UCD CENTRE FOR ECONOMIC RESEARCH

WORKING PAPER SERIES

2006

Social Learning in Continuous Time:
When are Informational Cascades
More Likely to be Inefficient?

Ivan Pastine, University College Dublin and
Tuvana Pastine, National University of Ireland Maynooth

WP06/21

November 2006

UCD SCHOOL OF ECONOMICS
UNIVERSITY COLLEGE DUBLIN
BELFIELD DUBLIN 4

Social Learning in Continuous Time: When are Informational Cascades More Likely to be Inefficient?

by

Ivan Pastine
University College Dublin
and CEPR

Tuvana Pastine*
National University of Ireland, Maynooth
and CEPR

July 22, 2006

Abstract

In an observational learning environment rational agents may mimic the actions of the predecessors even when their own signal suggests the opposite. In case early movers' signals happen to be incorrect society may settle on a common inefficient action, resulting in an inefficient informational cascade. This paper models observational learning in continuous time with endogenous timing of moves. This permits the analysis of comparative statics results. The effect of an increase in signal quality on the likelihood of an inefficient cascade is shown to be nonmonotonic. If agents do not have strong priors, an increase in signal quality may lead to a higher probability of inefficient herding. The analysis also suggests that markets with quick response to investment decisions, such as financial markets, may be more prone to inefficient collapses.

Keywords: Comparative Statics, Herding, Herd Manipulation.

We would like to thank Rabah Amir and Christophe Chamley for their assistance. This paper has been presented at the Workshop on Informational Herding Behavior 2005 arranged by the Dynamic Economic Theory Network based at the Department of Economics at the University of Copenhagen. We would like to thank the participants for their helpful comments. Responsibility for errors remains our own.

*Corresponding Author: Tuvana Pastine, NUI Maynooth, Economics Department, Maynooth, Co. Kildare, Ireland. fax:353-1-708 3934 e-mail: tuvana.pastine@nuim.ie

1. Introduction

Individuals with limited information often observe other people's actions before making their own decisions. The predecessors' actions tend to contain information about their private signals concerning the state of nature. In this social learning process the individual aggregates his own private signal with information collected from his observations of others' actions. This process may lead to herd behavior where the agent follows the crowd even when his private information suggests the opposite.[1] In case early movers' signals happen to be incorrect, agents may settle on a common inefficient action, resulting in an inefficient informational cascade. Gale (1996), Chamley (2004) and Bikhchandani, Hirshleifer and Welch (1998) give extensive lists of empirical phenomena that herding behavior may explain in both financial and real markets. Examples include balance-of-payments crises, R&D investment decisions, analysts' recommendation of stocks, bank runs and managers decisions to pay dividends.[2] It is often argued that conformist behavior in financial and real markets may lead to sudden booms and crashes.

In seminal herding papers by Banerjee (1992) and Bikhchandani, Hirshleifer and Welch (1992) and Welch (1992) there is an exogenously determined sequence in the moves. Chamley and Gale (1994) extend the literature to allow for endogenous timing of moves. Each agent has an incentive to wait in order to observe the actions of other players. However if everyone were to wait, the agent would rather move early in order to avoid costs of delay. Hence the timing decision is strategic. Chamley and Gale prove the existence of herd behavior with endogenous timing and characterize the equilibrium.

---

[1]An informational cascade implies herding behavior. See Smith and Sørensen (2000) for the distinction.

[2]There are a wide variety of markets where herding may arise. For instance, see Scharfstein and Stein(1990), Welch(1992), Devenow and Welch (1996), Avery and Zemsky(1998), Welch (2000), Chari and Kehoe (2003), Chamley (2003) for analysis of herd behavior in financial markets, Neeman and Orosel (1999) for analysis in auctions, Morton and Williams (1999) for herding in a political economy framework and Bose, Orosel and Vesterlund (2002), Choi, Dassiou and Gettings (2000), Kennedy (2002) and Levin and Peck (2005) for herding among firms.

There is a vast literature that examines the conditions under which informational cascades occur and analyzes the speed of cascade formation and the fragility of cascades.[3] There are however no studies that focus on the factors that affect the likelihood of erroneous mass behaviour. This paper studies the factors that influence the likelihood of inefficient cascades, either when there is an investment boom even though the true value of the project is low (an inefficient positive cascade), or when there is an investment collapse even though the true value is high (an inefficient negative cascade). The question the paper analyzes is not whether or not there will be cascade formation but the likelihood that the cascade outcome is inefficient.

Chamley and Gale offer an ex-ante welfare analysis on the inefficiency induced by delay and observational learning. The additional advantage of analyzing the likelihood of inefficient cascades arises when there are externalities from the market to society at large. The ex-ante welfare approach includes the expected discounted payoffs of the players directly involved in the game. However, bank panics, capital flight and stock market crashes have external consequences which may induce a social planner to place a greater weight on inefficient negative cascades than an individual investor. In other markets the party designing the structure of the market may not have an incentive to weigh all market participants equally. In the IPO market, for example, the features of the market are not controlled by a central planner, but rather by the firms offering companies for public sale. These companies may try to increase the probability of a positive cascade. Results on the probabilities of inefficient outcomes are a potentially useful building block for welfare and policy analysis in markets with such external effects.

The analysis of factors that influence the likelihood of inefficient herding – the discount rate, the signal quality, the prior, the expected value of the investment project – may be important for manipulating the outcome of the social learning game. For instance, the Securities Act of 1933 and the Securities Exchange Act of 1934 were formed in hopes of preventing catastrophic crashes like Black

[3]See Chamley (2004) for a review of theoretical advances in the study of informational cascades.

Thursday in 1929.[4] We would like to be able to analyse the implications of an improvement in accounting standards (signal quality) on the probability of an inefficient collapse. Likewise, we would like to know the effect of an improvement in trading technology on the probability of an inefficient outcome. This paper is the first to focus on comparative statics results for all key variables in an endogenous-timing framework.[5] Other than the directional effects of the factors that influence the probability of inefficient herding, the magnitude of the effect may be of importance as well. For instance, in an IPO decision reducing the price decreases the probability of collapse where the IPO fails. But we would like to know what the optimal price would be.

We find that the probability of an inefficient negative cascade goes up as agents get more patient.[6] Agents with a low rate of time preference are more inclined to wait to collect more information about the true value of the project. This conservative attitude makes inefficient negative cascades more likely. In the leading herding model of Bikhchandani, Hirshleifer and Welch (1992), inefficient negative and positive cascade probabilities go down as signal quality goes up, since early movers are more likely to take the correct action. We find that this not the case in general. While an increase in signal quality does unambiguously improve the expected utility of buyers in the market, if agents do not have strong priors an increase in the signal quality may lead to a higher probability of inefficient herding.

Our paper adapts the Chamley and Gale (1994) endogenous-timing information-revelation model of investment. The equilibrium is in the same spirit as in Chamley and Gale. However our agents

---

[4]Among other regulations, the act requires that investors receive financial and other significant information concerning securities being offered for public sale.

[5]In our paper, in equilibrium agents follow a critical mass. In Zhang's (1997) model with heterogenous signal quality, all imitate the one leader with the most precise signal. The probability of an inefficient cascade is then just equal to the probability of the most precise signal being incorrect.

[6]Chamley and Gale show that at $\delta=1$ only negative cascades will be inefficient. This paper extends their result to comparative statics over the full range of δ.

observe others’ actions in continuos time. The learning process can be approximated by a series of Brownian Motions and the familiar boundary crossing probabilities are employed to find the probabilities of inefficient positive and negative cascades. The merit of a continuous-time framework is that it enables us to approximate a closed-form solution to the probability of inefficient cascades. The cost is a number of ad-hoc assumptions on the market that will allow the model to be technically tractable.

2. Overview

In the Chamley-Gale model the higher the value of the project the more people there are with an option to invest. The project value and number of people with an option are unknown to the agents. Each agent with an investment option faces a tradeoff between investing and waiting. If the agent invests he collects the undiscounted payoff, but faces the risk of making a loss in case the true value of the project is low. If the agent waits he collects only the discounted payoff, but he can learn from others’ actions. In equilibrium the agent is just indifferent and randomizes between waiting and investing. Therefore, the rate of investment each period is stochastic. The agent tries to deduce from the rate of investment each period the number of people with an option and hence the value of the project. If the expected value of the project turns negative the agent strictly prefers to wait, as does everyone else, and so the game ends with an investment collapse. If the expected value of the project becomes so high that agents strictly prefer to invest, the game ends with a sudden investment boom. Since the learning process is stochastic, the outcome may be inefficient. Even when the true value of the project is high (low), there might be an investment collapse (boom).

Here we adapt a special case of the model in which there are only two possible values of the project allowing agents’ beliefs to be summarized in a single variable, the probability that the project value is high. It will be shown that a monotonic transformation of the subjective probability yields a learning process characterized by a simple Lévy process: $w_t = w_{t-1} - a + bk_t$ where w_t denotes the transformed subjective probability and k_t is the stochastic number of buyers observed in the period and it is given by the Poisson approximation to the binomial distribution. The problem is to determine the probability that w_t crosses the upper bound (where the agent strictly prefers to invest) before the lower bound (where the agent strictly prefers to wait) resulting in an investment boom and the probability that w_t crosses the lower bound before the upper bound, resulting in an investment collapse. This is a mathematical problem that arises in a number of contexts from psychology to queuing theory and hence has been extensively studied. Nevertheless, despite its seeming simplicity no closed-form solution has been found. Hence we depart from the Chamley and Gale framework and move to a continuous-time learning setting.

For the continuous-time adaptation, we make some institutional restrictions on the market to make the model technically tractable. We have an agency which takes state-contingent orders in discrete time and processes them in continuos time. Agents are allowed to make their invest and wait orders (placed in discrete time) contingent on the continuously evolving flow of order processing. The payoffs are received at the end of the period. If the expected level of investment were constant in all periods this would result in a continuous-time process for the transformed beliefs that is identical to the evolution of queue lengths in a standard M/D/1 queue. The standard approximation for this is Brownian motion. Our learning process is then a series of one-period Brownian motions.

For tractability it is also assumed that the processing agency has a capacity limit to the number of contingent orders it can accommodate. The capacity limit helps us keep the same basic incentives as in Chamley and Gale while ensuring that information transmission stops as soon as one of the boundaries is hit. Hence we can make use of the boundary crossing probabilities for Brownian Motion to find the probability of an inefficient informational cascade.

3. Framework

The true value of the investment project is denoted by $V \in \{V^H, V^L\}$ where $V^H > 0$ and $V^L < 0$. $V = V^H$ with prior probability $q^* \in (0,1)$. The number of rational players rN is unboundedly large, $r \rightarrow \infty$.[7] Players are risk neutral. They are ex-ante identical. However, only *rn* of them receive an investment option. Options are identical and indivisible. If the true value of the project is high, there are more agents with an investment option:

$$n = \begin{cases} n^H \ \ when\, V = V^H \\ n^L \ \ when\, V = V^L \end{cases} \tag{1}$$

Denote $\psi \equiv \frac{n^L}{n^H} < 1$. ψ is a measure for signal quality. The further apart n^L and n^H, the more information the agent has about the true value of the project from the fact that he has observed the investment option in the first place. If n^L were equal to n^H, the agent who receives the investment option would not update his prior. The agent will update his prior belief more heavily as ψ goes down.[8]

Each agent with an investment option can give an invest order at any date T=0,1,2,... of his choice. Payoffs from the exercised options are received at the end of the time period. $\delta \in (0,1)$ is the common discount factor. Investment is irreversible. If the player never invests, the payoff is 0. Whether or not the player has an option is private information. Only if the option is exercised is

[7]This paper adapts the *r-fold replica* game of Chamley and Gale (1994), Section 6.

[8]In order to compare this signal quality measure to the signal quality measure in Banerjee(1992) and Bikhchandani, Hirshleifer and Welch (1992), suppose that each of the rN potential investors receive a signal about the value of the project. The signal is correct with probability p>0.5. If $V=V^H$ then $prN=rn^H$ get a positive signal. If $V=V^L$ then $(1-p)rN=rn^L$ get the positive signal. Then $\psi = (1-p)/p$. One could easily formulate the problem in this way and the equilibrium would mirror that found here, with the agents who received a negative signal simply waiting to observe a positive cascade and then investing if and only if the true value is revealed to be V^H. Here we keep the Chamley and Gale structure in order to facilitate direct comparison.

information revealed.[9] Each player with an option chooses to either invest now or delay. When making their decisions, players can observe the history of other players' investments.

Here the value of the project is either high or low and there is a one-to-one mapping between V and n. The restriction to only two possible project values will allow us to summarize agents' beliefs at time t about the true state of nature via the probability that the project value is high, denoted by q_t. This mapping will prove to be very convenient in eventually formulating the learning process in a linear fashion.

3*.1. Trading Technology*

Agents place discrete-time state-contingent orders which get processed in continuous time. Orders are placed at the beginning of each period. They are processed randomly during the period. The exact time that an individual order is processed is distributed uniformly in the period. Payoffs on all orders processed in a period are received at the end of the period. Since information on others' actions will be arriving during the period, learning is continuous. Players are permitted to make their orders (both invest and wait orders) contingent on the flow of information. Each invest order comes with a state-contingent wait order. Investment cannot be reversed in case the invest order is already processed. Likewise each wait order comes with a state-contingent invest order.

We will eventually approximate the agents' learning process as a series of one-period Brownian Motions with absorbing boundaries. We can make use of the familiar boundary crossing probabilities only if the equilibrium is such that learning stops the first time the learning process hits either of the bounds. We introduce a limit on the capacity of the agency to process contingent orders so that the information flow ceases as soon as one of the boundaries are hit. During the interval $[T,T+1)$, if the state-contingent wait order is triggered then at most M invest orders are canceled. M is

[9]See Gossner and Melissas (2006) for a framework with cheap talk among agents.

a very large but finite number. During the interval [T,T+1), if the state of the state-contingent invest order is triggered, then at most M-κ of the newly triggered invest orders are processed randomly during the remainder of the period where κ is the number of invest orders placed at the beginning of the period.[10] After the description of the equilibrium strategies, the advantage of allowing for state-contingent orders and limiting the maximum capacity of the processing agency will become clear. It is discussed at the end of section 4.3.

3.2. Copycat Traders

The quality of the analytic approximation to the cascade probabilities will be improved if we will also include copycat traders. There are $\Upsilon=\phi rn^H$ people who are randomly assigned to this market and they simply imitate the probability of investment of rational traders with an investment option. The copycats are not needed for the equilibrium of model derived here. However it will be shown that the Υ people add noise to the information to be collected from the market. So they make it harder for the market participants to deduce the true value of the project from the observed purchases – resulting in a higher expected number of invest orders per period. In equilibrium, this will not alter the amount of information transmitted by observational learning, but it leads to a larger number of buyers required in equilibrium to transmit the same amount of information (to be discussed in Section 5.2). The larger the expected number of invest orders the better is the approximation, hence the existence of copycats improves the quality of the approximation to Brownian Motion.

---

[10]The capacity to accommodate state-contingent wait orders M does not need to be the same M as in the capacity to accommodate state-contingent invest orders M-κ. They just need to be very large finite numbers. One could make the limit on contingencies dependent on how busy the agency will be in the remainder of the period: A limit of $M\Delta$t where Δt is the amount of time remaining in the period. This would necessitate keeping track of more notation but would not change the results.

4. Equilibrium

The focus is restricted to symmetric Perfect Bayesian Equilibria. Before describing the equilibrium strategies, let us first introduce some critical values.

4.1. Critical Values

Since orders are processed in continuous time, q_t evolves in continuous time. The index of time for discrete-time decision nodes is denoted by T. While $t \in \mathbb{R}^+$, the index $T \in \mathbb{N}$. So, at time nodes when $t=T$, $q_t=q_T$. Bayes' rule is assumed to describe the agents' method of updating the probabilities. At the beginning of the game, the probability that the project has a high value $q_{T=0}$, conditional on having received an investment opportunity, is given by:

$$q_0 = \frac{\frac{rn^H}{rN} q^*}{\frac{rn^H}{rN} q^* + \frac{rn^L}{rN}(1-q^*)} \tag{2}$$

Since $\psi \equiv \frac{n^L}{n^H}$, (2) can be rewritten as,

$$q_0 = \frac{q^*}{q^* + \psi(1-q^*)} \tag{3}$$

The game is of interest if initially the expected value of the project is positive:

$$q_0 V^H + (1-q_0) V^L > 0 \tag{4}$$

Otherwise each agent would strictly prefer to wait and the game would end immediately with an investment collapse.

It will be useful to introduce two critical values for the subjective probability. Define $\underline{q}$ as the probability where the expected value of the project is zero:

$$\underline{q}\, V^H + (1-\underline{q})\, V^L = 0 \tag{5}$$

If q_t falls below $\underline{q}$, the agent strictly prefers to wait since the expected value of investment is negative. Since everyone who has not yet invested is identical they all prefer to wait and investment stops for good.

Define $\overline{q}$ as the probability where the agent is just indifferent between investing now and waiting when information about the true value of the project is to be fully revealed with certainty next period:

$$\overline{q}\, V^H + (1-\overline{q})\, V^L = \delta \overline{q}\, V^H \qquad (6)$$

If q_t rises above $\overline{q}$, the agent will strictly prefer to invest now. And so will all the identical players and the game ends with an investment boom where all players with an option invest. The game will be said to be *active* when $\underline{q} < q_t < \overline{q}$.

4.2. Endogenous Information Revelation and Learning

Let λ_T denote the probability that a player who has not yet exercised his option puts in an invest order at the decision node T. In the *active* phase of the game, it must be that $0<\lambda_T<1$. Assume for a moment that an agent expects all people with an investment opportunity to invest this period. Then he would strictly prefer to wait to be able to learn the value of the project for sure. But so would everyone else. Hence $\lambda_T \neq 1$. Let us now consider the case where nobody invests this period, $\lambda_T=0$. If nobody is expected to ever invest then no information will be revealed in the future so an individual would strictly prefer to invest since $\underline{q} < q_t$, a contradiction. If investment will resume sometime in the future then an individual contemplating being the first future investor would strictly prefer to invest now rather than wait since then he would have the same expected value of the game as in this period, but discounted. Hence he would strictly prefer to invest now, so $\lambda_T \neq 0$, by contradiction. Hence in the *active* phase of the game, there can be no pure-strategy equilibrium, $0<\lambda_T<1$. Players are just indifferent between waiting and investing now.

Notice that λ_T is the endogenous rate of information flow. If λ_T were zero, no information would be revealed. If λ_T were equal to one, the number of people who invest would fully reveal information about the value of the project.

In equilibrium, the history of the game will affect players' strategies only through the subjective probability q_t and therefore $\lambda_T=\lambda(q_T)$. While players' actions depend on the publicly observed history of the game, history could be payoff relevant for two reasons: *i)* It influences players' beliefs about the probability that the project has a high value, *ii)* As more and more people invest, the number of potential investors with unexercised options goes down and hence history could potentially alter the future flow of information. However the second argument cannot apply in equilibrium. Notice that the expected number of invest orders, either $\lambda_T(\psi+\phi)rn^H$ when $V=V^L$ or $\lambda_T(1+\phi)rn^H$ when $V=V^H$, must be finite. If it were infinite, the observation of the rate of investment in one period would reveal the true value of n, and hence V by the law of large numbers. In such a case all players would strictly prefer to wait, implying $\lambda_T=0$, a contradiction. As $r\to\infty$, the population of potential investors is very large so players are essentially sampling with replacement. For any finite number of exercised options there is still an infinitely large number of potential investors. Hence the history of the game is payoff relevant only because it influences the agents' belief that the project has a high value. In order to save on notation, we will denote $\lambda(q_T)$ *as* λ_T.

Since agents randomize with $0<\lambda_T<1$, the level of investment each period is stochastic. As $r\to\infty$, the number of people putting in invest orders at a decision node is given by the Poisson approximation to the binomial distribution. The parameter of the Poisson distribution is ($rn\lambda_T+\phi\lambda_T rn^H$), the mean number of invest orders by the investors $\lambda_T rn$ plus the expected number of invest orders by the copycats $\phi\lambda_T rn^H$. During the period these investors' orders will be processed randomly. Assuming no contingencies are triggered, the time between completed orders has the exponential distribution and the number of orders processed in any time interval $\Delta t\in(0,1)$ is distributed Poisson with parameter $(rn\lambda_T+\phi\lambda_T rn^H)\Delta t$. The possibility of contingencies being triggered is discussed in Section 4.3.

Take any $x \in [0,1)$ and any $\Delta t \in (0,1-x)$. When $V=V^H$, the probability that k players' investments are processed during the time Δt given λ_T is denoted by $f^H(k;\lambda_T,\Delta t)$:

$$f^H(k;\lambda_T,\ \Delta t) = \begin{cases} \dfrac{e^{-(1+\phi)\lambda_T rn^H \Delta t}\ [(1+\phi)\lambda_T rn^H \Delta t]^k}{k!} & \text{for } k \in \mathbb{Z} \cap [0,(1+\phi)rn^H] \\ 0 & \text{elsewhere} \end{cases} \tag{7}$$

and the p.d.f. is denoted by $f^L(k;\lambda_T,\Delta t)$ when $V=V^L$:

$$f^L(k;\lambda_T,\ \Delta t) = \begin{cases} \dfrac{e^{-(\psi+\phi)\lambda_T rn^H \Delta t}\ [(\psi+\phi)\lambda_T rn^H \Delta t]^k}{k!} & \text{for } k \in \mathbb{Z} \cap [0,(\psi+\phi)\mathrm{rn}^{\mathrm{H}}] \\ 0 & \text{elsewhere} \end{cases} \tag{8}$$

If n^L were equal to n^H, then $\psi=1$ and the two probability density functions would collapse together. In such an extreme case the quality of the information k contains would be nil and the observation of the rate of investment would not reveal any information.

The agent tries to deduce from the number of people who invest in each period which distribution the observation comes from. Define $k_{\Delta t}$ as the number of invest orders processed during the time Δt. Bayesian learning implies that at time $T+x+\Delta t$, when the agent observes $k_{\Delta t}$ people investing, the subjective probability will evolve following:

$$q_{T+x+\Delta t} = \frac{q_{T+x} f^H(k_{\Delta t};\lambda_T,\Delta t)}{q_{T+x} f^H(k_{\Delta t};\lambda_T,\Delta t) + (1-q_{T+x}) f^L(k_{\Delta t};\lambda_T,\Delta t)} \tag{9}$$

4.*3. Equilibrium Strategies*

Let us first assume that the institutional environment restricts the agents to only use $\underline{q}$ and $\overline{q}$ as their triggers for the contingent orders. In Appendix B, this assumption is relaxed. The equilibrium of the game with any finite set of possible trigger points which contains $\overline{q}$ and $\underline{q}$ is shown to yield the same cascade probabilities as this baseline model.

PROPOSITION 1: *The following equilibrium strategies support a Symmetric Perfect Bayesian Equilibrium:*

a) If the subjective probability is sufficiently low $q_T \leq \underline{q}$, put in a wait order with a state-contingent invest order. If in the time interval [T,T+1), $q_t \geq \overline{q}$, the state-contingent invest order is triggered.

b) If the subjective probability is sufficiently high $q_T \geq \overline{q}$, put in an invest order with a state-contingent wait order. If in the time interval [T,T+1), $q_t \leq \underline{q}$, the state-contingent wait order is triggered.

c) If the subjective probability is $\underline{q} < q_T < \overline{q}$, with probability λ_T, put in an invest order with a state-contingent wait order. If in the time interval [T,T+1), $q_t \leq \underline{q}$, the state-contingent wait order is triggered. With probability ($1-\lambda_T$) put in a wait order with a state-contingent invest order. If in the time interval [T,T+1), $q_t \geq \overline{q}$ the state-contingent invest order is triggered.

PROOF: a) By equation (5), when the subjective probability is $\underline{q}$, the expected value of the project is just equal to zero. All agents prefer to wait when $q_T \leq \underline{q}$. Hence the state $q_T \leq \underline{q}$ is absorbing. Investment stops for good. Information transmission terminates. There is no possible deviation from the equilibrium strategy that would make the agent better off. Off the equilibrium path, if in the time interval [T,T+1) new information arriving leads to an updated belief $q_t \geq \overline{q}$, the agent prefers to invest and the state-contingent invest order is triggered. Since $r \rightarrow \infty$, and only M-κ invest orders can be processed, the probability of an individual agent's state-contingent invest order being processed is zero.

b) When $q_T \geq \overline{q}$, by equation (6) the agent strictly prefers to invest now even if this period the true value of the project is to be revealed for sure. When $q_T \geq \overline{q}$, all with an option give invest orders. Since $r \rightarrow \infty$, during the time interval [T,T+1) the true value of n and hence V is revealed at once. If $V=V^H$ the subjective probability would remain above $\overline{q}$. If $V=V^L$, the subjective probability would drop below $\underline{q}$. All agents' state contingent wait orders would be triggered at once but only M of them would be able to stop investment.[11] The game would end with all investing except of those lucky M . Therefore, $q_T \geq \overline{q}$ is an absorbing state as well.

When $q_T \geq \overline{q}$, an individual agent could consider the following deviation from the equilibrium strategy: Giving a wait order with a state contingent invest order. If all follow the above described equilibrium strategy, information about the true value of the project would be revealed at once, and the agent's state contingent invest order would be triggered at once. However his order would be not served in the period, since at most M-κ of the newly triggered orders are processed during the period. In equilibrium, when $q_T \geq \overline{q}$ all give invest orders, since $r \to \infty$ so does κ. Hence the probability of a newly triggered invest order being processed is goes to zero. Therefore a deviation would lead to one period of discounting. By equation (6) the agent would strictly prefer to follow the equilibrium strategy.

c) If the subjective probability is $\underline{q} < q_T < \overline{q}$, the expected value from investment is positive but the agent will also consider waiting in order to learn about the true value of the project. In equilibrium, the agent is just indifferent between investing now and waiting. See the beginning of Section 4.2 for the discussion of the non-existence of pure-strategy equilibrium.

i) The agent with an investment option who has not yet exercised his option will put an invest order at time *T* with probability λ_T. If in the time interval $[T,T+1)$, q_t falls below $\underline{q}$, the agent would prefer to wait by equation (5). All unprocessed invest orders would convert into wait orders. Since *M* is a very large number, investment would stop for good.

ii) The agent with an investment option who has not yet exercised her option will put a wait order at time *T* with probability $(1-\lambda_T)$. If however in the time interval $[T,T+1)$, q_t rises above

[11]Because M is a large but finite number, the probability of an individual agent's state-contingent wait order being processed is zero.

$\overline{q}$ the agent would prefer to invest. *M* is very large but finite. *M-κ* newly arrived invest orders would be processed this period. All the rest would be processed next period. At time *T*, the agent realizes that there is an infinitely small probability that his invest order would be processed if the state is triggered.

□

At this point the advantage of allowing for contingent orders should be clear: In the active phase of the game, the subjective probability evolves in continuous time within the bounds $\underline{q}$ and $\overline{q}$. Because we let potential investors put in contingent orders, the game stops the first time q_t hits either of the bounds. Once either of the bounds is hit, information transmission stops. Without contingencies, within the time interval [T,T+1), the subjective probability could potentially cross one the bounds and then bounce back depending on the flow of the invest order processing since the exact timing of the processing of the individual order is a random variable. In the active phase of the game, if the process hits $\overline{q}$, all state-contingent invest orders are triggered.

If the processing agency did not have a capacity constraint, the true value of the project would be revealed at once and potentially the subjective probability could drop below $\underline{q}$. However the agency can process only M-κ newly triggered invest orders. So, no matter whether the true n is n^H or n^L the same number of invest orders will be processed. Therefore in the remainder of the period, the agent cannot gain further information about the true value of the project. Information flow stops the moment the subjective probability hits the upper bound. If the process hits $\underline{q}$, state-contingent wait orders are triggered and they get served since M is a very large number. Therefore information flow would stop.

5. Information Cascades

Agents' beliefs about the true value of the project evolve as a result of observational learning. If the subjective probability hits $\underline{q}$ before $\overline{q}$, the game ends with an investment collapse. If the subjective probability hits $\overline{q}$ before $\underline{q}$, the game ends with an investment boom. We are particularly interested in the probability of inefficient cascades. The measures of interest are then the probability that the process hits $\underline{q}$ before $\overline{q}$ when $V= V^H$, and the probability that the process hits $\overline{q}$ before $\underline{q}$ when $V= V^L$. The first is an inefficient negative cascade and the latter is an inefficient positive cascade.

While the paper will discuss both types of inefficient outcomes, notice that only inefficient negative cascades would be categorized as inefficient herding. Here agents that receive an investment option would invest if learning were not permitted, by (4)**.** Since by definition herding is acting against one's own signal, we can talk about inefficient herding only when the crowd chooses not to invest.

5.1. Transformation

In order to obtain the boundary crossing probabilities, we need to transform the problem into an equivalent problem that is tractable. Subjective probabilities evolve following (9), substitute $f^H(k;\lambda_T,\Delta t)$ and $f^L(k;\lambda_T,\Delta t)$ from (7) and (8) into (9). Cancel out k factorial from the numerator and denominator. Take the inverse of both the left and right hand side of the equality and subtract one from each side. Now plugging in ψ for $\frac{n^L}{n^H}$ yields,

$$\frac{1 - q_{t+\Delta t}}{q_{t+\Delta t}} = \frac{1 - q_t}{q_t} e^{(1-\psi)\lambda_T rn^H \Delta t} \left(\frac{\psi + \phi}{1+\phi} \right)^{k_{\Delta t}} \tag{10}$$

Taking the natural logarithm of both sides and multiplying both sides by minus one yields:

$$-ln\left(\frac{1 - q_{t+\Delta t}}{q_{t+\Delta t}} \right) = -ln\left(\frac{1 - q_t}{q_t} \right) - (1 - \psi)\lambda_T rn^H \Delta t - k_{\Delta t} ln\left(\frac{\psi + \phi}{1+\phi} \right) \tag{11}$$

where $k_{\Delta t}$ is distributed Poisson with the parameter $\lambda_T(1+\phi)rn^H\Delta t$ when the true value of the project is high and it is distributed Poisson with the parameter $\lambda_T(\psi+\phi)rn^H\Delta t$ when the true value of the project is low. Define a transformation w_t as:

$$w_t \equiv -ln\left(\frac{1-q_t}{q_t}\right) \tag{12}$$

Notice that w_t is an increasing monotonic transformation of q_t. We can rewrite (11) as:

$$w_{t+\Delta t} = (\psi - 1)\lambda_T rn^H \Delta t + w_t - ln\left(\frac{\psi+\phi}{1+\phi}\right)k_{\Delta t} \tag{13}$$

The transformed subjective probabilities evolve following (13), where $k_{\Delta t}$ is investment in Δt.

In this model individual learning follows a stochastic process with independent increments. It is interesting to note that this process is a well-known description of individual learning in cognitive psychology. That literature looks, for example, at how people identify objects looking at pictures. In much of the literature individuals are modeled as learning through random sampling. This characterization of the learning process is then used to explain laboratory evidence on individual response times and error rates. The present paper shows that even with fully rational agents, group behavior will resemble the individual behavior of boundedly rational agents of the type used in cognitive psychology.[12]

The transformation (12) of the lower bound given by (5), of the upper bound given by (6) and of the starting point given by (3) yield :

The lower bound: $\underline{q} \Rightarrow \underline{w}$ $\qquad \underline{w} = ln(-V^L) - lnV^H$ (14)

The upper bound: $\overline{q} \Rightarrow \overline{w}$ $\qquad \overline{w} = ln(-V^L) - lnV^H - ln(1-\delta)$ (15)

[12]See Luce (1986) for an introduction to this literature.

The starting point: $q_0 \Rightarrow w_0$ $\quad w_0 = lnq^* - ln(1-q^*) - ln\psi$ (16)

Since initially the expected value of the project is positive (4), $\underline{w} < w_0$. And $w_0 < \overline{w}$ examining (6) and (3) together.

5.2. Boundary Crossing Probabilities with constant λ

The transformed subjective probabilities follow equation (13) where the error term $k_{\Delta t}$ is distributed Poisson with the parameter $(rn\lambda_T + \phi\lambda_T rn^H)\Delta t$. Both the mean and the variance of the process depend on λ_T and hence they depend on the history of the game. They are not constant.

Now we are going to examine a different process. In this modified problem we will examine the processes described by equations (13) and (7) and (8) yet with a constant $\lambda \in (0,1)$, implying a constant Poisson parameter. Section 5.3 will prove that the process with the endogenously-determined non-constant λ will yield identical boundary crossing probabilities as in this modified problem with fixed λ.

With a constant λ, equation (13) implies that w_t evolves in the same way as the queue length in a standard M/D/1 queue. An M/D/1 queue has exponential arrivals, so the distribution of new customers over an interval Δt is Poisson, and one server who takes a deterministic amount of time to serve a customer. Here we will make use of the standard results for the queue length for heavy-traffic M/D/1 queues and approximate the evolution of w_t as a Brownian Motion[13]:

$$w_{t+\Delta t} \approx w_t + k^*_{\Delta t} \qquad (17)$$

[13]See Kleinrock (1976) for a textbook derivation and Gaver (1968) for the original contribution. This is known as the "heavy-traffic diffusion approximation."A heavy-traffic queue is one in which the average arrival rate of customers is close to the capacity of the server. In queuing theory this is important for the approximation because otherwise the queue length would have a mass point at zero length. This issue does not arise here because the game will effectively end when either of the boundaries are hit. Here we have absorbing boundaries rather than the reflective boundary of a queue.

where $k_{\Delta t}{}^{*}$ is distributed normal with mean $\mu\Delta t$ and variance $\sigma^2\Delta t$:

$$when\ V=V^H \qquad k^*_{\Delta t} \sim N\begin{cases} \mu^H = \left(\psi - 1 - (1+\phi)\ln\dfrac{\psi+\phi}{1+\phi}\right)\lambda rn^H, \\ (\sigma^2)^H = (1+\phi)\left(\ln\dfrac{\psi+\phi}{1+\phi}\right)^2 \lambda\, rn^H \end{cases} \tag{18}$$

$$when\ V=V^L \qquad k^*_{\Delta t} \sim N\begin{cases} \mu^L = \left(\psi - 1 - (\psi+\phi)\ln\dfrac{\psi+\phi}{1+\phi}\right)\lambda\, rn^H, \\ (\sigma^2)^L = (\psi+\phi)\left(\ln\dfrac{\psi+\phi}{1+\phi}\right)^2 \lambda rn^H \end{cases} \tag{19}$$

$\mu^H>0$ by Claim A1, and $\mu^L<0$ by Claim A2 in the Appendix.

It is important to emphasize the limitations of this approximation. It is a good approximation for a high expected rate of investment. This will happen with a high ϕ (high numbers of noise traders). Keeping the rate of information flow λ constant, an increase in ϕ leads to a weaker drift. That is the positive drift μ^H declines and the negative drift μ^L increases. The higher ϕ the less informative a single observation is about the true value of the project. Both $(\sigma^2)^H$ and $(\sigma^2)^L$ go down with an increase in ϕ, keeping λ constant. Each observation has less informational content, hence beliefs don't get updated as much. Therefore keeping λ constant, an increase in ϕ would make agents strictly prefer to wait. Since in equilibrium agents are just indifferent between investing and waiting, a higher ϕ must be associated with a higher λ. Moreover as $q_T \to \overline{q},\ \lambda_T rn \to \infty.$ Hence for a given Δt and q_T, there is a ϕ that will yield a high enough Poisson parameter $(rn\lambda_T + \phi\lambda_T rn^H)\Delta t$ so that the normal approximation is reasonable. For any finite ϕ there will be a range $q_T \in (\underline{q}, \tilde{q}]$ where the approximation is not reasonable. Notice however that λ only changes at $t \in \mathbb{Z}$, so by increasing ϕ and hence decreasing $\tilde{q}$ we can make the probability of ever encountering $q_T \in (\underline{q}, \tilde{q}]$ small. Brownian Motion also requires that the normal approximation holds as $\Delta t \to 0$. Just as in queuing theory there is no set of parameters where this is the case, hence as in queuing theory it will always be an approximation.

The standard boundary crossing probabilities for Brownian motion yield the following results.[14]

i) Probability of hitting $\underline{w}$ before $\overline{w}$ when $V=V^H$:

$$Prob(\underline{w}\ before\ \overline{w} \,|\, V = V^H) = 1 - \left[\left(e^{\frac{-2\mu^H \underline{w}}{(\sigma^2)^H}} - e^{\frac{-2\mu^H w_0}{(\sigma^2)^H}} \right) \Big/ \left(e^{\frac{-2\mu^H \underline{w}}{(\sigma^2)^H}} - e^{\frac{-2\mu^H \overline{w}}{(\sigma^2)^H}} \right) \right] \tag{20}$$

Multiply the numerator and the denominator of (20) by $exp\left(\frac{2\mu^H \underline{w}}{(\sigma^2)^H} \right)$. Plug in Equations (18), (14), (15) and(16) for $\underline{w}$, $\overline{w}$, w_0, μ^H and $(\sigma^2)^H$:

$$Prob(\underline{w}\ before\ \overline{w} \,|\, V = V^H) = 1 - \left(\frac{1 - e^{2\varphi^H \left[(ln(1-q^*) - lnq^* + ln\psi - lnV^H + ln(-V^L) \right]}}{1 - e^{2\varphi^H ln(1-\delta)}} \right) \tag{21}$$

where $\varphi^H \equiv \frac{\mu^H}{(\sigma^2)^H}$. By Claim A1 in Appendix A, $\mu^H > 0$. So $\varphi^H > 0$.

ii) Probability of hitting $\underline{w}$ before $\overline{w}$ when $V=V^L$:

$$Prob(\overline{w}\ before\ \underline{w} \,|\, V = V^L) = \left(e^{\frac{-2\mu^L \underline{w}}{(\sigma^2)^L}} - e^{\frac{-2\mu^L w_0}{(\sigma^2)^L}} \right) \Big/ \left(e^{\frac{-2\mu^L \underline{w}}{(\sigma^2)^L}} - e^{\frac{-2\mu^L \overline{w}}{(\sigma^2)^L}} \right) \tag{22}$$

Plug in (19), (14), (15), (16) into (22):

$$Prob(\overline{w}\ before\ \underline{w} \,|\, V = V^L) = \frac{1 - e^{2\varphi^L \left[(ln(1-q^*) - lnq^* + ln\psi - ln(V^H) + ln(-V^L) \right]}}{1 - e^{2\varphi^L ln(1-\delta)}} \tag{23}$$

---

[14]See for example Karlin and Taylor (1975) for the derivation.

where $\varphi^L \equiv \dfrac{\mu^L}{(\sigma^2)^L}$. By Claim A2 in Appendix A, $\mu^L < 0$. So $\varphi^L < 0$.

Before proceeding to the next subsection it is crucial to notice that φ^H and φ^L are independent of λ, from (18) and (19). While an increase in λ leads to an increase in μ, it also leads to an increase in σ^2. And these two effects counterbalance each other in the determination of the boundary crossing probabilities. Hence these probabilities are independent of λ.

5.3. Inefficient Cascade Probabilities for the Original Problem

PROPOSITION 2: *The boundary crossing probabilities of the original problem are equal to the boundary crossing probabilities found using a Brownian Motion, (21) and (23) of the modified problem.*

Proof:

In the actual learning process, the λ is updated at each decision node. The process is a series of one period Brownian Motions. The boundary crossing probabilities for this process can be reconstructed iteratively using Lemma B1 in Appendix B. Starting with the Brownian Motion with absorbing boundaries defined in (14) and (15) and the starting point given by (16), create a process where the parameter λ changes to λ' (which is stochastic) at T=1 and stays constant thereafter. From Lemma B1, this new process has the same transition probabilities as the original process. Iterating this argument yields the result.

□

Denote the probability of an inefficient negative cascade by *Prob(INC)* which is equal to the probability that the process hits $\underline{q}$ before $\overline{q}$ when $V = V^H$. By Propostion 2, *Prob(INC)* is given by equation (21).

Prob(IPC) denotes the probability of an inefficient positive cascade and it is the probability that the process hits $\overline{q}$ before $\underline{q}$ when $V = V^L$. By Proposition 2, *Prob(INC)* is given by equation (23).

6. Comparative Statics

6.1. The Prior Expected Value:

When there is potentially a lot to gain ($V^H\uparrow$) or little to loose ($V^L\uparrow$), much would be lost in expectation due to discounting while waiting. So the agent would be more prone to investing before he is certain it is a good project (hence $\overline{q}\downarrow$). And the belief about the odds of the project being a high value project does not need to be as high for the agent to strictly prefer to wait (so $\underline{q}\downarrow$). Therefore the probability of hitting the upper bound before hitting the lower bound increases. The likelihood of an inefficient positive cascade goes up and the likelihood of an inefficient negative cascade goes down.[15]

As the prior expected value of the project goes up due to an increase in q*, the upper bound $\overline{q}$ and the lower bound $\underline{q}$ are unaffected, by (6) and (5). Being closer to the upper bound to begin with, the agents require less evidence in their learning process to strictly prefer investment. Hence they are more likely to invest when the prior improves.

PROPOSITION 3: *As the prior expected value increases ($q^*\uparrow$ or $V^H\uparrow$ or $V^L\uparrow$), it becomes more likely that all agents with an option undertake the project. It becomes less likely that there is an investment collapse when the true value is high, Prob(INC)$\downarrow$. It becomes more likely that there is an investment boom when the true value is low, Prob(IPC)$\uparrow$.*

Proof: Appendix C.

[15] In an exogenous-timing herding framework Welch (1992) shows that as the prior expected value from investment goes up, there is a greater chance that society ends up in a positive cascade since early movers are more likely to invest.

6.2. Discounting:

Discounting doesn't play a role in exogenous-timing models. Examination of this issue requires an endogenous-timing model. The agent makes a choice between investing now or later. If the agent waits, he can learn by observing other people's actions, however the payoff gets discounted. All else constant, as people get more patient ($\delta \uparrow$), they will be more willing to wait. Since waiting induces learning, one might be tempted to conclude that higher δ would be associated with a smaller probability of an erroneous mass behavior. However this is not the case. In fact the probability of an inefficient negative cascade goes up. Chamley and Gale (1994) show that for δ=1there is a weakness of the investment process in the direction of underinvestment. Proposition 4 extends this result to comparative statics over the whole range of δ.

PROPOSITION 4: *As agents become more patient (δ↑), it becomes more likely that agents do not undertake the investment project. It becomes more likely that there is an investment collapse, Prob(INC) ↑, when the true value is high. It becomes less likely that there is an investment boom, Prob(IPC) ↓, when the true value is low.*

Proof:

The probability of an inefficient negative cascade goes up as δ goes up,

$$\frac{dProb(INC)}{d\delta} = \left(1 - Prob(INC)\right) \frac{2\varphi^{H} e^{2\varphi^{H} ln(1-\delta)}}{(1-\delta)\left(1 - e^{2\varphi^{H} ln(1-\delta)}\right)} > 0 \tag{24}$$

On the other hand, the probability of an inefficient positive cascade goes down as δ goes up.

$$\frac{dProb(IPC)}{d\delta} = - Prob(IPC) \frac{2\varphi^{L} e^{2\varphi^{L} ln(1-\delta)}}{(1-\delta)\left(1 - e^{2\varphi^{L} ln(1-\delta)}\right)} < 0 \tag{25}$$ □

An increase in δ has a two effects. The first effect is through the rate of information flow λ. As agents get more patient, at the ongoing information flow, they would strictly prefer to wait. So the rate of information flow goes down such that people are just indifferent between waiting and investing.[16]

However this first effect has no influence on the probability of inefficient cascades in this framework. When the true value of the project is high $V=V^H$, a weaker information flow implies a weaker drift velocity μ^H which simply increases the likelihood of a negative cascade. However at the same time the weaker information flow decreases the noise $(\sigma^2)^H$ in the learning process. Each observation will have a smaller influence on the updating process. This reduces the likelihood of a negative cascade. And these two opposing effects exactly counterbalance each other. λ cancels out from the probability of inefficient cascade (see equations (21) and (23)). The indirect effect through the information flow is therefore nullified. The spirit of the story is the same for the case when $V=V^L$.

The second effect of an increase in δ is through the upper bound. A higher δ yields a higher upper bound $\overline{q}$, leaving the starting point and the lower bound unchanged. Since investors are more patient, they are willing to wait until they are almost certain about the project before they buy. This makes an inefficient negative cascade more likely, and an inefficient positive cascade less likely.

This comparative statics result suggests that financial markets might be more prone to inefficient collapses than real markets. Once an investment order is given, the payoff can be collected only at the end of the period. Keeping the rate of time preference constant, as the time to process investment decisions increases so does the distance between the time periods in the model, leading to a lower δ. In financial markets, the administrative and technological systems may be faster to react to

---

[16]See Chamley (2004) for a simple two period two agent example. If $V=V^H$, both people have an option. If the $V=V^L$, only one agent has an option. In equilibrium, the agent if not invested in the first period would invest in the second period only if he observes investment in the first period. When δ goes up, the equilibrium probability if investment in the first period goes down, making an investment collapse more likely even if the true value is high.

agents investment decisions than in real markets. Hence in financial markets the relevant δ would be larger than in real markets leading to a higher likelihood of an inefficient collapse.

6.3. Quality of Information:

The leading model of Bikhchandani, Hirshleifer and Welch (1992) shows that as signal quality goes up, inefficient negative and positive cascade probabilities go down. The effect is monotonic. Here, this not the case. An increase in the signal quality does of course unambiguously improve the expected value of the game to the market participants.[17] But the probabilities of inefficient cascades may go up or down depending on the parameter values. Result 1 in Bikhchandani, Hirshleifer and Welch (1992) is closely related. It shows that all agents after the second are better off when the first agent's signal quality (expertise) is slightly decreased. This results in more information for later individuals as it decreases the probability that a cascade forms after just two individuals. A related issue arises here when the starting belief is close to one of the boundaries.

To understand the role of signal quality in this framework, first notice that agents who receive an investment option would all undertake the investment if there were no social learning. It is through social learning that the possibility of an investment collapse arises. The increase in signal quality affects the outcome of the game through two channels; i) *Self-confidence*: It increases the confidence of the agent in his own signal. Keeping the level of signal quality of the rest of the people constant, as the signal quality of the agent goes up, the agent is more likely to undertake the project, hence Prob(INC)$\downarrow$ and Prob(IPC)$\uparrow$. ii) *Confidence in observational learning*: It increases the confidence of the agent in observational learning since each individual has a high quality signal. Now keeping the signal quality of the agent constant, as the signal quality of the rest of the players increases the agent

[17]See equations (3) and (4).

becomes more likely not to undertake the project, hence Prob(INC)↑ and Prob(IPC)↓. These two channels with opposing forces can be examined below.

Let us first examine the probability of an inefficient negative cascade. Equation (20) can be rewritten as:

$$Prob(INC) = 1 - \frac{1 - e^{2\varphi^H(\underline{w} - w_0)}}{1 - e^{2\varphi^H(\underline{w} - \overline{w})}} \tag{26}$$

From (14) and (15) notice that $\underline{w}$ and $\overline{w}$ are independent of ψ. So,

$$\frac{d\,\Pr ob(INC)}{d\psi} = \left(\frac{\partial \Pr ob(INC)}{\partial w_0}\frac{\partial w_0}{\partial \psi}\right) + \left(\frac{\partial \Pr ob(INC)}{\partial \varphi^H}\frac{\partial \varphi^H}{\partial \psi}\right) \tag{27}$$

The first term relates to the first channel; self-confidence. It is the effect of ψ on the probability of an inefficient negative cascade through the agents's belief before the observational learning starts (w_0). It is positive by Claim C3 in Appendix C.[18] The second term relates to the second channel; confidence in observational learning. It is negative by Claim C1 and C4 in Appendix C. Therefore there are two forces working in opposite directions.

Let us now examine the probability of an inefficient positive cascade. Rewrite (22) as,

$$Prob(IPC) = \frac{1 - e^{2\varphi^L(\underline{w} - w_0)}}{1 - e^{2\varphi^L(\underline{w} - \overline{w})}} \tag{28}$$

[18]This is consistent with the discussion in the previous paragraph since an increase in ψ represents a deterioration in signal quality.

So,

$$\frac{d\Pr ob(IPC)}{d\psi} = \left(\frac{\partial \Pr ob(IPC)}{\partial w_0}\frac{\partial w_0}{\partial \psi}\right) + \left(\frac{\partial \Pr ob(IPC)}{\partial \varphi^L}\frac{\partial \varphi^L}{\partial \psi}\right) \tag{29}$$

The first term is negative by Claim C5 in Appendix C. It relates to the first channel. The second term relates to the second channel. It is positive by Claim C2 and Claim C6 in Appendix C. Hence there are two opposing effects.

PROPOSITION 5: *As the signal quality improves* $\left(\frac{1}{\psi}\uparrow\right)$, *the likelihood of an inefficient positive cascade and the likelihood of an inefficient negative cascade might go up or down depending on the parameter values.*

Proof: Appendix C

Proposition 5 is proven in two steps. The first step proves that $\frac{d\Pr ob(INC)}{d\psi}$ >0, when $w_0 \to \underline{w}$ and $\frac{d\Pr ob(INC)}{d\psi}$ <0 when w_0 is not to close to either of the bounds, $\underline{w} \ll w_0 \ll \overline{w}$. The second step proves that $\frac{d\Pr ob(IPC)}{d\psi}$ <0, when $w_0 \to \overline{w}$. $\frac{d\Pr ob(IPC)}{d\psi} > 0$ when w_0 is not to close to either of the bounds, $\underline{w} \ll w_0 \ll \overline{w}$. Depending on the initial belief of the agent with an option, the effect of an increase in signal quality through confidence in observational learning can dominate the effect through self-confidence. If the agent does not have a strong initial belief (such that w_0 is not too close to either of the bounds) the social learning channel becomes more important in determining the direction of the effect of signal quality on the outcome of the game. An increase in signal quality can lead to an increase in the probability of inefficient herding. Investment is stochastic by the very nature of the game. Even when the true value of the project is high, it is possible to get a few bad draws in a row. In such a case, when the signal quality is high, the agent will heavily update his beliefs possibly leading to an investment collapse.[19] If however the signal quality were lower, the agent would need to collect more evidence before he strictly preferred to wait.[20]

7. Conclusion

Knowledge of the direction and the magnitude of comparative statics results on the probability of a collapse is essential in herd manipulation. The pricing decision of a firm introducing a new technology, the advertising policy affecting the signal quality may be some of the key elements in manipulating the outcome of the social learning game. In financial markets, one of the objectives of regulatory bodies may be to reduce the probability of sudden crashes.[21] While the tools available to the regulatory body for herd manipulation are typically restricted there is some room for relevant regulation. Accounting standards affect the signal quality, transaction taxes affect the expected value, the closing and opening times as well as the time of delay for payments of the payoffs affect discounting.

The advantage of analyzing the likelihood of inefficient cascades arises when there are externalities from the market to society at large. In these cases looking at the ex-ante utility of the players directly involved in the game insufficient for welfare analysis. For example, bank panics, capital flight and stock market crashes have external consequences which may induce a social planner

---

[19]On the other hand, naturally if signal quality is high, it would be less likely to have bad draws when the true value is high.

[20]As the signal quality goes up, it is more likely that the first movers pick the correct choice and the confidence in observational learning increases. But this implies that the agent will be more likely to go with the "trustworthy" crowd which is bad news if the early movers happen to have picked the incorrect choice. If some agents' signal quality is unboundedly high then there is no chance of this occurring, see Rogers (2005). Also see De Vany and Lee (2001). Nelson (2002) looks at a related but different aspect of signal quality. The simulation results indicate that in a changing environment, the *frequency* of herding is non-monotone in signal quality.

[21]Avery and Zemsky (1998) show that herd behavior can lead to a significant short-run mispricing of financial assets. See Dremann, Oechssler and Roider (2005) for an experimental analysis of Avery and Zemsky (1998).

to place a greater weight on inefficient negative cascades than an individual investor. In other markets the party designing the structure of the market may not have an incentive to weigh all market participants equally. In the IPO market, for example, the features of the market are not controlled by a central planner, but rather by the firms offering companies for public sale. These companies may try to increase the probability of a positive cascade. Results on the probabilities of inefficient outcomes are a potentially useful building block for welfare and policy analysis in markets with such external effects.

REFERENCES

Avery and Zemsky (1998): “Multidimensional Uncertainty and Herd Behavior in Financial Markets,” *American Economic Review*, 88(4), 724-748.

Banerjee (1992), “A Simple Model of Herd Behavior,” *Quarterly Journal of Economics*, 107:3, 797-818.

Beatty and Ritter (1986): “Investment Banking, Reputation and the Underpricing of Initial Public Offerings,” *Journal of Financial Economics* 15, 213-232.

Bikhchandani, Hirshleifer and Welch (1992): “A Theory of Fads, Fashion, Custom, and Cultural Change as Informational Cascades,” *Journal of Political Economy,* 100:5, 992-1026.

______________ (1998): “Learning from the Behavior of Others: Conformity, Fads and Informational Cascades,” *Journal of Economic Perspective*, 12(3), 151-170.

Bose, Orosel and Vesterlund (2002), “Optimal Pricing and Endogenous Herding,” CESifo Working Paper Series No. 727.

Chamley (2003): “Dynamic Speculative Attacks”, *American Economic Review*, 93, No.3, 603-621.

______________ (2004): <u>Rational Herds, Economic Models of Social Learning</u>, Cambridge: Cambridge University Press.

Chamley and Gale (1994): “Information Revelation and Strategic Delay in a Model of Investment,” *Econometrica*, 62, No.5, 1065-1085.

Chari and Kehoe (2003): “Hot Money”, *Journal of Political Economy*, 111, No. 6, 1262-1292.

Choi, Dassiou and Gettings (2000): “Herding Behavior and the Size of Customer Base as a Commitment to Quality,” *Economica*, 67, 375-398.

De Vany and Lee (2001): “Quality Signals in Information Cascades and the Distribution of Motion Picture Box Office Revenues,” *Journal of Economic Dynamics and Control*, 25:3-4, 593-614.

Devenow and Welch (1996): “Rational Herding in Financial Economics,” *European Economic Review,* 40, 603-615.

Drehmann, Oechssler and Roider (2005): “Herding and Contrarian Behavior in Financial Markets: An Internet Experiment,” *American Economic Review,* 95(5), 1403-26.

Gale (1996): “What Have We Learned from Social Learning?” *European Economic Review*, 40(3-5), 617-628.

Gaver (1968): “Diffusion Approximations and Models for Certain Congestion Problems,”*Journal of Applied Probability*, 5, 607-623.

Gossner and Melissas (2006): “Informational Cascades Elicit Private Information,” *International Economic Review*, 47(1), 297-325.

Karlin and Taylor (1975): <u>A First Course in Stochastic Processes</u>, New York: Academic Press.

Kennedy (2002): “Strategy Fads and Competitive Convergence: An Empirical Test for Herd Behavior in Prime-time Television Programming,” *Journal of Industrial Economics*, 50(1), 57-84.

Kleinrock (1976): Queueing Systems, Volume II, New York: Wiley-Interscience Publication.

Levin and Peck (2005), "Investment Dynamics with Common and Private Values,"Ohio State University Economics Department, Working Paper.

Luce (1986): Response Times: Their Role in Inferring Elementary Mental Organization, New York: Oxford University Press.

Morton and Williams (1999): "Informational Asymmetries and Simultaneous versus Sequential Voting," *American Political Science Review*, 93(1), 51-68.

Neeman and Orosel (1999): "Herding and the Winner's Curse in Markets with Sequential Bids", *Journal of Economic Theory*, 85(1), 91-121.

Nelson (2002): "Persistence and Reversal on Herd Behavior: Theory and Application to the Decision to Go Public," *The Review of Financial Studies*, 15, No.1, 65-95.

Rogers (2005): "The Timing of Social Learning", mimeo. Caltech.

Smith and Sørensen (2000):"Pathological Outcomes of Observational Learning," *Econometrica*, 68(2), 371-398.

Scharfstein and Stein (1990): "Herd Behavior and Investment", *American Economic Review*, 80(3), 465-479.

Welch (1992): "Sequential Sales, Learning and Cascades," *The Journal of Finance*, 47(2), 695-732.

______________ (2000): "Herding Among Security Analysts," *Journal of Financial Economics*, 58, 369-396.

Zhang (1997): "Strategic Delay and the Onset of Investment Cascades," *Rand Journal of Economics*, 28, No. 1, 188-205.

APPENDIX

Appendix A:

Claim A1: $\mu^H > 0$ where $\mu^H = \left(\psi - 1 - (1+\phi)ln\frac{\psi+\phi}{1+\phi}\right)\lambda rn^H$.

Proof: *Define* $f(\psi) = \left(\psi - 1 - (1+\phi)ln\frac{\psi+\phi}{1+\phi}\right)$. *Note that f(1)=0.*

Since $f'(\psi) = \frac{\psi - 1}{\psi+\phi} < 0$ $\Rightarrow$ *f(ψ)>0 for 0<ψ<1. So,* $\mu^H > 0$. □

Claim A2: $\mu^L < 0$ where $\mu^L = \left(\psi - 1 - (\psi+\phi)ln\frac{\psi+\phi}{1+\phi}\right)\lambda rn^H$.

Proof: *Define* $f(\psi) = \left(\psi - 1 - (\psi+\phi)ln\frac{\psi+\phi}{1+\phi}\right)$. *Note that f(1)=0.*

Since $f'(\psi) = -\ln\frac{\psi+\phi}{1+\phi} > 0$ $\Rightarrow$ *f(ψ) <0 for 0<ψ<1. So,* $\mu^L < 0$. □

Appendix B:

Lemma B1: *Let* $w_t(\lambda)$ *be a Brownian Motion with absorbing boundaries as defined in (14) and (15),* μ *and* σ^2 *as defined in (18) and (19) and with starting point* $w_T \in (\underline{w}, \overline{w})$. *Let* $\tilde{w}_t$ *be another process with the same form and parameters as* w_t *up to some possibly stochastic time* $\tau > T$ *at which time the parameter* λ *is replaced by* λ', *which may also be stochastic. Both* $w_t(\lambda)$ *and* $\tilde{w}_t$ *yield the same probabilities of hitting the boundaries (21) and (23).*

Proof: *Define* $b(\tau, w_\tau, \lambda')$ *as the joint p.d.f. of* τ, w_τ *and* λ' *conditional on not hitting either boundary in* $t \le \tau$. *Define* $P_{w_T(\lambda)\to\overline{w}}$ *as the probability starting from* w_T *that process hits the boundary* $\overline{w}$ *before* $\underline{w}$. *Since* $w_t(\lambda)$ *is a standard Brownian Motion* $P_{w_0(\lambda)\to\overline{w}}$ *is given by*

$$P_{w_0(\lambda)\to\overline{w}} = 1 - \left[\left(e^{\frac{-2\mu\underline{w}}{\sigma^2}} - e^{\frac{-2\mu w_0}{\sigma^2}}\right) \Big/ \left(e^{\frac{-2\mu\underline{w}}{\sigma^2}} - e^{\frac{-2\mu\overline{w}}{\sigma^2}}\right)\right] \tag{30}$$

and $P_{w_t(\lambda)\to\underline{w}} = 1 - P_{w_t(\lambda)\to\overline{w}}$. *These depend on λ only through the ratio* μ/σ^2. *Hence the probabilities of* $w_t(\lambda)$ *hitting the boundaries do not depend on λ. Although the date* τ *as no special relevance to* *this process we can still decomposed this probability into the probability that it transitions before or at τ and the probability it transitions after τ:*

$$P_{w_0(\lambda)\to\overline{w}} = P_{w_0(\lambda)\to\overline{w}|t\le\tau} + \left[1 - P_{w_0(\lambda)\to\overline{w}|t\le\tau} - P_{w_0(\lambda)\to\underline{w}|t\le\tau}\right]\int_\tau\int_{w_\tau}\int_{\lambda'} P_{w_\tau(\lambda)\to\overline{w}}\, \mathrm{b}(\tau, w_\tau, \lambda')\cdot d\tau\cdot dw_\tau\cdot d\lambda' \tag{31}$$

While we know the left-hand side of this, the formulas for the conditional probabilities and p.d.f.s on the right-hand side are unknown. However, since $\tilde{w}_t$ starts off as the same process we can similarly decompose its probability as:

$$P_{\tilde{w}_0\to\overline{w}} = P_{w_0(\lambda)\to\overline{w}|t\le\tau} + \left[1 - P_{w_0(\lambda)\to\overline{w}|t\le\tau} - P_{w_0(\lambda)\to\underline{w}|t\le\tau}\right]\int_\tau\int_{w_\tau}\int_{\lambda'} P_{w_\tau(\lambda')\to\overline{w}}\, \mathrm{b}(\tau, w_\tau, \lambda')\cdot d\tau\cdot dw_\tau\cdot d\lambda' \tag{32}$$

Here both the left and right-hand side probabilities are unknown. Nevertheless, since it is the same process up to τ these conditional probabilities and p.d.f.s are the same as in (31) with the exception of the continuation probabilities in the integrals. Note however these are simply the probabilities for Brownian Motion starting from w_τ with parameter λ' and hence for each potential realization of w_τ and λ' the probability can be found from (30) by substituting w_τ for w_0. As before λ' cancels out from these probabilities. Therefore each $P_{w_\tau(\lambda')\to\overline{w}}$ in equation (32) is equal to the corresponding $P_{w_\tau(\lambda)\to\overline{w}}$ in (31) and hence $P_{w_0(\lambda)\to\overline{w}} = P_{\tilde{w}_0\to\overline{w}}$. The same argument shows that $P_{w_0(\lambda)\to\underline{w}} = P_{\tilde{w}_0\to\underline{w}}$, which completes the proof of the lemma.

□

Proposition B1: *The equilibrium of the game with any finite set Γ of possible contingency trigger points which contains $\overline{q}$ and $\underline{q}$, will yield the same transition probabilities as the baseline model.*

Proof: *From the baseline model where* $\Gamma = \{\underline{q}, \overline{q}\}$ *add one contingency trigger point q'. If $q' > \overline{q}$* or $q' < \underline{q}$ *then the state q' would never be reached in the baseline equilibrium. We can construct a parallel equilibrium where no agent chooses to have a contingency triggered at q'. Hence the edition of q' will not change the transition probabilities.*

If $q' \in (\underline{q}, \overline{q})$ then some agents may choose to set contingency triggers there. Let $\pi(q_T, t)$, henceforth π, be the probability that an individual agent chooses to set a contingency trigger at q'. This may be either to buy or to cancel an impending order. Note that π may depend on t since *for a given number of impending orders the time remaining in the period will determine the rate of information flow during the rest of the period which in tern influences the expected value of waiting. The same argument used for λ implies that $\pi<1$. If q' is a buy trigger and $\pi=1$ then each individual would prefer to wait since $q' < \overline{q}$. If it is a wait trigger and $\pi=1$ then each individual would prefer to buy since $q' > \underline{q}$.*

Moreover if the agents are using q' as a buy trigger then $r\pi n^H$ must be finite for all contingencies which may be hit with non-negligible probability. If not then if the contingency is triggered at any $t<T+1-\varepsilon$ the true value of n, and hence V, would be revealed with certainty by the end of the period. Since $q' < \overline{q}$ each individual would prefer to wait when the state will be revealed with certainty.

So either the addition of q' has no effect on the outcome in the period ($\pi=0$) or in equilibrium each individual will be indifferent between using it as a trigger or not and the number of agents with outstanding buy orders at each contingency trigger point will be drawn from a Poisson distribution with finite mean. So the addition of the contingency will cause λ to change when it is triggered. But by Lemma C1 this new process has the same transition probabilities as the original process. Iterating the argument allows us to add any finite number of contingency trigger points to the set Γ without altering the transition probabilities.

□

Appendix C:

Claim C1: $\dfrac{d\varphi^H}{d\psi} > 0$ *where* $\dfrac{d\varphi^H}{d\psi} = \dfrac{-2\psi + 2 + (1+\psi+2\phi)\ln\dfrac{\psi+\phi}{1+\phi}}{(\psi+\phi)(1+\phi)\left(\ln\dfrac{\psi+\phi}{1+\phi}\right)^3}$

Proof: *Define* $f(\psi) = -2\psi + 2 + (1+\psi+2\phi)\ln\dfrac{\psi+\phi}{1+\phi}$. *Note that $f(1)=0$.*

And $f'(\psi) = -2 + \dfrac{1+\psi+2\phi}{\psi+\phi} + \ln\dfrac{\psi+\phi}{1+\phi}$. *Note that* $f'(1) = 0$.

$f''(\psi) = \dfrac{\psi-1}{(\psi+\phi)^2} < 0$. *Hence* $f'(\psi) > 0$ *for $0<\psi<1$. So $f(\psi)<0$, so* $\dfrac{d\varphi^H}{d\psi} > 0$. □

Claim C2: $\dfrac{d\varphi^L}{d\psi} > 0$ *where* $\dfrac{d\varphi^L}{d\psi} = \dfrac{-2\psi + 2 + (1+\psi+2\phi)\ln\dfrac{\psi+\phi}{1+\phi}}{(\psi+\phi)^2\left(\ln\dfrac{\psi+\phi}{1+\phi}\right)^3}$

Proof: $\frac{d\varphi^L}{d\psi}$ *and* $\frac{d\varphi^H}{d\psi}$ *have the same numerator. In Lemma A3, it is shown that the numerator is negative. Since the denominator is negative as well,* $\frac{d\varphi^L}{d\psi} > 0$. □

Claim C3: *The first term of (27) is positive,* $\frac{\partial \Pr ob(INC)}{\partial w_0}\frac{\partial w_0}{\partial \psi} > 0$.

Proof: $\frac{\partial \Pr ob(INC)}{\partial w_0} = -\frac{2\varphi^H e^{2\varphi^H(\underline{w}-w_0)}}{1-e^{2\varphi^H(\underline{w}-\overline{w})}} < 0$, *since* $(1-e^{2\varphi^H(\underline{w}-\overline{w})}) > 0$. *And* $\frac{\partial w_0}{\partial \psi} = -\frac{1}{\psi} < 0$. □

Claim C4: $\frac{\partial \Pr ob(INC)}{\partial \varphi^H} < 0$ *where* $\frac{\partial \Pr ob(INC)}{\partial \varphi^H} = -\frac{1-e^{2\varphi^H(\underline{w}-w_0)}}{1-e^{2\varphi^H(\underline{w}-\overline{w})}}\left[\frac{2(\underline{w}-\overline{w})e^{2\varphi^H(\underline{w}-\overline{w})}}{1-e^{2\varphi^H(\underline{w}-\overline{w})}} - \frac{2(\underline{w}-w_0)e^{2\varphi^H(\underline{w}-w_0)}}{1-e^{2\varphi^H(\underline{w}-w_0)}}\right]$

Proof: *The term inside the brackets is positive. Examine the properties of the following function: Let* $g(x) = \frac{2(\underline{w}-x)e^{2\varphi^H(\underline{w}-x)}}{1-e^{2\varphi^H(\underline{w}-x)}}$ *where x>$\underline{w}$. Then notice that the term inside the brackets is equal to* $\left[g(\overline{w})-g(w_0)\right]$. *If* $\frac{dg(x)}{dx} > 0$, *then* $\left[g(\overline{w})-g(w_0)\right] > 0$ *and* $\frac{\partial \Pr ob(INC)}{\partial \varphi^H} < 0$.

$$\frac{dg(x)}{dx} = \frac{2e^{2\varphi^H(\underline{w}-x)}}{(1-e^{2\varphi^H(\underline{w}-x)})^2}\left(e^{2\varphi^H(\underline{w}-x)} - 1 - 2\varphi^H(\underline{w}-x)\right)$$

Let $z = 2\varphi^H(\underline{w}-x)$ *where z<0 for x∈(w_0, $\overline{w}$).*
The term in the parenthesis is then equal $f(z) = e^z - 1 - z$
f(z=0)=0. And $f'(z) = e^z - 1 < 0$ *for all z<0. Hence* $\frac{dg(x)}{dx} > 0$. *Therefore* $\frac{\partial \Pr ob(INC)}{\partial \varphi^H} < 0$. □

Claim C5: *The first term of (29) is negative,* $\frac{\partial \Pr ob(IPC)}{\partial w_0}\frac{\partial w_0}{\partial \psi} < 0$.

Proof: $\frac{\partial \Pr ob(IPC)}{\partial w_0} = \frac{2\varphi^L e^{2\varphi^L(\underline{w}-w_0)}}{1-e^{2\varphi^L(\underline{w}-\overline{w})}} > 0$ *since* φ^L*<0 and* $(1-e^{2\varphi^L(\underline{w}-\overline{w})}) < 0$. *And* $\frac{\partial w_0}{\partial \psi} = -\frac{1}{\psi} < 0$. □

Claim C6: $\frac{\partial \Pr ob(IPC)}{\partial \varphi^L} > 0$ *where* $\frac{\partial \Pr ob(IPC)}{\partial \varphi^L} = \frac{1 - e^{2\varphi^L(\underline{w}-w_0)}}{1 - e^{2\varphi^L(\underline{w}-\overline{w})}} \left[\frac{2(\underline{w}-\overline{w})e^{2\varphi^L(\underline{w}-\overline{w})}}{1-e^{2\varphi^L(\underline{w}-\overline{w})}} - \frac{2(\underline{w}-w_0)e^{2\varphi^L(\underline{w}-w_0)}}{1-e^{2\varphi^L(\underline{w}-w_0)}} \right]$

Proof: *The term inside the brackets is positive. Examine the properties of the following function:*

Let $g(x) = \frac{2(\underline{w}-x)e^{2\varphi^L(\underline{w}-x)}}{1-e^{2\varphi^L(\underline{w}-x)}}$ *where* $x>\underline{w}$ *then notice that the term inside the brackets is equal to* $\left[g(\overline{w}) - g(w_0) \right]$. *If* $\frac{dg(x)}{dx} > 0$, *then* $\frac{\partial \Pr ob(IPC)}{\partial \varphi^L} > 0$ *since* $\overline{w} > w_0$.

$$\frac{dg(x)}{dx} = \frac{2e^{2\varphi^L(\underline{w}-x)}}{(1-e^{2\varphi^L(\underline{w}-x)})^2}\left(e^{2\varphi^L(\underline{w}-x)} - 1 - 2\varphi^L(\underline{w}-x)\right)$$

Let $z = 2\varphi^L(\underline{w}-x)$ *where* $z>0$ *for* $x \in (w_0, \overline{w})$.

The term in the parenthesis is then equal $f(z) = e^z - 1 - z$

$f(z=0)=0$. *And* $f'(z) = e^z - 1 > 0$ *for all* $z>0$. *Hence* $\frac{dg(x)}{dx} > 0$. *Therefore* $\frac{\partial \Pr ob(IPC)}{\partial \varphi^L} > 0$

□

Proof of Proposition 3 (Prior Expected Value):

The prior expected value increases due to $q^* \uparrow$ *or* $V^H \uparrow$ *or* $V^L \uparrow$.

First notice that $\left(1 - e^{2\varphi^H \ln(1-\delta)}\right) > 0$ *since* $\varphi^H>0$ *and* $\left(1 - e^{2\varphi^L \ln(1-\delta)}\right) < 0$ *since* $\varphi^L<0$.

The comparative statics are as follows:

$$\frac{d \Pr ob(INC)}{dq^*} < 0 \quad and \quad \frac{d \Pr ob(IPC)}{dq^*} > 0$$

$$\frac{d \Pr ob(INC)}{dV^H} < 0 \quad and \quad \frac{d \Pr ob(IPC)}{dV^H} > 0$$

$$\frac{d \Pr ob(INC)}{dV^L} < 0 \quad and \quad \frac{d \Pr ob(IPC)}{dV^L} > 0$$

Comparative statics with respect to q:*

$$\frac{dProb(INC)}{dq^*} = \frac{-2\varphi^H e^{2\varphi^H\left[ln(1-q^*)-lnq^*+ln\psi-lnV^H+ln(-V^L)\right]}}{(1-q^*)q^*(1-e^{2\varphi^H ln(1-\delta)})} \ and \ \frac{dProb(IPC)}{dq^*} = \frac{2\varphi^L e^{2\varphi^L\left[ln(1-q^*)-lnq^*+ln\psi-lnV^H+ln(-V^L)\right]}}{(1-q^*)q^*(1-e^{2\varphi^L ln(1-\delta)})}$$

Comparative statics with respect to V^H:

$$\frac{dProb(INC)}{dV^L} = \frac{2\varphi^H e^{2\varphi^H\left[ln(1-q^*)-lnq^*+ln\psi-lnV^H+ln(-V^L)\right]}}{(1-e^{2\varphi^H ln(1-\delta)})V^L} \ and \ \frac{dProb(IPC)}{dV^H} = \frac{2\varphi^L e^{2\varphi^L\left[ln(1-q^*)-lnq^*+ln\psi-lnV^H+ln(-V^L)\right]}}{(1-e^{2\varphi^L ln(1-\delta)})V^H}$$

Comparative statics with respect to V^L:

$$\frac{dProb(INC)}{dV^L} = \frac{2\varphi^H e^{2\varphi^H\left[ln(1-q^*)-lnq^*+ln\psi-lnV^H+ln(-V^L)\right]}}{(1-e^{2\varphi^H ln(1-\delta)})V^L} \ and \ \frac{dProb(IPC)}{dV^L} = \frac{-2\varphi^L e^{2\varphi^L\left[ln(1-q^*)-lnq^*+ln\psi-lnV^H+ln(-V^L)\right]}}{(1-e^{2\varphi^L ln(1-\delta)})V^L}$$

□

Proof of Proposition 5 (Signal Quality):

Step 1: $\frac{d\,\Pr ob(INC)}{d\psi} > 0$*, when* $w_0 \rightarrow \underline{w}$. $\frac{d\,\Pr ob(INC)}{d\psi} < 0$ *when* w_0 *is not to close to either of the bounds,* $\underline{w} << w_0 << \overline{w}$.

Proof:

i) $\frac{d\,\Pr ob(INC)}{d\psi} > 0$*, when* $w_0 \rightarrow \underline{w}$.

w_0 is a function of q. All other terms,* $\underline{w}$, $\overline{w}$, μ^H and *$(\sigma^2)^H$ in (20) are invariant of q*.* $w_0 \rightarrow \underline{w}$ *is equivalent to* $\ln\frac{q*}{1-q*} \rightarrow \ln(-V^L) - \ln(V^H) + \ln\psi$ *. When* $w_0 \rightarrow \underline{w}$ *the second term of (27), goes to zero. The first term remains positive. Hence* $\frac{d\,\Pr ob(INC)}{d\psi} > 0$ *when* $w_0 \rightarrow \underline{w}$ *.*

ii) $\frac{d\Pr ob(INC)}{d\psi} < 0$ *when* w_0 *is not to close to either of the bounds,* $\underline{w} << w_0 << \overline{w}$.

$$\frac{d\Pr ob(INC)}{d\psi} = \left(\frac{\partial \Pr ob(INC)}{\partial w_0}\frac{\partial w_0}{\partial \psi}\right) + \left(\frac{\partial \Pr ob(INC)}{\partial \varphi^H}\frac{\partial \varphi^H}{\partial \psi}\right)$$

First examine,

$$\frac{\partial \Pr ob(INC)}{\partial \varphi^H} = -\frac{1 - e^{2\varphi^H(\underline{w}-w_0)}}{1 - e^{2\varphi^H(\underline{w}-\overline{w})}}\left[\frac{2(\underline{w}-\overline{w})e^{2\varphi^H(\underline{w}-\overline{w})}}{1-e^{2\varphi^H(\underline{w}-\overline{w})}} - \frac{2(\underline{w}-w_0)e^{2\varphi^H(\underline{w}-w_0)}}{1-e^{2\varphi^H(\underline{w}-w_0)}}\right]$$

Let $g(x) = \frac{2(\underline{w}-x)e^{2\varphi^H(\underline{w}-x)}}{1-e^{2\varphi^H(\underline{w}-x)}}$ *where* $x > \underline{w}$. *Notice that* $g(x) < 0$ *for* $x > \underline{w}$ *and* $\frac{dg(x)}{dx} > 0$.

Pick $\delta = \delta^*$ *such that* $g(\overline{w}) = \frac{1}{2}g(w_0)$. *Then* $\left[g(\overline{w}) - g(w_0)\right] = -\frac{1}{2}g(w_0) > 0$.

For δ^*:

$$\frac{\partial \Pr ob(INC)}{\partial \varphi^H} = \frac{1 - e^{2\varphi^H(\underline{w}-w_0)}}{1 - e^{2\varphi^H(\underline{w}-\overline{w})}}\left[\frac{(\underline{w}-w_0)e^{2\varphi^H(\underline{w}-w_0)}}{1-e^{2\varphi^H(\underline{w}-w_0)}}\right]$$

From Claim C1 and C3:

$$\frac{d\Pr ob(INC)}{d\psi} = \left\{\frac{2\varphi^H e^{2\varphi^H(\underline{w}-w_0)}}{\psi(1 - e^{2\varphi^H(\underline{w}-\overline{w})})} + \left(\frac{\partial \varphi^H}{\partial \psi}\right)\frac{(\underline{w}-w_0)e^{2\varphi^H(\underline{w}-w_0)}}{(1 - e^{2\varphi^H(\underline{w}-\overline{w})})}\right\}$$

Let us now factor out $\frac{e^{2\varphi^H(\underline{w}-w_0)}}{(1 - e^{2\varphi^H(\underline{w}-\overline{w})})} > 0$. *We then have,*

$$\frac{d\Pr ob(INC)}{d\psi} = \frac{e^{2\varphi^H(\underline{w}-w_0)}}{(1 - e^{2\varphi^H(\underline{w}-\overline{w})})}\left\{\frac{2\varphi^H}{\psi} + \left(\frac{\partial \varphi^H}{\partial \psi}\right)(\underline{w}-w_0)\right\}$$

The multiplicative term outside the parenthesis is positive. Inside the parentheses $\varphi^H > 0$ and $\frac{\partial \varphi^H}{\partial \psi} > 0$ *by Claim C1. By picking* q^* *we can have* w_0 *as big as we like and we can still maintain* $g(\overline{w}) = \frac{1}{2}g(w_0)$ *for a big enough* δ. *Hence* $\frac{d\Pr ob(INC)}{d\psi} < 0$ *for* $\underline{w} << w_0 << \overline{w}$.

□

Step 2: $\frac{d\,\mathrm{Prob}(IPC)}{d\psi}$ *<0, when* $w_0 \to \overline{w}$. $\frac{d\,\mathrm{Prob}(IPC)}{d\psi}$ *>0 when* w_0 *is not to close to either of the bounds,* $\underline{w} << w_0 << \overline{w}$.

Proof:

i) $\frac{d\,\mathrm{Prob}(IPC)}{d\psi}$ *<0, when* $w_0 \to \overline{w}$. w_0 *is a function of q*. All other terms,* $\underline{w}$, $\overline{w}$, μ^H and $(\sigma^2)^H$ *in (20) are invariant of q*.* $w_0 \to \overline{w}$ *is equivalent to:*

$\ln \frac{q^*}{1-q^*} \to \ln(-V^L) - \ln(V^H) - \ln(1-\delta) + \ln\psi$. *When* $w_0 \to \overline{w}$ *the second term of (29) goes to zero.*

The first term of (29) remains negative. Hence $\frac{d\,\mathrm{Prob}(IPC)}{d\psi}$ *<0.*

ii) $\frac{d\,\mathrm{Prob}(IPC)}{d\psi}$ *>0 when* w_0 *is not to close to either of the bounds,* $\underline{w} << w_0 << \overline{w}$.

$$\frac{d\,\mathrm{Prob}(IPC)}{d\psi} = \left(\frac{\partial\,\mathrm{Prob}(IPC)}{\partial w_0}\frac{\partial w_0}{\partial\psi}\right) + \left(\frac{\partial\,\mathrm{Prob}(IPC)}{\partial\varphi^L}\frac{\partial\varphi^L}{\partial\psi}\right)$$

First examine,

$$\frac{\partial\,\mathrm{Prob}(IPC)}{\partial\varphi^L} = \frac{1 - e^{2\varphi^L(\underline{w}-w_0)}}{1 - e^{2\varphi^L(\underline{w}-\overline{w})}}\left[\frac{2(\underline{w}-\overline{w})e^{2\varphi^L(\underline{w}-\overline{w})}}{1-e^{2\varphi^L(\underline{w}-\overline{w})}} - \frac{2(\underline{w}-w_0)e^{2\varphi^L(\underline{w}-w_0)}}{1-e^{2\varphi^L(\underline{w}-w_0)}}\right]$$

Let g(x)= $\frac{2(\underline{w}-x)e^{2\varphi^L(\underline{w}-x)}}{1-e^{2\varphi^L(\underline{w}-x)}}$ *where* $x > \underline{w}$ *.notice that g(x)>0 for* $x > \underline{w}$ *and* $\frac{dg(x)}{dx} > 0$.

Pick $\delta = \delta^*$ *such that* $g(\overline{w}) = \frac{1}{2}g(w_0)$. *Then* $\left[g(\overline{w}) - g(w_0)\right] = -\frac{1}{2}g(w_0) < 0$.

For δ:*

$$\frac{\partial \Pr ob(IPC)}{\partial \varphi^L} = \frac{1 - e^{2\varphi^L(\underline{w}-w_0)}}{1 - e^{2\varphi^L(\underline{w}-\overline{w})}}\left[\frac{(\underline{w}-w_0)e^{2\varphi^L(\underline{w}-w_0)}}{1-e^{2\varphi^L(\underline{w}-w_0)}}\right]$$

From Claim C2 and C4:

$$\frac{d \Pr ob(INC)}{d\psi} = \left\{-\frac{2\varphi^L e^{2\varphi^L(\underline{w}-w_0)}}{\psi(1 - e^{2\varphi^L(\underline{w}-\overline{w})})} + \left(\frac{\delta\varphi^L}{\delta\psi}\right)\frac{(\underline{w}-w_0)e^{2\varphi^L(\underline{w}-w_0)}}{(1 - e^{2\varphi^L(\underline{w}-\overline{w})})}\right\}$$

Let us now factor out $\frac{e^{2\varphi^L(\underline{w}-w_0)}}{(1 - e^{2\varphi^L(\underline{w}-\overline{w})})}$. *We then have:*

$$\frac{d \Pr ob(INC)}{d\psi} = \frac{e^{2\varphi^L(\underline{w}-w_0)}}{(1 - e^{2\varphi^L(\underline{w}-\overline{w})})}\left\{-\frac{2\varphi^L}{\psi} + \left(\frac{\delta\varphi^L}{\delta\psi}\right)(\underline{w}-w_0)\right\}$$

The term outside the parenthesis is negative. Inside the parentheses $\varphi^L < 0$, *and* $\frac{\delta\varphi^L}{\delta\psi} > 0$. *By picking q* we can have* w_0 *as large as we like and we can still maintain* $g(\overline{w}) = \frac{1}{2} g(w_0)$ *for a high enough δ.*

Hence $\frac{d \Pr ob(IPC)}{d\psi} > 0$ *for* $\underline{w} << w_0 << \overline{w}$.

□