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by

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Abstract Analysts should correct for distributional differences before undertaking local progressivity comparisons between income tax or tax and benefit schedules. A *transplant-and-compare* procedure is advocated, involving 'importation' of the schedule from one regime into another, or from both into a reference scenario. The residual progression ordering over transplanted schedules then assures a global ordering of original regimes by Lorenz or Suits curves. The algorithm is advocated for use only when transplantation functions are isoelastic, and is illustrated for the Canadian, Israeli and UK tax and benefit systems.

1. INTRODUCTION

A tax schedule with high marginal rates can have less progressivity than another with low marginal rates. This is because it matters for progressivity where the taxpayers are located: there may be few people actually being taxed at high published rates, or a substantial number; the upper tail of a schedule with mainly low rates could 'catch' most taxpayers, or none. The theorists recognize these facts of life, and have distinguished local measures of *structural* progression of an income tax or tax and benefit system from global measures of *effective* progression (also known as progressivity). The practitioners now typically conduct international and intertemporal comparisons in a split manner, examining schedules locally, using the elasticity measures known as residual and liability progression, and evaluating effective progression in summary, using indices of redistributive effect and disproportionality which subsume distributional differences rather than revealing their röle.

Despite the fact that close scrutiny of schedular differences can be misleading when judged in isolation from the income distributions to which the schedules apply, the landmark results of Jakobsson (1976) and Kakwani (1977), which predict unambiguous redistributive effects from

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residual progression comparisons, are still invoked. The predictions are mathematically valid, but they are only made by positing a *fixed and common distribution of pre-tax income* for all schedules being compared. Though this is plainly unrealistic for both international and intertemporal comparisons, the tradition of examining the residual progression of schedules along the income scale persists.

In this paper we advocate a new procedure which analysts could use, based on residual progression comparisons, to draw out correct distributional implications in international and intertemporal comparisons of income taxes and tax and benefit systems. This involves a *transplant-and-compare* procedure, in which the schedule is 'imported' from one regime into the other, or from both into a reference or baseline scenario. Comparisons of residual progression are then undertaken using what we shall call *log transplant curves*, and they lead, *via* the Jakobsson/Kakwani result, to valid and normatively significant redistributive conclusions predicated on actual income distributions.[2] We also reveal a link with the procedure suggested by Hayes et al. (1995), whereby the residual progression of different income tax schedules is compared at common percentile points in the relevant pre-tax income distributions. Under certain conditions to be revealed, this procedure is equivalent to ours and has the same normative significance.

Comparisons of effective progression based on Lorenz and Suits (1977) curves take account of pre-tax inequality.[3] The problems of reasoning from schedular changes were anticipated by Musgrave and Thin (1948): "... the less equal the distribution of income before tax, the more potent will be a (given) progressive tax structure in equalizing income" *(page 510).* The effect of pre-tax distributional change on progressivity comparisons in the Lorenz and Suits frameworks has been examined by Formby *et al*. (1981), Lambert and Pfähler (1992), Milanovic (1994) and Seidl (1994), but the reformulation of local progression comparisons to take pre-tax inequality differences into account is new.

The structure of the paper is as follows. Section 2 contains definitions and necessary preliminaries. In section 3, we define the concept of a *deformation* of an income tax or tax and benefit schedule, and outline the *transplant-and-compare* procedure, which is based on the

[2] Much of what we do will also apply to the liability progression ordering of income tax schedules, and thence to effects on the distribution of the income tax burden. See on.

[3] Partial orderings involve dominance configurations between such curves; complete orderings are provided by indices such as those of Suits (1977) and Reynolds and Smolensky (1977).

deformation concept. In section 4, we identify the class of admissible deformations for this procedure. In section 5, the transplant-and-compare procedure is illustrated using simple examples, and related to methodology to take account of distributional differences in the existing literature, including that of Hayes *et al* (1995). In section 6, issues of implementation are discussed, and the procedure is applied using micro-data from Canada, Israel and the UK. In the concluding section, we discuss the theoretical and empirical compromises involved in implementing the new procedure, and offer some concluding remarks.

2. ANALYTICAL FRAMEWORK AND EXISTING LITERATURE

Let x be pre-tax or original income. For convenience of presentation, pre-tax income distributions F(x) will be assumed continuous and strictly increasing on $\mathbb{R}_{++}$, and income tax or tax and benefit schedules T(x) will be assumed differentiable, with the property $T(x)<x$ and $0\leq T'(x)<1\ \forall x$. Net or final income $N(x) = x - T(x)$ thus satisfies $N(x)>0$ and $0<N'(x)\leq 1\ \forall x$.[4] The tax or tax and benefit system is progressive if and only if T(x)/x is increasing with x, equivalently, N(x)/x is decreasing with x. The basic unit for analysis will be the *regime* <N,F>, comprising the net income schedule and pre-tax income distribution pair.

The residual progression of schedule $N_i(x)$ is its elasticity with respect to income x, and will be denoted $RP_i(x)$. Hence $RP_i(x) = xN_i'(x)/N_i(x)$, the ratio of the marginal to average retention rates.[5] The partial ordering over schedules N by residual progression is denoted $\succeq_{RP}$:

$$N_1 \succeq_{RP} N_2 \Leftrightarrow RP_1(x) \leq RP_2(x)\ \forall x \qquad (1)$$

Given a pre-tax income distribution F(x), the Lorenz curve for pre-tax income will be denoted $L_X(p)$, $p=F(x)\in[0,1]$. The Lorenz and Suits curves for income after application of $N_i(x)$ will be denoted $L_{Ni}(p)$ and $R_{Ni}(q)$ respectively, where q is the income share ($q = L_X(p)$ and $R_{Ni}(q) = L_{Ni}(p)$). The Lorenz and Suits-based partial orderings of regimes <N,F> by their redistributive

[4] Typically, of course, there are jumps in the marginal tax rate, *e.g.* at the threshold in the case of an income tax, and, in the case of a tax and benefit system, between the rate of taper of benefits and first marginal tax rate. As Keen *et al.* (1999) have recently shown, there are implications for results in the standard literature of admitting an income tax threshold; we will note these at the relevant points but do not consider them important for the empirical procedure we advocate. Our assumptions also deny the presence of horizontal inequity (HI), evidenced in taxes which are not a function only of income x, and/or net income schedules which involve reranking, *i.e.* are non-monotonic. The thrust of much present-day research is to isolate the vertical (progressivity) and horizontal effects of tax systems, substituting counterfactual HI-free *smooth* schedules T(x) and N(x) in order to evaluate the former - we shall do this in the applications ahead - hence the non-importance of the differentiability assumption.

[5] Liability progression, $LP_i(x)$, is similarly defined, as the elasticity of the schedule $T_i(x)$, provided that $T_i(x)\neq 0$. $T_i(x)$ is progressive if and only if $RP_i(x) < 1$ for all x, equivalently if and only if $LP_i(x) > 1$ for all x.

effects will be denoted $\succeq_L$ and $\succeq_S$:

$$\langle N_1,F_1\rangle \succeq_L \langle N_2,F_2\rangle \Leftrightarrow L_{N1}(p) - L_{X1}(p) \geq L_{N2}(p) - L_{X2}(p)\ \forall p \tag{2}$$

$$\langle N_1,F_1\rangle \succeq_S \langle N_2,F_2\rangle \Leftrightarrow R_{N1}(q) \geq R_{N2}(q)\ \forall q \tag{3}$$

The Jakobsson (1976) and Kakwani (1977) result, already referred to, is this:

$$N_1 \succeq_{RP} N_2 \Leftrightarrow \langle N_1,F_0\rangle \succeq_L \langle N_2,F_0\rangle\ \forall F_o \Leftrightarrow \langle N_1,F_0\rangle \succeq_S \langle N_2,F_0\rangle\ \ \forall F_o \tag{4}$$

The local ordering of schedules is equivalent to the global ordering of regimes whenever the pre-tax income distribution is the same.[6]

3. THE TRANSPLANT AND COMPARE PROCEDURE

To 'correct' income tax or tax and benefit schedules for pre-tax distributional differences in regimes, prior to making local progression comparisons, we first define a deformation function:

Definition

Let $g:\mathbb{R}_+\rightarrow\mathbb{R}_+$ be any monotone increasing function. The deformation N^g of a net income schedule N is defined as:

$$N^g = g \circ N \circ g^{-1} \tag{5}$$

and the deformation $\langle N,F\rangle^g$ of a regime as:

$$\langle N,F\rangle^g = \langle N^g, F\circ g^{-1}\rangle \tag{6}$$

where $\circ$ is the composition operator.

The interpretation is straightforward. If F is the distribution function for x, $F\circ g^{-1}$ is the distribution function for g(x). If N maps an original income x into a final income y, N^g maps g(x) into g(y). Hence N^g is the schedule induced by N on deformed incomes, and $\langle N,F\rangle^g$ is the regime induced by $\langle N,F\rangle$ on the distribution of deformed incomes.

In mathematics, N^g is known as the conjugate by g of N within the group of real-valued functions under the composition operator (Budden 1972, esp. §20). Important properties of the

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[6] For income tax schedules $T_i(x)>0\ \forall x$ the liability progression partial ordering $\succeq_{LP}$ is defined as in (1), but over T_1 and T_2: $T_1 \succeq_{LP} T_2 \Leftrightarrow LP_1(x) \geq LP_2(x)\ \forall x$, and the corresponding link is with global orderings (of regimes $\langle T_1,F_1\rangle$ and $\langle T_2, F_2\rangle$) by Lorenz and Suits curves for the distribution of the income tax burden. For income tax schedules with thresholds, the content of $\succeq_{LP}$ (though not that of $\succeq_{RP}$) must be modified. Keen *et al.* (1999) show that the dominating schedule must have a higher threshold as well as higher elasticity above its threshold, and then the link is the same. For tax and benefit systems, the cases $T_i(x)>0$ and $T_i(x)<0$ (i=1,2) have to be treated separately in defining $\succeq_{LP}$, and there are problems in defining and comparing curves for distributions of net taxes (which could sum to zero, for example). In Ebert and Lambert (1999), residual and liability progression measures for tax and benefit systems are related to the corresponding measures for the tax and benefit components, and the respective orderings are discussed.

conjugate are that:

$$\{N^g\}^h = N^{h\circ g} \text{ and } \{\langle N,F\rangle^g\}^h = \langle N,F\rangle^{h\circ g} \quad \forall N, \forall F, \forall g, \forall h \tag{7}$$

For an equal absolute sacrifice income tax, with $u:\mathbb{R}_+\to\mathbb{R}_+$ the relevant utility of income function, N^u gives the associated tax on utilities, which is of course lump-sum. The deformation procedure could also be used to transform a nominally-specified income tax or tax and benefit system $\langle N,F\rangle$ into the corresponding regime in real income terms if the cost of living differed in different income groups: if the real value of a nominal income x is g(x), $\langle N,F\rangle^g$ is the real tax system.[7]

The deformation function g effects a variable stretch or shrink of pre-tax relative income differentials. When measured in the new units, g(x) rather than x, overall inequality is different. We can select a g-function to effect *any given inequality change* in an income distribution. In particular, deformations can be selected to transform tax regimes under comparison to the point where the (induced) pre-tax distributions are the same. This procedure can be thought of as transplanting the respective tax systems into a host distribution, a common baseline or reference situation, in which the standard Jakobsson-Kakwani results can be applied to draw out redistributive inferences from local progression comparisons.

To fix ideas, consider a comparison between the Israeli and British tax and benefit systems. We will actually undertake such a comparison later. The regimes $\langle N_{ISR},F_{ISR}\rangle$ and $\langle N_{UK},F_{UK}\rangle$ are the material for analysis. The idea is to identify the transformation function g(.) which would deform F_{ISR} into F_{UK}; and then to ask, how does the Israeli tax and benefit system act on this particular function of the shekel? Its action could be compared directly with that of N_{UK}, for both now apply to the same distribution, F_{UK}:

$$g = F_{UK}^{-1}\circ F_{ISR} \;\Rightarrow\; \langle N_{ISR},F_{ISR}\rangle^g = \langle N_{ISR}^g,F_{UK}\rangle \tag{8}$$

If, having identified g in this way, we should find that $N_{ISR}^g \succeq_{RP} N_{UK}$, for example, then we would know that *the Israeli transplant has more progressivity than UK schedule whenever both are applied to any common distribution*: in particular, when both are applied to F_{UK}, as is in fact the case. Normative significance comes by applying the Atkinson (1970) theorem. Having scaled N_{ISR}^g to have the same yield as N_{UK} (a residual-progression-neutral device to which we shall return), it follows from $N_{ISR}^g \succeq_{RP} N_{UK}$ that *welfare is higher, and inequality lower, after application of*

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[7] We return to this point later. For evidence in support of income-varying cost-of-living indices in the UK, see Crawford (1996), especially table 4.1 and figure 4.3.

the (scaled) Israeli transplant than after application of the actual UK schedule.[8] The procedure just described for achieving a normatively significant local-to-global comparison takes the UK as baseline; we might have chosen Israel instead. There is clearly an ‘independence of baseline’ issue to be resolved; we turn to this shortly.

In general terms, let $<N_1,F_1>$ and $<N_2,F_2>$ be two regimes. The deformation procedure can be used to transplant from both systems into a ‘reference’ distribution, call it F_o, by identifying the appropriate transformation functions g_i, i=1,2:

$$g_i = F_0^{-1} \circ F_i \;\Rightarrow\; <N_i,F_i>^{gi} = <N_i^{gi},F_o> \qquad (9)$$

An unambiguous local progression comparison between N_1^{g1} and N_2^{g2}, if such occurs, can be represented as a *partial progressivity ordering over regimes, conditioned by F_o*, call it $\succeq_{P|Fo}$:

$$<N_1,F_1> \succeq_{P|Fo} <N_2,F_2> \Leftrightarrow N_1^{g1} \succeq_{RP} N_2^{g2} \quad \text{where } g_i = F_0^{-1} \circ F_i,\ i=1,2 \qquad (10)$$

The defining characteristic of a successful ordering by $\succeq_{P|Fo}$ is that one transplant has unambiguously more progressivity than the other whenever both are applied to any common distribution of income:

$$<N_1,F_1> \succeq_{P|Fo} <N_2,F_2> \Leftrightarrow <N_1^{g1},F_3> \succeq_L <N_2^{g2},F_3> \Leftrightarrow <N_1^{g1},F_3> \succeq_S <N_2^{g2},F_3> \;\; \forall F_3 \qquad (11)$$

in particular, when both are applied to F_0, as is in fact the case. (For the result in (11), just combine (10) with (4)). The transplant-and-compare procedure ‘corrects’ N_1 and N_2 for inequality and size differences (if any) between the distributions to which they apply, and enables valid Lorenz and Suits findings to be drawn from a local comparison. Normative significance again comes by appropriate scaling, this time to create an equal yield comparison with respect to F_o, and inequality and welfare superiority of one scaled transplant over the other follows. We turn next to the independence of baseline question in this general scenario.

4. INDEPENDENCE OF BASELINE

Suppose we selected F_o as the reference distribution, and found $<N_1,F_1> \succeq_{P|Fo} <N_2,F_2>$ following the transplant-and-compare procedure. That is, we pronounce that regime 1 has more progressivity than regime 2 in the conditional sense. Had we instead made another choice of baseline, say G_o, would this conclusion be maintained, *i.e.* would we find $<N_1,F_1> \succeq_{P|Go} <N_2,F_2>$? Might this alternative comparison fail, or even be thrown into reverse? It turns out that the

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[8] A yield comparison between N_{UK} and the unscaled N_{ISR}^{g} would make little sense. For both international and intertemporal comparisons, normative significance can only come by standardizing yields. See Formby and Smith (1986) and Lambert and Pfähler (1987) on this.

comparison can only be guaranteed the same if F_o and G_o are linked in a very special way:

Theorem 1

The orderings $\succeq_{P|F_o}$ and $\succeq_{P|G_o}$ are the same $\Leftrightarrow$ $G_0^{-1} \circ F_0 = g$ is isoelastic ($\Leftrightarrow \exists$ A,b>0 : $g(x) = Ax^b$).

For the proof, see the Appendix. The implications of this result are significant. If x has distribution function F_0 then $g(x) = (G_0^{-1} \circ F_0)(x)$ has distribution function G_0, *i.e.* g maps from F_0 to G_0. Theorem 1 thus says that structural progression comparisons can be guaranteed invariant to the choice of baseline if and only if the candidate reference distributions are isoelastic transformations of one another.

This is not to say that, given two regimes $\langle N_1,F_1 \rangle$ and $\langle N_2,F_2 \rangle$, and alternative reference distributions F_0 and G_0 *not* isoelastically linked, the two structural progression comparisons will necessarily be found to be different. Independence of baseline can of course be checked empirically: we return to this issue later. The analyst's main interest would be in transplanting into one or other of the *actual* distributions, F_1 and F_2, as we suggested for the illustrative UK/Israeli case. Theorem 1 also tells us that these two must themselves be linked by an isoelastic transformation if the potential for dependency of conclusions on the baseline is to be avoided.

This admissibility condition becomes clearer upon transforming into logarithms. If F_1 and F_2 belong to a family of income distributions which, in logarithms, is location-and-scale-invariant (the lognormal family is a good example), then any member of this family can be selected as the host distribution for the transplant-and-compare procedure. If this is not (obviously) the case, the comparison can still be safely undertaken, using either F_1 or F_2 as host, if in logarithms these two distributions differ only in location and scale. If in logarithms F_1 or F_2 do *not* differ only in location and scale, then findings may depend upon which distribution is chosen as the baseline (but this can be checked empirically, see on). In terms of deformation functions, the following theorem summarizes the situation:

Theorem 2

Let $\langle N_1,F_1 \rangle$ and $\langle N_2,F_2 \rangle$ be two regimes.

(a) Let F_o be any income distribution such that $g_1 = F_0^{-1} \circ F_1$ and $g_2 = F_0^{-1} \circ F_2$ are both isoelastic. If $RP_1(g_1^{-1}(x)) \leq RP_2(g_2^{-1}(x))$ $\forall x$ then $\langle N_1,F_1 \rangle \succeq_{P|F_o} \langle N_2,F_2 \rangle$.

(b) Suppose that $g = F_1^{-1} \circ F_2$ is isoelastic. If $RP_1(g(x)) \leq RP_2(x)\ \forall x$ then $<N_1,F_1> \succeq_{P|F1} <N_2,F_2>$ and $<N_1,F_1> \succeq_{P|F2} <N_2,F_2>$.
(c) If $g = F_1^{-1} \circ F_2$ is not isoelastic, the partial orderings over regimes by $\succeq_{P|F1}$ and $\succeq_{P|F2}$ are different.[9]

Part *(a)* of this theorem applies in particular if F_1 and F_2 are (or can be fitted as) lognormal. The analyst could then transplant schedules into the standard lognormal, in which the logarithms of incomes are distributed as N(0,1), prior to undertaking comparisons. Other suitable families for such standardization using the transplant-and-compare procedure are the Pareto and Singh-Maddala.[10]

Part *(b)* may be applied when F_1 and F_2 themselves differ in logarithms by location and scale, but no family of which they are members is apparent. The criterion is a straightforward residual progression test; g(x) is the transformation mapping incomes from the distribution of regime 2 into the corresponding ones in regime 1; progression is compared at every distribution 2 income level; an unambiguous comparison ensures that, after transplanting into either 1 or 2, 1 has more global progressivity than 2.[11]

Part *(c)* of the theorem stresses that in the absence of a location-and-scale relation between distribution functions in logarithms, conclusions may be dependent on the baseline selected; this becomes an empirical question.

There are a number of additional grounds on which isoelasticity of the g-function linking the two distributions F_1 and F_2 is desirable for the transplant-and-compare procedure. Suppose, first, that the income tax is proportional (distributionally neutral) in each of two regimes 1 and 2. Then surely we would say that both regimes have equivalent (and zero) effective progression, no matter what the pre-tax income distributions F_1 and F_2 may be. *The transplant-and-compare procedure does not respect this principle unless $g = F_1^{-1} \circ F_2$ is isoelastic*. More generally, if the

[9] An analogous result holds for liability progression in the case of income tax schedules $T_i(x) > 0$ and $T_2(x) > 0\ \forall x$. Because of the complications which arise in making local and global comparisons of taxes with thresholds and tax and benefit schedules (see footnote 6) - and also because of the lesser normative interest in tax burden distributions - we do not discuss these results further here.

[10] The transformation $g(x) = e^{\lambda}x^{\eta}$ takes one from the lognormal distribution $LN(\theta,\sigma)$ to $LN(\eta\theta+\lambda,\eta\sigma)$, where θ and σ are the mean and variance of the logarithms of incomes; to transform from the Pareto distribution $P(\varepsilon,\alpha)$ to $P(\varepsilon,\beta)$, where ε is a common threshold, apply $g(x) = \varepsilon^{1-\alpha/\beta}x^{\alpha/\beta}$; and to go from the Singh-Maddala distribution $SM(\alpha,b,q)$ to $SM(\beta,b,q)$, use $g(x) = x^{\alpha/\beta}$.

[11] In fact, the conclusion remains the same after transplanting into *any* F_o which has an isoelastic link with F_1 and F_2: see part (a) of the theorem. By setting $g = \imath$ (the identity operator), we obtain a variant of the Jakobsson-Kakwani result from *(b)*: if $RP_1(x) \leq RP_2(x)\ \forall x$ then for any admissible distribution F, $<N_1,F> \succeq_{P|F_o} <N_2,F>$ for all admissible F_o.

transplant-and-compare procedure reveals structural progression equivalence upon transplantation from regime 2 into regime 1, then *equivalence will not be found using the reverse transplantation unless* $g = F_1^{-1} \circ F_2$ *is isoelastic*. We state these results formally in the following theorem, whose proof may be found in the Appendix:

Theorem 3

Given income distributions F_1 and F_2, it is necessary and sufficient that $g = F_1^{-1} \circ F_2$ be isoelastic in order that either of the following two properties should hold:

(a) N proportional $\Leftrightarrow N^g$ proportional

(b) $N_2^g \approx_{RP} N_1 \Leftrightarrow N_2 \approx_{RP} N_1^{g-1}$

Although flat taxes and structural progression equivalence are unlikely to obtain in realistic applications, these results could be argued to reveal a flaw in principle of the transplant-and-compare procedure when pre-tax income distributions are not isoelastically linked.

5. EXAMPLES AND RELATIONSHIP WITH EXISTING LITERATURE

Some simple numerical examples based on linear tax and benefit schedules demonstrate the insights to be gained using the transplant-and-compare procedure, and also the limitations of it. Let $N_1(x) = c_1x+d_1$ be net income in regime 1, for which residual progression is $RP_1(x) = x/(x+d_1/c_1)$, which reduces as income increases. Let $N_2(x) = c_2x + d_2$ be net income in regime 2, and suppose that this has more local progression at each income value x (i.e. $d_2/c_2 < d_1/c_1$). Let $<N_1,F_1>$ and $<N_2,F_2>$ be the two regimes, where $F_1 \neq F_2$. If we did not take account of the difference in income distribution in the two regimes, we would say that $N_2(x)$ engenders unambiguously more global progressivity than $N_1(x)$. *Suppose, though, that in logarithms,* F_2 *is obtained from* F_1 *by a rightward shift in location.* This moves the people to whom $N_2(x)$ applies into regions of low progression, lower, possibly, than the progression experienced by correspondingly-placed people in F_1. The global progressivity comparison might now be ambiguous or even reversed. The transplant-and-compare procedure will reveal what happens.

Taking $N_1(x_1) = 0.2 + 0.8x_1$ and $N_2(x_2) = .25 + 0.75x_2$ as the two schedules, we simulated income distribution F_1 as the standard lognormal, $ln\, x_1 \sim N(0,1)$, and took income distribution F_2 to be defined by the transformation $x_2 = g(x_1) = Ax_1^b$ or:

$$ln\ x_2 = a + b.ln\ x_1 \qquad (12)$$

(where a = *ln* A) with a=3 and b=1, that is, a simple rightward shift of all incomes in logarithms. We then transplanted from 2 into the standard lognormal distribution of regime 1 (using part *(b)* of theorem 2) to make the residual progression comparison.

The most convenient way to compare residual progression measures, which are elasticities, is of course to plot in logs and then inspect slopes. We therefore plotted on the same graph $ln\ N_1(x_1)$ and $[ln\ N_2(x_2)\text{-}a]/b$ against (in each case) $ln\ x_1$; we shall call such curves *log transplant curves* in the rest of the paper, for obvious reasons. These curves depict the logarithm of post-tax income as a function of the logarithm of pre-tax income, after transplantation from regime 2 into the standard lognormal income distribution of regime 1, and their slopes at each point are the relevant residual progression measures. See figure 1. As we suggested might happen, there is indeed *less* residual progression (and therefore progressivity) in the transplanted regime 2 than in regime 1 - despite 2 having *more* progression at each fixed and common income value than 1.[12]

[FIGURE 1 ABOUT HERE]

The transformation of (12) does not change the inequality in pre-tax incomes when b=1: incomes grow equiproportionately in the transition from F_1 to F_2. Because progression declines with income growth, the direction of change in effective progression is straightforward, and conveniently summarized by comparing residual progression values at the two mean incomes (Dilnot *et al.*, 1984). When $b \neq 1$, we can think of the transplant-and-compare procedure as "correcting for inequality", for it is precisely when the transformation $g(x) = Ax^b$ has $b \neq 1$ that the pre-tax distributions of the two regimes have different Lorenz curves.[13] In this case, there is scope for ambiguous global outcomes.[14]

To illustrate this, we formed another new income distribution, call it $F_2{}^\circ$, from F_1 by

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[12] We did not plot the curve for the raw (untransplanted) schedule 2 on this graph, alongside the transplant of 2 and the curve for 1, not least because the difference in location and scale between distributions 1 and 2 would place the two raw curves in different regions on the graph, making slope comparisons impossible (recall a=3 and b=1), but also because a comparison of residual progression between raw schedules is in any case devoid of global consequence, given the change of income distribution. We shall refrain from such comparisons in the rest of the paper too.

[13] One cannot transform isoelastically between distributions with intersecting Lorenz curves. The reason is that the transformation $g(x) = Ax^b$ compresses relative income differentials for b<1 and expands them for b>1; hence there is, in Moyes's (1994) terminology, 'dominance in relative differentials' between the untransformed and transformed distributions.

[14] In Norregaard (1990), estimates of liability progression of income tax systems (assumed constant at all income levels) are presented for a range of countries, alongside measures which depend upon the respective distributions of the tax burden. The correspondence of the rankings "*in broad terms*" is noted (page 94), but the differences which occur are attributed to factors which exclude pre-tax inequality differences.

selecting a°=3 and b°=5 for the transformation in (14), thereby producing a marked increase in pre-tax inequality. When N_2 was applied to $F_2°$ (call this regime 2°), and transplanted into 1, the residual progression comparison with 1 indeed became ambiguous: see figure 1 again. Progression became higher at the bottom, and lower at the top, in the transplant of 2° relative to 1; this takes account of the movement of people from the center ground of F_1 into the tails of $F_2°$ as inequality was increased. Whenever there is an inequality change between regimes, Dilnot *et al.*'s procedure of comparing residual progression values at mean income levels gives only a partial picture: an inspection of complete RP-profiles is needed to make an informative comparison (as in our figure 1).

Part *(c)* of theorem 2 and theorem 3 forewarn of baseline dependency problems if the deformation function g(.) is not isoelastic. Another numerical example illustrates how this can happen quite starkly. Let $N_1(x_1) = 1 + 0.5x_1$ and $N_2(x_2) = 1 + 0.25x_2$, let F_1 be any income distribution and let F_2 be defined from F_1 by $x_2 = g(x_1) = x_1 + 1$. The transplant of N_1 into distribution 2 is $N_1^g(x_2) = 1.5 + 0.5x_2$, and the transplant of N_2 into distribution 1 is $N_2^{g-1}(x_1) = 0.25 + 0.25\, x_1$. It follows that $N_2 \succeq_{RP} N_1^g$ and $N_1 \succeq_{RP} N_2^{g-1}$. The transplant-and-compare procedure would thus provide two diametrically opposed conclusions about progression and progressivity in this example, depending whether one selected the distribution of x_1 or that of x_2 as baseline. This has happened because F_1 and F_2 do not differ *in logarithms* by location and scale (but it is not bound to happen in all such cases, as we shall see later).

One can use the isoelastic specification in conjunction with numerical examples of schedules N(x) to examine the effects of distributional change *per se* on progressivity. For example, Musgrave and Thin's (1948, p. 510) speculation, already referred to, that "... the less equal the distribution of income before tax, the more potent will be a (given) progressive tax structure in equalizing income" can be put to the test for any chosen schedule N(x) by setting b < 1 and comparing N with N^g. We leave such experiments to the reader.[15]

It has been conventional practice in intertemporal studies to isolate the progressivity effects of tax policy changes from those due to distributional or population shifts. For example, Kasten *et al.* (1994) use so-called "income-fixed simulations" to evaluate the effects of U.S.

[15] Lambert and Pf¨ahler (1992) show that unambiguous results do not obtain for the Musgrave and Thin scenario. The case a ≠ 1 and b = , which could also be explored, is that of equiproportionate income growth. Moyes (1989) has shown that, given N(x), the effect of this sort of distributional change on global progressivity is neutral (unambiguously equalizing) if and only if N(x) has constant (everywhere increasing) residual progression.

federal income tax reforms, applying the tax laws after successive reforms all to data of the same year (*ibid.*, pages 10-11). They found substantially different interpretations of the policy effects of the reforms depending on the year to which the tax changes were applied, and accounted for this 'baseline dependence' property by pointing to the differential income growth and changes in the composition of income that took place between alternative base years.[16] In international comparisons, too, practitioners have sought to isolate the progressivity effects of tax schedule differences from those due to distributional differences. In Norregaard (1990), for example, the progressivity impact of superimposing each country's tax system on a "standard pre-tax income distribution" is presented, with Germany selected as the standard, but the possibility of baseline-dependent conclusions is not noted (*ibid.*, pages 98-99).

Although our own approach uses actual tax laws and actual distributions, without simulation or superimposition, it plainly can be adapted to the two-stage approach. At the first stage, the existing Jakobsson (1976) and Kakwani (1976) results allow global and normative implications to be drawn out of residual progression comparisons, since the change in tax law is applied to a fixed income distribution. The "transplant-and-compare" procedure can then be applied in the second stage, when the income distribution is adjusted given the new tax law, providing a means to draw out global and normative properties of pre-tax distributional change *per se* from an analysis of local progression (as between the new schedule and its transplant back into the old distribution).

In Hayes *et al.* (1995), an algorithm is given for computing residual progression at percentile points in the pre-tax income distribution when the data is in grouped form. This algorithm is used to derive and compare progression profiles for US federal income taxes at a grid of percentile points $p \in [0,1]$ for each year from t=1950 to t=1987; *inter alia*, a contour plot of progression $RP_t(F_t^{-1}(p))$ is derived with percentiles p on one axis and years t on the other. There is a link between this procedure and ours. In fact the two procedures are exactly equivalent when the distributions are isoelastically linked - and only then:

Theorem 4

Let $\langle N_1,F_1 \rangle$ and $\langle N_2,F_2 \rangle$ be two regimes. If and only if the function $g = F_1^{-1} \circ F_2$ is isoelastic are

[16] In our simplified model, the possibility that different income sources would be taxed differently is excluded, so that changes in the composition of income would have no effect on progressivity.

the following two statements equivalent:

(a) $RP_1(F_1^{-1}(p)) \leq RP_2(F_2^{-1}(p))\ \forall p \in [0,1]$;

(b) $\langle N_1,F_1\rangle \succeq_{P|F_o} \langle N_2,F_2\rangle$ for $F_0 = F_1, F_2$.[17]

The proof is in the Appendix, but the connection is easy to see. In theorem 2(a), the function g_i assigns to each income level x in distribution i= 1,2 the income g(x) in the reference distribution F_o which is at the same percentile as x.[18] Similarly, in theorem 2(b), the function g assigns to each income in distribution 2 the income in distribution 1 which occurs at the same percentile. Therefore the residual progression comparisons cited in parts (a) and (b) of theorem 2 can both be written in the form $RP_1(F_1^{-1}(p)) \leq RP_2(F_2^{-1}(p))\ \forall p$, which is the condition given in part (a) of theorem 4 and used by Hayes *et al.* (1995) without regard to the isoelasticity question. Global and normatively significant conclusions can thus be drawn from the Hayes *et al.* percentile-by-percentile comparison procedure only when the income distributions concerned are linked by isoelastic transformations; this is because their procedure comes down to the transplant-and-compare procedure in that case; Hayes *et al.* were wrong in giving the impression (on page 468) that *in general* the Jakobsson and Kakwani results establish global progressivity enhancement from local percentile dominance.[19]

6. IMPLEMENTATION: INTERTEMPORAL AND INTERNATIONAL COMPARISONS

Suppose that we have microdata for pre- and post-tax equivalized family or individual income, for two regimes, call them 1 and 2, in the form of samples, of sizes n_1 and n_2. We call the sample values simply ‘incomes’ in what follows.[20] Typically, the tax systems display horizontal inequity (HI), *i.e.* there is not perfect association between pre- and post-tax incomes. The first

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[17] And in fact for all other income distributions F_o such that $F_0^{-1}\circ F_2$ is isoelastic.

[18] This is because, with $g_i = F_o^{-1}\circ F_i$, we have $F_o(g_i(x)) = F_i(x)$.

[19] In fact, they noted the lack of an "outlier year": residual progression dominance did not actually obtain between any two of the years under study.

[20] We noted earlier that nominal tax systems can be converted into real terms using deformations. Theorem 2 forewarns of baseline dependency problems if cost-of-living conversion functions are not isoelastic. Assuming isoelasticity, there is in fact no need for the analyst to convert nominal tax systems into real terms before applying the transplant-and-compare procedure. Let $c_i(x)$ be the variable cost-of-living conversion function for regime $\langle N_i,F_i\rangle$ (i=1,2), so that $\langle N_i,F_i\rangle^{c_i} = \langle N_i^{c_i},F_i\circ c_i^{-1}\rangle$ is regime i in real terms. If h_i converts from the real income distribution $F_i\circ c_i^{-1}$ into a chosen baseline F_o, then $g_i = h_i\circ c_i$ would convert from the nominal F_i directly into F_o. Since by (7) we have $\{\langle N_i,F_i\rangle^{c_i}\}^{h_i} = \langle N_i,F_i\rangle^{g_i}$, the analyst will arrive at the same transplant whether or not cost-of-living conversions are first undertaken.

task is to construct net income schedules $N_i(x)$, i=1,2, to represent the progressivity (vertical stance) of the two regimes.

There are two prevailing views on how to do this. One approach, pioneered by Musgrave (1990) and Aronson *et al.* (1994), takes as starting point that the vertical stance of a tax system is defined by its *differential-narrowing effect on average between pre-tax unequals*, with effects around this average counted as classical HI. The analyst could construct the $N_i(x)$, i=1,2, as smoothed average relationships between pre- and post-tax incomes in the two samples, using either close equals groups (Lambert and Ramos, 1997) or the kernel (Duclos and Lambert, 2000).[21] According to the other approach, vertical equity is about the *choice of post-tax income distribution given the pre-tax income distribution* (and HI about the process of taxation). To accord with this view, the $N_i(x)$, i=1,2, should generate the sample post-tax distributions from the sample pre-tax distributions. The unique way to construct such functions $N_i(x)$ is simply to sort separately the pre- and post-tax income vectors in each sample i, breaking the disassociation which is present; $N_i(x)$ becomes the post-tax income whose rank is the same as the pre-tax rank of x (King, 1983, Jenkins, 1988, 1994).[22]

The sorting procedure is both sound and simple, whilst the close equals and kernel approaches, equally sound, are technically more demanding. For the illustrations to follow, we adopted the simpler sorting procedure, but nothing in this paper depends on how the analyst chooses to generate the net income schedules $N_i(x)$, i=1,2, in terms of which the progressivity analysis proceeds.[23] Either way, the analyst now has ordered sample values for pre-tax income x_i and post-tax income y_i in sample *i, i*=1,2, and can create the empirical distribution functions for pre-tax incomes, call these F_i. Denote by F_i* the distribution function for *ln* x_i.

The first thing to check is whether the F_i* belong to a location and scale invariant family. The obvious and easy one to check is the normal (corresponding to lognormality of income distributions). The goodness-of-fit test of D'Agostino and Pearson is straightforward, depending on the 3rd and 4th cumulants (Cox and Hinkley, 1974, p. 71-2). If the fit is good, the two

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[21] The Lorenz curves for net income $N_i(x)$, i=1,2, are then defined by accumulation points on the sample pre-tax ordered concentration curves for post-tax income.

[22] In this case, the Lorenz curves for net income $N_i(x)$, i=1,2 are of course the sample Lorenz curves.

[23] Howeever it is shown in Dardanoni and Lambert (2001) that for any given dataset with HI, the schedule obtained by smoothing will have a more progressive stance than the one obtained by sorting. Since the degree of HI affects this discrepancy in stance, comparisons between regimes with different degrees of HI can be affected by the choice between smoothing and sorting.

distributions can be transformed to N(0,1), for example, replacing *ln* x_i by $a_i + b_i.$*ln* x_i where a_i and b_i solve $a_i + b_i\mu_i = 0$ and $b_i\sigma_i = 1$, where μ_i and σ_i are the mean and standard deviation of logarithms in sample *i*. The residual progression comparison between transplanted tax systems can be made by inspecting slopes in the plots of $a_i + b_i.$*ln* y_i against $a_i + b_i.$*ln* x_i for i=1,2 on the same graph.

If the lognormal fit is unacceptable, then as in scenario *(b)* of theorem 2, the next step is to check for the closeness of the two vectors *ln* x_1 and a + b.*ln* x_2 (of lengths n_1 and n_2), searching for the a and b which minimize an appropriate distance measure. The straightforward approach is to create quantiles: let $x_i(p)$ be the income at rank p in sample *i*, where $p \in (0,1]$ indexes a set of equally-spaced positions.[24] One can then find the a and b which minimize the Euclidean distance between the vectors *ln* $x_1(p)$ and a + b.*ln* $x_2(p)$. This amounts to finding the point in the subspace of 2-dimensional Euclidean space spanned by a unit vector and *ln* $x_2(p)$ which is closest to *ln* $x_1(p)$; the familiar OLS estimators of the constant and slope in a regression of *ln* $x_1(p)$ on *ln* $x_2(p)$ provide the requisite a and b; the centered R^2 statistic is the appropriate measure of goodness-of-fit. If the fit is good, the tax function of regime 2 can be transplanted into the distribution of regime 1, using the entailed a and b, and the residual progression comparison then made, using log transplant curves as before; in this case, the curves will be of *ln* y_1 against *ln* x_1 and a + b.*ln* y_2 against a + b.*ln* x_2, plotted on the same graph. If the fit is poor, then in accord with part *(c)* of theorem 2, the transplant-and-compare procedure itself may be unsafe; the use of OLS estimators is then unsafe *a fortiori*. .

We illustrate these procedures for the tax-and-benefit systems of Canada, Israel and the UK. The Canadian data comprises 5000 cases drawn from the *Survey of Consumer Finances* for 1981 and 1990 of Statistics Canada. The Israeli data comprises 5212 cases drawn from the *Family Expenditure Survey* for 1992 of the Central Bureau of Statistics, and the UK data, comprising 2721 cases, is adapted from the *Family Expenditure Survey* for 1993 of the Central Statistical Office, using the package EBORTAX to compute tax liabilities according to the 1993/4 tax code.[25] From this data, we drew each household's original (pre-tax and pre-benefit) and final

[24] If $n_1 = n_2 = n$, select p = k/n, k = 1,2,.....n, and set $x_i(p)$ equal to the k^{th} value of x_i. If $n_1 \neq n_2$, select p = k/*min*{ n_1,n_2}, k = 1,2,....*min*{ n_1,n_2}, and use interpolation to locate $x_i(p)$ between two sample values of x_i.

[25] We thank Jean-Yves Duclos, Shlomo Yitzhaki and Alan Duncan for help with the provision of this data. EBORTAX computes tax liabilities and benefit entitlements on the basis of actual circumstances, rather than using recorded payments and receipts (which are prone to error), and has been developed by Alan Duncan, whom we thank for assistance.

(post-tax and post-benefit) money incomes, excluded all cases in which either was zero, and then deflated into units of equivalent income using a deflator of the form $m = (N_A + \varphi N_C)^\theta$ from the doubly-parametric family of Cutler and Katz (1992), where N_A and N_C are the numbers of adults and children in the family and $\varphi, \theta \in [0,1]$ tell the importance of children and economies of scale in determining the number of adult-equivalents. We selected parameter values $\varphi = \theta = ½$ for the illustrations, sorted both the original and final income values and trimmed the top ½ percent of these values from each sample in order to eliminate dependency of results on outliers.

[FIGURE 2 ABOUT HERE]

Figure 2*(a)* shows the distribution functions $F_{CAN81}*$ and $F_{CAN90}*$ for logged Canadian original incomes in the 1981 and 1990 samples. Normality is rejected by the D'Agostino and Pearson test in each case. But the second procedure reveals that these distributions differ essentially only by location and scale: when we fitted the 1990 distribution to that of 1981, the OLS estimates were a = .6645 and b = .9786, with an R^2 of 0.9975. Figure 2*(a)* shows the fit. In figure 2*(b)*, the log transplant curves are shown, whose slopes reveal the residual progression of the 1981 tax and benefit schedule, and of the 1990 one transplanted into the 1981 distribution. The 1990 transplant is flatter everywhere, which means that the system had become *locally more progressive at all income levels* by 1990. This finding corroborates that of Davidson and Duclos (1997), based on Lorenz curves, that the system had become *globally more progressive* by 1990.[26]

[FIGURE 3 ABOUT HERE]

Figure 3*(a)* shows the distribution functions $F_{ISR}*$ and $F_{UK}*$ for logged Israeli 1992 and UK 1993 original incomes. Normality is again rejected, and this time the second procedure reveals a less convincing location-and-scale relationship: when we fitted the UK distribution to that of Israel, the OLS estimates were a = 4.1177 and b = .7058, with an R^2 of 0.8489. Figure 3*(a)* shows the fit, which is visibly less good. In figure 3*(b)*, the log transplant curves are shown, predicated on the OLS values of a and b. Let us take these curves at face value and consider what they suggest about the two tax systems. A dotted line has been added to this figure, a "45^o line", above which net benefit recipients are located, and below which, net taxpayers. There are substantial income ranges, both for net benefit recipients and taxpayers, in which residual progression is constant in each system. Residual progression is also substantially higher for benefit recipients

[26] Davidson and Duclos used much larger samples than we did.

than for taxpayers in both systems.[27] The tapering of benefits is more gradual for Israel. The crossovers of the 45° line differ: Israel still gives benefits beyond the point where the UK starts to tax. In the UK benefit range, the UK system is more progressive than the Israeli one; in the UK tax range, the UK system is less progressive than the Israeli one.

These insights unfortunately cannot, as far as we know, be corroborated by reference to another study. The poorer R^2 gives us less confidence in the reliability of the isoelastic model in this case than in the Canadian one, but yields some intriguing facets of the two tax and benefit systems which could be further investigated. If the R^2 is rejected, then the two distributions cannot be held to differ in logarithms only in location and scale. In this case, conclusions might be dependent on the choice of Israel as baseline - and, of course, the use of the OLS estimators a and b to construct log transplant curves becomes inappropriate anyway. We should then check directly whether the transplants of the UK schedule into Israel and of the Israeli schedule into the UK yield qualitatively different conclusions.

It is straightforward, though cumbersome (relative to the very easy procedure using OLS estimators), to construct log transplant curves empirically.[28] In figure 4, parts *(a)* and *(b)*, we show these curves, first transplanting from the UK into Israel and then *vice versa*. The slopes contain the relevant information, and everything we said about slopes in figure 3 holds true in these two graphs also (as do the remarks about crossovers with the 45° line). The only significant difference between figures 3*(b)* and 4 is that the Israeli curve lies everywhere above the UK curve in each part of figure 4, whilst the two curves cross in figure *3(b)*. This difference would matter if we cared to compare tax yields but, as we have said, the yield of one schedule cannot meaningfully be compared with that of the transplant of the other in any case. We may conclude that baseline independence holds for the UK/Israeli comparison, and that in fact our acceptance of the R^2, and use of the OLS estimators for the isoelastic model, led to all significant facets of the two tax and benefit systems being captured.

[FIGURE 4 ABOUT HERE]

[27] This is a common feature of tax and benefit systems: withdrawal rates of benefits are typically high relative to tax rates.

[28] In the general case, with regimes $\langle N_1,F_1\rangle$ and $\langle N_2,F_2\rangle$ and $g = F_1^{-1}\circ F_2$ as before, so that $\langle N_2,F_2\rangle^g = \langle N_2^g,F_1\rangle$ is the transplant from regime 2 into the distribution of regime 1, the procedure to construct the log transplant curve for the schedule N_2^g is as follows. Plot against the log of a given pre-tax income x_1 in distribution 1, at quantile q say, the log of another income in distribution 1, call it y_1, which is such that its quantile in distribution 1 is the same as that to which a pre-tax income at quantile q in distribution 2 is mapped by N_2. Now vary x_1 to trace out the log transplant curve.

The isoelastic model might be more appropriate for intertemporal than for international comparisons, because distributional change is gradual in the former case whilst inequality differences can be appreciable between countries. If isoelasticity does hold between distributions in a country in a succession of years, then multiple log transplant curves can be graphed simultaneously, using successive OLS estimators, to detect trends. In the absence of isoelasticity, the task of making successive pairwise tests for baseline independence could be formidable.

7. CONCLUSIONS

Local progression comparisons are perilous when tax and benefit schedules operate on different income distributions. We advocate a 'transplant-and-compare' procedure designed to correct regimes for inequality and size differences and enable baseline comparisons of residual progression to be undertaken, using 'log transplant curves'. From an unambiguous comparison, a robust global ordering of original regimes by their redistributive effects in terms of Lorenz and Suits curves is implied. Even from a comparison in which the residual progression comparison varies with income, insights can be drawn out that would otherwise be inaccessible.

The crucial 'independence of baseline' property holds only when deformation functions are isoelastic. As we have seen, the transplant-and-compare procedure is not reliable unless the transformation function between distributions is isoelastic. In particular, problems are inevitable when the pre-tax Lorenz curves for the two regime cross,[29] but we can make approximations. Our procedures provide these approximations, along with appropriate goodness-of-fit indicators. If we accept the isoelastic fit, the transplant-and-compare procedure can be applied and illuminating results derived from the resulting log transplant curves. If we reject the isoelastic model, empirical transplant procedures can be applied, but it can be argued that the transplant-and-compare methodology itself is not wholly satisfactory in such cases (in view of our theorem 4, not least)
.

[29] Recall footnote 14.

APPENDIX

For for any schedule M and deformation function g, we have

$$RP_{N^g}(g(x)) = \frac{\ell_g(N(x))}{\ell_g(x)} \bullet RP_N(x) \tag{A}$$

where ℓ_g denotes the elasticity of g, $\ell_g(x) = xg'(x)/g(x)$. If g is isoelastic this reduces to

$$RP_{N^g}(g(x)) = RP_N(x) \tag{B}$$

The Proof of Theorem 1

$\Leftarrow$ Suppose $<N_1,F_1> \succeq_{P|F_o} <N_2,F_2>$, *i.e.* $N_1^{g1} \succeq_{RP} N_2^{g2}$ where $g_i = F_0^{-1}\circ F_i$, i=1,2. Let $h_i = G_o^{-1}\circ F_i$. To prove $<N_1,F_1> \succeq_{P|G_o} <N_2,F_2>$ when $g = G_0^{-1}\circ F_0$ is isoelastic we need to show $N_1^{h1} \succeq_{RP} N_2^{h2}$ in this case. Now $g\circ g_i = h_i$, so by (7), $N_i^{hi} = (N_i^{gi})^g$. Setting $N = N_i^{gi}$ in (B), isoelasticity of g implies that the residual progression of N_i^{hi} at g(x) equals that of N_i^{gi} at x. $N_1^{h1} \succeq_{RP} N_2^{h2}$ thus follows from $N_1^{g1} \succeq_{RP} N_2^{g2}$.

$\Rightarrow$ Select N_1 and N_2 such that $N_1^{g1} = \lambda N_2^{g2}$ for some $\lambda \neq 1$, *i.e.* $N_1^{g1} \approx_{RP} N_2^{g2}$. If the partial orderings $\succeq_{P|F_o}$ and $\succeq_{P|G_o}$ are the same, then $N_1^{h1} \approx_{RP} N_2^{h2}$. Now substitute $N = N_i^{gi}$ in (A) and conclude that $\ell_g(N_1^{g1}(x)) = \ell_g(N_2^{g2}(x))\ \forall x$. This forces N_1^{g1} and N_2^{g2} to be identical on any interval on which $\ell_g' \neq 0$. No such interval exists, hence g is isoelastic.

The Proof of Theorem 2

From (B), if g is isoelastic the residual progression of N_i^{gi} at x equals that of N_i at $g^{-1}(x)$, $\forall x$. The result in *(a)* follows. For the results in *(b)*, first set $F_o = F_1$ (so that g_1 is the identity operator and $g_2 = g$) and then set $F_o = F_2$ (so that that $g_1 = g^{-1}$ and g_2 is the identity operator). Everything follows. The result in *(c)* is immediate from theorem 1.

The Proof of Theorem 3

If g is isoelastic then from (B) N^g is proportional (has unit residual progression) if and only if N is, and $N_2^g \approx_{RP} N_1$ and $N_2 \approx_{RP} N_1^{g-1}$ are both equivalent to $RP_2(x) = RP_1(g(x))\ \forall x$. Hence *(a)* and *(b)* hold.

Suppose now that *(a)* holds and let $N(x) = \lambda x\ \forall x$ where $\lambda \neq 1$. From (A), $\ell_g(\lambda x) = \ell_g(x)$ $\forall x$. This forces λx and x to be identical on any interval on which $\ell_g' \neq 0$. No such interval can exist, so g is isoelastic.

Suppose now that *(b)* holds, *i.e.* $N_2^g \approx_{RP} N_1 \Leftrightarrow N_2 \approx_{RP} N_1^h$ where for convenience we write $h = g^{-1}$. Let $N_2 = \lambda.N_1^h$ for some $\lambda \neq 1$. Now apply (A). From $N_2^g \approx_{RP} N_1$ we obtain $\ell_g(N_2(x)).RP_2(x) = \ell_g(x).RP_1(g(x))$ $\forall x$ and from $N_2 \approx_{RP} N_1^h$ we obtain $\ell_h(N_1(y)).RP_1(y) = \ell_h(y).RP_2(h(y))$ $\forall y$. Substituting x = h(y) and y = g(x) in these, comparing the two values of $RP_2(x)/RP_1(y)$ which are implied and noting that $\ell_h(y) = 1/\ell_g(x)$, we find that $\ell_g(N_2(x)) = 1/\ell_h(N_1(y))$, equivalently $\ell_g(N_2(x)) = \ell_g(N_1^h(x))$ $\forall x$. This forces N_2 and N_1^h to be identical on any interval on which $\ell_g' \neq 0$. No such interval can exist, so g is isoelastic.

The Proof of Theorem 4

We prove that isoelasticity of g is equivalent to (a) ⇔ (b) for $F_0 = F_2$. We thus compare $<N_1,F_1>^g = <N_1^g,F_2>$ with $<N_2,F_2>$. The analysis for $F_0 = F_1$ is similar. Since $g = G_0^{-1} \circ F_0$, we can write $F_2(g(x)) = F_1(x) = p \in [0,1]$. It follows from (A) with $N = N_1$ that

$$\frac{RP_{N_1 g}(g(x))}{RP_{N_2}(g(x))} = \frac{\ell_g(N_1(x))}{\ell_g(x)} \bullet \frac{RP_{N_1}(x)}{RP_{N_2}(g(x))} \quad (9)$$

Hence, if g is isoelastic, the percentiles comparison and the residual progression comparison of $<N_1^g,F_2>$ and $<N_2,F_2>$ come down to the same thing. Suppose conversely that these two comparisons are always identical. Then $\ell_g(N_1(x)) = \ell_g(x)$ $\forall x$ for all possible N_1. Taking the case $N_1(x) = \lambda x$ $\forall x$ where $\lambda \neq 1$, we see that there can be no interval on which $\ell_g' \neq 0$, *i.e.* g must be isoelastic.

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Figure 1 : log transplant curves (numerical example)

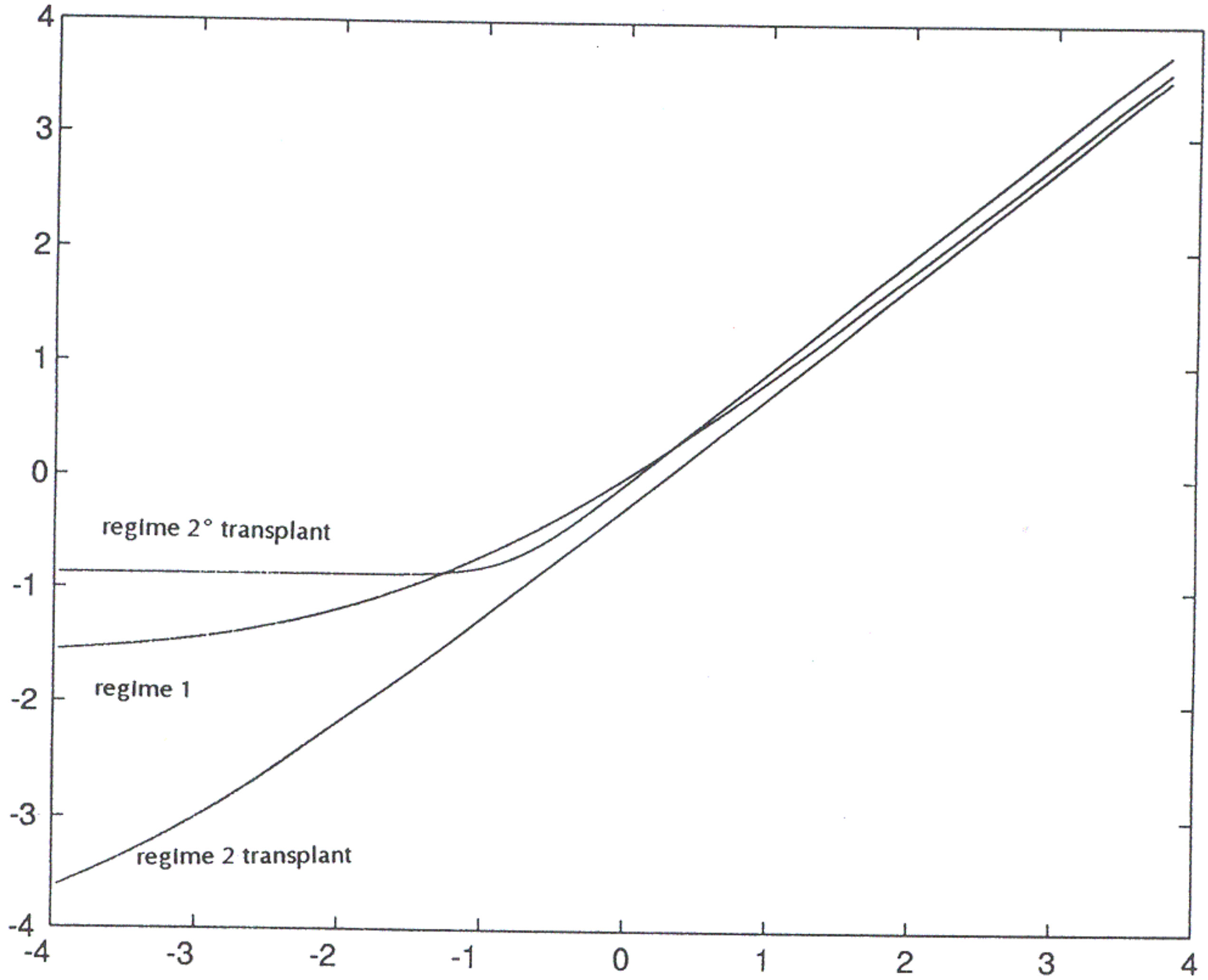
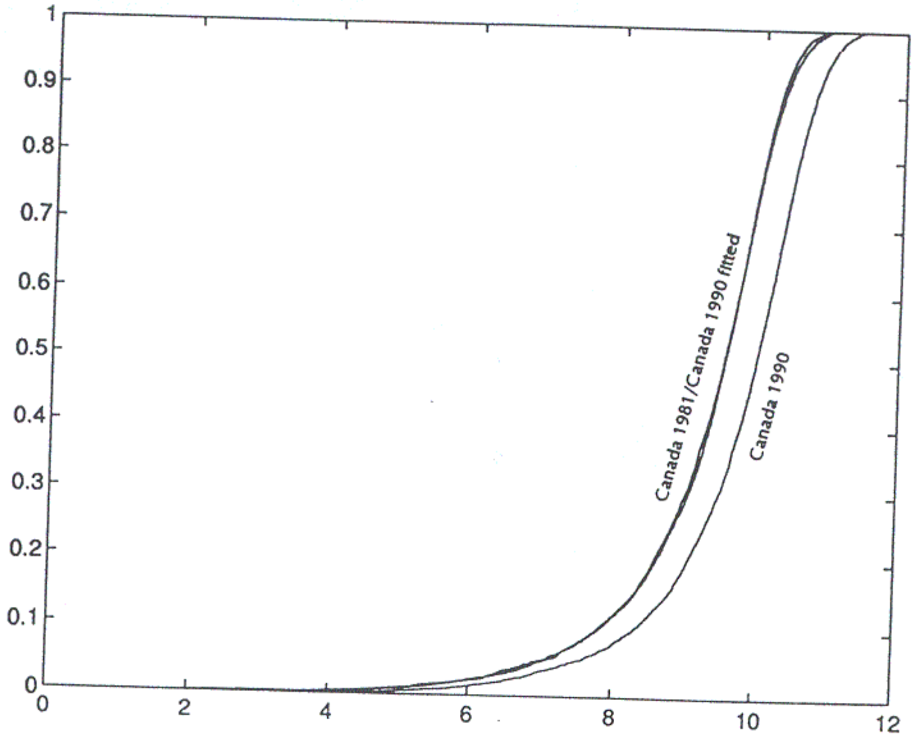


Figure 2 : Canada 1981 and 1990

(a) distribution functions



(b) log transplant curves

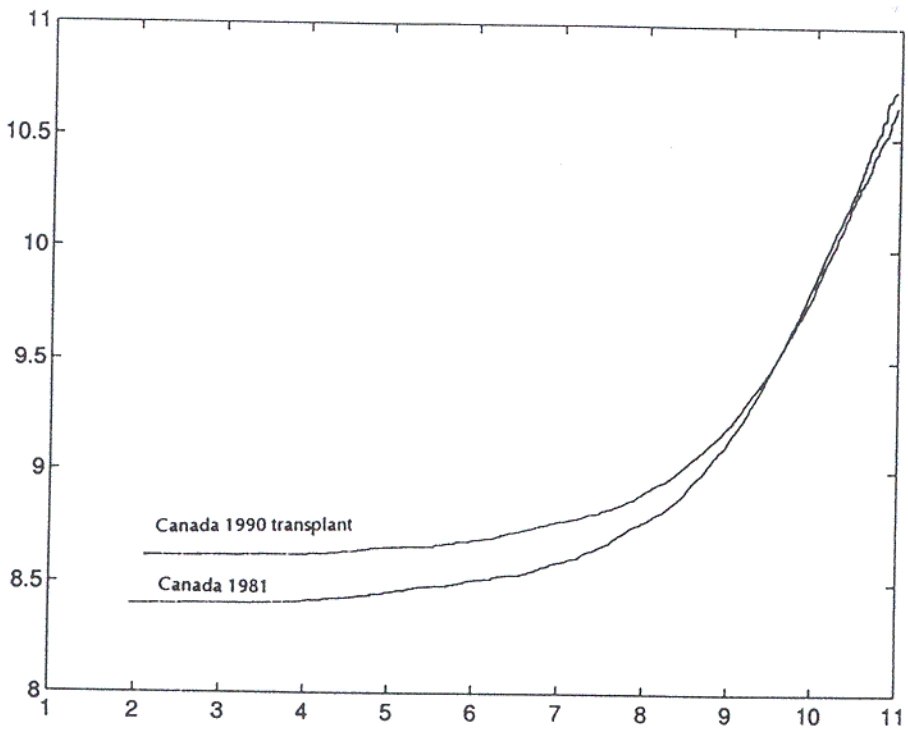
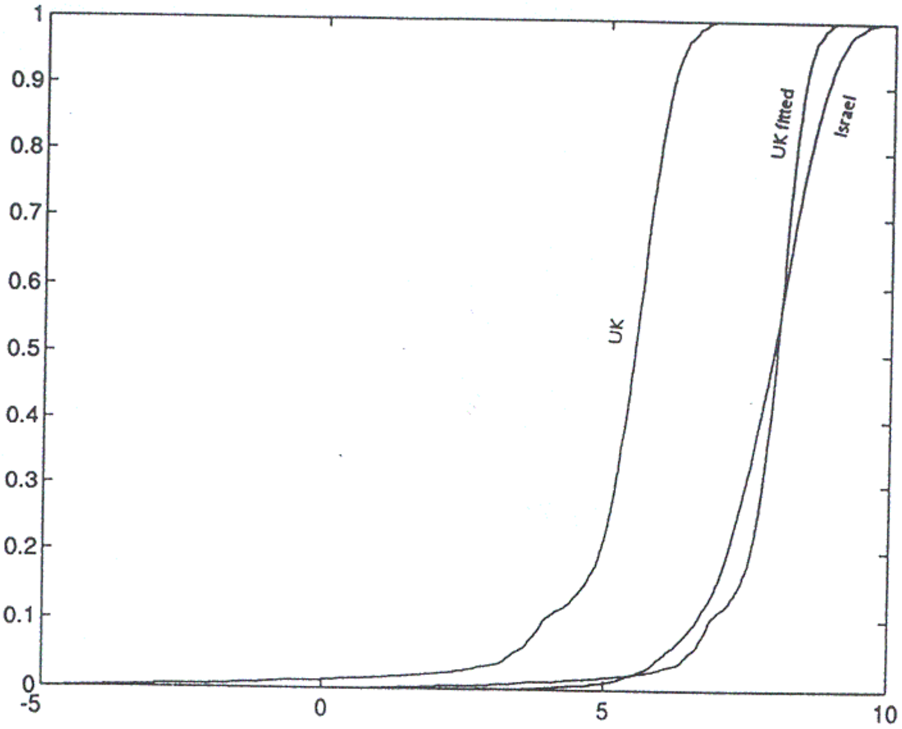


Figure 3 : Israel and the UK

(a) distribution functions



(b) log transplant curves

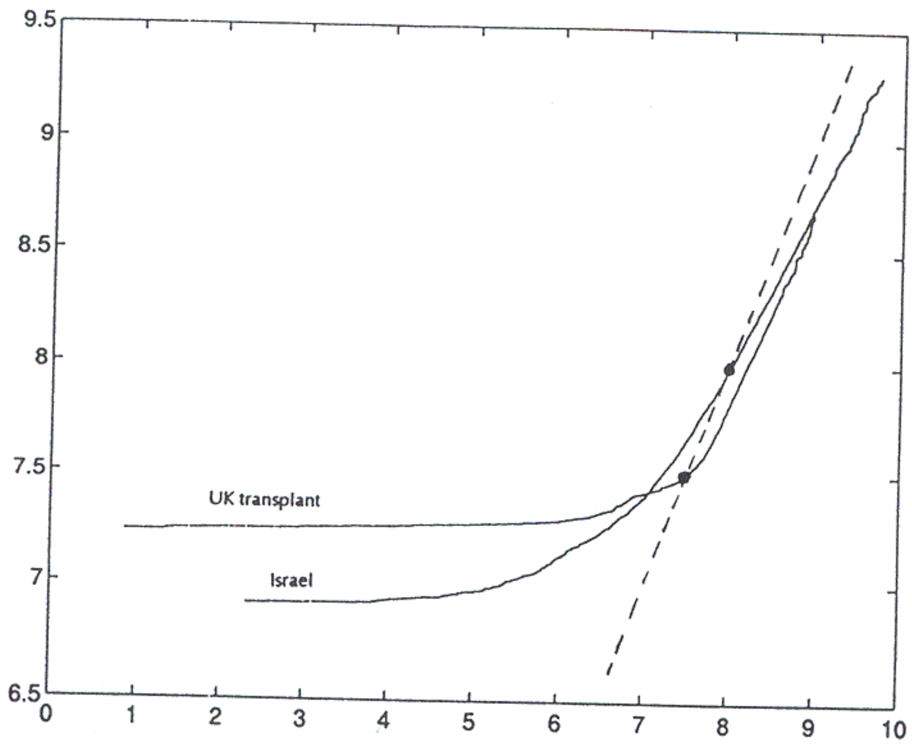
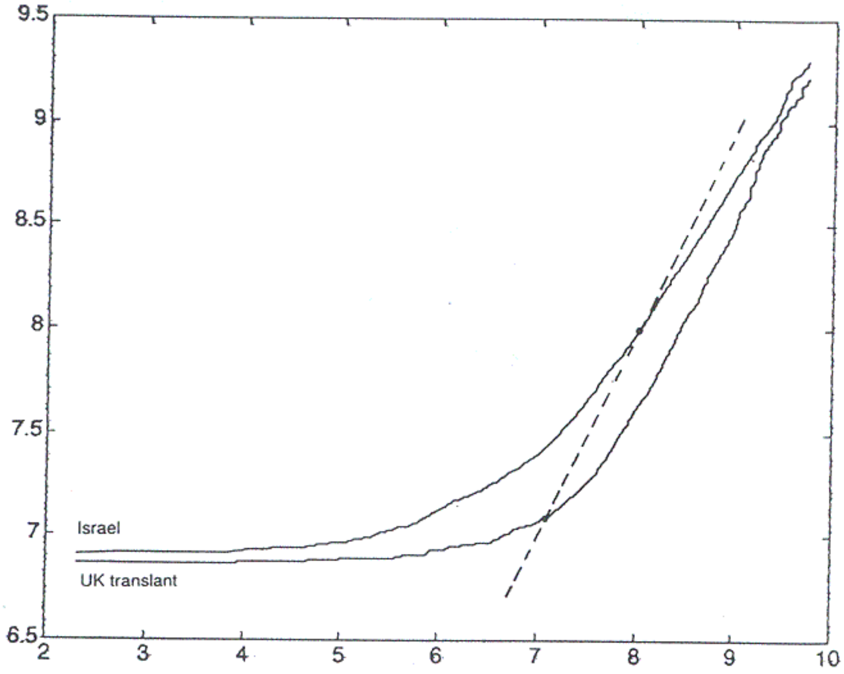


Figure 4: empirical log transplant curves

(a) transplantation into Israel



(b) transplantation into the UK