \subseteq L_2(X,\mu_x) \times L_2(Y,\mu_y)$ vary across markets. Here, μ_x and μ_y are Lebesgue measure on X and Y respectively and L_2 represents the space of square integrable functions. Suppose a multimarket sample exists from M markets with N_j observations on (z,y,x) from each market j. The marginal price and supply functions in each market will depend on $\left(f_x^j, f_y^j\right)$, the densities of observable x and y in each market. Dropping subscripts, write these functions as $P_z(z, f_x, f_y)$ and $s(x,\varepsilon, f_x, f_y)$.

From such a multimarket sample, the functions (f_x, f_y) and the functional $P_z(z, f_x, f_y)$ are identified. Using Matzkin (1999, 2003), the functional $s(x,\varepsilon,f_x,f_y)$ is nonparametrically identified. Hence, the multimarket data-set allows us to identify the distributions of observables and the dependence of the marginal price and the supply function on these distributions. We can use this information to identify the marginal utility function $U_z(z,x,\varepsilon)$.

Recall the workers' first order condition:

$$U_z(s(x,\varepsilon,f_x,f_y),x,\varepsilon) = P_z(s(x,\varepsilon,f_x,f_y),f_x,f_y)$$

where we have made explicit the dependence of P_z and s on f_x and f_y. In a single cross section, the price function and the supply function are fixed and we cannot independently vary the three arguments of U_z. With multimarket data, both P_z and s vary for each (x,ε) provided that f_x or f_y or both vary across markets.

[11] We can always adopt a specification rich enough to ensure this is true.

[12] The support of (x,z,ε) may be different in different markets. Thus, even under the conditions discussed in this section, we can identify $U_z(z,x,\varepsilon)$ only over the union of the supports across markets.

Our analysis provides a general approach to identification of U_z under weaker conditions than were required in using single cross section data. Use of multimarket data to identify hedonic models was proposed in Brown and Rosen (1982), Brown (1983) and Epple (1987). Our analysis is more general than theirs because we consider the nonseparable case where their analyses assume linearity of supply and price equations. Also, our approach brings out the point that the equilibrium price and the supply function depend on the distributions of observable characteristics of firms and workers. We now state and prove the theorem.

Theorem 5.1 *Pick an arbitrary point* (z, x, ε). *If the distribution of* ε *is constant across markets and there exists a pair* (f_x^*, f_y^*) *such that* $z = s(x, \varepsilon, f_x^*, f_y^*)$, *then* $U_z(z, x, \varepsilon)$ *is identified at the point* (z, x, ε).

Proof. Let (f_x^*, f_y^*) satisfy $z = s(x, \varepsilon, f_x^*, f_y^*)$. Then $U_z(z, x, \varepsilon) = P_z(s(x, \varepsilon, f_x^*, f_y^*), f_x^*, f_y^*)$. Thus, U_z is identified at all points (z, x, ε) such that z is an equilibrium choice for (x, ε) in some feasible equilibrium. ■

Under our conditions, we can independently vary the arguments of $U_z(z, x, \varepsilon)$ by varying $(f_x, f_y, x, \varepsilon)$. This is true for all (z, x, ε) such that an equilibrium with $z = s(x, \varepsilon)$ is feasible.

The theorem exploits the variation in price and supply functions induced by cross-market variation in the distributions of observables. The source of the variation is apparent from equilibrium equation (2.6):

$$\int_{\widetilde{X}} f_\varepsilon(\widetilde{s}(z, x)) f_x(x) \frac{\partial \widetilde{s}(z, x)}{\partial z} dx = \int_{\widetilde{Y}} f_\eta\left(\widetilde{d}(z, y)\right) f_y(y) \frac{\partial \widetilde{d}(z, y)}{\partial z} dy. \tag{5.1}$$

Consider two different markets with different distributions of observables (f_x^1, f_y^1) and (f_x^2, f_y^2). Because the distributions are different in the two markets, the inverse supply functions that satisfy equation (5.1) will generically be different across different markets. If $P(z, f_x^1, f_y^1)$ is the equilibrium price in market 1, then generically, $P(z, f_x^2, f_y^2) \neq k_0 + k_1 P(z, f_x^1, f_y^1)$ where $P(z, f_x^2, f_y^2)$ is the equilibrium price in market 2. This is an application of Theorem 1 in Ekeland, Heckman, and Nesheim (2004). Equivalently $s(x, \varepsilon, f_x^1, f_y^1)$ is not a linear function of $s(x, \varepsilon, f_x^2, f_y^2)$.

6 Estimation

We now consider how to convert the identification theorems of section 4 into estimation algorithms. We focus the discussion on a model satisfying the conditions of Theorem 4.3 because this is the most general of the three theorems. We then show how to alter the argument to define an estimator for a model satisfying the conditions of Theorem 4.1. Estimators for models satisfying conditions in Theorem 4.2 can be defined analogously.

The proofs of Theorems 4.1-4.3 suggest ways to nonparametrically estimate the supply function s, the marginal utility function U_z, and the distribution of ε. For example, under the conditions of Theorem 4.3, the supply function has the form $v(x_1, q_2(x, \varepsilon))$. To obtain an estimator for U_z, first estimate the distribution of ε and the supply function v using the conditional distribution function of z given (x_1, x_2) using the procedure described in Matzkin (2003). Then, use the estimated function $\widehat{v}$ and the known function q_1 to calculate the value x_1^* that satisfies

$$q_1(\widehat{v}(x_1^*, t_2), x_1^*) = t_1.$$

The estimator $\widehat{m}(t_1, t_2)$ of $m(t_1, t_2)$, is then given by the equation

$$\widehat{m}(t_1, t_2) = P_z\left(\widehat{v}(x_1^*(t_1, t_2), t_2)\right).$$

A similar procedure can be described using the steps in the proofs of Theorems 4.1 and 4.2. The statistical properties of the resulting estimator of m then arise from the statistical properties of $\widehat{v}$ filtered through this nonlinear equation.

To describe the estimators suppose that the equilibrium price function is known, and that the available data is $\{Z^i, X^i\}$ for each of N_1 workers. Let $f(z, x_1, x_2)$ and $F(z, x_1, x_2)$ denote, respectively, the joint pdf and cdf of (Z, X). Let $\hat{f}(z, x_1, x_2)$ and $\hat{F}(z, x_1, x_2)$ denote the corresponding kernel estimators. Let $\hat{f}_{Z|X=(x_1,x_2)}(z)$ and $\hat{F}_{Z|X=(x_1,x_2)}(z)$ denote the kernel estimators of, respectively, the conditional pdf and conditional cdf of Z given $X = (x_1, x_2)$. In this notation,

$$\hat{f}(z, x_1, x_2) = \frac{1}{N\sigma_N^3}\sum_{i=1}^{N} K\left(\frac{z - Z^i}{\sigma_N}, \frac{x_1 - X_1^i}{\sigma_N}, \frac{x_2 - X_2^i}{\sigma_N}\right),$$

$$\hat{F}(z, x_1, x_2) = \int_{-\infty}^{z}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2} \hat{f}_N(s, t_2, t_2)\, ds\, dt_1\, dt_2,$$

$$\hat{f}_{Z|X=(x_1,x_2)}(z) = \frac{\hat{f}_N(z, x_1, x_2)}{\int_{-\infty}^{\infty} \hat{f}_N(s, x_1, x_2)\, ds},$$

and

$$\hat{F}_{Z|X=(x_1,x_2)}(z) = \frac{\int_{-\infty}^{z} \hat{f}_N(s,x_1,x_2)ds}{\int_{-\infty}^{\infty} \hat{f}_N(s,x_1,x_2)ds}$$

where $K : R^3 \to R$ is a kernel function and σ_N is the bandwidth.[13] The above estimator for $F(z,x_1,x_2)$ was proposed in Nadaraya (1964). When $K(s,x_1,x_2) = k_1(s)k_2(x_1,x_2)$ for some kernel functions $k_1 : R \to R$ and $k_2 : R^2 \to R$,

$$\hat{F}_{Z|X=(x_1,x_2)}(z) = \frac{\int_{-\infty}^{z} \hat{f}_N(s,x_1,x_2)\, ds}{\int_{-\infty}^{\infty} \hat{f}_N(s,x_1,x_2)\, ds} = \frac{\sum_{i=1}^{N} \widetilde{k}_1(\frac{z-z^i}{\sigma_N})\; k_2(\frac{x_1-X_1^i}{\sigma_N}, \frac{x_2-X_2^i}{\sigma_N})}{\sum_{i=1}^{N} k_2(\frac{x_1-X_1^i}{\sigma_N}, \frac{x_2-X_2^i}{\sigma_N})}$$

where $\widetilde{k}_1(u) = \int_{-\infty}^{u} k_1(s)\, ds$. Note that the estimator for the conditional cdf of Z given X is different from the Nadaraya-Watson estimator for $F_{Z|X=x}(z)$ where $x = (x_1,x_2)$. The latter is the kernel estimator for the conditional expectation of $W \equiv 1[Z \leq z]$ given $X = x$. For any t and x, $\hat{F}^{-1}_{Z|X=x}(t)$ will denote the set of values of X for which $\hat{F}_{Z|X=x}(z) = t$. When the kernel function k_1 is everywhere positive, this set of values will contain a unique point.

6.1 Case 1

Theorem 4.3 assumes that the marginal utility function is weakly separable into two functions, each possessing one of the observable characteristics as one of its arguments. In other words, for some unknown function m we may represent the marginal utility function as

$$U_z(z,x_1,x_2,\varepsilon) = m\left(q_1(z,x_1), q_2(x_2,\varepsilon)\right)$$

where $q_1 : R^2 \to R$ and $q_2 : R^2 \to R$ are some *known* functions. Theorem 4.3 establishes the identifiability of the function m and the distribution of ε for this case. Following the statement of that theorem, normalize the value of the function m at one point by requiring that at some values $\overline{z}$ of z, $\overline{x}_1$ of x_1, and $\alpha \in R$,

$$m\left(q_1(\overline{z},\overline{x}_1),\alpha\right) = P_z\left(\overline{z}\right).$$

Define $s(x_1,x_2,\varepsilon)$ to be the function that satisfies, for each (x_1,x_2,ε), the FOC of the worker. As argued in the proof of Theorem 4.3, from the assumed structure of separability, we may

[13] For ease of exposition, we focus on the case where $x \in R^2$. The results extend readily to cases in which $x \in R^{n_x}$ where $n_x > 2$.

write:

$$s(x_1, x_2, \varepsilon) = v(x_1, q_2(x_2, \varepsilon))$$

for some unknown function v, which is strictly increasing in its second argument and satisfies the property that

$$v(\overline{x}_1, \alpha) = \overline{z}$$

where q_2 is a known function and $\alpha \in R$. Following Matzkin (2003), it follows that for any e

$$\widehat{F}_\varepsilon(e) = \widehat{F}_{Z|X=(\overline{x}_1, w(\alpha,e))}(\overline{z})$$

where $w(\alpha, e) = w^*$ is such that $q_2(w^*, e) = \alpha$, and for any $\widetilde{x}_1, \widetilde{x}_2, \widetilde{e}$

$$\widehat{v}(\widetilde{x}_1, q_2(\widetilde{x}_2, \widetilde{e})) = \widehat{F}^{-1}_{Z|X=(\widetilde{x}_1,\widetilde{x}_2)}\left(\widehat{F}_\varepsilon(\widetilde{e})\right).$$

Using the procedure described in the introduction to this section, to obtain an estimator for $m(t_1, t_2)$, we first calculate $\widehat{x}_1^*$ such that

$$q_1(\widehat{v}(\widehat{x}_1^*, t_2), \widehat{x}_1^*) = t_1$$

and then let

$$\widehat{m}(t_1, t_2) = P_z\left(\widehat{v}(\widehat{x}_1^*, t_2)\right).$$

The following theorem establishes the asymptotic properties of this estimator for the case where the function $q_1(z, x_1) = z \cdot x_1$ and the function $q_2(x_2, \varepsilon) = x_2 + \varepsilon$. We offer this analysis as a prototype. Similar results can be obtained for other specifications of the functions q_1 and q_2. Let $B(t, \xi)$ denote the neighborhood centered at t and with radius $\xi > 0$. Given t_2, let x_2 and e be such that $x_2 + e = t_2$. Let $\widetilde{x}_2 = \alpha + e$. We will make the following assumptions:

Assumption A.1: *The sequence* $\{Z^i, X^i\}$ *is i.i.d.*

Assumption A.2: $f(Z, X_1, X_2)$ *has compact support* $\widetilde{Z} \times \widetilde{X} \subset R^3$ *and is continuously differentiable of order* s'.

Assumption A.3: *The kernel function* $K(\cdot,\cdot,\cdot)$ *is differentiable of order* $\widetilde{s}$, *the derivatives of* K *of order* $\widetilde{s}$ *are Lipschitz,* K *vanishes outside a compact set, integrates to 1, and is of order* s'' where $\widetilde{s} + s'' + 1 \leq s'$.

Assumption A.4: *As* $N \to \infty$, $\sigma_N \to 0$, $\ln(N)/N\sigma_N^3 \to 0$, $\sqrt{N\sigma_N^2} \to \infty$, $\sqrt{N\sigma_N^{2+2s''}} \to 0$, *and* $\sqrt{N\sigma_N^2}\left(\sqrt{(\ln(N)/(N\sigma_N^3)} + \sigma_N^{s''}\right)^2 \to 0.$

Assumption A.5: $x^* \neq 0$; $0 < f(x_1^*, x_2), f(\overline{x}_1, \widetilde{x}_2) < \infty$; *there exist* $\delta, \xi > 0$ *such that* $\forall (x_1, x_2') \in B\left((x_1^*, x_2), \xi\right)$, $\forall \widetilde{v} \in B(v(x_1, x_2'), \xi)$, $f(x_1, x_2') \geq \delta$ and $f(\widetilde{v}, x_1, x_2') \geq \delta$; *there exist* $\delta', \xi' > 0$ *such that* $\forall (x_1, x_2') \in B\left((\overline{x}_1, \widetilde{x}_2), \xi'\right)$, $\forall \widetilde{v} \in B(v(x_1, x_2'), \xi')$, $f(x_1, x_2') \geq \delta'$ and $f(\widetilde{v}, x_1, x_2') \geq \delta'$; and $dF_{Z|X=x^*}(t_1/x^*)/dx \neq 0$.

Assumption A.6: t_1 belongs to the interior of the support of $q_1\left(v(x_1, t_2), x_1\right)$.

Let $\int K(z)^2 = \int \left(\int K(s,x)\, ds\right)^2 dx$, where $s \in R$. When Assumptions A.1-A.5 are satisfied, Theorems 1 and 2 in Matzkin (2003) imply that for $x_1 \neq \overline{x}_1$,

$$\sup_{e \in R} \left| \widehat{F}_\varepsilon(e) - F_\varepsilon(e) \right| \to 0 \quad \& \quad \widehat{v}(x_1, t_2) \to v(x_1, t_2) \quad \text{in probability}$$

$$\sqrt{N}\sigma_N \left(\widehat{F}_\varepsilon(e) - F_\varepsilon(e) \right) \to N\left(0, V_F\right)$$

and

$$\sqrt{N}\sigma_N \left(\widehat{v}(x_1, t_2) - v(x_1, t_2) \to N\left(0, V_v\right)\right.$$

where

$$V_F = \left\{ \int K(z)^2 \right\} \left[F_\varepsilon(e)\ (1 - F_\varepsilon(e)\right] \left[\frac{1}{f(\overline{x}_1, \widetilde{x}_2)} \right]$$

and

$$V_n = \left\{ \int K(z)^2 \right\} \left[\frac{F_\varepsilon(e)(1 - F_\varepsilon(e))}{f_{Z|X=x}(v(x_1, t_2))^2} \right] \left[\frac{1}{f(\overline{x}_1, \widetilde{x}_2)} + \frac{1}{f(x_1, t_2 + e)} \right].$$

Theorem 6.1 uses Assumptions A.1-A.6 to establish the asymptotic properties of $\widehat{m}(t_1, t_2)$. Let $x = (x_1^*, x_2)$ and $v^* = v(x_1^*, t_2)$. Let $\widetilde{x} = (\overline{x}_1, \widetilde{x}_2)$. Define the constant C by

$$C = \left(P_{zz}\left(\frac{t_1}{x_1^*} \right) \right)^2 \left(\frac{t_1}{(x_1^*)^2} \right)^2 \left[\frac{dF_{Z|X=(x_1^*, x_2)}\left(\frac{t_1}{x_1^*} \right)}{dx_1} \right]^{-2}$$

Theorem 6.1 *Suppose that Assumptions A.1-A.6 are satisfied. Then,* $\widehat{m}(t_1, t_2)$ *converges in probability to* $m(t_1, t_2)$ *and*

$$\sqrt{N\sigma_N^2}\left(\widehat{m}(t_1, t_2) - m(t_1, t_2) \right) \to N(0, V_m) \text{ in distribution,}$$

where

$$V_m = \left\{ \int K(z)^2 \right\} [C] \left(\frac{1}{f(\widetilde{x})} + \frac{1}{f(x)} \right) \left(F_{Z|X=\widetilde{x}}(\overline{z})(1 - F_{Z|X=\widetilde{x}}(\overline{z})) \right).$$

Proof. See Appendix. ■

6.2 Case 2

We next consider the situation where the assumptions of Theorem 4.1 are satisfied. In this case, $x \in R$ and we assume that for some unknown function m

$$U_z(z,x,\varepsilon) = m\left(q(z,x),\varepsilon\right)$$

where $q : R^2 \to R$ is a *known* function. We assume that F_ε, the distribution of ε, is known. Then, as argued in the proof of Theorem 4.1, the derived supply function satisfies

$$s\left(x,e\right) = F^{-1}_{Z|X=x}\left(F_\varepsilon(e)\right).$$

This can be estimated by

$$\widehat{s}\left(x,e\right) = \widehat{F}^{-1}_{Z|X=x}\left(F_\varepsilon(e)\right)$$

where $\widehat{F}_{Z|X=x}$ is calculated as in the above subsection. Next, to estimate $m\left(t_1,t_2\right)$ at specified values t_1, t_2, let $\widehat{x}$ be such that

$$q(\widehat{s}\left(\widehat{x},t_2\right), \widehat{x}) = t_1$$

Then,

$$\widehat{m}\left(t_1,t_2\right) = P_z\left(\widehat{s}\left(\widehat{x},t_2\right)\right).$$

Theorem 6.2 establishes the asymptotic properties of this estimator for the case where the function $q(z,x) = z \cdot x$. This analysis serves as a prototype for more general cases. The assumptions of the theorem are very similar to those of Theorem 6.1.

Let x^* be the value of x satisfying $q(v(x^*,t_2),x^*) = t_1$. In place of A.1-A.6, make the assumptions:

Assumption A.1′: *The sequence* $\{Z^i, X^i\}$ *is i.i.d.*

Assumption A.2′: $f(Z,X_1)$ *has compact support* $\widetilde{Z} \times \widetilde{X} \subset R^2$ *and is continuously differentiable of order* s'.

Assumption A.3′: *The kernel function* $K(\cdot,\cdot)$ *is differentiable of order* $\widetilde{s}$, *the derivatives of* K *of order* $\widetilde{s}$ *are Lipschitz,* K *vanishes outside a compact set, integrates to 1, and is of order* s'' where $\widetilde{s} + s'' + 1 \le s'$.

Assumption A.4′: *As* $N \to \infty$, $\sigma_N \to 0$, $\ln(N)/N\sigma_N^2 \to 0$, $\sqrt{N\sigma_N} \to \infty$, $\sqrt{N\sigma_N^{1+2s''}} \to 0$, *and* $\sqrt{N\sigma_N}\left(\sqrt{(\ln(N)/(N\sigma_N^2)} + \sigma_N^{s''}\right)^2 \to 0.$

Assumption A.5′: $x^* \neq 0$; $0 < f(x^*)$; *there exist* $\delta, \xi > 0$ *such that* $\forall x \in B(x^*,\xi)$

$\forall \widetilde{v} \in B(v(x,t_2),\xi),\ f(x) \geq \delta$ and $f(\widetilde{v},x) \geq \delta;\ dF_{Z|X=x^*}(t_1/x^*)/dx \neq 0$.

Assumption A.6′: t_1 is in the interior of the support of $q\,(v(x,t_2),x)$

Let $e = t_2$ and $x = x^*$. Theorems 1 and 2 in Matzkin (2003) imply that, under these assumptions,

$$\widehat{v}(x,e) \to v(x,e) \quad \text{in probability}$$

and

$$\sqrt{N\sigma_N}\,(\widehat{v}(x,e) - v(x,e)) \to N\,(0, V_{v'})$$

where

$$V_{v'} = \left\{ \int K(z)^2 \right\} \left[\frac{F_\varepsilon(e)(1-F_\varepsilon(e))}{f_{Z|X=x}(v(x,e))^2} \right] \left[\frac{1}{f(x)} \right].$$

The next theorem uses assumptions A.1′−A.6′ to establish the asymptotic properties of $\widehat{m}(t_1,t_2)$.

Theorem 6.2 *Suppose that Assumptions A.1′-A.6′ are satisfied. Then,* $\widehat{m}(t_1,t_2)$ *converges in probability to* $m(t_1,t_2)$ *and*

$$\sqrt{N\sigma_N^2}\,(\widehat{m}(t_1,t_2) - m(t_1,t_2)) \to N(0, V_{m'}) \text{ in distribution,}$$

where

$$V_{m'} = [C] \left\{ \int K(z)^2 \right\} \left(\frac{1}{f(x)} \right) (F_\varepsilon(t_2)(1-F_\varepsilon(t_2)))\,.$$

Proof. See Appendix. ■

The analysis for an estimator based on Theorem 4.2 is similar. We next present some Monte Carlo evidence on the performance of these estimators.

7 Simulations

We next present Monte Carlo experiments that illustrate the performance of the estimation techniques in Section 6. To obtain these results, we specify a nonadditive hedonic model and simulate data from this model using a range of parameter values. For each set of parameter values tested, we simulate 100 data-sets each with 500 observations. Then we estimate the utility function using each of the data-sets. The result of the estimation is an estimated marginal utility function. For each set of parameter values, we discuss the results of these simulations and present graphs which display the median (across the 100 data-sets) estimates of the method as well as the 5th and 95th percentile estimates. These results indicate that the techniques developed for estimating the nonadditive hedonic model work quite well.

We now document the specification of the simulation model that we studied and present and discuss these estimation results.

7.1 Model

The model specification is described in Table 1.

Table 1: Simulation Model Functional Forms

Firm	Technology	$\Gamma(z,\eta)$	$Az^{\alpha}\eta$
	Density of η	$f_{\eta}(\eta)$	$U[\eta_L,\eta_H]$
Worker	Utility	$U(z,x,\varepsilon)$	$Bz^{\beta}x^{\beta-1}\varepsilon^{-\delta}$
	Density of x	$f_x(x)$	$U[x_L,x_H]$
	Density of ε	$f_{\varepsilon}(\varepsilon)$	$U[\varepsilon_L,\varepsilon_H]$

We simulated and estimated this model for baseline parameter values described in Table 2 and for several other parameter values that illustrate the sensitivity of the results to the variance of the observable variables, the variance of the unobservable variables, and to the curvature of the utility and technology functions. The features of the model and the equilibrium that have the most significant impact on the performance of the estimators are the relative variance of observables and unobservables, and the equilibrium support of (zx,ε). As one would expect, increased variance of observables relative to unobservables reduces the sampling error of the estimator. Also, the estimator performs well on the interior of the support of (zx,ε) but less well near the boundary of the support where there are few observations in the data.

Table 2 presents the baseline parameter values and the alternative values that were tested.

Table 2: Baseline parameter values and alternative values

Parameter name	Baseline values	Feasible values	Alternative values
x_L	1.0	$x_L > 0$	n.a.
x_U	2.0	$x_U > x_L$	3.0
ε_L	1.0	$\varepsilon_L > 0$	n.a.
ε_U	2.0	$\varepsilon_U > \varepsilon_L$	n.a.
η_L	1.0	$\eta_L > 0$	n.a.
η_U	2.0	$\eta_U > \eta_L$	n.a.
α	0.25	$0 < \alpha < \beta$	n.a.
β	0.50	$\beta \neq 1$	$\{0.75, 0.9\}$
δ	1.0	$\delta > 0$	2.0
A	1.0	$A > 0$	n.a.
B	1.0	$B > 0$	n.a.

The baseline values were chosen to avoid numerical difficulties for parameter values near zero and to demonstrate the properties of the model. The alternative parameter values were chosen to illustrate interesting dependencies between model parameters and empirical results. We report results that illustrate the impact of variations in (x_U, β, δ). The parameter x_U affects the variance (and mean) of the observable variables and the size of the equilibrium support of (zx, ε). The parameters (β, δ) affect the degree of nonlinearity in the hedonic equilibrium, the shape of the hedonic pricing function and most importantly the shape of the equilibrium support of (zx, ε). We report results that illustrate how these features affect the empirical results.

We do not report results illustrating how the other parameters affect the empirical results. The parameters $(x_L, \varepsilon_L, \varepsilon_U, \eta_L, \eta_U)$ have impacts that are qualitatively similar to the impacts of x_U. The parameter x_L affects the mean and variance of x and the equilibrium support of (zx, ε). The parameters $(\varepsilon_L, \varepsilon_U)$ affect the mean and variance of ε and the equilibrium support of (zx, ε). Increases in the variance of ε reduce the precision of the estimates. The parameters (η_L, η_U) affect the equilibrium support of (zx, ε). We also do not report results for alternative values of α and for values of $\beta > 1$. These parameters affect the shape of the support of (zx, ε). In particular, when $\alpha < 1$ and $\beta > 1$, the support of (zx, ε) is confined to a very small region. These results are available from the authors upon request.

The model specification is a generalization of the model presented in section 3.1.1. The workers' marginal utility has the property that $U_z(z, x, \varepsilon) = m(q(z, x), \varepsilon)$ where $q(z, x) = zx$ and $m(q, \varepsilon) = \beta B q^{\beta-1} \varepsilon^{-\delta}$. We approximate equilibrium in this model computationally and present estimation results from data generated from this model.

7.2 Equilibrium

Computing equilibrium in this model is somewhat complicated by the fact that the supports of (x, ε, η) are compact. Because of this, the equilibrium supports of z and $q(z, x)$ are also compact and care must be taken to compute the supports properly. Nevertheless the equilibrium price, the equilibrium supply of workers $z = s(x, \varepsilon)$, and the equilibrium demand of firms $z = d(\eta)$ can be computed using numerical techniques. Details and computational algorithms are available from the authors upon request. We computed equilibrium in this model for the parameter values detailed in Table 2 and then for each set of parameter values simulated 100 data-sets each with 500 observations.

7.3 Estimation results

After generating data from the model described above, we used the procedure described in section 3 to estimate the supply function $z = s(x, \varepsilon)$. The results in that section show that it is impossible to recover the structural function unless additional structure such as $q(z, x) = zx$ and $\varepsilon \sim U[\varepsilon_L, \varepsilon_U]$. We make the assumption that these two facts are known. Under this assumption, we can compare the estimated values of m with the true value. We estimated the marginal utility function $m(q, \varepsilon)$ for a selected set of values of q and ε in the relevant domain. The domain upon which m is identified is both model dependent and data dependent. We illustrate this in the simulation results reported below. For each set of parameter values, we simulated 100 data-sets with 500 observations on (z, x, P_z). For each set of parameter values, we then estimated the model 100 times. The figures below display the median values of our estimation results as well as the 5th and 95th percentiles.

Figure 3 presents estimation results for the baseline model. The top two panels display the true function $m(q, \varepsilon)$ and the median of the estimates of that function. While m is well-defined for all positive values of q and ε, the function is only identified on the funnel shaped region underneath the graph in the figure. These limits of the region of identification are determined by the model; in particular they are determined by the fact that we assume (x, ε, η) are each uniformly distributed. The shape of the region is determined in equilibrium and depends strongly on the supports of (x, ε, η) and on the curvature parameters (α, β, δ).

The figure shows that the median of the estimates of m are very accurate. The two functions in the top two panels are nearly identical. The bottom two panels show this more clearly. They show the estimated values of m for fixed values of ε and q respectively. In these panels, the solid lines depict the true value of $m(q, \varepsilon)$, the dashed lines depict the median of the estimated values, the circles depict the 5th percentile estimates, and the plus signs depict the 95th percentile estimates. The solid lines and the dashed lines are indistinguishable. The 5th and 95th percentile values are also very close to the true values except near the boundaries of the supports. In the bottom left panel, the value of ε is fixed at 1.5. For this value of ε, the value of $m(q, \varepsilon)$ is accurately estimated for all values of q ranging from about 1 or 2 to about 24. The value of the function cannot be estimated for larger values of q. For other values of ε, the range of values of q that produce accurate estimates are different. In the bottom right panel, the variable q is fixed at the value 4.9564. For this value of q, $m(q, \varepsilon)$ is accurately estimated for values of ε ranging from about 1.3 to 1.9.

Figure 4 illustrates similar results when x_U is increased from 2.0 to 3.0. The precision of the estimates increase and the size of the region on which the function is identifiable increases. In Figure 3, the scale of the q axis ranges from 0 to 60. In contrast, in Figure 4, the q axis scale ranges from 0 to 150. In both Figures 3 and 4, the function m can be accurately estimated for all values of $\varepsilon \in [1.2, 1.8]$ when q is small. However, when q is large the interval in the ε dimension within which m can be accurately estimated is smaller.

Figure 5 illustrates the impact of increasing β to 0.75. This change has a dramatic impact on the support of (zx, ε) and hence on the region within which m is identified. The scale of the q axis in Figure 5 ranges from 0 to 2.5. Within this range, m can be estimated accurately. But, the equilibrium provides no information for values of q outside this region.

Figure 6 illustrates the impact of increasing β to 0.9. The support of (zx, ε) becomes smaller and the precision of the estimates decay. As β approaches 1, the performance of the estimator declines. In the limiting case where $\beta = 1$, x does not affect marginal utility.

Finally, Figure 7 illustrates the impact of increasing δ to 2.0. This change drastically increases the equilibrium support of z and hence of (zx, ε). Notice that the scale of the q axis in Figure 7 ranges from 0 to 500. The upper right panel of Figure 7 shows that the median of the estimates of m is very similar to the true value of m (depicted in the upper left panel). The lower panels show that when $\varepsilon = 1.5$, the value of m is accurately estimated for values of q ranging from 10 to 80. Similarly, the lower right show that the value of m is accurately estimated when $q = 25.07$ for all values of ε ranging from 1.4 to 1.7.

The figures illustrate that the estimator performs well on the interior of the support of (zx, ε). The estimator first estimates the supply function $z = s(x, \varepsilon)$ and then uses this estimated function, the marginal price function $P_z(z)$, and knowledge of the index structure $U_z(z, x, \varepsilon) = m(q, \varepsilon)$ where $q = zx$ to estimate m. Crucial determinants of the performance of the estimator of m are the relative variance of observables and unobservables and the equilibrium support of (zx, ε). In applications, since researchers must first estimate $z = s(x, \varepsilon)$, this first stage estimate can be used to construct a residual for each observation and estimate the joint density of (zx, ε). This joint density provides information as to the region in (zx, ε) where many observations are available and where it is possible to estimated m accurately.

The results from these simulations are prototypical. The estimator performs well on the interior of the support of (zx, ε). The performance decays in regions near the boundary where less data is available. In the simulations, the size and shape of the support of (zx, ε) are highly sensitive to the parameter values. This sensitivity is not a general feature of equilibria in hedonic models but is a specific feature of this example.

8 Summary

This paper considers hedonic equilibrium models where the marginal utility of each consumer and the marginal product of each firm are both nonadditive functions of the attribute and a random vector of individual characteristics, which are different for the consumers and firms. We demonstrate that this type of specification is capable of generating equilibria of different types, with and without bunching and analyze some properties of equilibria in these models. We develop conditions sufficient to identify the marginal utility and marginal product functions using both single market and multimarket data. In the single market data cases, we provide nonparametric estimators for these functions and show that they are consistent and asymptotically normal. Finally, we provide simulations that illustrate the performance of the estimators.

Appendix

Proof of Theorem 4.1. Let $s(x,\varepsilon)$ denote the supply function of a worker with characteristics (x,ε). By (4.4) and the first order conditions

$$m\left(q\left(s\left(x,\varepsilon\right),x\right),\varepsilon\right)=P_{z}\left(s(x,\varepsilon)\right).$$

By the second order conditions

$$\frac{\partial m}{\partial q}\frac{\partial q}{\partial z}-P_{zz}<0.$$

Hence,

$$\frac{\partial m}{\partial q}\frac{\partial q}{\partial z}\frac{\partial s}{\partial \varepsilon}+\frac{\partial m}{\partial \varepsilon}=P_{zz}\frac{\partial s}{\partial \varepsilon}.$$

By the monotonicity of m in ε,

$$\frac{\partial s}{\partial \varepsilon}=\frac{-\frac{\partial m}{\partial \varepsilon}}{\frac{\partial m}{\partial q}\frac{\partial q}{\partial z}-P_{zz}}>0.$$

Hence, s is a nonadditive function in ε which is strictly increasing in ε. Since ε is independent of X, it follows by Matzkin (1999) that

$$s\left(x,\varepsilon\right)=F_{Z|X}^{-1}\left(F_{\varepsilon}(\varepsilon)\right).$$

Since F_{ε} is given, s is identified. Let (t_1,t_2) be such that $t_1\in(q_l(t_2),q_u(t_2))$. Find x^* such that

$$q\left(s(x^*,t_2),x^*\right)=t_1.$$

Then,

$$m(t_1,t_2)=P_z\left(s(x^*,t_2)\right).$$

■

Proof of Theorem 4.2. By (4.5), in the statement of the theorem, it follows that the value of z that satisfies the FOC when $x=\overline{x}$ and $\varepsilon=t$ is $z=t$. Hence, the supply function, $s(x,\varepsilon)$, satisfies

$$s(\overline{x},\varepsilon)=\varepsilon.$$

By the SOC and the monotonicity assumption on m in terms of ε,

$$\frac{\partial s}{\partial \varepsilon} = \frac{-\frac{\partial m}{\partial \varepsilon}}{\frac{\partial m}{\partial q}\frac{\partial q}{\partial z} - P_{zz}} > 0.$$

Then, by Matzkin (1999)

$$F_{\varepsilon}(e) = F_{Z|X=\overline{x}}(e)$$

and

$$s(\widetilde{x}, e) = F^{-1}_{Z|X=\widetilde{x}}(F_{\varepsilon}(e)).$$

Next, to see that the function m is identified, let x^* denote the solution to

$$q\left(s(x^*, t_2), t_2\right) = t_1.$$

Hence,

$$m(t_1, t_2) = m\left(q\left(s(x^*, t_2), t_2\right), t_2\right)$$

and from the FOC

$$m(t_1, t_2) = P_z\left(s(x^*, t_2)\right).$$

■

Proof of Theorem 4.3. Since U_z is weakly separable in $q_2(x_2, \varepsilon)$, the function $z = s(x_1, x_2, \varepsilon)$, which satisfies the FOC is also weakly separable in $q_2(x_2, \varepsilon)$. Hence, for some unknown function v

$$s(x_1, x_2, \varepsilon) = v(x_1, q_2(x_2, \varepsilon)).$$

Let x_2 and ε be such that $q_2(x_2, \varepsilon) = \alpha$. Then, by separability and condition (4.7) in the statement of the theorem

$$U_z(\overline{z}, \overline{x}_1, q_2\left(x_2, \varepsilon\right)) = P_z(\overline{z})$$

where $\overline{z}$ satisfies the FOC when $x_1 = \overline{x}_1$ and $q_2(x_2, \varepsilon) = \alpha$. It then follows that

$$v(\overline{x}_1, \alpha) = s(\overline{x}_1, q_2\left(x_2, \varepsilon\right)) = \overline{z}.$$

By FOC and by (4.6) in the statement of the theorem it follows that

$$m\left[\ q_1(v(x_1, q_2(x_2, \varepsilon)), x_1)\ ,\ q_2(x_2, \varepsilon)\ \right] \ = \ P_z\left[\ v(x_1, q_2(x_2, \varepsilon))\ \right].$$

Differentiating with respect to q_2, since the SOC are satisfied, we have an interior solution and

$$\frac{\partial m}{\partial q_1}\frac{\partial q_1}{\partial z}\frac{\partial v}{\partial q_2}+\frac{\partial m}{\partial q_2}=P_{zz}\frac{\partial v}{\partial q_2}.$$

Hence,

$$\frac{\partial v}{\partial q_2}=\frac{-\frac{\partial m}{\partial q_2}}{\frac{\partial m}{\partial q_1}\frac{\partial q_1}{\partial z}-P_{zz}}.$$

From SOC the denominator is negative:

$$\frac{\partial m}{\partial q_1}\frac{\partial q_1}{\partial z}-P_{zz}<0.$$

From the strict monotonicity of m in its second argument the function v is strictly increasing in its second argument. Summarizing, the unknown function v that relates x_1, x_2, and ε to the value of z that satisfies the FOC is such that $z=v(x_1,q_2(x_2,\varepsilon))$, v is strictly increasing in its second argument and $v(\overline{x}_1,\alpha)=\overline{z}$. It then follows from the analysis of Matzkin (2003) that the function v and the distribution of ε are identified from the conditional distribution of Z given $X=(X_1,X_2)$.

To show that the function m is identified, let (t_1,t_2) be any vector such that $t_2\in\left[t_2^l,t_2^u\right]$ and $t_1\in[t_1^l(t_2),t_1^u(t_2)]$. Let x_1^* denote a solution to

$$q_1(v(x_1^*,t_2),x_1^*)=t_1.$$

Since q_1 is a known function and v can be recovered from the conditional cdf of z given (x_1,x_2), the only unknown in the above expression is x_1^*. Since $t_2\in\left[t_2^l,t_2^u\right]$ and $t_1\in[t_1^l(t_2),t_1^u(t_2)]$, x_1^* exists. Since $v(x_1^*,t_2)$ satisfies the FOC,

$$\begin{aligned} m(t_1,t_2) &= m\left(q_1\left(v(x_1^*,t_2),x_1^*\right),t_2\right) \\ &= P_z\left(v(x_1^*,t_2)\right) \\ &= P_z\left(s(x_1^*,x_2^*,\varepsilon^*)\right) \end{aligned} \tag{A.2}$$

for any x_2^* and ε such that $q_2(x_2^*,\varepsilon)=\alpha$. In (A.2), the first equality follows because $q(v(x_1^*,t_2),x_1^*)=t_1$; the second equality follows because when z is substituted by the value that satisfies the first order conditions, the value of the marginal utility function m equals the value of the marginal price function at the particular value of z that satisfies the first order conditions. The third equality follows by the restriction on the function s. Since the function P_z is known and the function s can be recovered without knowledge of m, $(A.2)$ implies that the function m is identified. ■

Proof of Theorem 6.1. We use a version of the Delta Method developed by Aït-Sahalia (1994) and Newey (1994). Let $F(z,x)$ denote the distribution function (cdf) of the vector of observable variables (Z,X), $f(z,x)$ denote its probability density function (pdf), $f(x)$ denote the marginal pdf of X, and $F_{Z|X=x}$ denote the conditional cdf of Z given $X=x$. Recall that $\widetilde{Z}\times\widetilde{X}$ is the compact support of (Z,X). Let $L=3$ be the dimension of $\widetilde{Z}\times\widetilde{X}$. For any function $G:R^L\to R$, define $g(z,x)=\partial^L G(z,x)/\partial z\partial x$, $g(x)=\int_{-\infty}^{\infty}g(s,x)ds$, $G_{Z|X=x'}(z')$ $=\left(\int_{-\infty}^{z'}g(s,x')ds\right)/g(x')$, and $\widetilde{G_Z}(z,x)=\int_{-\infty}^{z}g(s,x)ds=\int_{-\infty}^{\infty}1[s\le z]g(s,x)ds$ where $1[\cdot]=1$ if $[\cdot]$ is true and equals zero otherwise. Let $\underline{C}$ denote a compact set in R^L that strictly includes $\widetilde{Z}\times\widetilde{X}$. Let Q denote the set of all functions $G:R^L\to R$ such that $g(z,x)$ has bounded first order derivatives and vanishes outside $\underline{C}$. Let $\widetilde{Q}$ denote the set of all functions $\widetilde{G_Z}$ that are derived from some G in Q. Since there is a 1-1 relationship between functions in Q and functions in $\widetilde{Q}$, we can define a functional on Q or on $\widetilde{Q}$ without altering its definition. Let $\|G\|$ denote the maximum of the sup norms of $g(z,x)$ and the first order derivatives of $g(z,x)$. If $H\in Q$, there exists $\rho_1>0$ such that if $\|H\|\le\rho_1$ then, for some $0<a,b<\infty$, all x in a neighborhood of (x_1^*,x_2) and all $\widetilde{v}\in B(v(x_1,t_2),\xi)$, $|h(x)|\le a\,\|H\|$, $\left|\int^{\widetilde{s}}h(s,x)ds\right|\le a\,\|H\|$, $|f(x)+h(x)|\ge b\,|f(x)|$, and $f(\widetilde{s},x)+h(\widetilde{s},x)\ge b\,|f(\widetilde{s},x)|$. Let $\widetilde{x}=(\overline{x}_1,\widetilde{x}_2)$ and $v^*=v(x_1^*,t_2)$.

We will first derive the asymptotic behavior of $\widehat{x}_1$, defined as the value of x_1 that, given $\widehat{v}$, satisfies

$$q_1\left(\widehat{v}\left(\widehat{x}_1,t_2\right),\widehat{x}_1\right)=t_1.$$

Recall that

$$\widehat{v}\left(x_1,t_2\right)=\widehat{F}^{-1}_{Z|X=(x_1,x_2)}\left(\widehat{F}_{Z|X=\widetilde{x}}\left(\overline{z}\right)\right).$$

Hence, $\widehat{x}_1$ satisfies

$$\widehat{F}^{-1}_{Z|X=(\widehat{x}_1,x_2)}\left(\widehat{F}_{Z|X=\widetilde{x}}\left(\overline{z}\right)\right)\ \widehat{x}_1=t_1$$

or

$$\widehat{F}_{Z|X=\widetilde{x}}\left(\overline{z}\right)=\widehat{F}_{Z|X=(\widehat{x}_1,x_2)}\left(\frac{t_1}{\widehat{x}_1}\right),$$

and x_1^* satisfies

$$F_{Z|X=\widetilde{x}}\left(\overline{z}\right)=F_{Z|X=\left(x_1^*,x_2\right)}\left(\frac{t_1}{x_1^*}\right).$$

Define the functional $\rho(G, x_1)$ on $Q \times R$ by

$$\rho(G, x_1) = G_{Z|X=\widetilde{x}}(\overline{z}) - G_{Z|X=(x_1,x_2)}\left(\frac{t_1}{x_1}\right).$$

For any x_1 in a neighborhood of x_1^* and for H such that $\|H\|$ is small enough

$$\begin{aligned}
&\rho(F+H, x_1) - \rho(F, x_1) \\
= &\ \frac{\int_{-\infty}^{\overline{z}} (f(s,\widetilde{x}) + h(s,\widetilde{x}))\ ds}{(f(\widetilde{x}) + h(\widetilde{x}))} - \frac{\int_{-\infty}^{\overline{z}} f(s,\widetilde{x})\ ds}{f(\widetilde{x})} \\
&- \left[\frac{\int_{-\infty}^{(t_1/x_1)} (f(s,x_1,x_2) + h(s,x_1,x_2))\ ds}{(f(x_1,x_2) + h(x_1,x_2))} - \frac{\int_{-\infty}^{(t_1/x_1)} f(s,x_1,x_2)\ ds}{f(x_1,x_2)}\right] \\
= &\ \frac{f(\widetilde{x}) \int_{-\infty}^{\overline{z}} h(s,\widetilde{x})\ ds}{f(\widetilde{x})(f(\widetilde{x}) + h(\widetilde{x}))} - \frac{h(\widetilde{x}) \int_{-\infty}^{\overline{z}} f(s,\widetilde{x})\ ds}{f(\widetilde{x})(f(\widetilde{x}) + h(\widetilde{x}))} \\
&- \left[\frac{f(x_1,x_2) \int_{-\infty}^{(t_1/x_1)} h(s,x_1,x_2)\ ds}{f(x_1,x_2)(f(x_1,x_2) + h(x_1,x_2))} - \frac{h(x_1,x_2) \int_{-\infty}^{(t_1/x_1)} f(s,x_1,x_2)\ ds}{f(x_1,x_2)(f(x_1,x_2) + h(x_1,x_2))}\right] \\
= &\ \left[\frac{\int_{-\infty}^{\overline{z}} h(s,\widetilde{x})\ ds}{f(\widetilde{x})} - \frac{h(\widetilde{x}) F_{Z|X=\widetilde{x}}(\overline{z})}{f(\widetilde{x})^2}\right] \\
&- \left[\frac{\int_{-\infty}^{(t_1/x_1)} h(s,x_1,x_2)\ ds}{f(x_1,x_2)} - \frac{h(x_1,x_2) F_{Z|X=(x_1,x_2)}(t_1/x_1)}{f(x_1,x_2)^2}\right] \\
&- \left[\frac{\left[f(\widetilde{x}) \int_{-\infty}^{\overline{z}} h(s,\widetilde{x})\ ds - h(\widetilde{x}) F_{Z|X=\widetilde{x}}(\overline{z})\right] h[\widetilde{x}]}{f(\widetilde{x})^2 (f(\widetilde{x}) + h(\widetilde{x}))}\right] \\
&+ \left[\frac{\left[f(x_1,x_2) \int_{-\infty}^{(t_1/x_1)} h(s,x_1,x_2)\ ds - h(x_1,x_2) \int_{-\infty}^{(t_1/x_1)} f(s,x_1,x_2)\ ds\right] h(x_1,x_2)}{f(x_1,x_2)^2 (f(x_1,x_2) + h(x_1,x_2))}\right].
\end{aligned}$$

Define

$$D_F\rho(x_1;H) = \left[\frac{\int_{-\infty}^{\overline{z}} h\left(s,\widetilde{x}\right)\ ds}{f\left(\widetilde{x}\right)} - \frac{h\left(\widetilde{x}\right) F_{Z|X=\widetilde{x}}\left(\overline{z}\right)}{f\left(\widetilde{x}\right)^2}\right] - \left[\frac{\int_{-\infty}^{(t_1/x_1)} h\left(s,x_1,x_2\right)\ ds}{f\left(x_1,x_2\right)} - \frac{h\left(x_1,x_2\right) F_{Z|X=(x_1,x_2)}\left(t_1/x_1\right)}{f\left(x_1,x_2\right)^2}\right]$$

and

$$R_F\rho(x_1;H) = -\left[\frac{\left[f\left(\widetilde{x}\right)\int_{-\infty}^{\overline{z}} h\left(s,\widetilde{x}\right)\ ds - h\left(\widetilde{x}\right) F_{Z|X=\widetilde{x}}\left(\overline{z}\right)\right] h\left[\widetilde{x}\right]}{f\left(\widetilde{x}\right)^2\left(f\left(\widetilde{x}\right)+h\left(\widetilde{x}\right)\right)}\right] + \left[\frac{\left[f\left(x_1,x_2\right)\int_{-\infty}^{(t_1/x_1)} h\left(s,x_1,x_2\right)\ ds - h\left(x_1,x_2\right)\int_{-\infty}^{(t_1/x_1)} f\left(s,x_1,x_2\right)\ ds\right] h\left(x_1,x_2\right)}{f\left(x_1,x_2\right)^2\left(f\left(x_1,x_2\right)+h\left(x_1,x_2\right)\right)}\right].$$

Then, for some $a_1 < \infty$ and all x_1 in the neighborhood of x_1^*,

$$|D_F\rho(x_1;H)| \leq a_1\left\|H\right\| \quad \text{and} \quad |R_F\rho(x_1;H)| \leq a_1\left\|H\right\|^2$$

and, for all such x_1

$$\rho\left(F+H,x_1\right) - \rho\left(F,x_1\right) = D_F\rho(x_1;H) + R_F\rho(x_1;H).$$

Next, for any x_1 close enough to x_1^*, for any small enough $\Delta x_1 \neq 0$, and for any G such that $\|G-F\|$ is small enough

$$
\begin{aligned}
& \rho\left(G, x_1+\Delta x_1\right)-\rho\left(G, x_1\right) \\
= & -(G)_{Z \mid X=\left(x_1+\Delta x_1, x_2\right)}\left(\frac{t_1}{x_1+\Delta x_1}\right)+(G)_{Z \mid X=\left(x_1, x_2\right)}\left(\frac{t_1}{x_1}\right) \\
= & -\frac{\int_{-\infty}^{t_1 /\left(x_1+\Delta x_1\right)} g\left(s, x_1+\Delta x_1, x_2\right)\ ds}{g\left(x_1+\Delta x_1, x_2\right)} \\
& +\frac{\int_{-\infty}^{t_1 / x_1} g\left(s, x_1, x_2\right)\ ds}{g\left(x_1, x_2\right)} \\
= & -\frac{\int_{-\infty}^{t_1 /\left(x_1+\Delta x_1\right)} g\left(s, x_1, x_2\right)+\frac{\partial g\left(s, x_1, x_2\right)}{\partial x_1} \Delta x_1+R_{f, 1}\ ds}{g\left(x_1+\Delta x_1, x_2\right)} \\
& +\frac{\int_{-\infty}^{t_1 / x_1} g\left(s, x_1, x_2\right)\ ds}{g\left(x_1, x_2\right)}
\end{aligned}
$$

where for some $a_2<\infty$, $\left|R_{g, 1}\right| \leq a_2\left|\Delta x_1\right|^2$, and where the last equality follows by Taylor's Theorem. Using again Taylor's Theorem, it follows that for some $a_3<\infty$, and for $R_{g, 2}$ and $R_{g, 3}$ with $\left|R_{g, 2}\right| \leq a_3\left|\Delta x_1\right|^2$ and $\left|R_{g, 3}\right| \leq a_3\left|\Delta x_1\right|^2$

$$
\begin{aligned}
& \rho\left(G, x_1+\Delta x_1\right)-\rho\left(G, x_1\right) \\
= & -\frac{\left[\int_{-\infty}^{t_1 /\left(x_1\right)}\left(g\left(s, x_1, x_2\right)+\frac{\partial g\left(s, x_1, x_2\right)}{\partial x_1} \Delta x_1+R_{g, 1}\right)\ ds\right]\ g\left(x_1, x_2\right)}{g\left(x_1+\Delta x_1, x_2\right) g\left(x_1, x_2\right)} \\
& -\frac{\left[g\left(\frac{t_1}{x_1}, x_1, x_2\right) \Delta x_1+\frac{\partial g\left(\frac{t_1}{x_1}, x_1, x_2\right)}{\partial x_1}\left(\Delta x_1\right)^2+R_{g, 3}\right] g\left(x_1, x_2\right)}{g\left(x_1+\Delta x_1, x_2\right) g\left(x_1, x_2\right)} \\
& +\frac{\left[\int_{-\infty}^{t_1 / x_1} g\left(s, x_1, x_2\right)\ ds\right]\left[g\left(x_1, x_2\right)+\frac{\partial g\left(x_1, x_2\right)}{\partial x_1} \Delta x_1+R_{g, 2}\right]}{g\left(x_1+\Delta x_1, x_2\right) g\left(x_1, x_2\right)} .
\end{aligned}
$$

Hence,

$$\rho\left(G, x_1+\Delta x_1\right)-\rho\left(G, x_1\right)$$

$$= -\frac{\left[g\left(x_1, x_2\right) \int_{-\infty}^{t_1/(x_1)} g\left(s, x_1, x_2\right)\ ds + g\left(x_1, x_2\right) \int_{-\infty}^{t_1/(x_1)} \frac{\partial g(s, x_1, x_2)}{\partial x_1} ds\ \Delta x_1\right]}{g\left(x_1+\Delta x_1, x_2\right) g\left(x_1, x_2\right)}$$

$$-\frac{\left[\int_{-\infty}^{t_1/(x_1)} R_{g,1}\ ds\right]\ g\left(x_1, x_2\right)}{g\left(x_1+\Delta x_1, x_2\right) g\left(x_1, x_2\right)}$$

$$-\frac{\left[g\left(x_1, x_2\right)\ g\left(\frac{t_1}{x_1}, x_1, x_2\right) \Delta x_1\right]}{g\left(x_1+\Delta x_1, x_2\right) g\left(x_1, x_2\right)}$$

$$-\frac{\left[\frac{\partial g\left(\frac{t_1}{x_1}, x_1, x_2\right)}{\partial x_1}\left(\Delta x_1\right)^2 + R_{g,3}\right] g\left(x_1, x_2\right)}{g\left(x_1+\Delta x_1, x_2\right) g\left(x_1, x_2\right)}$$

$$+\frac{\left[g\left(x_1, x_2\right)\left[\int_{-\infty}^{t_1/x_1} g\left(s, x_1, x_2\right)\ ds\right] + \frac{\partial g(x_1, x_2)}{\partial x_1}\left[\int_{-\infty}^{t_1/x_1} g\left(s, x_1, x_2\right)\ ds\right] \Delta x_1\right]}{g\left(x_1+\Delta x_1, x_2\right) g\left(x_1, x_2\right)}$$

$$+\frac{\left[\int_{-\infty}^{t_1/x_1} g\left(s, x_1, x_2\right)\ ds\right]\left[R_{g,2}\right]}{g\left(x_1+\Delta x_1, x_2\right) g\left(x_1, x_2\right)}.$$

Let

$$D_{x_1} \rho\left(G ; \Delta x_1\right)=$$

$$= -\frac{\int_{-\infty}^{t_1/(x_1)} \frac{\partial g(s, x_1, x_2)}{\partial x_1} ds\ \Delta x_1}{g\left(x_1, x_2\right)}$$

$$-\frac{g\left(\frac{t_1}{x_1}, x_1, x_2\right) \Delta x_1}{g\left(x_1, x_2\right)}$$

$$+\frac{\frac{\partial g(x_1, x_2)}{\partial x_1}\left[\int_{-\infty}^{t_1/x_1} g\left(s, x_1, x_2\right)\ ds\right] \Delta x_1}{g\left(x_1, x_2\right)^2}$$

and

$$
\begin{aligned}
& R_{x_1}\rho\left(G;\Delta x_1\right) \\
= \; & -D_{x_1}\rho\left(G;\Delta x_1\right)\left[\frac{g\left(x_1+\Delta x_1,x_2\right)-g\left(x_1,x_2\right)}{g\left(x_1,x_2\right)^2 g\left(x_1+\Delta x_1,x_2\right)}\right] \\
& -\frac{\left[\int_{-\infty}^{t_1/(x_1)} R_{g,1}\ ds\right]\ g\left(x_1,x_2\right)}{g\left(x_1+\Delta x_1,x_2\right)g\left(x_1,x_2\right)} \\
& -\frac{\left[\frac{\partial g\left(\frac{t_1}{x_1},x_1,x_2\right)}{\partial x_1}\left(\Delta x_1\right)^2+R_{g,3}\right]g\left(x_1,x_2\right)}{g\left(x_1+\Delta x_1,x_2\right)g\left(x_1,x_2\right)} \\
& +\frac{\left[\int_{-\infty}^{t_1/x_1} g\left(s,x_1,x_2\right)\ ds\right]\left[R_{g,2}\right]}{g\left(x_1+\Delta x_1,x_2\right)g\left(x_1,x_2\right)}.
\end{aligned}
$$

Then, for some $a_4<\infty$,

$$
\left|D_{x_1}\rho\left(G;\Delta x_1\right)\right|\leq a_4\left|\Delta x_1\right|,\quad \left|R_{x_1}\rho\left(G;\Delta x_1\right)\right|\leq a_4\left|\Delta x_1\right|^2
$$

and

$$
\rho\left(G,x_1+\Delta x_1\right)-\rho\left(G,x_1\right)=D_{x_1}\rho\left(G;\Delta x_1\right)+R_{x_1}\rho\left(G;\Delta x_1\right).
$$

Moreover, for some $a_5<\infty$ and for all H such that $\|H\|$ is small enough

$$
\left|D_{x_1}\rho\left(F+H;\Delta x_1\right)-D_{x_1}\rho\left(F;\Delta x_1\right)\right|\leq a_5\left\|H\right\|\left|\Delta x_1\right|.
$$

By assumption, for any $\Delta x_1\neq 0$, $D_{x_1^*}\rho\left(F;\Delta x_1\right)\neq 0$. Since Q is a Banach space, it follows from the Implicit Function Theorem of Hildebrandt and Graves (1927) (see Zeidler (1991) p. 150), that there exists a unique functional $\kappa\left(G\right)$ and a small $r<\infty$ such that for all H with $\|H\|$ small enough,

$$
\left|\kappa\left(F+H\right)-\kappa\left(F\right)\right|\leq r
$$

and

$$\rho\left(F+H,\kappa\left(F+H\right)\right)=\left(F+H\right)_{Z|X=\widetilde{x}}\left(\overline{z}\right)-\left(F+H\right)_{Z|X=\left(\kappa\left(F+H\right),x_{2}\right)}\left(\frac{t_{1}}{\kappa\left(F+H\right)}\right)=0.$$

Since

$$\begin{aligned}
& \rho\left(F+H,\kappa\left(F+H\right)\right)-\rho\left(F,\kappa\left(F\right)\right) \\
= & \left(F+H\right)_{Z|X=\widetilde{x}}\left(\overline{z}\right)-F_{Z|X=\widetilde{x}}\left(\overline{z}\right) \\
& -\left(F+H\right)_{Z|X=\left(\kappa\left(F+H\right),x_{2}\right)}\left(\frac{t_{1}}{\kappa\left(F+H\right)}\right)+F_{Z|X=\left(\kappa\left(F\right),x_{2}\right)}\left(\frac{t_{1}}{\kappa\left(F\right)}\right) \\
& -\left(F+H\right)_{Z|X=\left(\kappa\left(F\right),x_{2}\right)}\left(\frac{t_{1}}{\kappa\left(F\right)}\right)+\left(F+H\right)_{Z|X=\left(\kappa\left(F\right),x_{2}\right)}\left(\frac{t_{1}}{\kappa\left(F\right)}\right) \\
= & D_{F}\rho(x_{1};H)+D_{x_{1}}\rho\left(F;\Delta x_{1}\right) \\
& +R_{F}\rho(x_{1};H)+\left(D_{x_{1}}\rho\left(F+H;\Delta x_{1}\right)-D_{x_{1}}\rho\left(F;\Delta x_{1}\right)\right)+R_{x_{1}}\rho\left(G;\Delta x_{1}\right) \\
= & 0
\end{aligned}$$

it follows that

$$\kappa\left(F+H\right)-\kappa\left(F\right)=-\left[\frac{dF_{Z|X=\left(x_{1}^{*},x_{2}\right)}\left(\frac{t_{1}}{x_{1}^{*}}\right)}{dx_{1}}\right]^{-1}\left[D_{F}\rho(x_{1}^{*};H)\right]+R\kappa$$

where

$$\left|R\kappa\right|\leq a_{3}\left\|H\right\|^{2}.$$

By the Delta method in Newey (1994), it follows that

$$\sqrt{N\sigma^{2}}\left(\widehat{x}_{1}-x_{1}^{*}\right)\rightarrow N(0,V_{\widehat{x}})$$

where

$$V_{\widehat{x}} = \left[\frac{dF_{Z|X=\left(x_1^*,x_2\right)}\left(\frac{t_1}{x_1^*}\right)}{dx_1} \right]^{-2} \left\{ \int K(z)^2 \right\} \left(\frac{1}{f(\widetilde{x})} + \frac{1}{f(x)} \right) \left(F_{Z|X=\widetilde{x}}(\widetilde{z})(1 - F_{Z|X=\widetilde{x}}(\widetilde{z})) \right).$$

Since

$$\begin{aligned} \widehat{m}\left(t_1, t_2\right) &= P_z\left(\widehat{v}\left(\widehat{x}_1, t_2\right)\right) \\ &= P_z\left(\frac{t_1}{\widehat{x}_1}\right) \end{aligned}$$

it follows by the standard Delta method that

$$\sqrt{N\sigma_N^2}\left(\widehat{m}\left(t_1, t_2\right) - m\left(t_1, t_2\right)\right) \rightarrow N(0, V_m)$$

where

$$V_m = C\left\{ \int K(z)^2 \right\} \left(\frac{1}{f(\widetilde{x})} + \frac{1}{f(x)} \right) \left(F_{Z|X=\widetilde{x}}(\widetilde{z})(1 - F_{Z|X=\widetilde{x}}(\widetilde{z})) \right)$$

and

$$C = \left(P_{zz}\left(\frac{t_1}{x_1^*} \right) \right)^2 \left(\frac{t_1}{(x_1^*)^2} \right)^2 \left[\frac{dF_{Z|X=\left(x_1^*,x_2\right)}\left(\frac{t_1}{x_1^*}\right)}{dx_1} \right]^{-2}.$$

■

Proof of Theorem 6.2. The method of proof is very similar to that of Theorem 6.1. The only difference is that $\widehat{F}_{Z|X=\widetilde{x}}(\widetilde{z})$ and $F_{Z|X=\widetilde{x}}(\widetilde{z})$ in the proof of Theorem 6.1 are now replaced by $F_\varepsilon(t_2)$. Following the same steps as in the proof of Theorem 6.1, it is then easy to show that

$$\sqrt{N\sigma_N^2}\left(\widehat{m}\left(t_1, t_2\right) - m\left(t_1, t_2\right)\right) \rightarrow N(0, V_{m'})$$

where

$$V_{m'} = C\left\{ \int K(z)^2 \right\} \left(\frac{1}{f(x)} \right) \left(F_\varepsilon(t_2)(1 - F_\varepsilon(t_2)) \right)$$

and C is as in the proof of Theorem 6.1. ■

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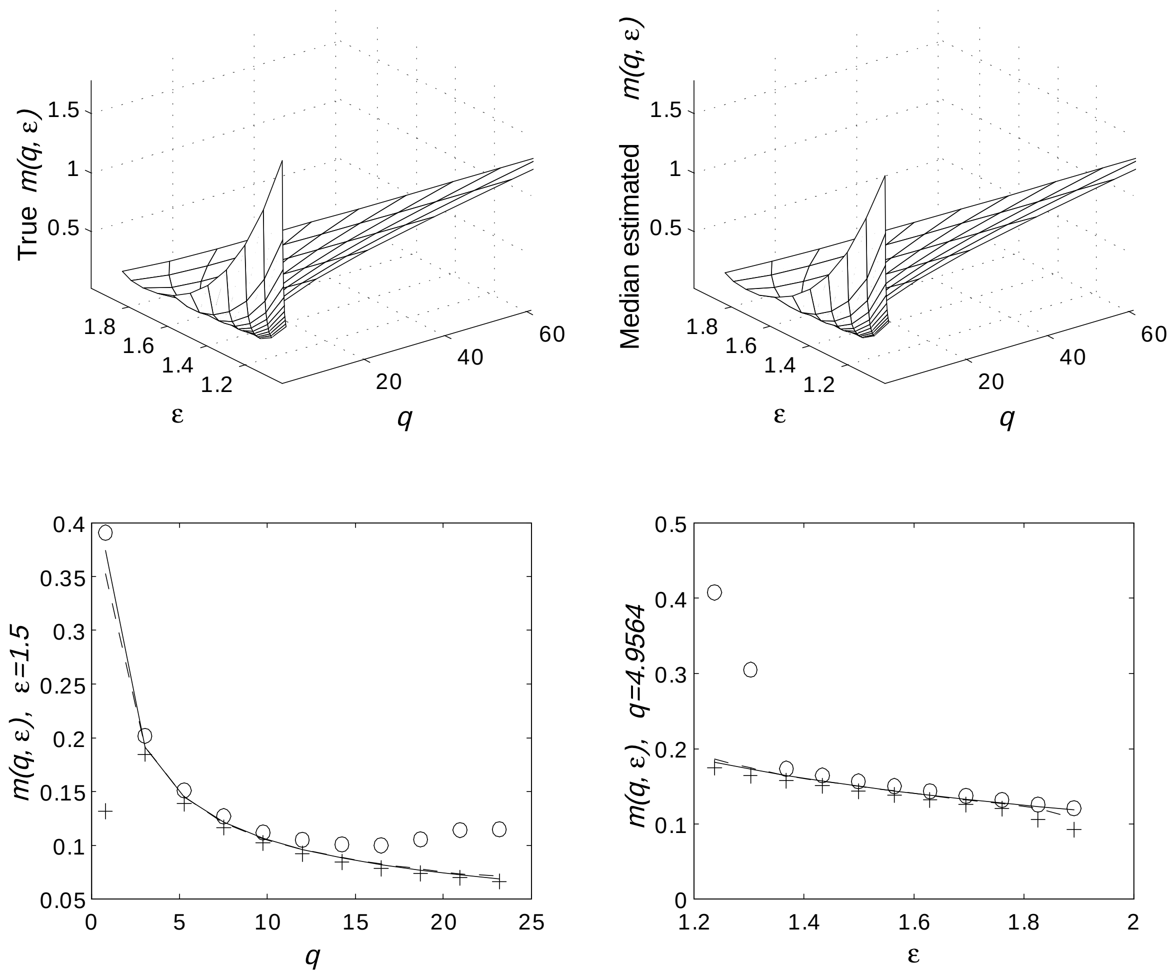
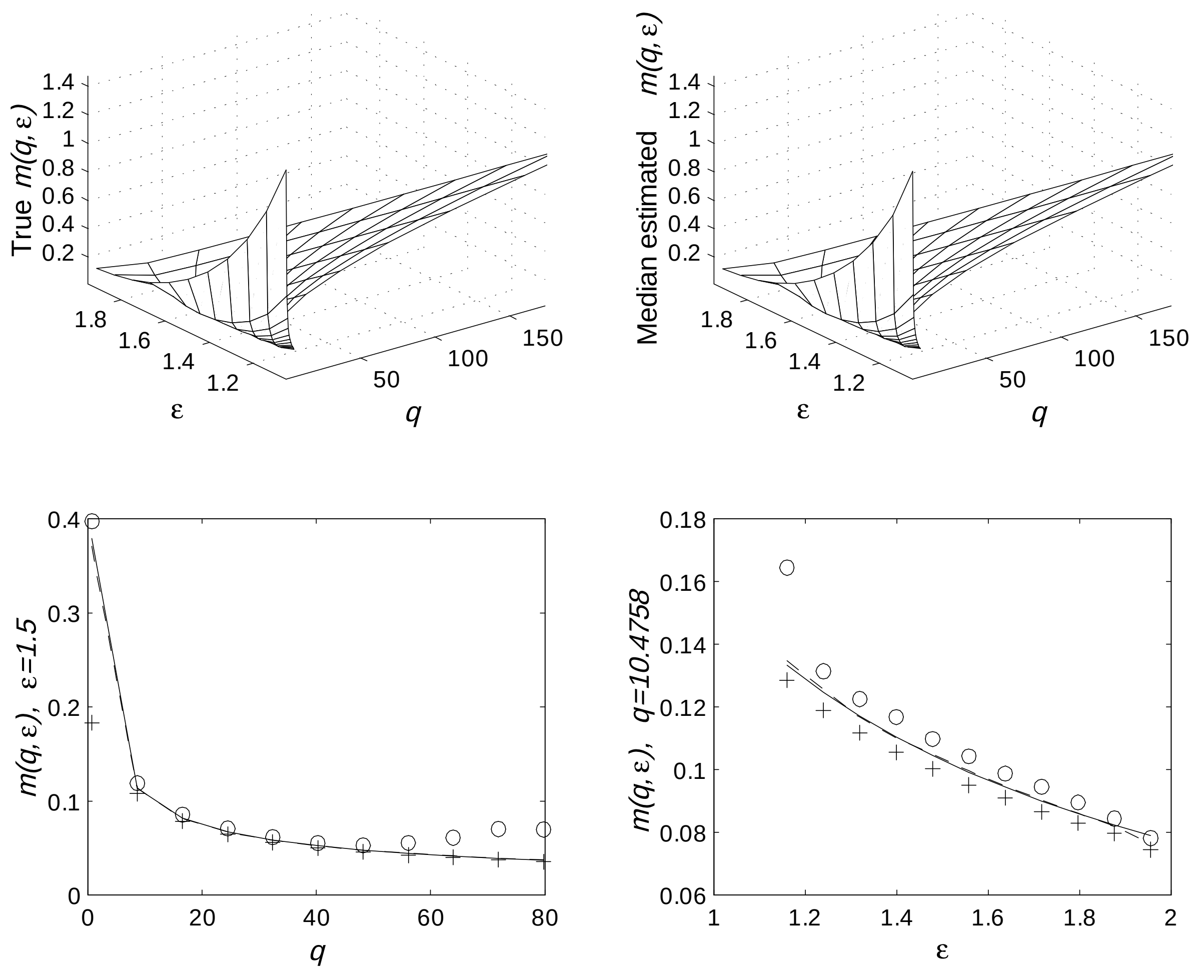
Figure 3: Simulation results: baseline parameter values



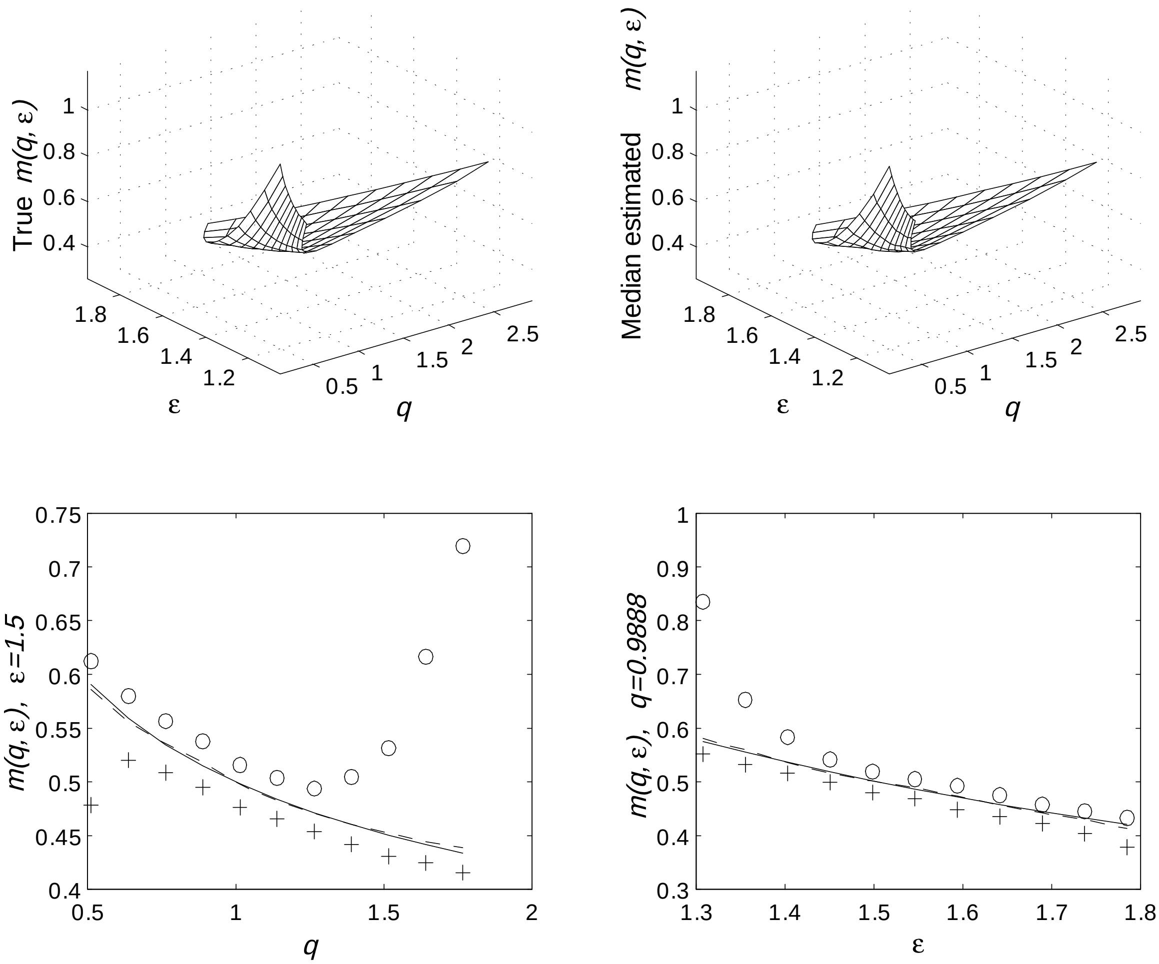
Note: The upper left panel plots the true values of $m(q, \varepsilon)$ where $q = zx$. The upper right panel plots the median of the estimates of $m(q, \varepsilon)$ (Sample size 500, 100 Monte Carlo replications). The supports of the graphs indicate the equilibrium region on which the function m is identified. The lower left panel plots the true and estimated values of $m(q, \varepsilon)$ when $\varepsilon = 1.5$. The lower right panel plots the true and estimated values of $m(q, \varepsilon)$ when $q = 4.9564$. The solid lines plot the true function values, the dashed lines plot the medians of the estimated values, the circles plot the 5th percentile estimates, and the plus symbols plot the 95th percentile estimates. True baseline parameter values are given in Table 2.

Figure 4: Simulation results: $x_U = 3.0$



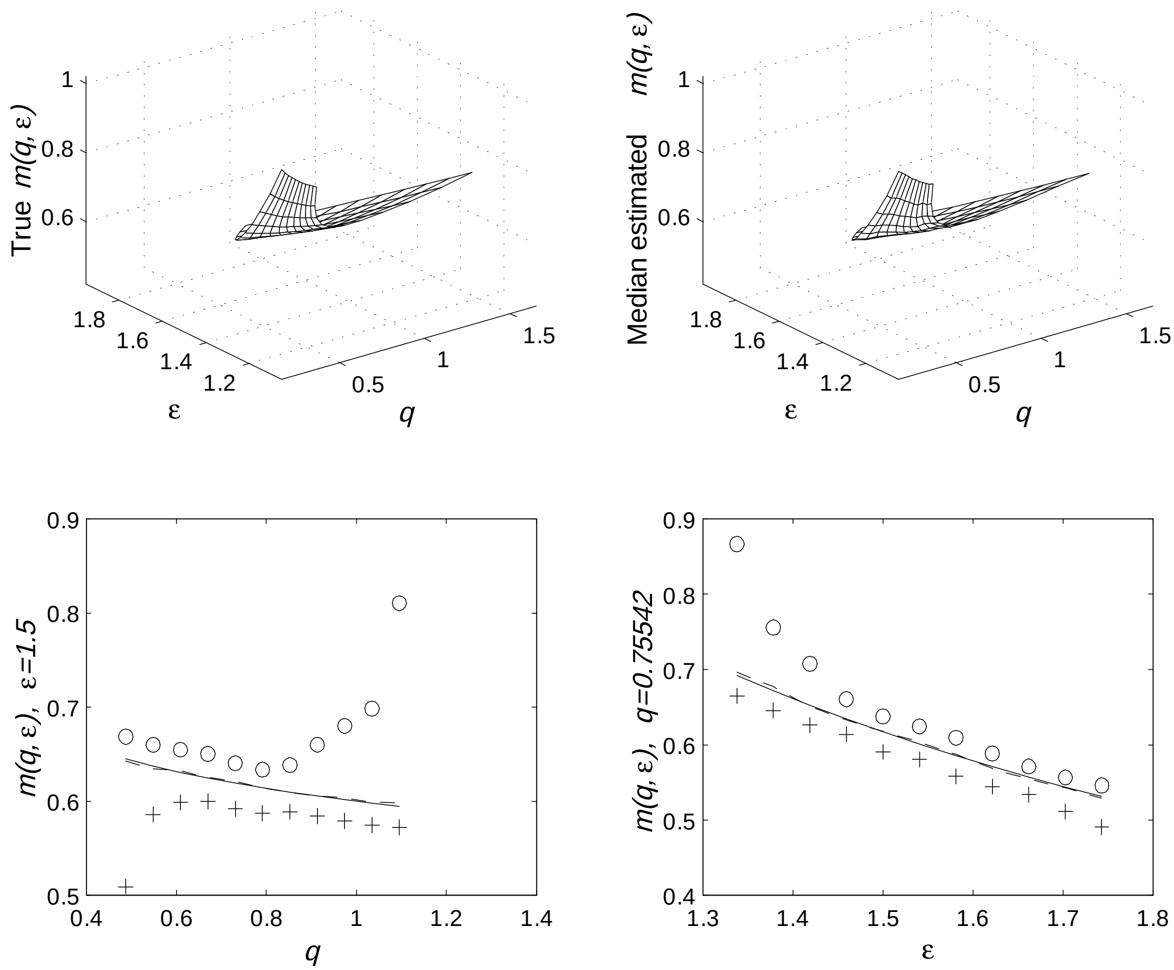
Note: The upper left panel plots the true values of $m(q, \varepsilon)$ where $q = zx$. The upper right panel plots the median of the estimates of $m(q, \varepsilon)$ (Sample size 500, 100 Monte Carlo replications). The supports of the graphs indicate the equilibrium region on which the function m is identified. The lower left panel plots the true and estimated values of $m(q, \varepsilon)$ when $\varepsilon = 1.5$. The lower right panel plots the true and estimated values of $m(q, \varepsilon)$ when $q = 10.4758$. The solid lines plot the true function values, the dashed lines plot the medians of the estimated values, the circles plot the 5th percentile estimates, and the plus symbols plot the 95th percentile estimates. All true parameter values except x_U are identical to the baseline parameter values. This case used the value $x_U = 3.0$.

Figure 5: Simulation results: $\beta = 0.75$



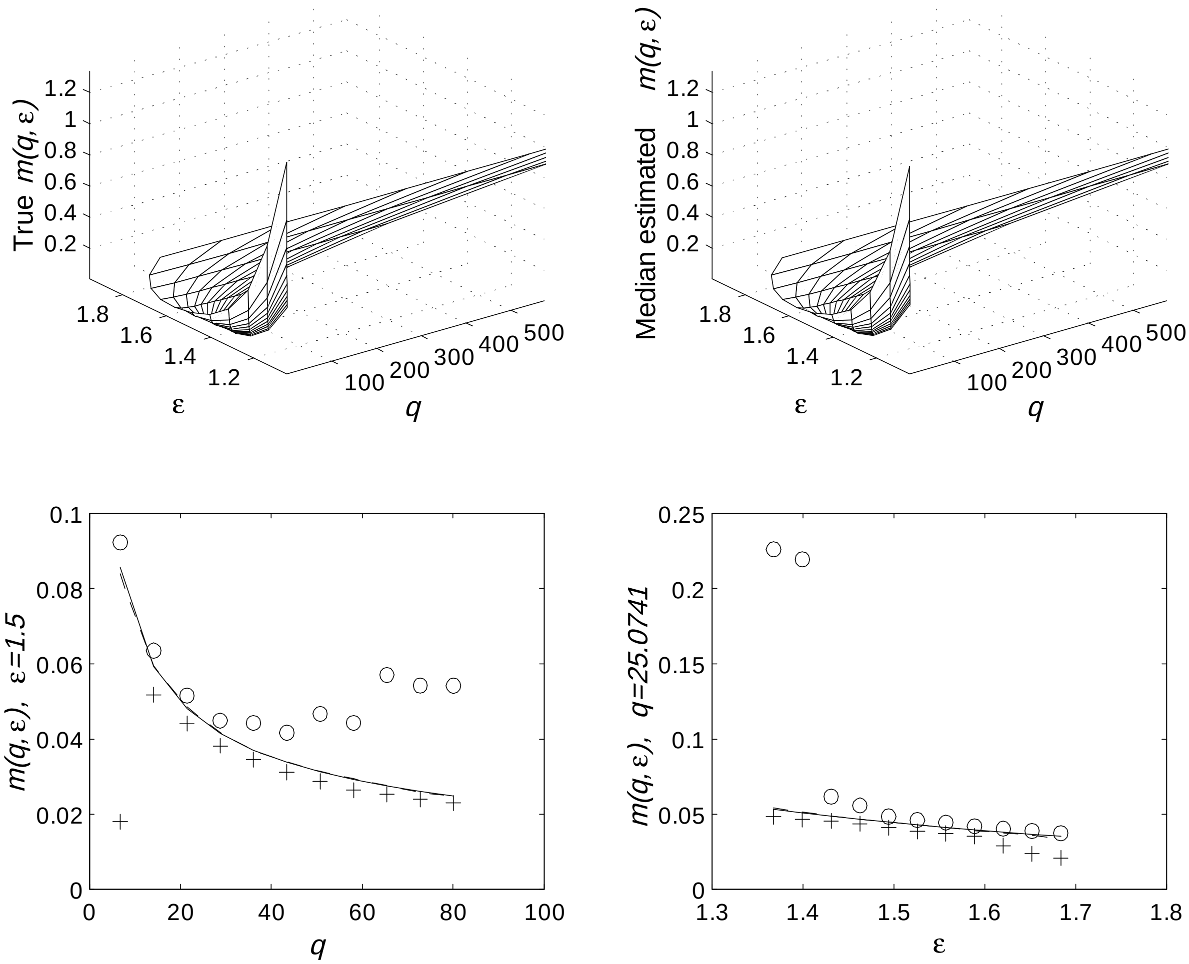
Note: The upper left panel plots the true values of $m(q, \varepsilon)$ where $q = zx$. The upper right panel plots the median of the estimates of $m(q, \varepsilon)$ (Sample size 500, 100 Monte Carlo replications). The supports of the graphs indicate the equilibrium region on which the function m is identified. The lower left panel plots the true and estimated values of $m(q, \varepsilon)$ when $\varepsilon = 1.5$. The lower right panel plots the true and estimated values of $m(q, \varepsilon)$ when $q = 0.9888$. The solid lines plot the true function values, the dashed lines plot the medians of the estimated values, the circles plot the 5th percentile estimates, and the plus symbols plot the 95th percentile estimates. All true parameter values except β are identical to the baseline parameter values. This case used the value $\beta = 0.75$.

Figure 6: Simulation results: $\beta = 0.9$



Note: The upper left panel plots the true values of $m(q,\varepsilon)$ where $q = zx$. The upper right panel plots the median of the estimates of $m(q,\varepsilon)$ (Sample size 500, 100 Monte Carlo replications). The supports of the graphs indicate the equilibrium region on which the function m is identified. The lower left panel plots the true and estimated values of $m(q,\varepsilon)$ when $\varepsilon = 1.5$. The lower rightpanel plots the true and estimated values of $m(q,\varepsilon)$ when $q = 0.75542$. The solid lines plot the true function values, the dashed lines plot the medians of the estimated values, the circles plot the 5th percentile estimates, and the plus symbols plot the 95th percentile estimates. All true parameter values except β are identical to the baseline parameter values. This case used the value $\beta = 0.9$.

Figure 7: Simulation results: $\delta = 2.0$



Note: The upper left panel plots the true values of $m(q, \varepsilon)$ where $q = zx$. The upper right panel plots the median of the estimates of $m(q, \varepsilon)$ (Sample size 500, 100 Monte Carlo replications). The supports of the graphs indicate the equilibrium region on which the function m is identified. The lower left panel plots the true and estimated values of $m(q, \varepsilon)$ when $\varepsilon = 1.5$. The lower rightpanel plots the true and estimated values of $m(q, \varepsilon)$ when $q = 25.0741$. The solid lines plot the true function values, the dashed lines plot the medians of the estimated values, the circles plot the 5th percentile estimates, and the plus symbols plot the 95th percentile estimates. All true parameter values except δ are identical to the baseline parameter values. This case used the value $\delta = 2.0$.