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Commuting Involution Graphs for $\tilde{A}_n$

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Abstract

In this article we consider the commuting graphs of involution conjugacy classes in the affine Weyl group $\tilde{A}_n$. We show that where the graph is connected the diameter is at most 6.
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1 Introduction

Let G be a group and X a subset of G. The commuting graph on X, denoted $\mathcal{C}(G, X)$, has vertex set X and an edge joining $x, y \in X$ whenever $xy = yx$. If in addition X is a set of involutions, then $\mathcal{C}(G, X)$ is called a commuting involution graph. Commuting graphs have been investigated by many authors. Sometimes they are tools used in the proof of a theorem, or they may be studied as a way of shedding light on the structures of certain groups (as in [1]). Commuting involution graphs for the case where X is a conjugacy class of involutions were studied by Fischer [4] – in that case X was the class of 3-transpositions of a 3-transposition group. These groups include all finite simply laced Weyl groups, in particular the symmetric group.

Commuting involution graphs for arbitrary involution conjugacy classes of symmetric groups were considered in [2]. The remaining finite Coxeter groups were dealt with in [3]. In this article we consider commuting involution graphs in the affine Coxeter group of type $\tilde{A}_n$. As in [2] and [3], we will focus on the diameter of these graphs. We show that if X is a conjugacy class of involutions, then either the graph is disconnected or it has diameter at most 6.

For the rest of this paper, let G_n denote $\tilde{A}_{n-1}$, for some $n \geq 2$, writing G when n is not specified, and let X be a conjugacy class of involutions of G. We write Diam $\mathcal{C}(G, X)$ for the diameter of $\mathcal{C}(G, X)$ (when it is connected). Let $\hat{G}$ be the underlying Weyl group A_{n-1} of G. It will be shown that every conjugacy class X of G corresponds to a certain conjugacy class $\hat{X}$ of $\hat{G}$. We may now state our main results (notation will be explained in Section 3).

Theorem 1.1 *Let $G = G_n \cong \tilde{A}_{n-1}$ and $a = (12)(34)\cdots(2m-1\ 2m) \in \hat{X}$. Then $\mathcal{C}(G, X)$ is disconnected if and only if either $n = 2m+1$, or $m = 1$ and $n \in \{2, 4\}$.*

Theorem 1.2 *Suppose $\mathcal{C}(G, X)$ is connected. If $n > 2m$ or m is even, then*

$$\text{Diam } \mathcal{C}(G, X) \leq \text{Diam } \mathcal{C}(\hat{G}, \hat{X}) + 2.$$

If $n = 2m$ and m is odd, then Diam $\mathcal{C}(G, X) \leq$ Diam $\mathcal{C}(\hat{G}, \hat{X}) + 3$.

Using results about commuting involution graphs in A_{n-1} (see Section 2) we can then deduce the following result.

Corollary 1.3 *Let $G = G_n \cong \tilde{A}_{n-1}$ and $a = (12)(34)\cdots(2m-1\ 2m)$. Suppose $\mathcal{C}(G, X)$ is connected.*
(i) If $n \neq 2m+2$ or $n > 10$, then Diam $\mathcal{C}(G, X) \leq 5$.
(ii) If $n = 2m+2$ and $n = 6, 8$ or 10 then Diam $\mathcal{C}(G, X) \leq 6$.

In Section 2 we will establish notation, describe the conjugacy classes of involutions in G and state results which we will require. Section 3 is devoted to proving Theorem 1.2. In Section 4 we give examples of commuting involution graphs which show that the bounds of Theorem 1.2 are strict.

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Remark In the case of finite Weyl groups, given any conjugacy class X of a finite Weyl group W, it was shown in [3] that if $\mathcal{C}(G,X)$ is connected, then Diam $\mathcal{C}(G,X) \leq 5$. It is natural to ask whether there is a similar bound in the case of affine Weyl groups. The answer is no. Let $W \cong \tilde{B}_n$, and let W_I be a standard parabolic subgroup of G such that W_I has type B_{n-1}. Let w_I be the central involution of W_I, and set $X = w_I^W$. It can be shown that Diam $\mathcal{C}(G,X) = n$. Thus the set of diameters of commuting involution graphs is unbounded.

2 The group $G_n \cong \tilde{A}_{n-1}$

Let W be a finite Weyl group with root system Φ and let $\check{\Phi}$ denote the set of coroots. (For full details, see for example [5].) The affine Weyl group $\tilde{W}$ is the semidirect product of W with the translation group Z of the coroot lattice $\mathbb{Z}\check{\Phi}$ of W.
Elements of $\tilde{W}$ are written as pairs (w,z), for $w \in W, z \in Z$. Multiplication is given by

$$(\sigma,\mathbf{v})(\tau,\mathbf{u}) = (\sigma\tau, \mathbf{v}^\tau + \mathbf{u}).$$

We now fix $W = A_{n-1}$. Then $W \cong \mathrm{Sym}(n)$, the symmetric group of degree n. W acts on $\mathbb{R}_n = \langle \varepsilon_1, \varepsilon_2, \dots \varepsilon_n\rangle$ by permuting the subscripts of the basis vectors. The root system Φ of W is the set $\{\pm(\varepsilon_i - \varepsilon_j) : 1 \leq i < j \leq n\}$, and in this case $\check{\Phi} = \Phi$. Writing a translation by $\sum_{i=1}^n \lambda_i\varepsilon_i$ as $(\lambda_1, \dots, \lambda_n)$, we see that

$$\begin{aligned} Z &= \langle (0,\dots,0,1,0,\dots,0,-1,0,\dots,0)\rangle \\ &= \langle (u_1,\dots,u_n) : \sum_{i=1}^n u_i = 0\rangle. \end{aligned}$$

2.1 Involutions in G_n

By the definition of group multiplication in G_n, we see that the element $(\sigma,\mathbf{v})$ of G is an involution precisely when $(\sigma^2, \mathbf{v}^\sigma + \mathbf{v}) = (1,\mathbf{0})$. So σ is an involution of $\mathrm{Sym}(n)$ and for appropriate a_i, b_i, c_i and m,

$$\sigma = (a_1b_1)\cdots(a_mb_m)(c_{2m+1})(c_{2m+2})\cdots(c_n).$$

Setting $\mathbf{v} = (v_1, \cdots, v_n)$ we must have

$$v_{a_1} + v_{b_1} = \cdots = v_{a_m} + v_{b_m} = 2v_{c_{2m+1}} = \cdots = 2v_{c_n} = 0.$$

Hence we have the following lemma:

Lemma 2.1 *Any involution in G_n is of the form $(\sigma,\mathbf{v})$, where*

$$\sigma = (a_1b_1)\cdots(a_mb_m)(c_{2m+1})(c_{2m+2})\cdots(c_n),$$

with $v_{b_i} = -v_{a_i}$ for $1 \leq i \leq m$ and $v_{c_i} = 0$ for $2m+1 \leq i \leq n$.

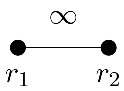
It will be convenient to use a more compact notation for involutions of G. Let $g = (\prod_{i=1}^m(\alpha_i\beta_i), \mathbf{v})$ with $\alpha_i, \beta_i \in \{1,\dots,n\}$ for $1 \leq i \leq m$. Then, by Lemma 2.1, $v_{\beta_i} = -v_{\alpha_i}$, and if $j \notin \{\alpha_1,\dots,\alpha_m,\beta_1,\dots,\beta_m\}$, then $v_j = 0$. Thus $\mathbf{v}$ is determined from the set $\lambda_i := v_{\alpha_i}$, $1 \leq i \leq m$. We may therefore write

$$g = \prod_{i=1}^m \overset{\lambda_i}{(\alpha_i\beta_i)}.$$

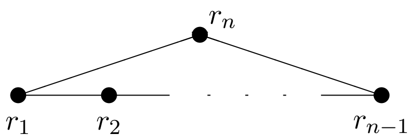
2.2 Conjugacy classes of Involutions

We now describe the conjugacy classes of involutions in G_n. Conjugacy classes of involutions in Coxeter groups are well understood and in order to use the known results we must give another description of G_n, this time in terms of its Coxeter graph. A Coxeter group W has a generating set R of involutions (known as the fundamental reflections), where the only relations are $(rs)^{m_{rs}} = 1$ $(r, s \in R)$, with $m_{rr} = 1$ and, for $r \neq s$, $m_{rs} = m_{sr} \geq 2$. This information is encoded in the Coxeter graph $\Gamma = \Gamma(W)$. The vertex set of Γ is R, where vertices r, s are joined by an edge labelled m_{rs} whenever $m_{rs} > 2$. By convention the label is omitted when $m_{rs} = 3$. The Coxeter graphs of $G_2 \cong \tilde{A}_1$ and $G_n \cong \tilde{A}_{n-1}, n \geq 3$ are as follows:

G_2



$G_n (n \geq 3)$



We may define $r_n = (\overset{1}{1n})$, and for $1 \leq i \leq n-1$, $r_i = (\overset{0}{i\; i+1})$ (using the notation defined in Section 2.2). It is not difficult to see that the appropriate relations hold.
The symmetric group $\mathrm{Sym}(n)$ is a Coxeter group of type A_{n-1}, with Coxeter graph

$\mathrm{Sym}(n) \cong A_{n-1}$

We may set $r_i = (i\; i+1)$ for $1 \leq i \leq n-1$.

Definition 2.2 *Let W be an arbitrary Coxeter group, with I, J two subsets of R. We say that I, J are W-equivalent if there exists $w \in W$ such that $I^w = J$.*

Any subset I of R generates a Coxeter group in its own right, denoted W_I. Such subgroups are called standard parabolic subgroups of W. If W_I is finite then it has a unique longest element, denoted w_I. Richardson [6] proved

Theorem 2.3 *Let W be an arbitrary Coxeter group, with R the set of fundamental reflections. Let $g \in W$ be an involution. Then there exists $I \subseteq R$ such that w_I is central in W_I, and g is conjugate to w_I. In addition, for $I, J \subseteq R$, w_I is conjugate to w_J if and only if I and J are W-equivalent.*

It will be useful to narrow down the possible elements in the conjugacy class of involutions $(a, \mathbf{u})$ in the case where a is an involution of $\mathrm{Sym}(n)$ with no fixed points.

Lemma 2.4 *Suppose $n = 2m$. Let $a = \prod_{i=1}^{m}(\alpha_i\, \beta_i)$ and $b = \prod_{i=1}^{m}(\gamma_i\, \delta_i)$. Suppose $g = (a, \mathbf{u})$ and $h = (b, \mathbf{v})$ are conjugate involutions of G_n. Then $\sum_{i=1}^{m} u_{\alpha_i} \equiv \sum_{i=1}^{m} v_{\gamma_i} \mod 2$.*

Proof Let $g = (a, \mathbf{u})$, and suppose $h = (b, \mathbf{v})$ is conjugate to g in G_n via $(c, \mathbf{w})$. Reordering if necessary, assume that $c(\alpha_i) = \gamma_i$ and $c(\beta_i) = \delta_i$ for $1 \leq i \leq m$. We see that

$$\begin{aligned}(b, \mathbf{v}) &= (a, \mathbf{u})^{(c,\mathbf{w})} \\ &= (c^{-1}ac, (\mathbf{w}^{-1})^{c^{-1}ac} + \mathbf{u}^c + \mathbf{w}).\end{aligned}$$

Thus $b = c^{-1}ac$ and $\mathbf{v} = \mathbf{w} - \mathbf{w}^b + \mathbf{u}^c$. Hence, for $1 \leq i \leq m$, $v_{\gamma_i} = w_{\gamma_i} - w_{b(\gamma_i)} + [\mathbf{u}^c]_{\gamma_i}$. Since $c(\alpha_i) = \gamma_i$, it follows that $[\mathbf{u}^c]_{\gamma_i} = u_{\alpha_i}$. Hence, recalling that $\sum_{j=1}^{n} w_j = 0$,

$$
\begin{aligned}
\sum_{i=1}^{m} v_{\gamma_i} &= \sum_{i=1}^{m} (w_{\gamma_i} - w_{\delta_i} + u_{\alpha_i}) = \sum_{i=1}^{m} (w_{\gamma_i} + w_{\delta_i} - 2w_{\delta_i} + u_{\alpha_i}) \\
&= \sum_{j=1}^{n} w_j - 2\sum_{i=1}^{m} w_{\delta_i} + \sum_{i=1}^{m} u_{\alpha_i} \equiv \sum_{i=1}^{m} u_{\alpha_i} \mod 2.
\end{aligned}
$$

Therefore $\sum_{i=1}^{m} u_{\alpha_i} \equiv \sum_{i=1}^{m} v_{\gamma_i} \mod 2$, and the result holds. □

We use Theorem 2.3 to establish the next result.

Proposition 2.5 *Let $g \in G$ be an involution. Then there is $m \in \mathbb{Z}^+$ such that g is conjugate to exactly one of the following:*

$$
\begin{aligned}
&\overset{0}{(12)} \cdots \overset{0}{(2m-1\ 2m)};\ \textit{or} \\
&\overset{1}{(12)}\overset{0}{(34)} \cdots \overset{0}{(2m-1\ 2m)}\ (\textit{and } n = 2m).
\end{aligned}
$$

If $n = 2m$ and $g = \prod_{i=1}^{m} \overset{\lambda_i}{(\alpha_i\, \beta_i)}$, then g is conjugate to $\overset{0}{(12)} \cdots \overset{0}{(2m-1\ 2m)}$ if and only if $\sum_{i=1}^{m} \lambda_i \equiv 0$ mod 2.

Proof By Theorem 2.3, g is conjugate to w_I for some finite standard parabolic subgroup W_I of W in which w_I is central. Note that if g is also conjugate to w_J for some $J \subseteq R$ then $|I| = |J|$, so that $|I|$ only depends on g and not the particular choice of I. For any proper subset $I \subsetneq R$, we see that W_I is isomorphic to a direct product of symmetric groups. The only symmetric group with non-trivial centre is $\mathrm{Sym}(2) \cong A_1$. So for w_I to be central, W_I must be a direct product of symmetric groups of degree 2, and $w_I = r_{i_1} r_{i_2} \cdots r_{i_l}$ for some l, where $r_{i_j} r_{i_k} = r_{i_k} r_{i_j}$ for $1 \leq j < k \leq l$. This immediately implies that $|I| \leq n/2$. Suppose that w_I and w_J are central in W_I, W_J respectively, and, in addition, that there exists $r \in R \backslash (I \cup J)$. Set $K = R \backslash \{r\}$. Then K is isomorphic to $\mathrm{Sym}(n)$. It is well known that conjugacy classes in the symmetric group are parameterised by cycle type, so that w_I is conjugate to w_J precisely when $|I| = |J|$. Therefore, in the case $|I| < n/2$, we may assume that $I = \{r_1, r_3, \cdots, r_{2m-1}\}$ for some $m < n/2$, so that $w_I = \overset{0}{(12)} \cdots \overset{0}{(2m-1\ 2m)}$, with $2m < n$.

It only remains to consider the case $|I| = n/2$ (and then of course n must be even). We quickly see that there are only two possibilities for I such that w_I is central in W_I. Either $I = \{r_1, r_3, \ldots, r_{n-1}\}$ and $w_I = g_1 := \overset{0}{(12)} \cdots \overset{0}{(2m-1\ 2m)}$, or $I = \{r_2, r_4, \ldots, r_n\}$ and $w_I = g_2 := \overset{1}{(1n)}\overset{0}{(23)} \cdots \overset{0}{(2m-2\ 2m-1)}$. By Lemma 2.4, g_1 is not conjugate to g_2. Hence g is conjugate to exactly one of g_1 and g_2. By Lemma 2.4, $g_3 := \overset{1}{(12)}\overset{0}{(34)} \cdots \overset{0}{(2m-1\ 2m)}$ must be conjugate to g_2. Hence $g = \prod_{i=1}^{m} \overset{\lambda_i}{(\alpha_i\, \beta_i)}$ is conjugate to exactly one of g_1 and g_3. Furthermore g is conjugate to g_1 if and only if $\sum_{i=1}^{n} \lambda_i \equiv 0 \mod 2$. We have now proved Proposition 2.5. □

If $n = 2m$, there are two conjugacy classes. However, there is an automorphism of the Coxeter graph of G mapping $\{r_1, r_3, \ldots, r_{n-1}\}$ to $\{r_2, r_4, \ldots, r_n\}$. This induces an automorphism of G which maps one conjugacy class to the other. Hence the two conjugacy classes have isomorphic commuting involution graphs. Thus we may assume that g is conjugate to $\overset{0}{(12)} \cdots \overset{0}{(2m-1\ 2m)}$.
We end this section by stating some results from [2] concerning the diameters of commuting involution graphs in $\mathrm{Sym}(n)$.
Let $a = (12)(34) \cdots (2m-1\ 2m) \in \mathrm{Sym}(n)$ and write $Y = a^{\mathrm{Sym}(n)}$.

Theorem 2.6 *(Theorem 1.1 of [2]) $\mathcal{C}(\mathrm{Sym}(n), Y)$ is disconnected if and only if $n = 2m + 1$ or $n = 4$ and $m = 1$.*

Proposition 2.7 *(Corollary 3.2 of [2]) If $n = 2m$, then $\mathcal{C}(\mathrm{Sym}(n), Y)$ is connected and* Diam $\mathcal{C}(\mathrm{Sym}(n), Y) \leq 2$, *with equality when $n > 4$.*

Theorem 2.8 *(Theorem 1.2 of [2]) Suppose that $\mathcal{C}(\mathrm{Sym}(n), Y)$ is connected. Then one of the following holds:*

(i) Diam $\mathcal{C}(\mathrm{Sym}(n), Y) \leq 3$; *or*

(ii) $2m + 2 = n \in \{6, 8, 10\}$ *and* Diam $\mathcal{C}(\mathrm{Sym}(n), Y) = 4$.

3 Proof of Theorems 1.1 and 1.2

From now on, fix $a = (12)\cdots(2m-1\ 2m)$, where $2m \leq n$, and set $t = (a, \mathbf{0}) = \overset{0}{(12)}\cdots\overset{0}{(2m-1\ 2m)}$ and $X = t^G$. As we have observed, every commuting involution graph of G is isomorphic to $\mathcal{C}(G, X)$ for an appropriate choice of m. Write $\hat{G} = \mathrm{Sym}(n)$ and $\hat{X} = a^{\hat{G}}$. Finally, if $g = \prod_{i=1}^{m} \overset{\lambda_i}{(\alpha_i\beta_i)} \in X$, then set $\hat{g} = \prod_{i=1}^{m}(\alpha_i\beta_i) \in \hat{G}$. Clearly if $g, h \in X$, then $\hat{g}, \hat{h} \in \hat{X}$. We begin with the following lemma.

Lemma 3.1 *Suppose $g, h \in X$. If $d(\hat{g}, \hat{h}) = k$, then $d(g, h) \geq k$. If $\mathcal{C}(\hat{G}, \hat{X})$ is disconnected, then $\mathcal{C}(G, X)$ is disconnected.*

Proof Observe that if σ commutes with τ in G_n then $\hat{\sigma}$ commutes with $\hat{\tau}$ in $\mathrm{Sym}(n)$. The lemma follows. □

Lemma 3.2 *Let $g_1 = \overset{\lambda_1}{(\alpha\beta)}\overset{\lambda_2}{(\gamma\delta)}$, $g_2 = \overset{\mu_1}{(\alpha\gamma)}\overset{\mu_2}{(\beta\delta)}$, $g_3 = \overset{\lambda_1}{(\alpha\beta)}$, $g_4 = \overset{\lambda_2}{(\alpha\beta)}$ for distinct $\alpha, \beta, \gamma, \delta$ in $\{1, \ldots, n\}$ and integers λ_i, μ_i. Then*

(a) $g_1g_2 = g_2g_1$ if and only if $\mu_1 - \lambda_1 = \mu_2 - \lambda_2$;

(b) $g_3g_4 = g_4g_3$ if and only if $\lambda_1 = \lambda_2$;

(c) If $h \in G$ is an involution such that $\hat{h}(\alpha) = \alpha$ and $\hat{h}(\beta) = \beta$, then $g_3h = hg_3$ for all $\lambda_1 \in \mathbb{Z}$.

Proof For part (a), we lose no generality by assuming, for ease of notation, that $n = 4$, $g_1 = \overset{\lambda_1}{(12)}\overset{\lambda_2}{(34)}$ and $g_2 = \overset{\mu_1}{(13)}\overset{\mu_2}{(24)}$. That is, $g_1 = ((12)(34), (\lambda_1, -\lambda_1, \lambda_2, -\lambda_2))$ and $g_2 = ((13)(24), (\mu_1, \mu_2, -\mu_1, -\mu_2))$. Hence

$$\begin{aligned} g_1g_2 &= ((14)(23), (\lambda_1, -\lambda_1, \lambda_2, -\lambda_2)^{(13)(24)} + (\mu_1, \mu_2, -\mu_1, -\mu_2)) \\ &= ((14)(23), (\mu_1 + \lambda_2, \mu_2 - \lambda_2, -\mu_1 + \lambda_1, -\mu_2 - \lambda_1)). \end{aligned}$$

Now g_1 and g_2 commute if and only if g_1g_2 is an involution. This occurs if and only if $\mu_1 + \lambda_2 = -(\mu_2 - \lambda_1)$ and $\mu_2 - \lambda_2 = -(-\mu_1 + \lambda_1)$. Rearranging gives $\mu_1 - \lambda_1 = \mu_2 - \lambda_2$, as required.
For part (b), we may assume that $n = 2$, $g_3 = ((12), (\lambda_1, -\lambda_1))$ and $g_4 = ((12), (\lambda_2, -\lambda_2))$. Then $g_3g_4 = (1, (-\lambda_1 + \lambda_2, \lambda_1 - \lambda_2))$. Hence $g_3g_4 = g_4g_3$ if and only if $\lambda_1 = \lambda_2$.
For part (c), we again assume that $g_3 = ((12), (\lambda_1, -\lambda_1))$, and write $h = (b, (v_1, \ldots, v_2))$. Since b fixes 1 and 2 and h is an involution, we must have $v_1 = v_2 = 0$. Hence $hg_3 = ((12)b, (\lambda_1, -\lambda_1, v_3, \ldots, v_n) = g_3h$. This completes the proof of Lemma 3.2. □

We may now dispose of the case $n = 2$.

Proposition 3.3 *Let $G = G_2 \cong \tilde{A}_1$. Then there are two conjugacy classes of involutions, representatives of which are $\overset{0}{(12)}$ and $\overset{1}{(12)}$. In either case $\mathcal{C}(G, X)$ is completely disconnected (the graph has no edges).*

A double transposition $\overset{\lambda_1}{(\alpha_1 \beta_1)}\overset{\lambda_2}{(\alpha_2 \beta_2)}$ for which $\lambda_1 + \lambda_2$ is even is called an *even pair*. Otherwise it is an *odd pair*.

Proposition 3.4 *Suppose $n > 2$ and that $\mathcal{C}(\hat{G}, \hat{X})$ is connected. Let $g \in X$. If $n > 2m$ or m is even, then there exists $h = (c, \mathbf{0}) \in X$ such that $d(g, h) \leq 2$.*

Proof Let $g \in X$. We will find it useful to split g into various components. Write $g = \prod_{j=1}^{m} \overset{\rho_j}{(x_j\, y_j)}$, for some $x_j, y_j \in \{1, \ldots, n\}$ and $\rho_j \in \mathbb{Z}$. Suppose that $n = 2m$. Then m is even, and by Lemma 2.4, $\sum_{j=1}^{m} \rho_j$ is even. We may therefore split g into a product of even pairs.
Suppose that $n > 2m$. Since $\mathcal{C}(\hat{G}, \hat{X})$ is connected, by Theorem 2.6 there are at least two fixed points. Splitting g into as many even pairs as possible, we will be left with either no transpositions (if m is even and $\sum_{i=1}^{j} \rho_j$ is even), an odd pair (if m is even and $\sum_{i=1}^{j} \rho_j$ is odd), or a single transposition (if m is odd).
Hence, in all cases, we may write g in the following form:

$$g = P_1 P_2 \cdots P_k Q$$

where $P_1, \ldots, P_k$ are even pairs and Q is either the identity, a single transposition along with at least two fixed points, or an odd pair along with at least two fixed points.
Let $P_i = \overset{\mu_i}{(\alpha_i \beta_i)}\overset{\nu_i}{(\gamma_i \delta_i)}$ with $\mu_i + \nu_i = 2\lambda_i$ even. Now set $P_i' = \overset{\lambda_i}{(\alpha_i \delta_i)}\overset{\lambda_i}{(\gamma_i \beta_i)}$ and $P_i'' = \overset{0}{(\alpha_i \gamma_i)}\overset{0}{(\delta_i \beta_i)}$. Note that each of P_i, P_i' and P_i'' is an even pair. It is clear by Lemma 3.2(a) that $P_i' P_i'' = P_i'' P_i'$. But Lemma 3.2(a) also implies that $P_i P_i' = P_i' P_i$, because we may rewrite $P_i = \overset{\mu_i}{(\alpha_i \beta_i)}\overset{-\nu_i}{(\delta_i \gamma_i)}$ and $P_i' = \overset{\lambda_i}{(\alpha_i \delta_i)}\overset{-\lambda_i}{(\beta_i \gamma_i)}$.

If Q is the identity, then let $Q' = Q'' = Q$. If Q is a single transposition along with at least two fixed points, then we may write $Q = \overset{\lambda}{(\alpha \beta)}\overset{0}{(\varepsilon_1)} \cdots \overset{0}{(\varepsilon_l)}$ for some $l \geq 2$. Let $Q' = Q'' = \overset{0}{(\varepsilon_1 \varepsilon_2)}\overset{0}{(\alpha)}\overset{0}{(\beta)}\overset{0}{(\varepsilon_3)} \cdots \overset{0}{(\varepsilon_l)}$.
If Q is an odd pair, along with at least two fixed points, then we may write $Q = \overset{\mu}{(\alpha \beta)}\overset{\nu}{(\gamma \delta)}\overset{0}{(\varepsilon_1)} \cdots \overset{0}{(\varepsilon_l)}$ for some $l \geq 2$. Let $Q' = \overset{0}{(\varepsilon_1 \varepsilon_2)}\overset{\nu}{(\gamma \delta)}\overset{0}{(\alpha)}\overset{0}{(\beta)}\overset{0}{(\varepsilon_3)} \cdots \overset{0}{(\varepsilon_l)}$ and let $Q'' = \overset{0}{(\alpha \beta)}\overset{0}{(\varepsilon_1 \varepsilon_2)}\overset{0}{(\gamma)}\overset{0}{(\delta)}\overset{0}{(\varepsilon_3)} \cdots \overset{0}{(\varepsilon_l)}$.
Then set $g' = P_1' \cdots P_k' Q'$ and $h = P_1'' \cdots P_k'' Q''$. Let $c = \hat{h}$. Then by choice of h, $h = (c, \mathbf{0})$. If $n = 2m$, note that g' and h consist entirely of even pairs, so $g', h \in X$. If $n \neq 2m$, then g' and h are obviously in X. By construction, and Lemma 3.2, g commutes with g' and g' commutes with h. Therefore $d(g, h) \leq 2$. This completes the proof of the proposition. $\square$

Proposition 3.5 *Suppose that $n = 2m$ with $m > 1$ odd. Let $g = (b, \mathbf{v}) \in X$. Then there exists $h = (c, \mathbf{0}) \in X$ such that $d(g, h) \leq 3$.*

Proof We may write $g = PQ$ where P is a product of k even pairs and Q is an 'even triple' with $Q = \overset{\mu_1}{(\alpha_1 \beta_1)}\overset{\mu_2}{(\alpha_2 \beta_2)}\overset{\mu_3}{(\alpha_3 \beta_3)}$ and $\mu_1 + \mu_2 + \mu_3 = 2\lambda$ for some $\lambda \in \mathbb{Z}$. Set $\rho = \mu_1 - \lambda$.
Now define $Q_1 = \overset{\mu_1}{(\alpha_1 \beta_1)}\overset{\rho+\mu_2}{(\alpha_2 \alpha_3)}\overset{\rho+\mu_3}{(\beta_2 \beta_3)}$, $Q_2 = \overset{0}{(\alpha_1 \alpha_2)}\overset{\mu_2-\lambda}{(\beta_1 \alpha_3)}\overset{\rho+\mu_3}{(\beta_2 \beta_3)}$ and $Q_3 = \overset{0}{(\alpha_1 \alpha_2)}\overset{0}{(\beta_1 \beta_3)}\overset{0}{(\alpha_3 \beta_2)}$. By repeated use of Lemma 3.2(a), we see that $QQ_1 = Q_1 Q$, $Q_1 Q_2 = Q_2 Q_1$ and $Q_2 Q_3 = Q_3 Q_2$. Note also that Q_1, Q_2 and Q_3 are all even triples.

By Proposition 3.4, there exist P', P'', both products of k even pairs, such that $P'' = \prod_{i=1}^{2k} \overset{0}{(\gamma_i \delta_i)}$ for some γ_i, δ_i and $PP' = P'P$, $P'P'' = P''P'$. In addition, we may assume that $\mathrm{Fix}(\hat{P}) = \mathrm{Fix}(\hat{P}') = \mathrm{Fix}(\hat{P}'')$. Define $g_1 = PQ_1$, $g_2 = P'Q_2$ and $h = P''Q_3$. Then by construction g_1 commutes with g and g_2, and g_2 commutes with h. So $d(g, h) \leq 3$. Furthermore, by construction g_1, g_2 and h are all elements of X, and $h = (\hat{h}, \mathbf{0})$. We have now proved Proposition 3.5. $\square$

Lemma 3.6 *Suppose $g_1 = (b_1, \mathbf{0}), g_2 = (b_2, \mathbf{0}) \in X$. If $\mathcal{C}(\hat{G}, \hat{X})$ is connected, then $d(g_1, g_2) = d(b_1, b_2)$.*

We are now able to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1 The case $n = 2, m = 1$ is Proposition 3.3. If $n = 2m + 1$ or $n = 4, m = 1$, then $\mathcal{C}(G, X)$ is disconnected by Lemma 3.1 and Theorem 2.6. If $n > 2$ and $\mathcal{C}(\hat{G}, \hat{X})$ is connected, then the fact that $\mathcal{C}(G, X)$ is connected is an easy consequence of Propositions 3.4 and 3.5, and Lemma 3.6. □

Proof of Theorem 1.2 By Proposition 3.4, if $n > 2m$ or m is even, and $\mathcal{C}(G, X)$ is connected, then there exists $h = (c, \mathbf{0}) \in X$ such that $d(g, h) \leq 2$. If $n = 2m$ and m is odd, then by Proposition 3.5 there exists $h = (c, \mathbf{0}) \in X$ such that $d(g, h) \leq 3$. By Lemma 3.6, $d(h, t) \leq \mathrm{Diam}\ \mathcal{C}(\hat{G}, \hat{X})$. Thus $d(g, t) \leq \mathrm{Diam}\ \ \mathcal{C}(\hat{G}, \hat{X}) + 2$ if $n > 2m$ or m is even, and $d(g, t) \leq \mathrm{Diam}\ \ \mathcal{C}(\hat{G}, \hat{X}) + 3$ otherwise. Theorem 1.2 follows immediately. □

Corollary 1.3 now follows from Theorem 1.2 in conjunction with Proposition 2.7 and Theorem 2.8.

4 Two Examples

In this section we give $\mathcal{C}(G, X)$ for two examples: $n = 4, m = 2$ and $n = 6, m = 3$. These graphs, of diameters 3 and 5 respectively, illustrate the fact that the bounds in Theorem 1.2 are tight, because the respective diameters of $\mathcal{C}(\mathrm{Sym}(4), (12)(34)^{\mathrm{Sym}(4)})$ and $\mathcal{C}(\mathrm{Sym}(6), (12)(34)(56)^{\mathrm{Sym}(6)})$ are 1 and 2. Figure 1 shows $\mathcal{C}(G, X)$ for $n = 4$, $m = 2$. The variable(s) above a transposition can be taken to be any integers. So for example $\overset{\lambda}{(13)}\overset{\lambda}{(24)}$ commutes with $\overset{\mu}{(12)}\overset{\mu}{(34)}$ for any integers λ, μ.

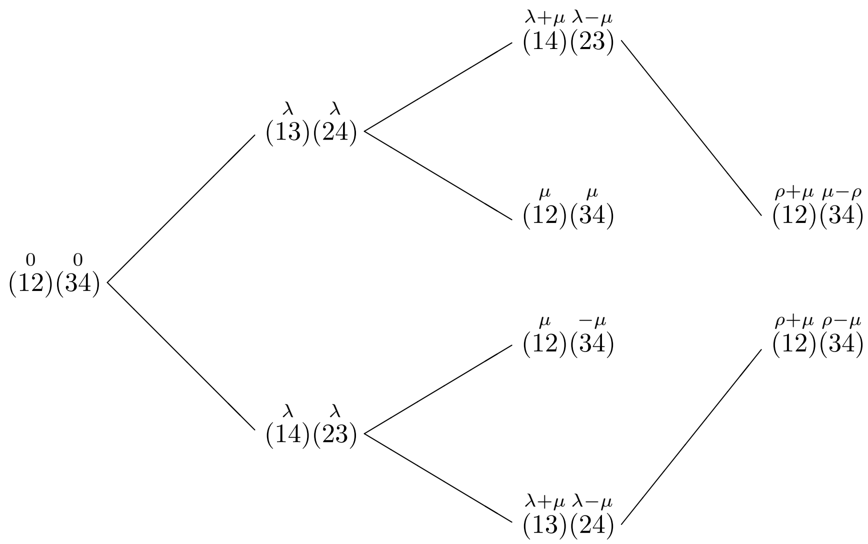


Figure 1: $n = 4$, $m = 2$

Figure 2 shows the collapsed adjacency graph in the case $n = 6, m = 3$. If $g, h \in G$ are in the same orbit of the centralizer $C_G(t)$ of t in G, then clearly $d(g, t) = d(h, t)$. The vertices of the graph in Figure 2 are the $C_G(t)$-orbits of $\mathcal{C}(G, X)$. We give one representative for each $C_G(t)$-orbit.

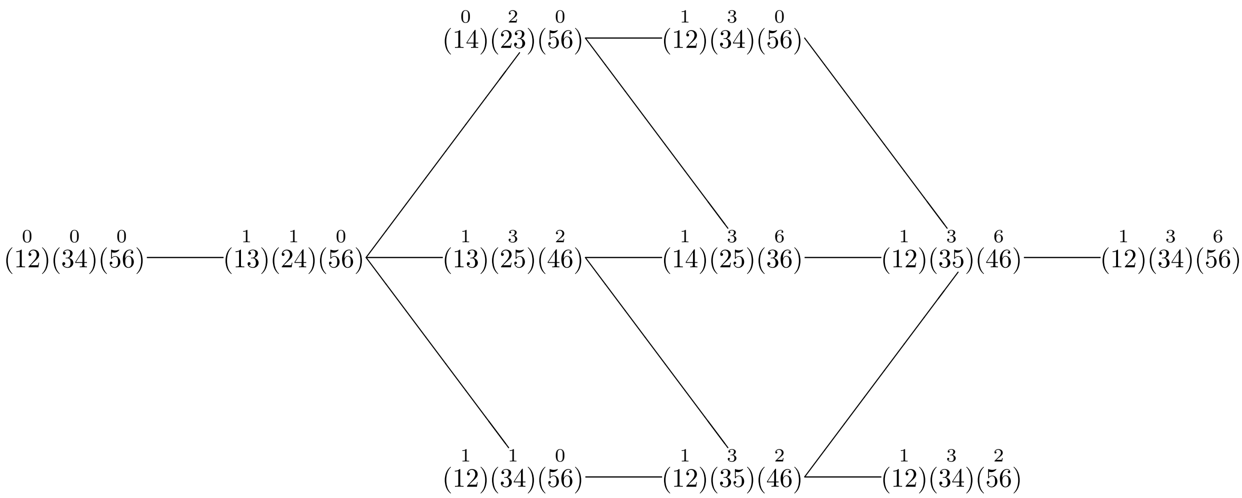


Figure 2: $n = 6, m = 3$

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