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Aditya Goenka and Odile Poulsen

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Department of Economics
Aarhus School of Business

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Aditya Goenka†
(National University of Singapore &
University of Essex)

Odile Poulsen‡
(Aarhus School of Business)

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Abstract: This paper studies a two-sector endogenous growth model with labour augmenting externalities or Harrod-Neutral technical change. The technologies are general and the preferences are of the CES class. If consumers are sufficiently patient, ergodic chaos and geometric sensitivity to initial conditions can emerge if either (1) there is factor intensity reversal; or (2) if the consumption goods producing sector is always capital intensive. The upper bound on the discount rate is determined only by the transversality condition. If utility is linear, there can be chaos only if there is factor intensity reversal.

Keywords: Ergodic Chaos, Two-sector endogenous growth model, Factor intensity reversal, Labor-augmenting externalities.
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†Correspondence to A. Goenka, Department of Economics, National University of Singapore, AS2, Level 6, 1 Arts Link, Singapore 117570, EMail: ecsadity@nus.edu.sg & goenka@essex.ac.uk

‡Department of Economics, Aarhus School of Business, Silkeborgvej 2, 8000 Aarhus C, Denmark, EMail: odp@asb.dk

1 Introduction

Recent research has established that economies with fully rational agents and market clearing can be subject to endogenous fluctuations (see the surveys by Baumol and Benhabib (1989), Day and Panigiani (1991), Boldrin and Woodford (1990)). Equilibrium trajectories in dynamic general equilibrium models can exhibit not only deterministic cycles but also chaotic behavior. This line of research provides an explanation of business cycle fluctuations that is fully self-contained in the style of the axiomatic tradition of general equilibrium theory as all fluctuations are equilibrium consequences of fully specified general equilibrium models. However, an issue with this literature is that chaotic trajectories emerge for parametric values that do not conform with the calibrated values that are usually used in the macroeconomic literature.

In this paper we show that chaotic dynamics can arise in a model of economic growth for a wider class of productions functions than the ones used in the literature so far. Secondly, chaotic dynamics (ergodic chaos and geometric sensitivity[1]) can occur for low levels of impatience in general two-sector economic growth models. Thirdly, ergodic chaos can occur for different configuration of the factor intensity. We show that ergodic chaos can occur when there is factor intensity reversal, and in addition in the case of non-linear utility if the consumption good sector is capital intensive for all values of the growth factor.

The model we study is a two sector growth model with Harrod-Neutral technical change or labour augmenting externalities. This is the same framework used by Drugeon and Venditti (1998), Drugeon, Poulsen and Venditti (2003) and Goenka and Poulsen (2004) to study indeterminacy. In the two-sector model, sector 1 produces a pure consumption good while sector 2 produces a pure investment good. In addition to the labor and capital inputs provided by the representative consumer, each sectors' productivity is affected by the aggregate capital stock. This provides a positive externality in the production of both sectors through learning by doing. The class of utility functions is the CES family. This specification is consistent with the existence of a balanced growth path.

To establish ergodic chaos we show that a suitably defined map of the

[1]In our model there is ergodic chaos if and only if there is geometric sensitivity. Thus, for brevity, we refer to the chaotic dynamics as ergodic chaos.

growth factor of capital (growth rate minus 1) can be unimodal and expansive. Different sufficient conditions are obtained depending on whether the utility function is non-linear or non-linear. A sufficient condition in the first case for unimodality is that either there is factor intensity reversal or the consumption good sector is capital intensive. We show that a necessary and sufficient condition for the law of motion to be expansive is that the rate of impatience of the representative consumer is not too low. In the second case, the sufficient condition for unimodality holds only if there is factor intensity reversal.

Our results extend what is known in the literature as we work with general functional forms. Nishimura and Yano (1995, 2000) and Nishimura et.al. (1994) also establish ergodic chaos in a two-sector growth model but the production functions in both sectors are Leontief; in Boldrin et.al. (2001) the production function in the consumption good sector is Cobb-Douglas and the production function in the investment good sector is linear. All we need to assume is that the production functions in both sectors are homogenous of degree one with respect to private inputs and that external effects are labor augmenting.

The result that ergodic chaos is compatible with low impatience is similar to the results obtained by Nishimura and Yano (1995, 2000) and Nishimura et al. (1994). (This work moves away from the so-called "minimum impatience theorems" (Mitra (1996) and Nishimura and Yano (1996), Sorger (1992, 1994)) which show that topological chaos is not feasible in optimal growth models unless the representative consumer discounts future utilities very heavily.) They derive sufficient conditions on the parameters of a family of optimal growth models that give rise to an ergodic chaotic optimal policy function for all discount factor inside the unit interval. Nishimura and Yano's (2000) have a constructive example establishing chaos. Nishimura, Sorger and Yano (1994) have a more general approach which relies on wealth effects. The utility of the representative consumer depends on the consumption good as well as current stock. This is similar to Majumdar and Mitra (1994) who have a general result based on wealth effects. In our model we do not have wealth effects but have the classical learning by doing externality which is Harrod Neutral or labour augmenting. There are some other differences in the nature of results. First, we derive an upper bound on the rate of impatience of the representative consumer. The Transversality Condition

(TVC) imposes a restriction on the discount factor[2] relating to the maximum feasible growth factor, the depreciation rate, and the exponent of the utility function. If growth is bounded and there is full depreciation, the model exhibits ergodic chaos for all discount factors arbitrarily close to unity. If growth is unbounded, there exists, as in Boldrin et al. (2001), an inverse relationship between discounting and growth. As long as the discount factor is inside the range of values allowed by the TVC, ergodic chaos can be obtained without further restrictions on the level of impatience of the representative consumer. In Boldrin et al. (2001) the upper bound on the discount rate can be lower than the bound derived in our paper for certain range of the external effects. Second, we derive a lower bound on the rate of impatience that is related to the underlying economic fundamentals.

The result on capital intensity configurations allows us to establish a link between the results of Boldrin et al. (2001), Nishimura and Yano (1995, 2000), Nishimura et al. (1994) and the earlier literature on chaos in two-sector neoclassical growth models. In all these papers the capital intensity in both sectors are arbitrarily chosen. The unimodal map is also derived from specific functional forms when no factor intensity reversal takes place between sectors. In the earlier literature on neoclassical growth, factor intensity reversal is a necessary condition for the emergence of topological chaos, as shown by Deneckere and Pelikan (1986), Boldrin (1989), Boldrin and Deneckere (1990). In our model when utility is nonlinear, unless the consumption sector is always capital intensive, capital intensity reversal is necessary for the emergence of chaos. If utility is linear capital intensity reversal is necessary for chaos. Furthermore, unlike the earlier literature on neoclassical growth models which had topological chaos, we have ergodic chaos in our model. In Nishimura and Yano (1995, 2000) and Nishimura et al. (1994) the consumption goods sector is always capital intensive and there is ergodic chaos.

It is worth noting that the work of Nishimura and Yano (1995, 2000) and Nishimura et al. (1994) uses corner solutions to establish a non-monotonic map that is essential for chaos. In our model the boundary behaviour is not the driving force to generate the non-monotonic dynamics.

We study the *backward dynamics* of the model as the map describing the dynamics is non-invertible in the forward dynamics. This problem also arises

[2]The discount factor can be rewritten as $\beta = 1/(1+\rho)$, where $\rho > 0$ is the rate of time preference or the rate of impatience of the representative consumer.

in overlapping generations models (see e.g. Grandmont (1985), and Goenka, Kelly and Spear (1998) which has a discussion of this issue).

The rest of the paper is organized as follows. In Section 2 we present the model and do some preliminary analysis. Section 3 collects some results on chaos to make the discussion self-contained. Then we study the case of linear utility in section 4. Section 5 contains the results for non-linear utility. The intuition of the results and further discussion of the results are in section 6.

2 The Model

The model is a discrete time, two–sector growth model with Harrod-Neutral technical change. Consumers are indexed by h and are distributed along the unit interval. The utility function of the representative consumer is of the CES class.

Assumption 1 :

$$u(c_t) = \frac{c_t^\alpha}{\alpha}, \ \ 0 < \alpha \leq 1$$

where c_t is per capita consumption at time t.

Consumers discount lifetime utility by the discount factor β, where, $0 < \beta < 1$. Each consumer $h \in [0, 1]$ is initially endowed with an equal fraction of the aggregate capital stock $\overline{k}$. Consumers also supply a single unit of labor inelastically. This labor is allocated between the two productive sectors of the economy. The aggregate labor supply in the economy is unity.

There are two productive sectors in the economy. Sector 1 produces the consumption good, c_t. Sector 2 produces the investment good, y_t. Inputs, capital, k, and labor, l, are freely mobile between sectors. Capital is assumed to depreciate at rate $\delta \in (0, 1]$. Thus, $k_{t+1} = y_t + (1 - \delta)k_t$. Market clearing in the two sectors is given by:

$$k_t = k_t^1 + k_t^2,$$
$$1 = l_t^1 + l_t^2,$$

where k_t^i and l_t^i denotes the amount of private inputs used in sector, $i = 1, 2$. We will omit the time subscripts whenever they are not necessary. The stationary production functions in the two sectors are given by:

$$c = F^1(k^1, l^1, X), \;\; y = F^2(k^2, l^2, X). \tag{1}$$

The productivity of the private inputs is affected by the aggregate capital stock X, where $X = \int_0^1 k(h)dh$. Thus, it is a two-sector version of the learning-by-doing model (Arrow (1961), Sheshinski (1967), Romer (1986)). We make the standard assumption that for a given level of the aggregate capital stock the marginal productivities of both inputs are positive and there are diminishing marginal productivities in private inputs.

Assumption 2: For $i = 1, 2$, $F^i : \Re^3_+ \to \Re_+$, are continuous functions. For a given $X \in \Re_+$:

1. $F^i(.,.,.)$ is C^3 on $\Re_{++} \times \Re_{++} \times \Re_+$;
2. $F^i(.,.,X)$ is homogenous of degree one and strictly increasing over $\Re_{++} \times \Re_{++}$;
3. $F^i_{11}(., l^i, X) < 0$, for all $k^i \in \Re_{++}$ and $\lim\limits_{k^i \to 0} F^i_1(k^i, l^i, X) = \infty$;
4. $F^i_{22}(k^i, ., X) < 0$, for all $l^i \in (0, 1]$ and $\lim\limits_{l^i \to 0} F^i_2(k^i, l^i, X) = \infty$.

While we work with general production functions, the key restriction in our model is the nature of the externality.

Assumption 3 $(Harrod - Neutrality)$: External effects in sector i are Harrod-Neutral (labor augmenting):

$$F^i(k^i, l^i, X) = \mathcal{F}^i(k^i, l^i X), \text{ where } i = 1, 2,$$

where $\mathcal{F}^i(.,.)$ is homogenous of degree 1 in k^i and $l^i X$.

We define the growth factor of capital γ_t as $\gamma_t = k_{t+1}/k_t$. Under Assumption 3, it follows that the maximum feasible growth factor $\overline{\gamma}$ is equal to $\mathcal{F}^2(1,1) + (1-\delta)$. As capital may not depreciate fully in every period, the minimum feasible growth factor $\underline{\gamma}$ is $(1-\delta)$. Hence for all feasible sequences of capital $\{k_t, k_{t+1}\}$ given by $(1-\delta)k_t \le k_{t+1} \le \mathcal{F}^2(k_t, X_t) + (1-\delta)k_t$, $\gamma_t \in [(1-\delta), \mathcal{F}^2(1,1) + (1-\delta)] \equiv \Gamma$.

The representative consumer's behavior is described by the following optimization problem:

$$\max_{\{c_t, k_t^1, l_t^1, k_t^2, l_t^2,\}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^\alpha}{\alpha} \tag{2}$$

subject to

$$\begin{aligned}
c_t &= \mathcal{F}^1(k_t^1, l_t^1 X_t), \\
y_t &= \mathcal{F}^2(k_t^2, l_t^2 X_t), \\
k_{t+1} &= y_t + (1-\delta)k_t, \\
k_t &= k_t^1 + k_t^2, \\
1 &= l_t^1 + l_t^2, \\
& k_t^i \ge 0,\ l_t^i \ge 0,\ \{X_t\}_{t=0}^{\infty},\ k_0 \text{ given.}
\end{aligned}$$

The constraints can be collapsed using the production possibility frontier (PPF from here on), $T(k, y, X)$, given by $c_t = \mathcal{F}^1(k_t - k^2(k,y,X), X_t(1 - l^2(k,y,X)))$. The above maximization problem reduces to:

$$\max_{\{k_t\}} \sum_{t=0}^{\infty} \beta^t \left[\frac{T(k_t, k_{t+1}, X_t)^\alpha}{\alpha} \right] \tag{3}$$

subject to

$$\begin{aligned}
&(1-\delta)k_t \le k_{t+1} \le \mathcal{F}^2(k_t, X_t) + (1-\delta)k_t, \\
&\{X_t\}_{t=0}^{\infty},\ k_0 \text{ given.}
\end{aligned}$$

Let $k_t\{X_t\}_{t=0}^{\infty}$ denote the solution to this problem. If $\{k_t\}_{t=0}^{\infty}$ satisfies $k_t\{X_t\}_{t=0}^{\infty} = X_t$ for all $t \ge 0$, then the path $\{k_t\}_{t=0}^{\infty}$ will be referred to as an equilibrium path. This fixed point problem is solved in a different framework in detail by Romer (1983) and Mitra (1998). We do not address this issue here. We assume for simplicity that there exists an equilibrium path $\{k_t\}_{t=0}^{\infty}$ such that $k_t\{X_t\}_{t=0}^{\infty} = X_t$ for all $t \ge 0$.

As we are working with general technologies, we make the following regularity assumption[3]

Assumption 4 : $T(k_t, y_t, X_t)$ is C^3 on $\Re_{++} \times \Re_{++} \times \Re_+$.

On defining the indirect utility function as $V(k_t, k_{t+1}, X_t) = [T(k_t, k_{t+1}, X_t)^\alpha/\alpha]$, the maximization problem of the representative agent reduces to:

$$\max_{\{k_t\}} \sum_{t=0}^{\infty} \beta^t V(k_t, k_{t+1}, X_t) \tag{4}$$

subject to

$$(k_t, k_{t+1}) \in D(X_t),$$
$$\{X_t\}_{t=0}^{\infty}, \quad k_0 \text{ given.}$$

The set of feasible solutions to (4) $D(X_t)$ is defined as follows:

$$D(X_t) = \{(k_{t,}k_{t+1}) \in \Re_+ \times \Re_+ : (1-\delta)k_t \leq k_{t+1} \leq \mathcal{F}^2(k_t, X_t) + (1-\delta)k_t\}.$$

This set is non-empty and convex. Following Boldrin et. al (2001), $\{k_t\}_{t=0}^{\infty}$ is an *interior solution* to Problem 4 if the following conditions are satisfied:

$$V_2(k_t, k_{t+1}, X_t) + \beta V_1(k_{t+1}, k_{t+2}, X_{t+1}) = 0, \tag{5}$$

$$\lim_{t\to\infty} \beta^t k_t V_1(k_t, k_{t+1}, X_t) = 0, \tag{6}$$

$$\sum_{t=0}^{t=\infty} \beta^t V(k_t, k_{t+1}, X_t) < \infty. \tag{7}$$

Equation (5) is the Euler equation. Equation (6) is the transversality condition. Equation (7) is the summability condition. Conditions (5) to (7) are the same as conditions (3.1)-(3.3) given in Boldrin et. al (2001) with the exception that $u(c_t)$ is non-linear. To insure that the transversality condition (TVC) is satisfied we impose the following restriction

[3]It is satisfied in the case of Cobb-Douglas production functions, among others.

Assumption 5: $\beta(\mathcal{F}^2(1,1) + (1-\delta))^\alpha \equiv \beta\overline{\gamma}^\alpha < 1$.

For unbounded growth we need to assume that $\overline{\gamma} > 1$. In that case we see that the TVC will be satisfied for all $\beta \in (0, 1/\overline{\gamma}^\alpha)$. For bounded growth $\overline{\gamma} < 1$, and if there is full depreciation, then TVC holds for all $\beta \in (0,1)$. If α is small enough then the TVC will also hold for all β even if there isn't full depreciation. For a fixed $\overline{\gamma}$, there is a inverse relationship between α and the upper-bound on β for the TVC to be satisfied.

The simplest dynamics in the model arise in when there is a balanced growth path or when capital grows at a constant rate.

Definition 1 *An equilibrium path $\{k_t\}$ is a balanced growth path (BGP) if there exists a growth factor $\gamma \in [0, \overline{\gamma}]$ such that for all $t \geq 0$, $k_t = \gamma^t k_0$, where $k_0 \neq 0$.*

The issue is whether there exists an interior solution with such a property. If $\mathcal{F}_1^2(k^2(1,1), l^2(1,1)) < 1$, i.e., there is bounded growth, there exists a BGP. For the case of unbounded growth is the following additional assumption has to be made :

Assumption 6 : $\beta\left[\mathcal{F}_1^2(k^2(1,\delta,1), l^2(1,\delta,1)) + 1 - \delta\right] > 1$.

Proposition 1 *Let Assumptions 1-6 be satisfied. Then there exists an interior BGP with unbounded growth, $\widetilde{\gamma} \in (1, \overline{\gamma})$.*

Proof. See Goenka and Poulsen (2004). ■

We are interested in the dynamics of the model. These are given by (5) - (7). To study general dynamics in the model, instead of working with this system of equations, we want to transform these to obtain a one dimensional difference equation which will be more amenable to analysis.

In order to do so, it is helpful to investigate properties of the Production Possibility Frontier. Under Harrod-Neutrality, Drugeon and Venditti (1998) show that the input demand functions $k^i(k_t, y_t, X_t)$ and $l^i(k_t, y_t, X_t)$, $i = 1, 2$, are homogenous of degree 1 and 0 respectively and that $T(k_t, y_t, X_t)$ is homogenous of degree 1. Then using the homogeneity properties of the production functions, Drugeon, Poulsen and Venditti (2003) establish the following:

Lemma 1 *Let Assumptions 2-3 be satisfied. Then, function of $T(k, y, X)$, is homogenous of degree one. Furthermore if Assumption 5 holds then,*

$$T_{21} = \frac{\mathcal{F}^1_{12}\mathcal{F}^2_{12} q \mathcal{F}^2 l^1}{\Omega k^2 k^1} \left(\frac{k^1}{l^1} - \frac{k^2}{l^2} \right), \tag{8}$$

$$T_{23} = -T_{21} \frac{k^1}{l^1 X} + \frac{2l^2}{\mathcal{F}^2_1} \left(\mathcal{F}^1_{12} + q\mathcal{F}^2_{12} \right), \tag{9}$$

$$T_{22} = \frac{l^2}{\mathcal{F}^2} \left(\frac{k^1}{l^1} - \frac{k^2}{l^2} \right) T_{21}, \tag{10}$$

where

$$\Omega = -\frac{\mathcal{F}^1_{12}(\mathcal{F}^1)^2(\mathcal{F}^2_1)^2}{(\mathcal{F}^1_1)^2 k^1 l^1 X} - \frac{\mathcal{F}^2_{12}(\mathcal{F}^2)^2\mathcal{F}^1_1}{\mathcal{F}^2_1 k^2 l^2 X} < 0.$$

Proof. See Drugeon and Venditti (1998) and Drugeon, Poulsen and Venditti (2003). ■

We also see from this result that the sign of T_{21} depends on the factor intensity. With factor intensity reversal, T_{21} will change sign. A similar result can also be found in Benhabib and Nishimura (1985) in a two-sector optimal growth model without externalities.

Under *Harrod -Neutrality*, the Euler equation can now be represented in terms of the growth factor, γ_t as the homogeneity properties of the PPF imply that the indirect utility function is homogenous of degree α.

Lemma 2 *Let Assumptions 1-3 be satisfied. Then $V(k_t, k_{t+1}, X_t)$ is homogenous of degree α. Furthermore, the Euler equation (5) reduces to:*

$$\gamma_t^{1-\alpha} V_2(1, \gamma_t, 1) + \beta V_1(1, \gamma_{t+1}, 1) = 0. \tag{11}$$

Proof. See Drugeon and Venditti (1998). ■

Recall that the set of feasible growth factors is $\Gamma = [(1-\delta), \mathcal{F}^2(1,1) + (1-\delta)]$. We want to represent the general forward dynamics as map $\tau : \Gamma \to \Gamma$,

$$\gamma_{t+1} = \tau(\gamma_t) \tag{12}$$

or, in backward dynamics as a map, $\theta : \Gamma \to \Gamma$:

$$\gamma_t = \theta(\gamma_{t+1}). \tag{13}$$

From (11) we can define the following two functions:

$$
\begin{aligned}
f(\gamma_{t+1}) &= \beta V_1(1, \gamma_{t+1}, 1), \\
g(\gamma_t) &= -\gamma_t^{1-\alpha} V_2(1, \gamma_t, 1),
\end{aligned}
\tag{14}
$$

and so (11) can be written as

$$
-g(\gamma_t) + f(\gamma_{t+1}) = 0.
$$

To obtain a first order difference equation like (12) or (13), we need to show that either g or f has a well defined inverse on $[\underline{\gamma}, \overline{\gamma}]$. This is taken up in section 4 for the linear utility and in section 5 for non-linear utility.

3 Results on chaos

To make the discussion self contained, some standard results and definitions on the theory of chaos in dynamical systems are collected here (see Collet and Eckmann (1980), Day (1994) and Eckmann and Ruelle (1985) for more details). We focus on chaos in the sense of ergodic oscillations and geometric sensitivity (GS). In other words, the law of motion is generated by a nonlinear difference equation with no random terms such that *(i)* the long run behavior of the system is characterized by extremely complicated aperiodic dynamics, *(ii)* small differences in the initial conditions are magnified at a geometric rate for arbitrary finite lengths of time (GS) and *(iii)* the time average of the orbit can be replaced by the space average (ergodic chaos).

A dynamical system is a pair (I, θ) where I is a compact interval on the real line and is called the state space and θ a function describing the law of motion of the state variable $x \in I$. Thus, if x_t is the state of the system in period t, then $x_{t+1} = \theta(x_t)$ is the state of the system in period $t+1$. If we denote by x_0 the initial state of the system, and $\theta^0(x) = x$ for all $x \in I$ then $\theta^{t+1}(x) = \theta(\theta^t(x))$ for all $t \geq 0$ and all $x \in I$ where θ^t is the $t-th$ iterate of θ, $t = 0, 1, 2, ...$

The notion of chaos that is often used in the economics literature is that of topological chaos. In many models this is easy to establish using the theorem of Li and Yorke (1975) that a "cycle of period 3 implies chaos." (also see Mitra (2001) for a different set of conditions to establish topological chaos).

Definition 2 *Let $\theta : I \to I$ define a dynamical system. We say that θ exhibits topological chaos if:*

1. *For every period N, there exist points $x_N \in I$ such that $\theta^N(x_N) = x_N$.*

2. *There exists an uncountable set $S \subset I$ and an $\epsilon > 0$ such that every pair x, y in S with $x \neq y$:*

   (a) $\lim_{n\to\infty} \sup |\theta^n(x) - \theta^n(y)| \geq \epsilon$.

   (b) $\lim_{n\to\infty} \inf |\theta^n(x) - \theta^n(y)| = 0$.

   (c) *For every periodic point z and $x \in S$:* $\lim_{n\to\infty} \sup |\theta^n(x) - \theta^n(z)| \geq \epsilon$.

While topological chaos can often be easy to show, S may have Lebesgue measure zero. For example, for the quadratic map $\theta(x) = \mu x(1-x)$ with $\mu \in [1, 4]$, there is topological chaos if $\mu = 3.828427$, but almost all initial conditions lead to a cycle of period 3. A different notion of chaos is the following.

Definition 3 *(Nishimura and Yano (2000)). The dynamical system (I, θ), exhibits Geometric Sensitivity (GS) if there exists a constant $h > 1$ such that for any $\tau \geq 0$ there exists $\varepsilon > 0$ such that for all x and $x' \in I$ with $|x_t - x'_t| < \varepsilon$ and for all $t \in \{0, 1, ..., \tau\}$*

$$\left|\theta^t(x) - \theta^t(x')\right| \geq h^t \left|x - x'\right|.$$

In this case small differences in initial conditions magnify geometrically over time. As I is bounded, the geometric magnification of the effects of a small perturbation cannot last indefinitely. Furthermore, the dynamical system (I, θ) has no locally stable cyclical path.

There is also the notion of *ergodic chaos*. Ergodic chaos is a stronger property than topological chaos in the sense that it is "observable chaos".

Let Υ be a σ-algebra on I. [4] Define a probability measure $\mu : \Upsilon \to \Re^+$ such that (i) $\mu(\varnothing) = 0$ and (ii) $\mu(\cup_{n=0}^{\infty} Y_n) = \Sigma_{n=0}^{\infty}\mu(Y_n)$, (iii) $\mu(I) = 1$, where $\{Y_t\}_{n=0}^{\infty}$ is a countable collection of disjoint sets in Υ. The probability measure μ is said to be invariant with respect to θ if $\mu(\theta^{-1}(Y)) = \mu(Y)$ for all $Y \in \Upsilon$. Invariant measures have an important property.

[4] A $\sigma-$algebra is a collection of subsets Υ of I such that (i) I is inside Υ, (ii) the complement of any set Y included in Υ is also in Υ, (iii) the union of any countable collection of subsets in Υ is inside Υ.

Theorem 1 ***(Poincaré Recurrence Theorem:)*** *Let $\{I, \Upsilon, \mu\}$ be a probability space and let μ be invariant under θ. Let Y be any measurable set of positive measure. Then all points of Y return to Y infinitely often.*

Consider $x \in Y$. Then, $x \in \theta^{-1}(x) \Rightarrow \theta(x) \in Y$. Thus, the trajectory will stay in Y forever. The invariant probability measure μ is said to be ergodic if for some $Y \in \Upsilon$ then $\theta^{-1}(Y) = Y$ implies either $\mu(Y) = 0$ or $\mu(Y) = 1$. In other words, the system cannot split into non-trivial parts. For ergodic measures, a fundamental result is:

Theorem 2 ***(The Birkhoff-von Neuman Mean Ergodic Theorem:)*** *Let (I, θ) be a dynamical system. If μ is an invariant and ergodic probability measure then, for any $\mu-$integrable function g,*

$$\lim_{t\to\infty} \frac{1}{t} \sum_{\tau=0}^{t-1} g(\theta^{\tau}(x)) = \int_I g d\mu \tag{15}$$

for almost all $x \in I$.

The left hand side of (15) is the average value of g along the orbit $\{x,\ \theta(x), ...\}$. The right hand side of (15) is the expected value of g evaluated on I. In other words, the time averages along an orbit $\{x,\ \theta(x), ...\}$ can be replaced by space averages. Furthermore, for almost all $x \in I$, an orbit $\{x,\ \theta(x), ...\}$ will visit every measurable set proportionally to its measure. The system is "chaotic" if the support of the measure is a "large set." However, the measure could be concentrated on a point as in the case of a fixed point, or on a finite subset of points in the case of a cycle. Absolute continuity of a measure avoids this problem.

Definition 4 *The probability measure μ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}$ on I if there exists an integrable function f such that*

$$\mu(Y) = \int_Y f dm$$

for all measurable sets Y in $\sigma-$algebra Υ.

Absolute continuity implies that for all $Y \in \Upsilon$, if $\mathcal{L}(Y) = 0$ then $\mu(Y) = 0$. Thus, the support of μ cannot be a set of measure zero.

We can now define the concepts of ergodic chaos in the following way:

Definition 5 *The dynamical system (I, θ) exhibits ergodic chaos if there exists a probability measure μ on I which is absolutely continuous, invariant and ergodic.*

Lasota and Yorke (1973) establish that if θ is a *piecewise* C^2 and *expansive* mapping then there exists an absolutely continuous invariant measure

Definition 6 *A mapping θ defined on $[a, b]$ is piecewise C^2 and expansive if:*

1. *There exists a finite set $x_0 = a < x_1 < x_2 < ... < x_n = b$,*
2. *For all $j = 0, 1, ..., n$, θ is C^2 on (x_j, x_{j+1}) and can be extended as a C^2 function to $[x_j, x_{j+1}]$,*
3. *$|\theta'(x)| \geq h > 1$ for all $x \in (x_j, x_{j+1})$.*

Li and Yorke (1978) show that if θ is also a unimodal map then this measure is ergodic.

Definition 7 *Assume there exits a constant $c \in [a, b]$, $a < c < b$. Then a mapping θ, defined on $[a, b]$ is unimodal if*

1. *θ is continuous on $[a, b]$,*
2. *θ is strictly increasing on (a, c) and strictly deceasing on (c, b).*

Nishimura and Yano (2000) establish that a map that is expansive is also chaotic in the sense of GS. Thus, we have:

Theorem 3 *(Lasota and Yorke (1973), Li and Yorke (1978), Nishimura and Yano (2000)):*
Let (I, θ) be a dynamical system. If $\theta : I \rightarrow I$ is expansive and unimodal then θ is chaotic in the sense of ergodic oscillations and GS.

4 Linear Utility

We first study the case where $\alpha = 1$ or utility is linear and there is full depreciation, i.e. $\delta = 1$. In this case charaterizing the dynamics simplifies considerably and factor intensity reversal is necessary for the occurrence of chaos. Full depreciation will imply that $[\underline{\gamma}, \overline{\gamma}] = [0, \mathcal{F}^2(1, 1)]$.

In the model, there can in general be several capital intensities where factor intensity reversal takes place. We make the simplifying assumption that there exists a unique value of the growth factor $\widehat{\gamma}$, where $0 < \widehat{\gamma} < \overline{\gamma}$, at which factor reversal takes place. [5]

Assumption 7: *(Unicity of factor intensity reversal)* There exists a unique $\widehat{\gamma} \in (0, \overline{\gamma})$ such that

$$\frac{k^1(1, \widehat{\gamma}, 1)}{l^1(1, \widehat{\gamma}, 1)} = \frac{k^2(1, \widehat{\gamma}, 1)}{l^2(1, \widehat{\gamma}, 1)}.$$

Under linear utility, $\{k_t\}_{t=0}^{\infty}$ is an equilibrium path if and only if the following conditions are satisfied

$$T_2(k_t, k_{t+1}, k_t) + \beta T_1(k_{t+1}, k_{t+2}, k_{t+1}) = 0, \tag{16}$$

$$\lim_{t \to \infty} \beta^t k_t T_1(k_t, k_{t+1}, k_t) = 0, \tag{17}$$

$$\sum_{t=0}^{t} \beta^t T(k_t, k_{t+1}, k_t) < \infty. \tag{18}$$

Conditions (16) to (18) are identical to conditions (3.1)-(3.3) given in Boldrin et. al (2001). Given assumptions 2 and 3, these can be satisfied only for an interior trajectory, i.e., with $k_t^1 > 0$, $k_t^2 > 0$.

The homogeneity properties of the PPF imply that the left hand side of equation (16) can be rewritten as

$$\psi(\gamma_t, \gamma_{t+1}) \equiv T_2(1, \gamma_t, 1) + \beta T_1(1, \gamma_{t+1}, 1) \tag{19}$$

or, as

$$\psi(\gamma_t, \gamma_{t+1}) = 0. \tag{20}$$

[5] One can relax this assumption and still show that θ is chaotic in the sense of ergodic oscillations and GS. To do so, one would need to (i) establish that θ is piecewise strictly monotonic, piecewise C^2 and expansive on the following intervals $(0, \widehat{\gamma}_1), (\widehat{\gamma}_1, \widehat{\gamma}_2), ...$and $(\widehat{\gamma}_n, \overline{\gamma})$, where $\widehat{\gamma}_j$, $j = 1, ...n$, denotes a growth factor at which the ith factor reversal occurs. (ii) Show that $\theta(\widehat{\gamma}_j)$, , $j = 1, ...n$, is a turning point and that $\widehat{\gamma}_{j-1} < \widehat{\gamma}_j < \widehat{\gamma}_{j+1}$. For more details for this case, see Theorem 8.5 and Corollaries 8.3 and 8.4 in Day (1994) .

From (19), we can define the following two functions:

$$\begin{aligned} f(\gamma_{t+1}) &= \beta T_1(1, \gamma_{t+1}, 1), \\ g(\gamma_t) &= -T_2(1, \gamma_t, 1) \end{aligned} \tag{21}$$

and hence, (19) can be written as

$$-g(\gamma_t) + f(\gamma_{t+1}) = 0. \tag{22}$$

To obtain a first order difference equation such as (12) or (13) either g or f has to have a well defined one-to-one inverse on $[0, \overline{\gamma}]$. As we can see from (8) in Lemma 1, T_{12} changes sign if factor intensity reversal takes place at $\widehat{\gamma}$. Hence f does not have a well defined one-to-one inverse. The same is not true for the existence of g^{-1}.

Proposition 2 *Let Assumptions 1-5 and 7 be satisfied. Then there exists a* $1-1$ *map* $\theta : [0, \overline{\gamma}] \to [0, \overline{\gamma}]$ *such that*

$$\gamma_t = \theta(\gamma_{t+1}). \tag{23}$$

Proof.
Define

$$f(\gamma_{t+1}) = \beta T_1(1, \gamma_{t+1}, 1), \tag{24}$$
$$g(\gamma_t) = -T_2(1, \gamma_t, 1), \tag{25}$$

where $f : [0, \overline{\gamma}] \to J \sqsubseteq \Re_+^*$ and $g : [0, \overline{\gamma}] \to I \sqsubseteq \Re_+^*$.
Using (24) and (25), the Euler equation (19) can be written as $-g(\gamma_t) + f(\gamma_{t+1}) = 0$. Assumptions 2, 3 and the *unicity of factor intensity reversal (*Assumption 7) imply that $g' > 0$ for all $\gamma_t \in (0, \widehat{\gamma}) \cup (\widehat{\gamma}, \overline{\gamma})$. Furthermore, Assumption 7 implies there is a unique $\widehat{\gamma}$, $0 < \widehat{\gamma} < \overline{\gamma}$ such that $g'(\widehat{\gamma}) = 0$. Hence g is one to one. From the definition of g we see that range$(g) = I$. So g has a well defined inverse g^{-1}. We can rewrite (20) as $\gamma_t = \theta(\gamma_{t+1})$, where $g^{-1}\left[f(\gamma_{t+1})\right]$. From this definition, it follows that $\theta : [0, \overline{\gamma}] \to [0, \overline{\gamma}]$. ■

Grandmont (1985), Goenka, Kelly and Spear (1998), amongst others, also study backward foresight dynamics. In Grandmont (1985) the same problem about the non-existence of the inverse of f. In his overlapping generations model, the excess demand function of the young consumers is a unimodal function if the Arrow-Pratt relative degree of risk aversion of the old is a non-decreasing function of their wealth. See Goenka, Kelly and Spear (1998) for a discussion of forward and backward dynamics.

The issue of showing chaos in the model thus reduces to showing that the map θ derived in section 4 is unimodal and expansive.

4.1 Unimodality of θ

To establish the unimodality of θ we need to show θ is continuous on $[0, \overline{\gamma}]$, strictly monotonically increasing on $(0, \widehat{\gamma})$ and strictly monotonically decreasing on $(\widehat{\gamma}, \overline{\gamma})$. It can be shown that the sign of $\theta^{'}(\gamma_t)$ depends on the sign of T_{12}. As T_{12} changes sign at the point of factor intensity reversal, under *unicity of factor intensity reversal* $\theta^{'}(\gamma_t)$ changes sign only once at $\widehat{\gamma}$. Hence θ is either strictly monotonically increasing on $(0, \widehat{\gamma})$ and strictly monotonically decreasing on $(\widehat{\gamma}, \overline{\gamma})$ or strictly monotonically decreasing on $(0, \widehat{\gamma})$ and strictly monotonically increasing on $(\widehat{\gamma}, \overline{\gamma})$. Which of these two cases prevails depends on which sector is the most capital intensive on $(0, \widehat{\gamma})$. Given the definition of unimodality we need the capital good sector to be more capital intensive for all $\gamma_{t+1} \in [0, \widehat{\gamma})$. In other words:

Assumption 8 :

$$\frac{k^1(1, \gamma_{t+1}, 1)}{l^1(1, \gamma_{t+1}, 1)} < \frac{k^2(1, \gamma_{t+1}, 1)}{l^2(1, \gamma_{t+1}, 1)},$$

for all $\gamma_{t+1} \in [0, \widehat{\gamma})$.

Proposition 3 *Let Assumptions 1-5 and 8 be satisfied. Let $\theta : [0, \overline{\gamma}] \to [0, \overline{\gamma}]$ be defined as in (23). Then, unicity of factor intensity reversal (Assumption 7) is a necessary and sufficient condition for θ to be unimodal.*

Proof.

(i) Continuity of θ. From the definition of θ given in Proposition 2, θ is continuous as the composite function of two continuous functions.
(ii) Monotonicity of θ. Let Assumptions 1-8 be satisfied. Then using the inverse function theorem we can derive $\theta^{'}(\gamma_{t+1})$ on $(0, \widehat{\gamma})$ and $(\widehat{\gamma}, \overline{\gamma})$ as

$$\theta^{'}(\gamma_{t+1}) = -\frac{\beta T_{21}(1, \gamma_{t+1}, 1)}{T_{22}(1, \gamma_t, 1)}. \tag{26}$$

Upon inspection of (26) we see that the sign of the denominator depends on the sign of $-T_{22}$. Looking at (10), we see that under Assumption 7 $-T_{22} > 0$ for all $\gamma_t \in (1, \widehat{\gamma}) \cup (\widehat{\gamma}, \overline{\gamma})$. Hence $\theta^{'}(\gamma_{t+1}) \gtrless 0$ if and only if $T_{12}(1, \gamma_{t+1}, 1) \gtrless 0$. Assumptions 7 and 8, together with the results of Lemma 1, imply that

$$\theta^{'}(\gamma_{t+1}) > 0 \text{ for all } \gamma_{t+1} \in (0, \widehat{\gamma}) \text{and } \theta^{'}(\gamma_{t+1}) < 0 \text{ for all } \gamma_{t+1} \in (\widehat{\gamma}, \overline{\gamma}).$$

■

The result that factor intensity reversal is a necessary condition for the occurrence of cycle or chaotic dynamics is not new. Benhabib and Nishimura (1985) establish that factor intensity reversal is a necessary condition for the emergence of a cycle of period two in an optimal growth model. Deneckere and Pelikan (1986), Boldrin (1989) and Boldrin and Deneckere (1990) show that intensity factor reversal is a necessary condition for the occurrence of topological chaos. However, to our knowledge, no such result has yet been established in models with endogenous growth or in model of optimal growth in which ergodic oscillations occur for low levels of impatience.

4.2 Expansiveness of θ and chaos

In the next Proposition we show that a necessary and sufficient condition for θ to be expansive is that the discount factor lies within a given interval. As explained above, the lower bound on the discount factor means that the representative consumer must not be too impatient. The upper bound on the discount factor imposed by the TVC is only binding if one assumes that growth is unbounded.

Proposition 4 *Let Assumptions 1-5, 7, and 8 be satisfied.*
Then a necessary condition for $\theta : [0, \overline{\gamma}] \rightarrow [0, \overline{\gamma}]$ to be expansive is

$$\left| -\frac{T_{22}(1, \theta(\gamma_{t+1}), 1)}{T_{21}(1, \gamma_{t+1}, 1)} \right| < 1 \textit{ for all } \gamma_{t+1} \in (0, \widehat{\gamma}) \cup (\widehat{\gamma}, \overline{\gamma}). \tag{27}$$

A necessary and sufficient condition for $\theta : [0, \overline{\gamma}] \rightarrow [0, \overline{\gamma}]$ to be expansive is that $\beta \in (\beta_{\min}, \beta_{\max})$, where

$$\beta_{\min} = \max_{\gamma_{t+1} \in (0, \widehat{\gamma}) \cup (\widehat{\gamma}, \overline{\gamma})} \left| -\frac{l^2}{\mathcal{F}^2} \left(\frac{k^1}{l^1} - \frac{k^2}{l^2} \right) \frac{T_{21}(1, \theta(\gamma_{t+1}), 1)}{T_{21}(1, \gamma_{t+1}, 1)} \right|,$$
$$\beta_{\max} = \frac{1}{\mathcal{F}^2(1,1)}.$$

Proof.

From the definition of θ it follows that θ is twice continuously differentiable on both $(0, \widehat{\gamma})$ and $(\widehat{\gamma}, \overline{\gamma})$. Then, θ is expansive if $\left|\theta'(\gamma_{t+1})\right| > 1$ for all $\gamma_{t+1} \in (1, \widehat{\gamma}) \cup (\widehat{\gamma}, \overline{\gamma})$. Or, equivalently, using (10) if

$$\left| -\frac{l^2}{\mathcal{F}^2} \left(\frac{k^1}{l^1} - \frac{k^2}{l^2} \right) \frac{T_{21}(1, \theta(\gamma_{t+1}), 1)}{T_{21}(1, \gamma_{t+1}, 1)} \right| < \beta \text{ for all } \gamma_{t+1} \in (1, \widehat{\gamma}) \cup (\widehat{\gamma}, \overline{\gamma}).$$

It follows that

$$\beta_{\min} = \max_{\gamma_{t+1} \in (1, \widehat{\gamma}) \cup (\widehat{\gamma}, \overline{\gamma})} \left| -\frac{l^2}{\mathcal{F}^2} \left(\frac{k^1}{l^1} - \frac{k^2}{l^2} \right) \frac{T_{21}(1, \theta(\gamma_{t+1}), 1)}{T_{21}(1, \gamma_{t+1}, 1)} \right|$$

Since, by assumption, we have $\beta < 1$, a necessary condition for expansiveness is

$$\left| -\frac{T_{22}(1, \theta(\gamma_{t+1}), 1)}{T_{21}(1, \gamma_{t+1}, 1)} \right| < 1.$$

For the transversality condition to be satisfied, $\beta\overline{\gamma} < 1$. As $\overline{\gamma} = \mathcal{F}^2(1, 1)$, we have $\beta_{\max} = \dfrac{1}{\mathcal{F}^2(1, 1)}$. ■

Condition (27) can be interpreted as follows. Applying the envelope theorem to the first order conditions of Problem (3) we obtain $T_2 = -q$. Hence $T_{21} = -\partial q/\partial k$ and $T_{22} = -\partial q/\partial y$. So, (27) states that the rate of change of the shadow price of investment with respect to the current capital capital stock must be greater than the rate of change of the shadow price of investment with respect to last period's stock of investment.

Collecting the previous two propositions we have the main result.

Proposition 5 *Let Assumptions 1-5, 7, and 8 be satisfied. $\theta : [0, \overline{\gamma}] \to [0, \overline{\gamma}]$ is chaotic both in the sense of ergodic oscillations and geometric sensitivity if*
(i) $\left| -\frac{T_{22}(1, \theta(\gamma_{t+1}), 1)}{T_{21}(1, \gamma_{t+1}, 1)} \right| < 1$ *for all* $\gamma_{t+1} \in (0, \widehat{\gamma}) \cup (\widehat{\gamma}, \overline{\gamma})$ *holds.*
(ii) The discount factor β satisfies $\beta \in (\beta_{\min}, \beta_{\max})$.

5 Non Linear Utility

To the study the non-linear utility case ($\alpha < 1$), we first establish the following results.

Lemma 3 *Let Assumptions 1-5 be satisfied. Then:*

$$V_{21} = (\alpha-1)T^{\alpha-2}[T_1 - (1-\delta)T_2]T_2 + T^{\alpha-1}[T_{21} - (1-\delta)T_{22}], \quad (28)$$
$$V_{23} = T^{\alpha-2}\left[(\alpha-1)T_2T_3 + TT_{23}\right]. \quad (29)$$

Furthermore, no matter which sector is the most capital intensive, $V_{21}+V_{23} \geq 0$ *for all* $\gamma_t \in (\underline{\gamma}, \overline{\gamma})$.

Proof. By definition $V(k_t, k_{t+1}, X_t) = \left[T(k_t, k_{t+1} - (1-\delta)k_t, X_t)\right]^\alpha/\alpha$. Under Assumption 4 we can compute the following derivative: $V_2 = T^{\alpha-1}T_2$, $V_{21} = (\alpha-1)T^{\alpha-2}[T_1 - (1-\delta)T_2]T_2 + T^{\alpha-1}[T_{21} - (1-\delta)T_{22}]$ and $V_{23} = T^{\alpha-2}\left[(\alpha-1)T_2T_3 + TT_{23}\right]$. Using the expressions of V_{21} and V_{23}, we get:

$$V_{21} + V_{23} = (\alpha-1)T^{\alpha-2}[T_1 - (1-\delta)T_2 + T_3]T_2 + T^{\alpha-1}[T_{21} - (1-\delta)T_{22} + T_{23}]$$

Under Assumptions 2-3 we can show that $T_3 = \mathcal{F}^1_1\frac{\partial k^1}{\partial X} + X\mathcal{F}^1_2\frac{\partial l^1}{\partial X} + l^1\mathcal{F}^1_2$. By definition, $y = \mathcal{F}^2(k^2, l^2X)$. Hence,

$$0 = \mathcal{F}^2_1\frac{\partial k^2}{\partial X} + \mathcal{F}^2_2\frac{\partial l^2}{\partial X} + l^2\mathcal{F}^2_2. \quad (30)$$

Under the full employment of productive resources we have

$$\frac{\partial k^1}{\partial X} = -\frac{\partial k^2}{\partial X}, \quad (31)$$
$$\frac{\partial l^1}{\partial X} = -\frac{\partial l^2}{\partial X}. \quad (32)$$

Furthermore, the envelope theorem tells us that $T_2(k, y, X) = -q$, and by definition,

$$q = \frac{\mathcal{F}^1_1}{\mathcal{F}^2_1} = \frac{\mathcal{F}^1_2}{\mathcal{F}^2_2}. \quad (33)$$

Substituting (31), (32) and (33) into (34), and using (30), we get

$$T_3 = q\mathcal{F}^2_2 > 0. \quad (34)$$

We now determine the sign of $T_{21} + T_{23}$. Using (10), we have

$$T_{21} + T_{23} = T_{21}\left(1 - \frac{k^1}{l^1X}\right) + \frac{2l^2(\mathcal{F}^1_{12} + q\mathcal{F}^2_{12})}{\mathcal{F}^2_1}.$$

Under Assumption 2, $2l^2(\mathcal{F}_{12}^1 + q\mathcal{F}_{12}^2)/\mathcal{F}_1^2 > 0$. It follows that $V_{21} + V_{23} > 0$ if $T_{21} + T_{23} > 0$. And $T_{21} + T_{23} > 0$ if $T_{21}(1 - k^1/l^1 X) \geq 0$. So, if $T_{21}(1 - k^1/l^1 X) \geq 0$, then $V_{21} + V_{23} > 0$. Along a BGP,

$$1 - \frac{k^1}{l^1 X} = \frac{l^1 k - k^1}{l^1 k} = \frac{l^2}{k}\left(\frac{k^2}{l^2} - \frac{k^1}{l^1}\right).$$

Hence, on using (8), we obtain

$$T_{21}\left(1 - \frac{k^1}{l^1 X}\right) = \frac{\mathcal{F}_{12}^1 \mathcal{F}_{12}^2 q \mathcal{F}^2 l^1 l^2}{\Delta k^1 k^2 k}\left(\frac{k^2}{l^2} - \frac{k^1}{l^1}\right)^2 \geq 0.$$

■

We can see from (8) that when the consumption good sector is more capital intensive for all $\gamma_t \in [0, \overline{\gamma}]$, then $T_{12} < 0$. The sign of V_{21} is ambiguous. So, if V_{21} changes sign at some value of $\gamma \in [0, \overline{\gamma}]$, then f does not have a well defined inverse. If one assumes that factor intensity reversal takes place V_{21} may also change sign. [6, 7]

Lemma 4 *Let Assumptions 1-5 be satisfied. Then there exists a map θ defined from $[\underline{\gamma}, \overline{\gamma}]$ onto $[\underline{\gamma}, \overline{\gamma}]$, such that*

$$\gamma_t = \theta(\gamma_{t+1}). \tag{35}$$

Proof. Define

$$f(\gamma_{t+1}) = V_1(1, \gamma_{t+1}, 1)/\beta, \tag{36}$$
$$g(\gamma_t) = -\gamma_t^{1-\alpha} V_2(1, \gamma_t, 1), \tag{37}$$

[6]With full depreciation, i.e. for $\delta = 1$ looking at the expression for V_{21} derived in Lemma 3 we see that V_{21} can become negative if utility is not too concave, i.e. for α close to unity.

With partial depreciation, i.e. for $\underline{\gamma} < \delta < 1$ $V_{21} = T^{\alpha-2}(\alpha - 1)T_2[T_1 - (1-\delta)T_2] + T^{\alpha-1}[T_{21} - (1-\delta)T_{22}]$.

For $V_{21}$to be negative we need both:

(i) $-T_{21} > (1-\delta)T_{22}$ and

(ii) $-T[T_{21} - (1-\delta)T_{22}] > (\alpha - 1)T_2[T_1 - (1-\delta)T_2]$.

[7]Assumption 6 holds for a simple economic example. If $c = (k^1)^\sigma (l^1 X)^{1-\sigma}$ and $y = (k^2)^{1/2}(l^2 X)^{1/2}$ then $\widehat{\gamma} = 1$. In this case it can be shown that if $k^1/l^1 > k^2/l^2$ there exists $\overset{0}{\gamma} \in [0, \overline{\gamma}]$ such that $\mathrm{V}_{21}(1, \overset{0}{\gamma}, 1) = 0$. Full request of this statement can be obtained on request.

where $f : [\underline{\gamma}, \overline{\gamma}] \rightarrow J \sqsubseteq \Re_+^*$ and $g : [\underline{\gamma}, \overline{\gamma}] \rightarrow I \sqsubseteq \Re_+^*$.
Using (36) and (37), the Euler equation (5) can be written as $-g(\gamma_t) + f(\gamma_{t+1}) = 0$.

$$g' = (1-\alpha)\gamma_t^{-\alpha} V_2(1, \gamma_t, 1) + \gamma^{1-\alpha} V_{22}(1, \gamma_t, 1). \tag{38}$$

From the Euler theorem on homogenous functions we have

$$(\alpha - 1)V_2(1, \gamma_t, 1) = V_{21}(1, \gamma_t, 1) + V_{22}(1, \gamma_t, 1)\gamma_t + V_{23}(1, \gamma_t, 1).$$

Hence

$$g' = V_{21}(1, \gamma, 1) + V_{23}(1, \gamma, 1) > 0$$

g is one to one. From the definition of g we see that range$(g) = I$. Thus, g has a well defined inverse g^{-1}. We can rewrite (38) as $\gamma_t = \theta(\gamma_{t+1})$, where $g^{-1}\left[f(\gamma_{t+1})\right]$. From this definition, it follows that $\theta : [\underline{\gamma}, \overline{\gamma}] \rightarrow [\underline{\gamma}, \overline{\gamma}]$. ■

It can be shown that the sign of $\theta'(\gamma_t)$ depends on the sign of V_{12}.

Lemma 5 *Let Assumptions 1-5 be satisfied. Let* $\theta : [\underline{\gamma}, \overline{\gamma}] \rightarrow [\underline{\gamma}, \overline{\gamma}]$ *be defined as in (35). Then,*

$$\theta'(\gamma_{t+1}) = \frac{\beta \gamma_t^{\alpha} V_{21}(1, \gamma_{t+1}, 1)}{V_{21}(1, \gamma_t, 1) + V_{23}(1, \gamma_t 1,)}.$$

Under the results of Lemma 3 it follows that $\theta'(\gamma_{t+1}) \gtreqless 0$ if and only if $V_{12}(1, \gamma_{t+1}, 1) \gtreqless 0$. If the investment good sector is more capital intensive, we see from Lemmas 1 and 3 that $V_{12}(1, \gamma_{t+1}, 1) > 0$ for all $\gamma_{t+1} \in [\underline{\gamma}, \widehat{\gamma})$. However the results of Lemmas 1 and 3 indicate that the sign of V_{12} is ambiguous for $\gamma_{t+1} \in [\underline{\gamma}, \widehat{\gamma})$. Factor intensity reversal does not guarantee the unimodality of $\theta'(\gamma_t)$. We first make the following assumption:

Assumption 9 : There exists a unique $\overset{0}{\gamma}$, where $0 < \overset{0}{\gamma} < \overline{\gamma}$, such that V_{21} changes sign at $\overset{0}{\gamma}$.

5.1 Unimodality of θ

To establish the unimodality of θ we need to show θ is continuous on $[\underline{\gamma}, \overline{\gamma}]$, strictly monotonically increasing on some open interval $(\underline{\gamma}, \gamma^*) \subset [\underline{\gamma}, \overline{\gamma}]$ and strictly monotonically decreasing on the interval $(\gamma^*, \overline{\gamma}) \subset [\underline{\gamma}, \overline{\gamma}]$.

Lemma 6 *Let Assumptions 1-5 and 9 be satisfied. Let $\theta : [\underline{\gamma}, \overline{\gamma}] \to [\underline{\gamma}, \overline{\gamma}]$ be defined as in (35). Then, θ is unimodal if one of the following holds.*
(i) Factor reversal occurs once at some $\hat{\gamma} \in (\underline{\gamma}, \overline{\gamma})$.
(ii) For all $\gamma_t \in (\underline{\gamma}, \overline{\gamma})$ the consumption good sector is capital intensive.

Proof. :

From the definition of θ given in Lemma 4, θ is continuous as a composite function of two continuous functions.

(i) Suppose first that for all $\gamma_{t+1} \in [\underline{\gamma}, \widehat{\gamma})$ the capital good sector is more capital intensive, i.e. suppose that for all $\gamma_{t+1} \in [\underline{\gamma}, \widehat{\gamma})$

$$\frac{k^1(1, \gamma_{t+1}, 1)}{l^1(1, \gamma_{t+1}, 1)} < \frac{k^2(1, \gamma_{t+1}, 1)}{l^2(1, \gamma_{t+1}, 1)},$$

From Lemma 5 we know that $\theta'(\gamma_{t+1}) \gtrless 0$ if and only if $V_{12}(1, \gamma_{t+1}, 1) \gtrless 0$. If for all $\gamma_{t+1} \in [\underline{\gamma}, \widehat{\gamma})$

$$\frac{k^1(1, \gamma_{t+1}, 1)}{l^1(1, \gamma_{t+1}, 1)} < \frac{k^2(1, \gamma_{t+1}, 1)}{l^2(1, \gamma_{t+1}, 1)},$$

then it follows that $T_{21} > 0$, $V_{21} > 0$ and $\theta'(\gamma_{t+1}) > 0$ for all $\gamma_{t+1} \in [\underline{\gamma}, \widehat{\gamma})$. Under Assumption 9 there exists a unique $\overset{0}{\gamma}$, where $\underline{\gamma} < \overset{0}{\gamma} < \overline{\gamma}$, such that V_{21} changes sign at $\overset{0}{\gamma}$. It follows that $V_{12}(1, \gamma_{t+1}, 1) < 0$ and that $\theta'(\gamma_{t+1}) < 0$ for all $\gamma_{t+1} \in (\overset{0}{\gamma}, \overline{\gamma})$. A necessary condition to be true is that $T_{21}(1, \overset{0}{\gamma}, 1) < 0$. Under the result so Lemma 1 this implies that

$$\frac{k^1(1, \overset{0}{\gamma}, 1)}{l^1(1, \overset{0}{\gamma}, 1)} > \frac{k^2(1, \overset{0}{\gamma}, 1)}{l^2(1, \overset{0}{\gamma}, 1)}.$$

Factor reversal has happened at $\overset{0}{\gamma}$. Under Assumption 9 this means that

$$\frac{k^1(1, \gamma_{t+1}, 1)}{l^1(1, \gamma_{t+1}, 1)} > \frac{k^2(1, \gamma_{t+1}, 1)}{l^2(1, \gamma_{t+1}, 1)} \text{ for all } \gamma_{t+1} \in (\overset{0}{\gamma}, \overline{\gamma}).$$

(ii) Suppose now that for all $\gamma_{t+1} \in [\underline{\gamma}, \overline{\gamma}]$ the consumption good sector is more capital intensive, i.e. suppose that for all $\gamma_{t+1} \in [\underline{\gamma}, \overline{\gamma}]$

$$\frac{k^1(1, \gamma_{t+1}, 1)}{l^1(1, \gamma_{t+1}, 1)} > \frac{k^2(1, \gamma_{t+1}, 1)}{l^2(1, \gamma_{t+1}, 1)},$$

From Lemma 5 we know that $\theta^{'}(\gamma_{t+1}) \gtrless 0$ if and only if $V_{12}(1, \gamma_{t+1}, 1) \gtrless 0$. Under Assumption 9 there exists a unique $\overset{0}{\gamma}$, where $\underline{\gamma} < \overset{0}{\gamma} < \overline{\gamma}$, such that V_{21} changes sign at $\overset{0}{\gamma}$. It follows that $V_{12}(1, \gamma_{t+1}, 1) \gtrless 0$ for $\gamma_{t+1} \in (\overset{0}{\gamma}, \overline{\gamma}]$. Suppose that $V_{12}(1, \gamma_{t+1}, 1) < 0$ for $\gamma_{t+1} \in (\overset{0}{\gamma}, \overline{\gamma}]$.and that $V_{12}(1, \gamma_{t+1}, 1) > 0$ for $\gamma_{t+1} \in [\underline{\gamma}, \overset{0}{\gamma})$. It follows that $\theta^{'}(\gamma_{t+1}) < 0$ for all $\gamma_{t+1} \in (\overset{0}{\gamma}, \overline{\gamma})$ and that $\theta^{'}(\gamma_{t+1}) > 0$ for all $\gamma_{t+1}[\underline{\gamma}, \overset{0}{\gamma})$. The same argument can be used to prove that $\theta^{'}(\gamma_{t+1}) > 0$ for all $\gamma_{t+1} \in (\overset{0}{\gamma}, \overline{\gamma})$ and that $\theta^{'}(\gamma_{t+1}) < 0$ for all $\gamma_{t+1}[\underline{\gamma}, \overset{0}{\gamma})$ if one assume that $V_{12}(1, \gamma_{t+1}, 1) > 0$ for $\gamma_{t+1} \in (\overset{0}{\gamma}, \overline{\gamma}]$ and that $V_{12}(1, \gamma_{t+1}, 1) < 0$ for $\gamma_{t+1} \in [\underline{\gamma}, \overset{0}{\gamma})$.

■

In other words, unicity of factor reversal is no longer a necessary and sufficient condition for the unimodality of θ. An alternative sufficient condition for unimodality is that the consumption good sector is capital intensive for all $\gamma_t \in [0, \overline{\gamma}]$ and that Assumption 9 is satisfied. A similar result is obtained in Nishimura and Yano (1995, 2000) and Nishimura et.al. (1994).

5.2 Expansiveness of θ, and chaos

In this section we show that if the discount factor is bounded below then the slope of θ is everywhere greater than unity for all feasible values of the growth factor. We then establish the main result on ergodic oscillations and geometric sensitivity.

Lemma 7 *Let Assumptions 1-5 and 9 be satisfied. Then a necessary condition for $\theta : [\underline{\gamma}, \overline{\gamma}] \to [\underline{\gamma}, \overline{\gamma}]$ to be expansive is*

$$\left| \frac{V_{21}(1, \theta(\gamma_{t+1}), 1) + V_{23}(1, \theta(\gamma_{t+1}), 1)}{\theta(\gamma_{t+1})^{\alpha} V_{21}(1, \gamma_{t+1}, 1)} \right| < 1 \text{ for all } \gamma_{t+1} \in (\underline{\gamma}, \widehat{\gamma}) \cup (\widehat{\gamma}, \overline{\gamma}). \quad (39)$$

A necessary and sufficient condition for $\theta : [\underline{\gamma}, \overline{\gamma}] \to [\underline{\gamma}, \overline{\gamma}]$ *to be expansive is that* $\beta \in (\beta_{\min}, \beta_{\max})$*, where*

$$\beta_{\min} = \max_{\gamma_{t+1} \in (0,\widehat{\gamma}) \cup (\widehat{\gamma}, \overline{\gamma})} \left| \frac{V_{21}(1, \theta(\gamma_{t+1}), 1) + V_{23}(1, \theta(\gamma_{t+1}), 1)}{\theta(\gamma_{t+1})^{\alpha} V_{21}(1, \gamma_{t+1}, 1)} \right|,$$
$$\beta_{\max} = \frac{1}{[\mathcal{F}^2(1,1)]^{\alpha}}.$$

Proof. From the definition of θ it follows that θ is twice continuously differentiable on both $(\underline{\gamma}, \widehat{\gamma})$ and $(\widehat{\gamma}, \overline{\gamma})$. θ is expansive if $\left|\theta'(\gamma_{t+1})\right| > 1$ for all $\gamma_{t+1} \in (1, \widehat{\gamma}) \cup (\widehat{\gamma}, \overline{\gamma})$. Or, equivalently, if

$$\left| -\frac{V_{21}(1, \theta(\gamma_{t+1}), 1) + V_{23}(1, \theta(\gamma_{t+1}), 1)}{\theta(\gamma_{t+1})^{\alpha} V_{21}(1, \gamma_{t+1}, 1)} \right| < \beta \text{ for all } \gamma_{t+1} \in (\underline{\gamma}, \widehat{\gamma}) \cup (\widehat{\gamma}, \overline{\gamma}).$$

It follows that

$$\beta_{\min} = \max_{\gamma_{t+1} \in (1,\widehat{\gamma}) \cup (\widehat{\gamma}, \overline{\gamma})} \left| -\frac{V_{21}(1, \theta(\gamma_{t+1}), 1) + V_{23}(1, \theta(\gamma_{t+1}), 1)}{\theta(\gamma_{t+1})^{\alpha} V_{21}(1, \gamma_{t+1}, 1)} \right|.$$

Since, by assumption, we have $\beta < 1$, a necessary condition for expansiveness is

$$\left| -\frac{V_{21}(1, \theta(\gamma_{t+1}), 1) + V_{23}(1, \theta(\gamma_{t+1}), 1)}{\theta(\gamma_{t+1})^{\alpha} V_{21}(1, \gamma_{t+1}, 1)} \right| < 1.$$

For the transversality condition to be satisfied, $\beta \overline{\gamma}^{\alpha} < 1$. As $\overline{\gamma} = \mathcal{F}^2(1,1)$, we have

$$\beta_{\max} = \frac{1}{\mathcal{F}^2(1,1)^{\alpha}}.$$

■

Collecting these results together we have the following result.

Proposition 6 *Let Assumptions 1-5 and 9 be satisfied.* $\theta : [\underline{\gamma}, \overline{\gamma}] \to [\underline{\gamma}, \overline{\gamma}]$ *is chaotic both in the sense of ergodic oscillations and geometric sensitivity if the following four conditions hold:*
(i) Either there is factor intensity reversal atleast once or the consumption sector is always capital intensive,
(ii) V_{21} *changes sign once,*
(iii) (39) is satisfied for all $\gamma_{t+1} \in (0, \widehat{\gamma}) \cup (\widehat{\gamma}, \overline{\gamma})$ *and*
(iv) The discount factor β *satisfies* $\beta \in (\beta_{\min}, \beta_{\max})$*.*

6 Conclusion

The intuition of the result is as follows (see Benhabib and Nishimura (1985) and Reichlin (1997)). It is easiest to see it in the case of full discounting. At time t, set $k = k_t$ and $y = k_{t+1}$. An oscillating trajectory will have: $k_t < y_t = k_{t+1} > y_{t+1} = k_{t+2}$. This possibility can arise when the consumption sector is more capital intensive or if there is capital intensity reversal. To induce the oscillation, we need

1. An increase of the capital stock from k_t to $k_{t+1} > k_t$ (a shift outward of the PPF)
2. to generate a fall of the investment output from y_t to $y_{t+1} < y_t$
3. and a rise of consumption from c_t to $c_{t+1} > c_t$.

The Rybczynski Theorem tells us that an ncrease in a sector's endowment of a factor will cause an increase in output of the good which uses that factor intensively, and a decrease in the output of the other good. Under the Rybczynski theorem, this can happen if the consumption sector is more capital intensive than the investment sector.

What about factor intensity reversal? Consider two periods where $k_{t+1} > k_t$. Then the Rybczynski theorem will imply that the capital stock will be increasing. At some time market clearing and static efficiency will imply that the it is desirable to increase consumption output and decrease investment output. This will happen when there is a factor intensity reversal.

This is the intuition for having a 2-period cycle. Having externalities introduces an additional non-linearity so that there can be chaos where there were only cycles. The Harrod-Neutrality is important as the homogeneity properties enable us to get a characterization of the dynamics.

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