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Abstract

Recent research has suggested that natural selection in financial markets may be a very slow process, taking hundreds of years. We show in a general equilibrium model that it may be much faster in markets with large state spaces. In many cases, the time it takes to wipe out irrational investors is inversely proportional to the number of stocks in the market, i.e., if it takes about 500 years with one stock, it takes about one year with 500 stocks. Thus, theoretically, natural selection can be very efficient even when there is high market uncertainty. The speed of the natural selection process is a known function of irrational investors' sentiment and of the real characteristics of the stock market. According to a calibration to U.S. stock data, it takes about fifty years for an irrational investor to be wiped out. This is in line with studies of individual investor underperformance.

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1 Introduction

The idea of natural selection, that rational investors will outperform irrational investors and eventually dominate the market, dates back to Alchian (1950) and Friedman (1953). However, although the idea is simple and intuitive, it has been shown to be false under several conditions. Over-optimistic investors may invest a larger share of their wealth in risky assets and ultimately dominate the market when prices are set exogenously (DeLong, Shleifer, Summers, and Waldman, 1990; Blume and Easley, 1992). Similarly, irrational investors with a lower consumption-to-savings ratio than rational investors may dominate the market. Moreover, even when rational investors eventually dominate the market measured by fraction of wealth, irrational investors may still have nonnegligible impact on prices (Kogan, Ross, Wang, and Westerfield, 2006).

However, when rational and irrational investors have identical utilities, natural selection will occur except under special conditions. In general equilibrium with complete markets, Sandroni (2000) shows that rational investors will eventually dominate the market under general conditions if agents have identical intertemporal discount factors (although Blume and Easley 2006 recently showed that in incomplete markets, this result may not hold). Loewenstein and Willard (2006) point out that models of the type of DeLong, Shleifer, Summers, and Waldman (1990) implicitly have to allow for real transfers of production (between risk-less storage and risky technology) and for changes in aggregate consumption.

An important *quantitative* question in markets where natural selection occurs is: How long will it take to wipe out irrational investors? The answer to this question is crucial to our understanding of stock markets. If it takes a limited amount of time, say less than a decade, this may warrant a rational equilibrium view of the stock market. Rational equilibrium pricing prevail most of the time, except for in periods of temporary disequilibrium. If, on the other hand, it takes hundreds of years to wipe out irrational investors, the correct model must be one of disequilibrium pricing.

A recent strand of research suggests that the natural selection process may be *very* slow. Building on the general equilibrium literature with heterogeneous investors (see, e.g., Detemple and Murthy 1994 and Basak 2000), Yan (2006) analyzes a Lucas model with one risky and one risk-free asset, and shows that it may take several hundred years before a rational representative investor dominates an irrational one. Similar results are derived in Dumas, Kurshev, and Uppal (2005), under slightly different assumptions, and used in Branger, Schlag, and Wu (2006).

A slow selection process is somewhat alarming for believers in rational asset pricing theory.

Moreover, it does not fit well with studies documenting underperformance by unsophisticated investors in the market. For example, individual investors in the study by Barber, Lee, Liu, and Odean (2005) underperform institutional investors by about 2.1% per year, which implies a 50% underperformance in a 30-year horizon. Therefore, the quantitative question may not be adequately addressed by current theoretical models.

One property of these current models is that they are based on severely restricted state spaces, i.e., they have only one stock and one bond. One may ask if the results would change in a more realistic model, with a large state space (e.g., with many stocks). Of course, in conventional finance theory, with one single representative investor, little is changed by taking into consideration additional states beyond what is spanned by the market portfolio. However, the situation is different when agents disagree and there is no-longer a representative investor.

What might change in a richer model with respect to the speed of natural selection? A priori this is not clear. On the one hand, one could argue that if it is difficult to take advantage of irrational traders quickly in a simplified model of the stock market, it must be almost impossible in a more complex market, where the irrationality is spread out over a huge state space. On the other hand, one could argue that it is exactly in complex markets that rationality will pay off, as a larger state space allows rational investors to separate their strategies from irrational traders to a higher degree.

In this paper we show that the latter intuition is correct, and that natural selection may indeed be a much faster process than that suggested by models with only one risky asset. The intuition is simple: Consider a one-factor model with multiple firms, in which irrational investors are slightly bullish about the prospects of half of the firms and slightly bearish about the other half, leading to slight overpricing in half of the stocks and underpricing in the other half. The natural arbitrage strategy for a rational investor in this case is to form a long-short portfolio, eliminating all the market risk and almost all the idiosyncratic risk. Thereby, the investor obtains a slight excess return with almost no risk. The reason why the first intuition fails is that the rational investors have no need to "find" the markets in which the irrational investors are trading. The irrational traders' sentiments are automatically revealed by the prices of the different assets.

The main contribution of this paper is to formalize this idea in a general equilibrium framework and study its implications for the speed of natural selection. We call this speed the market's *arbitrageability*,[1] and we show that it can be conveniently measured in our model. We study

[1]There are never pure arbitrage opportunities in our economy, but the arbitrageability quantifies how "close"

a simplified model of value creation compared with the standard Lucas model. Our economy is observationally equivalent to a Lucas economy with a modified value processes. Under quite general conditions, the time it takes to wipe out irrational investors is inversely proportional to the number of stocks. For example, if it takes 500 years in a market with one stock, it takes less than one year in a market, with the same market Sharpe ratio, but with 500 stocks. Thus, although the model with one risky asset qualitatively gives the same result as the multi-asset model (extinction of irrational traders), the quantitative difference is striking. This result is robust to various assumptions about the sentiments of irrational investors and the structure of the stock market. The only cases for which the natural selection process is not faster is when there is no spread of investor sentiment across stocks, or when stock returns are uncorrelated. In these cases, the model collapses to the one stock model.

A second contribution of this paper is to calibrate the model to U.S. stock data, and estimate the arbitrageability of the market. Under the ideal conditions of the theoretical model, trading in the S&P 500 universe is extremely hazardous for unsophisticated investors. At the other extreme, with effectively only one stock, prices are informationally quite inefficient, as sentiment investors influence prices for a long time. Our simple calibration points to somewhere in-between. The time it takes to wipe out irrational investors is about fifty years, in line with studies of individual investor behavior in the stock market (Barber and Odean, 2001; Barber, Lee, Liu, and Odean, 2005).

The paper is organized as follows. In the next section we introduce the model. For expositional reasons we begin with a one-stock, one representative agent set-up, which we then generalize to multiple stocks and two agents. We then derive the results for the speed of the natural selection process in Section 3. In Section 4, we do a simple calibration of the model to the U.S. stock market. Finally, in Section 5, we make some concluding remarks. Details and proofs are left to the appendix.

2 The model

2.1 One stock and one investor

For expositional simplicity, we first study the case with one stock and infinitely lived representative investor with time-separable log utility. This case will show how the general equilibrium set-up works and we will subsequently use it for comparison with the multi-stock, multi-investor

the economy is to allowing pure arbitrage.

results.

In a standard manner, we assume a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and an $\mathcal{F}_t$-adapted standard Brownian motion B_t, satisfying the usual conditions. The instantaneous return of investing in the stock at price P is:

$$\tilde{\mu} = \mu_t^F dt + \sigma_t^F dB_t, \qquad \text{where } \mu_t^F \stackrel{\text{def}}{=} \frac{g(c_t)}{P_t}, \quad \sigma_t^F \stackrel{\text{def}}{=} \frac{\sigma(c_t)}{P_t}. \tag{1}$$

Here g and σ are exogenously given functions, P_t is the stock price at time t, c_t is the instantaneous consumption per unit time, which is equal to the firm's production, and B_t is a standard Brownian motion. In the appendix we give a motivation for such a stochastic return process using a simple production economy. Unless stated, all processes are assumed to be $\mathcal{F}_t$-adapted. Moreover, all conditions are assumed to hold almost surely.

The representative investor has time-separable log expected utility over consumption:

$$U = E\left[\int_0^\infty e^{-\rho t} \log(c_t) dt\right]. \tag{2}$$

There is also a risk-free bond available in zero net supply, offering an instantaneously risk-free interest rate of r_t. For notational compactness we suppress dependencies when obvious, e.g., writing r instead of r_t for interest rates, etc. The investors wealth, W_t then satisfies the stochastic process:

$$dW_t = -c_t dt + W_t\big(r dt + \alpha_t(\mu^F dt + \sigma^F dB_t - r dt)\big), \tag{3}$$

where α_t is the fraction of wealth invested in the risky asset at time t. We make the natural restriction to only consider feasible investment strategies, i.e., strategies for which $W_t \geq 0$.

The consumption market, per definition, clears at each point in time. An equilibrium can therefore be described by the following three conditions: At each point in time

1. The representative investor solves the optimal consumption problem:

$$\max_{c_t, \alpha_t} E\left[\int_t^\infty e^{-\rho s} \log(c_s) ds\right], \text{ subject to (3)},$$

2. The demand for the risk-free asset is zero: $W_t = P_t$,

3. The stock market clears: $\alpha_t W_t = P_t$, (i.e. — by 2. — $\alpha_t = 1$).

The following proposition characterizes the unique equilibrium in this economy:

Proposition 1 *Given an investor with initial wealth W_0, there is a unique equilibrium, in which the wealth is:*

$$dW_t = \big(g(c_t) - \rho W_t\big)dt + \sigma(c_t)dB_t, \tag{4}$$

the interest rate is

$$r_t = \frac{g(c_t)W_t - \sigma(c_t)^2}{W_t^2},$$

the price is $P_t = W_t$, and the instantaneous consumption is $c_t = \rho W_t$.

In the standard Lucas set-up, g and σ are linear in c_t. We deviate from this set-up by assuming that g and σ are constants. This leads to an Ornstein-Uhlenbeck process for wealth.[2] The deviation is needed for us to be able to solve for the case with multiple stocks. The reason is that the dynamic systems become prohibitively difficult to solve in the standard set-up: With N stocks, the general equilibrium formulation leads to a system of N coupled non-linear parabolic PDEs. Only in special cases can the solution be found.[3] We shall see that only relative wealth levels of different investor groups are important for the speed of the natural selection process, so this is no major restriction. This should come as no surprise as investors have logarithmic utility, so the total wealth level is obviously unimportant.

Thus, in our set-up, wealth and price oscillate around the steady state wealth level $\bar{W} = g/\rho$ and the corresponding consumption $\bar{c} = g$, depending on the realization of the real economy. Using the relation $\bar{W} = g/\rho$, it is easy to see that the steady-state interest rate is $\bar{r} = \rho - \rho^2\sigma^2/g^2$ and the steady state Sharpe ratio is $\bar{S} = \rho\sigma/g$.

For subsequent comparison, we introduce a numerical example: An intertemporal discount factor of $\rho = 10\%$, value creation drift of $g = 1$ and volatility $\sigma = 2$ lead to the following steady state solution $\bar{W} = \bar{P} = 10$, $\bar{r} = 6\%$, and a steady state Sharpe ratio of $\bar{S} = 0.2$.

[2]For very low levels of c_t, we assume that $\sigma(c_t)$ decreases to zero, to ensure that wealth always is strictly positive. For example, we can assume that $\sigma(c_t) = \sigma$, for $c_t \geq \epsilon$ and $\sigma(c_t) = c_t\sigma/\epsilon$ for $c_t < \epsilon$. This ensures that the SDE (4), with initial condition $W_0 = w$ has a unique strictly positive strong solution, see Karatzas and Shreve (1998), pp. 287-289. This modification has negligible impact for the quantitative questions we wish to analyze, as discussed in the appendix.

[3]We know of only two papers that analyze such cases: Cochrane, Longstaff, and Santa-Clara (2005) who assume one representative investor with log utility and two stocks and Walden (2006), who assumes an OLG structure with short-lived identical CARA investors. Neither approach is applicable to the problem we wish to analyze. Our approach leads to an asymptotically stationary distribution of total wealth, as opposed to growing expected total wealth in standard set-ups. In a different (slightly more complicated) set-up, we obtain identical results with nonstationary, growing expected total wealth.

We make three remarks: First, as the Sharpe ratio is *not* time-scale invariant, it can be used to calibrate the model to the true stock market. For example, if the real stock market has an annual Sharpe ratio of 0.2, then the interpretation of t in our numerical example is that it measures years.

Second, for wealth levels below σ^2/g, the interest rate is negative, reflecting the fact that the representative investor's risk aversion is so high, that the risk-free asset must offer negative rates of return for him invest all his money in the stock. As we are working with real variables, this could occur if there is inflation.

Third, the economy is observationally equivalent to a Lucas economy with a tree paying a dividend stream following the Ornstein-Uhlenbeck process $dD = \rho(g - D)dt + \rho\sigma dB$. The representative investor with wealth W_t holds the claim to the whole technology output, which provides him with dividend flow to consume. At any time t, this claim costs exactly P_t, since it allows the investor to keep the production assets, consume Ddt, and be left with P_{t+dt}.

2.2 Multiple stocks and two investors

We generalize the model to multiple stocks and investors. We assume a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, and N-dimensional $\mathcal{F}_t$-adapted standard Brownian motions $\mathbf{B}_t = (B_{1,t}, \dots, B_{N,t})'$, (where $'$ denotes transpose) satisfying the usual conditions, and $Cov(dB_{i,t}, dB_{j,t}) = \rho_{ij}\, dt$.

There are N firms, $i = 1, \dots, N$, with stock selling at price P_i. Without loss of generality we assume that each stock is in unit supply. Similar to the one-stock case, the instantaneous return of each stock is

$$\tilde{\mu}_i = \mu_i^F dt + \sigma_i^F dB_{i,t}, \qquad \text{where} \qquad \mu_i^F \stackrel{\text{def}}{=} \frac{g_i}{P_i} \text{ and } \sigma_i^F \stackrel{\text{def}}{=} \frac{\sigma_i}{P_i}, \tag{5}$$

and where g_i and σ_i are firm-specific drift and volatility terms respectively. We define the vectors $\mathbf{g} \stackrel{\text{def}}{=} (g_1, \dots, g_N)'$ and $\mu^F = (\mu_1^F, \dots, \mu_N^F)'$, and the matrix $\Sigma = [\sigma_{i,j}]$, with $\sigma_{i,j} = \sigma_i \sigma_j \rho_{ij}$. The pair $(\mathbf{g}, \Sigma)$ thereby completely characterizes the real part of the economy. We assume that Σ is invertible. The price at time t is represented by the price vector $\mathbf{P}_t = (P_1, \dots, P_N)'$.

There are two investors with time-separable log expected utility of consumption (2) and the same intertemporal discount rate, ρ. These investors are enumerated by $k \in \{1, 2\}$. Investor 1 is rational and knows the correct $(\mathbf{g}, \Sigma)$. His belief is therefore $\mathbf{g}^1 = \mathbf{g}$.

Investor 2 is irrational. He agrees with investor 1 about the correct Σ (which can be motivated by it being easy to infer volatilities and covariances in an arbitrary short time period in continuous time). However, he mistakes the drift term for $\mathbf{g}^2 = \mathbf{g} + \delta$, where $\delta \in \mathbb{R}^N$ is a constant. Moreover, the irrational investor does not update his beliefs over time (that is why he is irrational). The term δ is the irrational investor's *sentiment vector.* It represents the multi-stock version of the irrationality assumption made by Yan (2006). We do not formally model the source of the irrational investor's sentiment, but refer to the vast literature on potential sources for such irrationality, see e.g., De Bond and Thaler (1985) and De Bond (1993). We use the notation $g_i^k = [\mathbf{g}^k]_i$. At each point in time, the two investors, of course, agree on the price, but they disagree about the return prospects of investing in stock i, each investor believing it is:

$$\tilde{\mu}_i^k = \mu_i^{Fk} dt + \sigma_i^F dB_{it}^k, \qquad \text{where } \mu_i^{Fk} \stackrel{\text{def}}{=} \frac{g_i^k}{P_i}, \qquad k \in \{1, 2\}, \tag{6}$$

(the k superscript over the Brownian processes is added, as they will not agree about the realization of the random term as time progresses). We denote the different expectations for the two investors by E_k, $k \in \{1, 2\}$. Finally, there is a risk-free bond in zero net supply offering instantaneous return r. We define the vector of drifts $\mu^{Fk} = (\mu_1^{Fk}, \ldots, \mu_N^{Fk})'$, and the volatility matrix $\mathbf{S} = diag(\sigma_1^F, \ldots, \sigma_N^F)'$.[4] Finally, let the vectors $\alpha_{1,t}, \alpha_{2,t} \in \mathbb{R}^N$ denote the fraction of wealth invested in different stocks by investors 1 and 2 respectively.

The investors' wealth processes, $W_{k,t}$ follow the true stochastic processes:

$$dW_{k,t} = -c_{k,t} dt + W_{k,t}(rdt + \alpha_{k,t}'(\mu^F dt + \mathbf{S} d\mathbf{B}_t - r\mathbf{1}_N dt)), \tag{7}$$

whereas their *perceived* wealth processes are

$$dW_{k,t} = -c_{k,t} dt + W_{k,t}(rdt + \alpha_{k,t}'(\mu^{F,k} dt + \mathbf{S} d\mathbf{B}_{k,t} - r\mathbf{1}_N dt)), \tag{8}$$

i.e., the irrational investor perceives the wrong drift-term and the wrong realization of the Brownian motion. Here, $\mathbf{1}_N$ is the unity vector with N elements, $\mathbf{1}_N = (1, \ldots, 1)'$.

An equilibrium is described by the following conditions: At each point in time,

1. Investor $k \in \{1, 2\}$ solves the optimal consumption problem:

$$\max_{c_{k,t}, \alpha_{k,t}} E_k \left[\int_t^\infty e^{-\rho s} \log(c_s) ds \right], \text{ subject to (8).}$$

[4] Here, $diag(x_1, \ldots, x_N)$ denotes a diagonal $N \times N$ matrix with x_i as its ith diagonal element.

2. The net demand for the risk-free asset is zero: $W_{1,t} + W_{2,t} = \mathbf{1}'_N \mathbf{P}_t$.

3. The stock market clears: $\alpha_{1,t} W_{1,t} + \alpha_{2,t} W_{2,t} = \mathbf{P}_t$.

We define the total wealth process $W_t = W_{1,t} + W_{2,t}$. The following proposition characterizes the equilibrium completely:

Proposition 2 *The unique equilibrium wealth process for two investors (with initial wealth $W_{1,0}$ and $W_{2,0}$ respectively), satisfies*

$$\frac{dW_{1,t}}{W_{1,t}} = -\rho dt + \frac{A}{W_t} dt + \frac{1}{W_t^2}\Big(- BW_{2,t} + CW_{2,t}^2 \Big) dt + \frac{1}{W_t}\Big(dZ_{1,t} - W_{2,t} dZ_{2,t} \Big), \tag{9}$$

$$\frac{dW_{2,t}}{W_{2,t}} = -\rho dt + \frac{A}{W_t} dt + \frac{1}{W_t^2}\Big(BW_{1,t} - CW_{1,t}W_{2,t} \Big) dt + \frac{1}{W_t}\Big(dZ_{1,t} + W_{1,t} dZ_{2,t} \Big), \tag{10}$$

the interest rate is

$$r_t = \frac{AW_t - D}{W_t^2} + \frac{BW_{2,t}}{W_t^2}, \tag{11}$$

the price vector is

$$\mathbf{P}_t = \frac{1}{r_t}\left(\mathbf{g} + \frac{1}{W_t}\big(W_{2,t}\delta - \Sigma \mathbf{1}_N \big) \right), \tag{12}$$

and the consumption is $c_{k,t} = \rho W_{k,t}$, $k \in \{1, 2\}$. The constants A, B, C and D depend on the real economy characteristics $(\mathbf{g}, \Sigma)$ and the sentiment (δ):

$$\begin{aligned} A &= \mathbf{1}'_N \mathbf{g}, \\ B &= \mathbf{1}'_N \delta, \\ C &= \delta' \Sigma^{-1} \delta, \\ D &= \mathbf{1}'_N \Sigma \mathbf{1}_N. \end{aligned}$$

The stochastic processes, Z_1 and Z_2 are defined by $Z_{1,t} = \mathbf{P}'_t \mathbf{S} \mathbf{B}_t$ and $Z_{2,t} = \delta' \Sigma^{-1} diag(\mathbf{P}_t) \mathbf{S} \mathbf{B}_t$, thereby satisfying: $Var(dZ_1) = D\,dt$, $Var(dZ_2) = C\,dt$, $Cov(dZ_1, dZ_2) = B\,dt$.

The constants A, B, C, D characterize how well rational investors take advantage of irrational investors. Clearly, A is the aggregate drift term of the real economy, corresponding to g in the one-firm case, and D is its total variance, corresponding to σ^2. It is natural to call B the *market sentiment*, as it is the sum of all stock sentiments. As we shall see below, it is natural to call C

the *arbitrageability* of the market.

The way the terms enter into the deterministic part of the wealth equations (9-10) provides some immediate information. First, bullish market sentiment ($B > 0$) will lead to an expected wealth transfer from rational to irrational investors. This is the effect noted in DeLong, Shleifer, Summers, and Waldman (1990) that bullish investors will over-invest in the stock market thereby having higher expected returns than rational investors. Similarly, if they are bearish, they will under-perform. Second, the arbitrageability coefficient, C, will lead to an expected wealth transfer from irrational to rational investors. This is true regardless of the sign of the sentiment, as Σ^{-1} is strictly positive definite, so C is always strictly greater than zero, as long as there is any sentiment in the market.

By setting $W_{2,0} = 0$, we get the N-stock version of the representative agent model of the previous section with the following dynamics:

$$dW_t = (A - \rho W_t)dt + dZ_t$$

$$r_t = \frac{AW_t - D}{W_t^2},$$

$$\mathbf{P}_t = \frac{1}{r_t}\left(\mathbf{g} - \frac{1}{W_t}\Sigma \mathbf{1}_N\right), \tag{13}$$

consumption is $c_t = \rho W_t$, and the stochastic process, Z is defined by $Z_t = \mathbf{P}_t' \mathbf{S} \mathbf{B}_t$ with $VAR(dZ) = D\,dt$. The steady state wealth is $\bar{W} = A/\rho$, the instantaneous market Sharpe ratio is $S_t = \sqrt{D}/W_t$ and the steady state market Sharpe ratio is $\bar{S} = \rho\sqrt{D}/A$. In our analysis, we will mainly study the steady state market Sharpe ratio, which we will simply call the market Sharpe ratio.

Several insights arise from comparing the static structures of the equilibria, with and without rational investors, for a specific wealth realization $W = W_1 + W_2$. We can, for example, use equations (12,13) to analyze the price impact of the irrational investors. Let P_i^{R} and P_i^{I} represent the price of stock i, for two identical economies, with the only difference that only rational investors are present in the R-economy ($W_2 = 0$), whereas irrational investors are present in the I-economy ($W_2 > 0$). We have

$$\frac{P_i^{\mathrm{I}}}{P_i^{\mathrm{R}}} = \frac{1}{1+\gamma}\left(1 + \frac{W_2 \delta_i}{r^{\mathrm{R}} W P_i^{\mathrm{R}}}\right), \qquad \gamma = \frac{BW_2}{AW - D}.$$

Thus, the price impact of irrational investors has two components. First, a market component,

γ, that depends on the market sentiment (B) and wealth (W_2) of the irrational investors. This component reflects the general equilibrium structure of the model: sentiment in one stock will influence the prices of all other stocks. If the market sentiment is zero ($B = 0$), then the market price error will also be zero. The second component ($W_2\delta_i/(r^{\mathrm{R}}WP_i^{\mathrm{R}})$) is stock specific: it only depends on the sentiment in stock i, δ_i.

We also note that when W_2 is small, both these components are small. Thus, contrary to the results in the model of Kogan, Ross, Wang, and Westerfield (2006), when irrational investors make up a small part of the market, their price impact (as well as their impact on interest rates) is small in our model. The reason for this difference lies in the assumptions about the real economy. Whereas aggregate production at a specific point in time is exogenous in Kogan, Ross, Wang, and Westerfield (2006), it is endogenous in our model. This gives increased opportunities for consumption smoothing, and consumption "squeezes" can be avoided in bad states of the world, resulting in less extreme behavior of the irrational investor in such states.

3 High-speed natural selection

We use the results in the previous section to analyze how the speed of natural selection process depends on the number of stocks. Clearly, the constants $A-D$ are crucial in deciding the wealth developments. We saw that B and C determine expected wealth changes. It turns out that the probability distribution for different relative wealth levels only depends on the arbitrageability-term, C. We define the stochastic process for the fraction of the total wealth held by the rational investor

$$f_t \stackrel{\text{def}}{=} \frac{W_{1t}}{W_t}.$$

We also define the random stopping time

$$\tau_f \stackrel{\text{def}}{=} \inf_t \{t : f_t \geq f\}.$$

This is the time it takes until the rational investor controls a fraction f of the market wealth. The following result characterizes the expected time it takes to reach a certain fraction of market wealth:

Proposition 3 *If the initial fraction of wealth owned by the rational investor is f_0, then*

$$E(\tau_f) = \frac{2\eta}{C}, \tag{14}$$

and

$$Var(\tau_f) = \frac{4\eta}{C^2}, \tag{15}$$

where $\eta = \log\left(\frac{f}{1-f}\right) - \log\left(\frac{f_0}{1-f_0}\right)$.

We see that C is indeed the crucial parameter for describing how fast natural selection occurs.

3.1 A one-factor economy

To see how C depends on N, let us study a simple model with one systematic risk and one idiosyncratic risk driving returns in each stock. We call this the market model. We wish to keep the total market uncertainty constant as N grows. To have the same risk at the market level, we set

$$\mathbf{g} = \mathbf{1}_N/N, \qquad \Sigma = [\sigma_{ij}], \text{ with } \sigma_{ii}^2 = 8/(N(N+1)), \text{ and } \sigma_{ij} = 4/(N(N+1)), \ i \neq j. \tag{16}$$

This implies that $A = 1$ and $D = 4$, so with $\rho = 10\%$, the steady state market Sharpe ratio is the same as in the one-stock case $\overline{S} = \rho\sqrt{D}/A = 0.2$, independently of N. We further assume that the irrational investors are slightly bullish about the value creation process for exactly half of the stocks and slightly bearish about the other half (assuming that N is even), i.e., for a $q > 0$, $\delta_i = q/N$ for $i = 1, \ldots, N/2$ and $\delta_i = -q/N$ for $i = N/2+1, \ldots, N$. We use Proposition 2 to get

$$B = 0, \qquad \text{and} \qquad C = \frac{N+1}{4}\, q^2.$$

Thus, by (14-15), as the number of stocks grows, the time it takes to reach any fixed wealth fraction for the rational investor is inversely proportional to the number of stocks, $N + 1$. For the case where $N = 1$, we get[5] $C = q^2/4$. Therefore, as with two stocks $C = 3q^2/4$, the speed-up of going from one to two stocks is even more drastic — it is 3 times faster with two stocks than with one. If it takes, in expectation, 1800 years for the rational investor to capture 90% of the market with one stock, it takes 600 years (i.e., 1800/3 years) with 2 stocks, and about 3 years

[5] In the case with only one stock, the market sentiment is not zero, as the irrational investor has to be either bullish or bearish about this one stock.

with 600 stocks (i.e., 1800/601 years). In Figure 1 we show the expected time to reach different wealth fractions for different number of stocks, $N = 1, 2, 50, 600$, for the case where the rational and irrational investor have the same initial wealth, $f_0 = 1/2$. The numbers for $f = 90\%$ are in line with the argument just made.

The distribution of τ_f is thin-tailed. In fact, as shown in the proof of Proposition 3, it is the first passage time distribution of a Brownian motion with drift $C/2$ and variance C per unit time. This distribution is known, and decreases faster than a normal distribution for any fixed f. In Figure 2, we show the time distribution to reach $f = 90\%$, with $f_0 = 0.5$, for $N = 2$, $N = 10$ and $N = 50$ stocks.

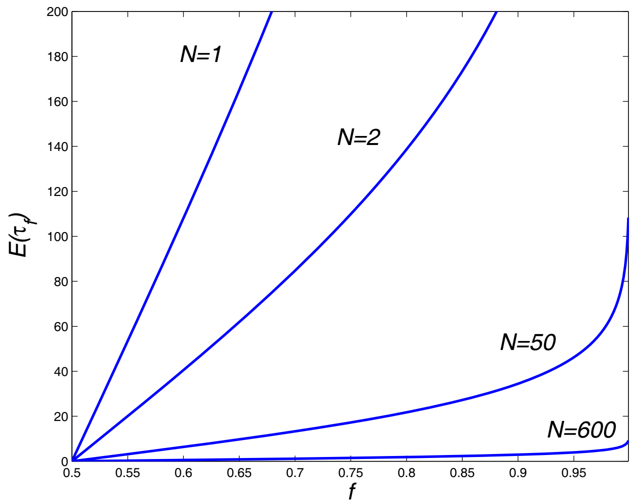

Figure 1: *Expected time,* $E[\tau_f]$*, for rational investors to reach a specific fraction of total wealth,* $f = W_1/(W_1 + W_2)$*, for* $N = 1$*,* $N = 2$*,* $N = 50$ *and* $N = 600$ *stocks, when irrational investors' sentiment is* $q = 10\%$*.*

We also note that even though the natural selection process is faster, the irrational investors are never totally wiped out, regardless of the number of stocks in the market and the time passed, as they become excessively risk-averse once they become poor. We still use the terminology of the group being "wiped out," denoting that their fraction of wealth becomes small, e.g., 10% of total wealth.

How does the rational investor take advantage of the irrational investor so fast? By studying the rational investor's portfolio choice (given in the appendix) it is clear that the rational investor will choose exactly the type of long-short position suggested in the introduction. In the case of equal wealth for the rational and irrational investor, for example, in addition to holding his part

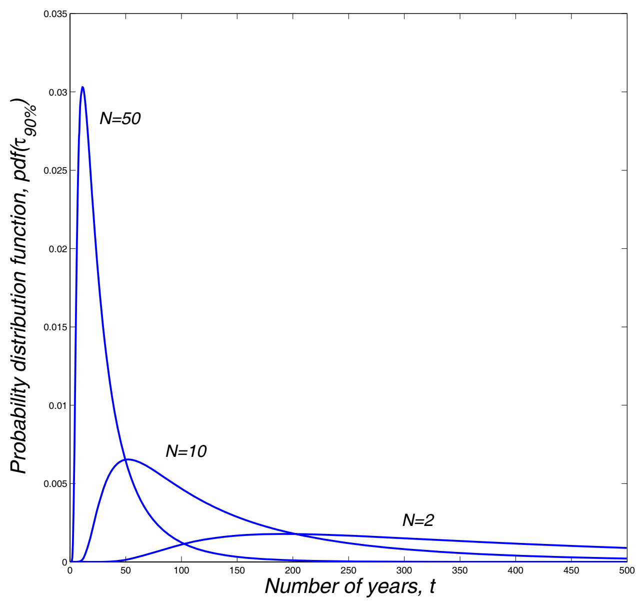

Figure 2: *Probability distribution of the time it takes for rational investors to reach* 90% *of total wealth for* $N = 2$, $N = 10$ *and* $N = 50$ *stocks, when irrational investors' sentiment is* $q = 10\%$.

of the market portfolio, corresponding to the case with only one rational investor, the rational investor will speculate against the irrational investor, by longing $(N+1)q/2$ shares of each stock that the irrational investor is pessimistic about, and short-selling the same amount of each stock the irrational investor is optimistic about. This gets rid of the market risk, so investor 1 is only exposed to a low level of idiosyncratic risk, and therefore earns an excess gross payoff of about q per unit time. The strategy closely resembles ideas for how to form market neutral portfolios with excess return in multi-factor economies, proposed e.g., in Rosenberg and Rudd (1982) and Rosenberg, Reid, and Lanstein (1985). As shown in the appendix, the rational investor's Sharpe ratio is essentially proportional to $q\sqrt{N}$, whereas the irrational investor on the other hand will have a Sharpe ratio proportional to $-q\sqrt{N}$. The market Sharpe ratio is almost constant, so the wealth transfer can not be inferred from market level data.

3.2 General economies

The main objectives of this section are to show that natural selection will be fast in markets with large state spaces under general conditions and that if a large arbitrary random economy is chosen, the natural selection process will almost always be fast.

How representative is the market model of Section 3.1? It turns out that high-speed natural selection occurs almost universally in our model, even though the natural selection process may not be as fast as in the market model. For example, sentiments need not to be symmetric. Consider the market model with the same real-economy parameters as before (implying that $A = 1$ and $D = 4$), but with asymmetric sentiments: being q_1 for a fraction, a, of the stocks and q_2 for the rest, $1 - a$.[6] Without loss of generality, we assume that $0 < a < 1$, (as the cases $a = 0$ and $a = 1$ are covered by taking $q_1 = q_2 = q$). We call this the asymmetric market model. From the definition of C in Proposition 2, it is straightforward to show that

$$C = \frac{1}{4}\Big(q_1^2 a + q_2^2(1-a) + \big(q_1^2 a + q_2^2(1-a) - (q_1 a + q_2(1-a))^2\big) \times N\Big).$$

Thus, by Proposition 3, high-speed natural selection will occur unless the term multiplying N equals zero. It is easy to show that the term equals zero if and only if $q_1 = q_2$.[7]

To study a more general set-up, we introduce a sequence of markets $\mathcal{M} = (\mathcal{M}_1, \ldots, \mathcal{M}_N, \ldots)$ where $\mathcal{M}_N = (\mathbf{g}_N, \Sigma_N, \delta_N)$, is a market with N stocks. We do not make any restrictions on $\mathcal{M}$. However, we will mainly be interested in sequence of markets where the market Sharpe ratio, size and total sentiment are roughly constant when N increases. We define

Definition 1 *The* total sentiment*, Δ, in a market with sentiment vector δ is defined as*

$$\Delta = \sum_i |(\delta)_i|.$$

Thus contrary to the market sentiment (B), the total sentiment aggregates unsigned sentiment information. We could for example have zero market sentiment, although the total sentiment is high. We use the notation $a_N \sim b_N$, if there are strictly positive constants, $0 < c_0 \leq c_1 < \infty$, such that for N large enough, $c_0 b_N \leq a_N \leq c_1 b_N$.

Definition 2 *A sequence of markets, $\mathcal{M} = (\mathcal{M}_1, \mathcal{M}_2, \ldots)$, is said to be* asymptotically well-behaved *if market Sharpe ratios $\bar{S}_N \sim 1$, sizes of markets $A_N \sim 1$ and total sentiments $\Delta_N \sim 1$.*

For a sequence of markets, $\mathcal{M}$, we say that

[6] We restrict our study to the cases when aN and $(1-a)N$ are both integers, to avoid issues rounding issues.
[7] If $q_1 = q_2$, then $C = q^2/4$ regardless of N and the whole model collapses into the one-stock case.

Definition 3 High-speed natural selection *occurs if, in market* $\mathcal{M}_N$*, the expected time to reach the fraction of wealth* f*, when initial wealth distribution is* f_0*, satisfies* $E(\tau_f) \leq G(f_0, f, N)$ *for some function* $G : (0,1) \times (0,1) \times \mathbb{Z}_{++} \to \mathbb{R}_{++}$[8]*, which for all* f_0 *and* $f > f_0$ *satisfies*

$$\lim_{N\to\infty} G(f_0, f, N) = 0.$$

Definition 4 High-speed natural selection of order ν *(where* $\nu > 0$*) occurs if the function* G *in Definition 3 can be written in the form* $G(f_0, f, N) = H(f_0, f)/N^\nu$*.*

We let C_N denote the arbitrageability term in market $\mathcal{M}_N$. Clearly, Proposition 3 implies that high-speed natural selection of order ν occurs if and only if

$$c \stackrel{\text{def}}{=} \liminf_{N\to\infty} \frac{C_N}{N^\nu}, \qquad \text{where} \qquad C_N = \delta_N' \Sigma_N^{-1} \delta_N,$$

is not zero, i.e., $0 < c \leq \infty$. We define $\rho(\Sigma)$ to be the spectral radius of the matrix Σ. We have:

Proposition 4 *For a sequence of markets,* $\mathcal{M}$*, high-speed natural selection of order* ν *occurs if there are strictly positive constants* c_1*,* c_2 *and* N_0*, and a function,* $f : \mathbb{Z}_{++} \to \mathbb{R}_{++}$ *such that the following two conditions are satisfied for all* $N > N_0$*:*

- $\delta_N' \delta_N \geq c_1 f(N)$*,*
- $\rho(\Sigma) \leq c_2 N^{-\nu} f(N)$*.*

Proposition 4 cannot be applied to prove high-speed natural selection of order one in the market model of Section 3.1.[9] However, the following proposition ensures that when real economy randomness is symmetric, in the sense that $\mathbf{1}_N$ is an eigenvector to Σ,[10] then the only cases where high-speed natural selection of order one will *not* occur is when there is no spread of sentiment across stocks or when the covariance matrix is asymptotically not well-behaved. For an arbitrary vector, $\mathbf{q}$, we define $Q_\mathbf{q}$ to be the Euclidean projection operator onto the orthogonal complement of $\mathbf{q}$.[11] We have

[8]Here, and subsequently, $\mathbb{Z}_{++}$ represents the set of strictly positive integers and $\mathbb{R}_{++}$, the set of strictly positive real numbers.

[9]This is easily seen: The first condition implies that $f(N) \sim N^{-1}$, which leads to a violation of the second condition, as $\rho(\Sigma_N) \sim N^{-1}$.

[10]The symmetry here is that each row of the variance covariance matrix sums to the same constant, for any specific N. This is of course equivalent to $\mathbf{1}_N$ being an eigenvector. Clearly, it holds for the market model, as well as for several other models, e.g., with industry-specific risks.

[11]That is, with matrix notation, $Q_\mathbf{q} = I - \frac{\mathbf{q}\mathbf{q}'}{\mathbf{q}'\mathbf{q}}$, where I is the identity matrix.

Proposition 5 *For a sequence of markets, $\mathcal{M}$, high-speed natural selection of order ν occurs if there are strictly positive constants c_1, c_2, N_0, and a function, $f : \mathbb{Z}_{++} \to \mathbb{R}_{++}$, such that the following three conditions are satisfied for all $N > N_0$:*

- $\mathbf{1}_N$ *is an eigenvector of* Σ_N,
- $\delta'_N \delta_N - \frac{(\mathbf{1}'_N \delta_N)^2}{N} \geq c_1 f(N)$,
- $\rho(Q'_{\mathbf{1}_N} \Sigma_N Q_{\mathbf{1}_N}) \leq c_2 N^{-\nu} f(N)$.

It is straightforward to check that both the market model and the asymmetric market model satisfy the conditions of Proposition 5 with $\nu = 1$, except for the asymmetric market model with $q_1 = q_2$.[12] This is in line with our analysis so far. The only other case when high-speed natural selection may fail for the market model is when the third condition is violated (the first condition is obviously always satisfied for the market model). For general variance covariance matrices of the form $\Sigma_N = (aI_N + b\mathbf{1}'_N\mathbf{1}_N)/(Na + bN^2)$, $a > 0$, $b \geq 0$, this happens if and only if $b = 0$. In this case, there are effectively N separate financial markets, and there is no gain to the rational investor of being able to form long-short portfolios.

Under the conditions of Proposition 5, as the number of stocks increases, natural selection works faster. In the limit, as the number of stocks N grows, the arbitrageability approaches infinity and the natural selection becomes instantaneous. It is worth pointing out, however, that for any finite N, there are never any *pure* arbitrage opportunities in the model, as investors agree on zero-probability events. It is shown in the appendix that for an asymptotically well-behaved sequence of markets, high-speed natural selection implies the presence of an *asymptotic* arbitrage opportunity (Ross, 1976). However, this opportunity is not *scalable* in our general equilibrium model, contrary to the assumptions of the Arbitrage Pricing Theory. This makes our model suitable for the dynamic study of the natural selection process.

Propositions 4 and 5 can be used to prove high-speed natural selection for a specific sequence of markets but do not say how "often" high-speed natural selection breaks down. Is high-speed natural selection the norm, or are the previous examples just exceptional special cases? To approach this question, we use theory of random quadratic forms. Specifically, we study how often high-speed natural selection occurs in randomly generated markets. We look at a special case of a random market (one factor) model. In the appendix we discuss how the results can be generalized to K-factor models and to even more general random economies.

[12] If $q_1 = q_2$ in the asymmetric market model, the second condition of Proposition 5 fails.

We make the following assumptions about the randomness of the markets, $\mathcal{M}_N = (\mathbf{g}_N, \Sigma_N, \delta_N)$. We assume that $(\mathbf{g}_N)_i = \tilde{p}_i^N/N$, where $\tilde{p}_i^N$ are i.i.d. random variables, $E(\tilde{p}_1^1) = \bar{p} > 0$ and $Var(p_1^1) = \sigma_p^2 > 0$. Similarly, $(\delta_N)_i = \tilde{q}_i^N/N$, where $\tilde{q}_i^N$ are i.i.d. random variables, $E(\tilde{q}_1^1) = \bar{q}$ and $Var(\tilde{q}_1^1) = \sigma_q^2 > 0$. Furthermore, we assume that the randomness of the ith asset, $\sigma_i dB_{it}$, is of the form $\sigma_i dB_{it} = \frac{1}{N}(\beta_i^N d\xi_{0t} + \alpha_i^N d\xi_{it})$, where ξ_{it} are i.i.d. jointly independent standard Brownian motions, and α_i^N, β_i^N are i.i.d. random variables: $E(\alpha_1^1) = \bar{\alpha} > 0$, $Var(\alpha_1^1) = \sigma_\alpha^2 > 0$, $E(\beta_1^1) = \bar{\beta}$ and $Var(\beta_1^1) = \sigma_\beta^2 > 0$. All random variables are jointly independent. For simplicity, we assume that all random variables are absolutely continuous (with respect to Lebesgue measure) and that the β's are (a.s.) bounded below by a strictly positive constant, $\epsilon > 0$.[13] Obviously, we require the $\tilde{p}$'s to be positive. For a fixed N, the market $\mathcal{M}_N$ will thus be characterized by $\mathbf{g}_N = (\tilde{p}_1^N, \ldots, \tilde{p}_N^N)'/N$, $\delta_N = (\tilde{q}_1^N, \ldots, \tilde{q}_N^N)'/N$ and $\Sigma_N = (diag(\alpha_1^N, \ldots, \alpha_N^N)^2 + \mathbf{b}_N \mathbf{b}_N')/N^2$, where $\mathbf{b}_N = (\beta_1^N, \ldots, \beta_N^N)'$. This corresponds to sequences of generalized market models, with random loadings on the market factor and idiosyncratic factors for each stock, with random value creation and random sentiment. We have

Proposition 6 *A sequence of markets, $\mathcal{M}$, satisfying the previous assumptions is asymptotically well-behaved (almost surely), and has high-speed natural selection of order one (almost surely).*

Thus, high-speed natural selection is really the norm in such markets, and the exception is when it breaks down.

3.3 Variations

In this section, we discuss several variations under which high-speed natural selection occurs that provide additional insight. Specifically, we show that high-speed natural selection can arise even if sentiment is uniform, if both investors are irrational and that the order of the natural selection process can be higher than one.

Uniform sentiments: Propositions 2 – 5 can be used for additional analysis of specific markets. Let us, for example, analyze a market with firms of different sizes and show that, in this case, even uniform sentiment can lead to high speed natural selection.

Suppose that there are N firms, $i = 1, ..., N$, each with s_i stocks outstanding. In the previous analysis, we had unit supply of each stock. However, it is easy to incorporate variable supply. Let us define the vector $\mathbf{s} = (s_1, ...s_N)'$, and the diagonal matrix $\Lambda_\mathbf{s} = diag(\mathbf{s})$. Consider the market

[13] These assumptions can be relaxed in several directions, but at the expense of increased complexity.

$\mathcal{M} = (\hat{\mathbf{g}}, \hat{\Sigma}, \hat{\delta}) \stackrel{\text{def}}{=} (\Lambda_{\mathbf{s}}\mathbf{g}, \Lambda_{\mathbf{s}}\Sigma\Lambda_{\mathbf{s}}, \Lambda_{\mathbf{s}}\delta)$. This can be thought of as a market with each stock in unit supply and firm characteristics and sentiments defined by $\hat{\mathbf{g}}$, $\hat{\Sigma}$ and $\hat{\delta}$ respectively. Alternatively, it can be interpreted as a market with stock supply given by $\mathbf{s}$ and firm characteristics and sentiments *per unit supply of stock* given by $\mathbf{g}$, Σ and δ respectively.

Now consider a sequence of markets with

$$\mathbf{s}_N = (s_1, ...s_N)', \qquad \mathbf{g}_N = g\mathbf{1}_N, \qquad \Sigma_N = \sigma^2 I_N, \qquad \delta_N = q\mathbf{1}_N, \tag{17}$$

where, $g > 0$, $q \neq 0$, $\sigma^2 > 0$, $s_i > 0$ for all i, I_N is the $N \times N$ identity matrix, and we require that

$$T_1 \stackrel{\text{def}}{=} \sum_{i=1}^{\infty} s_i < \infty \qquad \text{and} \qquad T_2 \stackrel{\text{def}}{=} \sqrt{\sum_{i=1}^{\infty} s_i^2} < \infty.$$

Without loss of generality, we can assume that the s_i's are decreasing, $s_1 \geq s_2 \geq \cdots$. According to our previous discussion, the market with N stocks is characterized by $\hat{\mathbf{g}}_N = g\mathbf{s}_N$, $\hat{\Sigma} = \sigma^2\Lambda_{\mathbf{s}}^2$ and $\hat{\delta} = q\mathbf{s}_N$. Irrational investors are equally bullish about all stocks, so the (relative) market sentiment is equal to (relative) individual stock sentiment. This is one of the two cases when high-speed natural selection collapsed in the market model in Section 3.1.

Applying formula (12) to derive the vector of capitalizations leads to

$$\hat{\mathbf{P}}_t = \frac{1}{r_t}\left(g\mathbf{s} + \frac{1}{W_t}\big(W_{2,t}q\mathbf{s} - \sigma^2\Lambda_{\mathbf{s}}^2\mathbf{1}_N\big)\right),$$

or, in scalar notation, the capitalization of firm i at time t, $\hat{P}_{i,t}$ is

$$\hat{P}_{it} = \frac{1}{r_t}\left(gs_i + \frac{1}{W_t}\big(W_{2,t}qs_i - \sigma^2 s_i^2\big)\right). \tag{18}$$

In order to make comparisons between different assets and their returns we introduce the following notation. Recall that μ_{it}^F denotes the instantaneous expected return of asset i. For each asset, i, let us define $\gamma_{i,t} \stackrel{\text{def}}{=} r_t/\mu_{i,t}^F$, i.e., $\gamma_{i,t}$ measures the cost of r_t units of expected surplus by investing in asset i:

$$\gamma_{i,t} = \hat{P}_{it}\frac{r_t}{s_i g}.$$

The risky assets with $\gamma_{i,t} < 1$ have positive expected excess return. The assets with $\gamma_{i,t} > 1$ have negative expected excess return.

Using equation (18), we obtain

$$\gamma_{i,t} = \left(\mathbf{1} + \frac{1}{gW_t} \big(W_{2,t} q - s_i \sigma^2 \big) \right). \tag{19}$$

The above expression for prices implies that (actual) expected excess return for each company is positive if and only if

$$s_i > \frac{W_{2t} q}{\sigma^2}.$$

This in turn means that when sentiment is optimistic (positive) and identical across all stocks, the actual risk premium for small stocks is low and even becomes negative when the above inequality fails to hold. For rational investors, such stocks are overpriced (negative excess return and positive market β), so rational investors will short-sell such small stocks. As the irrational investors are driven out of the market, stock prices adjust to their fundamental level. This stylized argument fits qualitatively well with what was observed during the New Economy boom. Young growth companies had high stock prices, measured with standard indicators, and their prices eventually collapsed, whereas mature value companies were hit less severely by the market downturn.

Next, let us study the speed of the natural selection process in this kind of market. To make our point, we study sequences of markets that have similar market Sharpe ratios, regardless of the choice of the s_i's. To achieve this, we scale with T_1 and T_2, i.e., we study sequences of markets with

$$\mathbf{s}_N = (s_1, ...s_N)', \qquad \mathbf{g}_N = \frac{g}{T_1} \mathbf{1}_N, \qquad \Sigma_N = \frac{\sigma^2}{T_2^2} I_N, \qquad \delta_N = \frac{q}{T_1} \mathbf{1}_N. \tag{20}$$

Regardless of the choice of s_i's, the market Sharpe ratio, $\bar{S} = \rho \sqrt{D_N}/A_N$, converges to $\rho\sigma/g$, as N becomes large, so this is the correct scaling when comparing sequences of markets with different s's.

Proposition 4 immediately implies that high-speed natural selection of order one will always occur. In fact, the expression for arbitrageability is fairly simple:

$$C_N = \left(\frac{q}{T_1} \mathbf{s} \right)' \left(\frac{\sigma^2}{T_2^2} \Lambda_{\mathbf{s}} I_N \Lambda_{\mathbf{s}} \right)^{-1} \left(\mathbf{s} \frac{q}{T_1} \right) = \frac{T_2^2 q^2}{T_1^2 \sigma^2} N.$$

Thus, as long as $T_1 < \infty$, high-speed natural selection of order one will occur, even though irrational investors' sentiments are uniform across stocks. For example, if the sizes of the companies decrease geometrically: $s_k = s^k$, where s is a constant between zero and one ($0 < s < 1$), then it is straightforward to show that

$$C_N = \frac{(1-s)q^2}{(1+s)\sigma^2}N,$$

so the faster the decrease in firm size (s closer to zero), the faster the natural selection. When s approaches one, high speed natural selection disappears, in line with our previous result: When sizes of firms are equal and sentiment is uniform across stocks, rational investors can do nothing better then hold the market portfolio.

Both investors are irrational: If both investor groups are irrational, then high speed natural selection will still occur, favoring the investor group that is least irrational in a metric weighted by the inverse of the covariance matrix. Under the same assumptions as before, but with the first and second investors, having sentiment vectors $\delta_1 \in \mathbb{R}^N$ and $\delta_2 \in \mathbb{R}^N$ respectively, we get the following generalization of Proposition 3.

Proposition 7 *If the initial fraction of wealth owned by investor group 1 is* f_0, $C_1 \stackrel{\text{def}}{=} \delta_1'\Sigma^{-1}\delta_1$, $C_2 \stackrel{\text{def}}{=} \delta_2'\Sigma^{-1}\delta_2$, $C_{12} \stackrel{\text{def}}{=} \delta_1'\Sigma^{-1}\delta_2$, *and* $C_1 < C_2$, *then*

$$E(\tau_f) = \frac{2\eta}{C_2 - C_1}, \tag{21}$$

and

$$Var(\tau_f) = \frac{4\eta(C_2 + C_1 - 2C_{12})}{(C_2 - C_1)^3}, \tag{22}$$

where $\eta = \log\left(\frac{f}{1-f}\right) - \log\left(\frac{f_0}{1-f_0}\right)$.

Higher-order natural selection: Our examples so far have led to high-speed natural selection of order one. Natural selection of other orders may also occur. The definition of $C = \delta'\Sigma^{-1}\delta$, shows that the closer Σ is to singular, the faster we can expect the natural selection process to take place. For example, for a constant $\nu > -1$, the sequence of markets with $\mathcal{M}_N = (\mathbf{g}_N, \Sigma_N, \delta_N)$ defined by

$$\mathbf{g}_N = \frac{\mathbf{1}_N}{N}, \quad (\Sigma_N)_{ii} = \frac{1 + N^{-\nu}}{N^2}, \quad (\Sigma_N)_{ij} = \frac{1}{N^2}, \; i \neq j, \quad \text{and} \quad (\delta_N)_i = (-1)^i q, \tag{23}$$

will lead to high-speed natural selection of order $1+\nu$. This can be shown by a direct application of Proposition 5. Thus, an efficient way for sophisticated investors to take advantage of unsophisticated ones is to introduce highly correlated assets. For example, we could interpret the economy, as one where some firms have “dot.com” names and irrational investors have positive sentiment about these, although they are effectively identical to other firms. If such assets are introduced, then even a slight degree of sentiment can lead to fast natural selection.

High-speed natural selection can thus be achieved in various ways. To summarize: According to our model, sophisticated investors can efficiently take advantage of unsophisticated ones in markets with large spread of firm sizes, in markets with many financial (i.e., zero-net supply) assets, in markets with firms that have highly correlated value creation and in markets where there is a large sentiment dispersion across assets.

3.4 Discussion

An alternative interpretation of our results is that investors with systematic sentiments die out slowly, whereas any idiosyncratic sentiment will be punished very quickly. For example, in the market model of Section 3.1, only uniform sentiment ($\delta \propto \mathbf{1}$) survives in the intermediate term. Such sentiment is parallel to market risk.[14] This interpretation is in line with the results in Daniel, Hirshleifer, and Subrahmanyam (2001), where the authors, using a one period model, find that arbitrageurs will remove idiosyncratic mispricing but not systematic mispricing. In our model, it *is* possible for irrational investors to survive a long time in the market, but only if they have a very restricted type of irrationality: In a market with 500 stocks and five factors there are 495 types of irrationality that will be punished very quickly and five that will be punished slowly.

In our examples so far, rational investors have been short-selling overpriced stocks. Do the results survive in markets with short-sale constraints? In the market model of Section 3.1, high-speed natural selection will not occur if short-sale constraints are present. However, short-sale constraints do not seem to be binding in the market (see, e.g., Diether, Lee, and Werner 2006), so we do not view this as a major concern. In the appendix we discuss why high-speed natural selection breaks down with short-sale constraints for market model. We also show other examples for which high-speed natural selection occurs even with short-selling constraints present. For example, in rapidly expanding markets, high-speed natural selection can occur even with short-sale constraints, as the payoff of investing in the winners may be very high.

[14]The principal component of Σ is (proportional to) $\mathbf{1}$, representing market risk in the economy.

4 Arbitrageability in the U.S. stock market

How applicable is our model to the U.S. stock market? Clearly, we do not see the extreme cases where unsophisticated traders are almost immediately wiped out of the market, so the market model of Section 3.1 does not seem to calibrate well with the real world.

However, studies of individual investor performance *do* find quite severe underperformance by unsophisticated investors, so one-stock models, in which it takes hundreds of years for irrational investors to be wiped out, seem equally ill-calibrated. For example, individual investors in the study by Barber, Lee, Liu, and Odean (2005) underperform institutional investors by about 2.1% per year, which implies a 50% underperformance in a 30-year horizon. Thus, neither the market model of Section 3.1, nor the one-stock model seem to quantitatively be in line with arbitrageability in the real stock market.

Propositions 4–6 are of little help here, as they provide asymptotic results and do not tell us about constants involved. However, qualitatively, the requirements of Proposition 5 for speed-up of the natural selection process in the stock market, i.e., spread in stock sentiment and correlated stock returns, seem to be met in real markets. For example, returns in boom periods have been driven by specific sectors, like the Internet and high-tech sectors in the New Economy boom (Ofek and Richardson, 2002; Lamont and Thaler, 2003). This suggests that if sentiment was a driver in the boom, it was not uniform across stocks. Moreover, numerous studies estimate the number of risks present in the stock market to be higher than one, but much lower than the dimensionality of the stock market, e.g. Connor and Korajczyk (1993).

We can, in principle, estimate the arbitrageability of a market, C, using observable data. This immediately implies the speed of the natural selection process through Proposition 3. We do this in two ways: first using stock returns, and then using trading volume. The results are quite similar: The first method suggests that it takes about 45 years to wipe out irrational investors, whereas the second measure suggests that it takes about 58 years. This corresponds to an average annual underperformance of 3.9%–5.0% per year, which is slightly larger than, but of the same order of magnitude as, the results in Barber, Lee, Liu, and Odean (2005).

These results are indicative but should not be oversold. The empirical estimates we use are rough and do not control for other factors driving returns and trading volume. A detailed empirical analysis, although clearly of interest, is not within the scope of this paper.

4.1 Method

A return-based estimate: Using the expected return vector μ^F, the return covariance matrix, $\Sigma^F = [\bar{\sigma}_{ij}]_{ij}$, the expected excess return vector, $\mu_e = \mu^F - r_t \mathbf{1}_N$, and the relative firm size $\mathbf{s} = \mathbf{P}/\mathbf{1}_N'\mathbf{P}$, it is straightforward to show that the following relationship holds:

$$X \stackrel{\text{def}}{=} \left(\frac{W_{2,t}}{W_t}\right)^2 C = (\Sigma^F \mathbf{s} - \mu_e)'(\Sigma^F)^{-1}(\Sigma^F \mathbf{s} - \mu_e). \tag{24}$$

Although this does not totally nail down C, it captures the main source of wealth transfer to the rational investor in equations (9-10). We make the simplifying assumption that $f_0 = 50\%$.[15] At any point in time, $\mathbf{s}$ is observable, whereas Σ^F and μ_e must be estimated. To estimate the covariance matrix, Σ^F, we used both the sample covariance matrix, and the weighted covariance estimator suggested by Litterman and Winkelmann (1998).

A weakness of this measure is the difficulty of empirically estimating stock-wise expected excess returns, μ_e. To get around this issue, we note that (24) can be rewritten as:

$$X = \sigma_m^2 - 2\mu_m + Y, \qquad \text{where } Y = \mu_e'(\Sigma^F)^{-1}\mu_e.$$

Here, σ_m is the volatility of the market portfolio and μ_m is the market risk-premium. A lower bound for Y is given by $Y \geq N\mu_m^2/R$, where N is the number of stocks and $R = \rho(\Sigma^F)$ is the spectral radius of the covariance matrix. Thus, market level data for volatility and risk-premium, together with an estimate of the spectral radius of the covariance matrix, lead to a lower bound for arbitrageability

$$C \geq 4\hat{X}, \qquad \text{where } \hat{X} = \sigma_m^2 - 2\mu_m + N\frac{\mu_m^2}{R}. \tag{25}$$

This is our return-based arbitrageability estimate.

A volume-based estimate: As noted, the return-based estimate, $\hat{X}$, suffers from the difficulties in measuring expected returns. Another approach that does not suffer from this drawback is to use data for trading volumes.

Trading in our model arises when there is wealth transfer between rational and irrational investors: the higher the degree of wealth transfer, the higher the volume. We define the difference in portfolio holdings between the rational and irrational investor

$$\mathbf{z}_t \stackrel{\text{def}}{=} \frac{1}{2}\Lambda_t^{-1}\left(\alpha_{1,t}W_{1,t} - \alpha_{2,t}W_{2,t}\right), \tag{26}$$

[15] Analogue results for other values, $\hat{f}_0$, are immediately obtained by the mapping $C \mapsto (2 - 2\hat{f}_0)^{-2}C$.

where $\Lambda_t = diag(\mathbf{P}_t)$. Then, $d\mathbf{z}_t$ is the signed instantaneous relative trading volume, measured as a fraction of market capitalization. We let $\mathbf{q}_t$ denote the observed (unsigned) relative trading volume in the real stock market. We assume that investors' rebalancing interval is Δt, which leads to the relation $\mathbf{q}_t = |\int_{t-\Delta t}^{t} d\mathbf{z}_t|/\Delta t$.[16] We assume that $\Delta t = 1/12$, i.e., that investors rebalance once per month. We define $\bar{\mathbf{q}}_t = E[\mathbf{q}_t]$, the expected relative turnover, which we will approximate with the sample mean. Finally, we assume that the market sentiment is small compared with the total sentiment, i.e., that $B << \Delta$ and that $f_0 = 0.5$.

Under these assumptions it can be shown (see appendix) that

$$\int_t^{t+\Delta t} (d\mathbf{z}_t)'\Lambda_t\mathbf{S}^2\Lambda_t(d\mathbf{z}_t) \approx \left(\frac{W_1W_2}{W^2}\right)^2 \big(\mathbf{P}_t'\mathbf{S}^2\mathbf{P}_t + D\big)C\Delta t$$
$$= \frac{1}{4}\Big(\sum_{i=1}^N (\mathbf{P}_t)_i^2\bar{\sigma}_{ii} + \sum_{i=1,j=1}^N (\mathbf{P}_t)_i\bar{\sigma}_{ij}(\mathbf{P}_t)_j\Big)C\Delta t, \tag{27}$$

and that

$$\int_t^{t+\Delta t} (d\mathbf{z}_t)'\Lambda_t\mathbf{S}^2\Lambda_t(d\mathbf{z}_t) = \frac{\pi}{2}\bar{\mathbf{q}}_t'\Lambda_t\mathbf{S}^2\Lambda_t\bar{\mathbf{q}}_t\Delta t + o(\Delta t) = \frac{\pi}{2}\sum_{i=1}^N(\bar{\mathbf{q}}_t)_i^2(\mathbf{P}_t)_i^2\bar{\sigma}_{ii}\Delta t + o(\Delta t),^{17} \tag{28}$$

where, as before, $\mathbf{S} = diag(\bar{\sigma}_{11}, \ldots, \bar{\sigma}_{NN})$. Altogether, this leads us to

$$C \approx 2\pi\hat{Z}, \qquad \text{where } \hat{Z} = \frac{\sum_{i=1}^N(\bar{\mathbf{q}}_t)_i^2(\mathbf{P}_t)_i^2\bar{\sigma}_{ii}}{\sum_{i=1}^N(\mathbf{P}_t)_i^2\bar{\sigma}_{ii} + \sum_{i,j=1}^N(\mathbf{P}_t)_i\bar{\sigma}_{ij}(\mathbf{P}_t)_j}. \tag{29}$$

This is our volume-based arbitrageability estimate.[18]

4.2 Data

Both our estimates use the return covariance matrix. Ideally, we would wish to estimate the arbitrageability of the whole market, but this is not feasible due to the difficulties of accurately estimating $N(N+1)/2$ variances and covariances for N stocks. We therefore use a randomly selected subset of S&P 500 stocks.

We used the Center for Research in Security Prices (CRSP) to get daily data for returns and turnover of 40 randomly chosen S&P 500 stocks, over the ten-year period 1996-2005. We

[16] This avoids the issue of infinite trading volumes due to the unbounded variation of Brownian motions.

[17] Here $y(x) = o(x)$ denotes that $\lim_{x\searrow 0} y(x)/x = 0$.

[18] Our estimate is based on the assumption that trading volume mostly is driven by speculation. A generalization of the volume-based measure would be to divide trading volume into into a speculative and a nonspeculative component. The implied arbitrageability would be lower with such a decomposition.

used the one-month T-bill rate adjusted to a daily basis as the risk-free rate. The identity of, and summary statistics for, the individual companies are shown in Table 1. The end-of-period total market capitalization of companies in the sample was USD 893 Billion, corresponding to an average company size of USD 22 Billion. The median company size was USD 11 Billion. The largest company was Pfeizer Inc. (PFE) with a market capitalization of USD 172 Billion. The smallest company was Steak N Shake (SNS) with a market capitalization of USD 0.47 Billion. The annualized value-weighted average excess return for the companies in the sample was 6.1% per year and the annualized portfolio volatility was 16.5%. The average annualized turnover was about 95%.

4.3 Results

We estimate the arbitrageability from the return-based statistic, $\hat{X}$, for 3-40 stocks. The results are shown in Figure 3. We see that for above 15 stocks, the estimated arbitrageability is almost

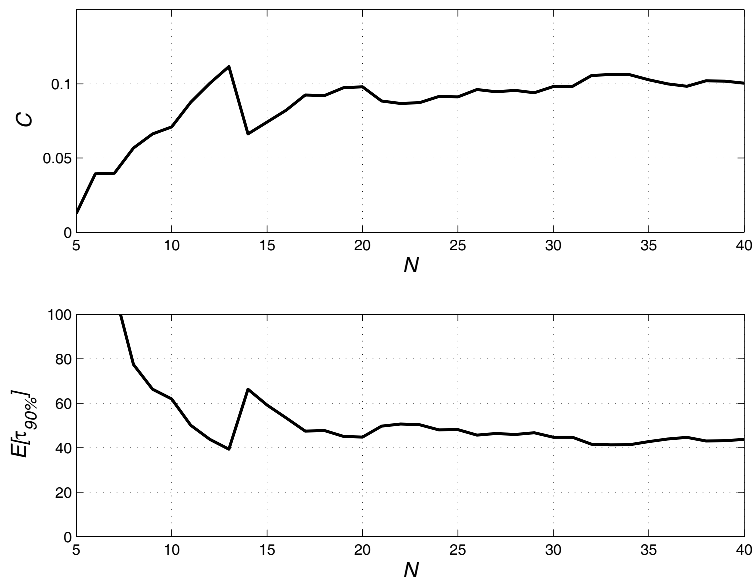

Figure 3: *Estimated arbitrageability from return-based measure, $\hat{X}$, as a function of number of stocks, N. Above: Estimated arbitrageability coefficient, C. Below: Expected time (in years) for rational investors to capture 90% of wealth.*

constant. With all 40 stocks, we have $\hat{X} = 0.025$, which corresponds to an arbitrageability of $C = 0.10$. This implies that it takes, in expectation, 44 years for rational investors to capture 90% of the market if both groups start with the same wealth. Equivalently, it corresponds to an average underperformance of 5.0% per year for irrational investors. This suggests a faster process compared with the 2.1% underperformance shown in Barber, Lee, Liu, and Odean (2005), but clearly it is more in line with these results than the hundreds of years elsewhere suggested.

The results are quite similar when using the volume-based measure, shown in Figure 4. Again, when more than 15 stocks are included in the sample, the estimated arbitrageability is almost constant. With all 40 stocks, $\hat{Z} = 0.012$, corresponding to $C = 0.076$. This implies that it takes, in expectation, 58 years for rational investors to capture 90% of the market if both groups start with the same wealth, corresponding to an underperformance of 3.9% per year for irrational investors.

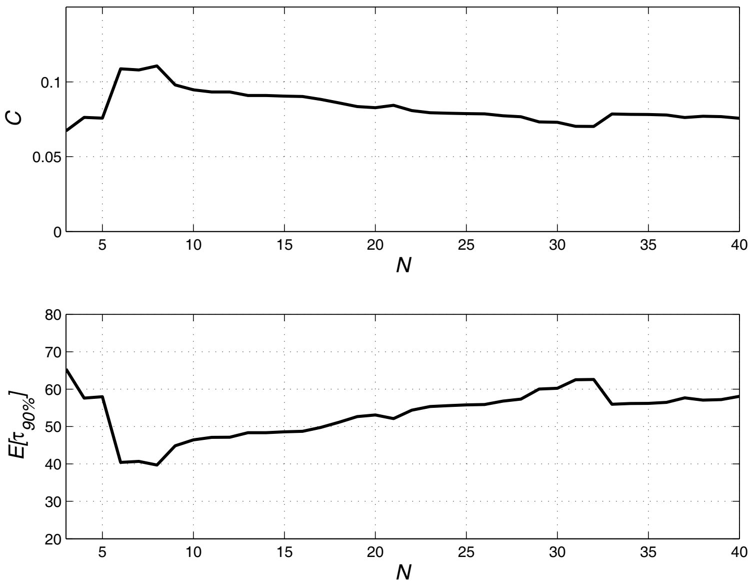

Figure 4: *Estimated arbitrageability from volume-based measure, $\hat{Z}$, as a function of number of stocks, N. Above: Estimated arbitrageability coefficient, C. Below: Expected time (in years) for rational investors to capture 90% of wealth.*

5 Concluding remarks

Theory alone does not tell us how fast irrational investors will be wiped out from the stock market. As we have shown, the time it takes may vary by several orders of magnitude, ranging from a few months to hundreds — or even thousands — of years. The speed depends on the so-called arbitrageability of the market. The arbitrageability is a function of both real stock market characteristics and the *stock-wise* sentiment of irrational investors. High-speed natural selection occurs in stock markets with large state spaces, which allow rational investors to take advantage of irrational investors more effectively than in markets with only one stock and one bond. We show that for very large markets, high-speed natural selection is the norm rather than the exception.

Our analysis is connected to pure noarbitrage theory and to the arbitrage pricing theory. The original theory provides bounds on the degree of irrationality allowed for a trader not to be immediately wiped out. We show that, if the market is right, even a much lower degree of irrationality (compared with the pure arbitrage case) may lead to a fast punishment. In our model there are no real arbitrage opportunities — neither for pure arbitrage nor for scalable asymptotic arbitrage. The arbitrage is statistical and the general equilibrium features of the model allow it to be dynamically quantified.

The results have implications for investor performance in stock markets. A rough empirical calibration of the model to the US stock market suggests that it takes about fifty years to wipe out an irrational investor. The result is equivalent to an average underperformance of 3.9-5.0% per year for unsophisticated investors, compared with sophisticated ones. This is in line with recent studies of individual investor performance.

No.	Ticker	Company Name	Excess return	Volatility	Turnover	Market cap.
1	FO	FORTUNE BRANDS INC	0.103	0.26	0.81	11.3
2	KO	COCA COLA CO	-0.012	0.26	0.47	95.9
3	DD	DU PONT E I DE NEMOURS CO	0.012	0.30	0.76	39.0
4	GM	GENERAL MOTORS CORP	-0.070	0.34	1.84	10.9
5	ITT	I T T INDUSTRIES INC	0.132	0.28	1.12	9.49
6	COP	CONOCOPHILLIPS[19]	0.121	0.26	0.92	82.8
7	CR	CRANE CO	0.063	0.30	0.83	2.12
8	LMT	LOCKHEED MARTIN CORP	0.028	0.30	1.13	27.7
9	PFE	PFIZER INC	0.060	0.31	0.68	171.9
10	MMM	3M CO	0.078	0.26	0.96	58.8
11	HNZ	HEINZ H J CO	0.0099	0.23	0.74	11.2
12	SNS	STEAK N SHAKE CO	0.042	0.37	0.58	0.47
13	K	KELLOGG CO	0.0034	0.27	0.54	17.9
14	COHU	COHU INC	0.033	0.58	1.98	0.50
15	PLL	PALL CORP	-0.012	0.31	0.93	3.34
16	MDP	MEREDITH CORP	0.067	0.27	0.94	2.07
17	MCD	MCDONALDS CORP	0.015	0.29	0.95	42.4
18	MMC	MARSH & MCLENNAN COS INC	0.068	0.32	0.89	17.2
19	BNI	BURLINGTON NORTHERN CP	0.084	0.28	0.88	26.4
20	GWW	GRAINGER W W INC	0.057	0.30	1.01	6.35
21	CTX	CENTEX CORP	0.210	0.40	2.75	9.13
22	ITW	ILLINOIS TOOL WORKS INC	0.089	0.28	0.70	24.6
23	STA	ST PAUL TRAVELERS COS INC	0.040	0.30	1.18	30.9
24	CTL	CENTURYTEL INC	0.058	0.30	1.19	4.3
25	FDX	FEDEX CORP	0.147	0.33	1.25	31.4
26	DLX	DELUXE CORP	0.038	0.25	1.15	1.52
27	CSX	C S X CORP	-0.0035	0.32	0.99	11.0
28	TMK	TORCHMARK CORP	0.085	0.26	0.76	5.77
29	STI	SUNTRUST BANKS INC	0.065	0.26	0.64	26.2
30	MYL	MYLAN LABS INC	0.036	0.42	1.73	4.29
31	BBK	B B & T CORP	0.117	0.25	0.45	22.7
32	GFR	GREAT AMERICAN FINANCIAL RES	0.020	0.23	0.06	0.93
33	BJS	B J SERVICES CO	0.217	0.49	3.61	11.8
34	AZO	AUTOZONE INC	0.083	0.35	1.99	7.03
35	EP	EL PASO CORP	-0.026	0.51	1.77	8.01
36	KSS	KOHLS CORP	0.179	0.37	1.60	16.7
37	PX	PRAXAIR INC	0.096	0.31	1.15	17.0
38	ABC	AMERISOURCEBERGEN CORP	0.134	0.41	2.55	8.63
39	WOR	WORTHINGTON INDUSTRIES INC	-0.0041	0.37	1.12	1.69
40	ROK	ROCKWELL AUTOMATION INC	0.119	0.33	0.90	10.5
All			0.0607	0.165	0.938	893.24

Table 1: *Summary statistics for companies in sample. Time period: 1996-2005. Excess return denotes annualized excess return over 30-day T-bill rate, using geometric means. Variance is defined as annualized sample variance of daily returns. Turnover denotes annual number of shares traded, divided by shares outstanding. Market capitalization is measured on end-date (12/30 2005). Source: CRSP.*

[19] Until 2002, Phillips Petroleum (P). In 2002, Conoco and Phillips merged to form ConocoPhillips (COP).

Appendix

The production economy

We present the model in discrete time, and then take the continuous time limit. One firm produces a consumption good. The firm exists as a sequence of one-period entities and we include the time scale Δt. The firm uses a concave production technology as follows: for small production levels, the marginal cost of producing an additional unit increases, but above a specific level, it is constant. The firm uses marginal cost pricing, e.g. motivated by competition.[20] The production process is reversible, so there is no waste related to overproducing. There is also a fixed cost to producing in each time period, $k = q\Delta t + \xi_{n\Delta t}\sigma\sqrt{\Delta t}$, where the first part is deterministic and the second part is stochastic and $\xi_{n\Delta t} \sim Normal(0,1)$ are i.i.d. shocks over time.

The expected total surplus of producing and selling goods in one period is therefore bounded by $g\,\Delta t$ for g defined as follows:

$$g = z - q, \qquad \text{where} \qquad z = \int_0^{C(t)} (p - mc(s))ds \tag{30}$$

is the total variable part of the firm's profit per unit time. That is, if the demand for consumption at time t is $C(t)$, the variable cost of producing an additional good is $mc(s)$, and the price for the good is p, then the total surplus generated is $z - k$ (revenue – variable cost – fixed cost), and $z - q$ is the deterministic part. The idea is that $mc(s) = p(s)$ above a small threshold, so even if $C(t)$ varies, the stochastic process for the total profit will be the same. As we shall see, under market clearing, this endogenously leads to an Ornstein-Uhlenbeck process for consumption per unit time, which fits well with the notion of a business cycle.

We assume that the probability that demand will be below the point of constant marginal cost is negligible, so this is exactly the surplus created per unit time.[21] For timing purposes, we assume that the good is produced and sold immediately (at t), whereas the cash flow of holding the stock is realized at time $t + \Delta t$, i.e., is paid out as an end-of-period liquidating dividend.

In each period, the firm is set up and one divisible share, representing full ownership in the firm, is sold at the market clearing price. Then the product is produced, sold and profits are realized. Finally, the firm is liquidated and the value is paid out. The short term cash-flows of investing in the firm at time t if the market clearing price is P are shown in Table 2 and the one-period return for the stock is then

$$\tilde{\mu} \overset{\text{def}}{=} \frac{\Delta P}{P} = \frac{g}{P}\Delta t + \frac{\sigma}{P}(-\xi)\sqrt{\Delta t}. \tag{31}$$

Time:	t	$t + \Delta t$
Cash flows:	$-P$	$P + g\Delta t + \sigma(-\xi)\sqrt{\Delta t}$

Table 2: Cash flows from investing in firm.

Proof of Proposition 1: Special case of Proposition 2.

[20] The assumption of flat marginal costs to producing above a certain level can be viewed as an approximation of a market in which marginal costs are steep for small production quantities, and almost flat for large production quantities. The key assumption that allows us to simplify the analysis is that the gross profit of the firm is insensitive to demand shocks.

[21] This is assumed for simplicity, and imposes no major restriction. The model can also be formulated without the assumption, with ξ_t's having compact distributional support. In this case, the equilibrium conditions derived in what follows will be the same as, long as consumption demand is above the threshold (which by assumption can be arbitrarily small). This formulation complicates the notation considerably without offering additional insight about the objects for our attention, the wealth processes, so we avoid it.

Proof of Proposition 2:

As shown in Merton (1969), solving the optimal consumption problem for infinitely lived log-investors is specifically simple: Regardless of the future investment opportunity set, the investors behave myopically: instantaneously consuming $\rho W_{k,t}dt$, and only caring about instantaneous returns when making portfolio decisions. Also, investors will instantaneously choose mean-variance efficient portfolios. Define $\Lambda_t = diag(\mathbf{P}_t)$.

The perceived instantaneous return of the stocks at time t for investor k is $\tilde{\mu}^k = \mu^{Fk}dt + \mathbf{S}d\mathbf{B}^k$, and the investors' optimal relative portfolio choices will therefore be:

$$\alpha_{k,t} = \Lambda\Sigma^{-1}\Lambda(\mu^{Fk} - r_t\mathbf{1}_N) = \Lambda\Sigma^{-1}(\mathbf{g}^k - r_t\mathbf{P}_t). \tag{32}$$

The stock market clearing condition then becomes:

$$\begin{aligned}
&\alpha_{1,t}W_{1,t} + \alpha_{2,t}W_{2,t} = \mathbf{P}_t \Rightarrow \\
&W_{1,t}\Sigma^{-1}(\mathbf{g}^1 - r_t\mathbf{P}_t) + W_{2,t}\Sigma^{-1}(\mathbf{g}^2 - r_t\mathbf{P}_t) = \Lambda^{-1}\mathbf{P}_t \Rightarrow \\
&W_{1,t}(\mathbf{g}^1 - r_t\mathbf{P}_t) + W_{2,t}(\mathbf{g}^2 - r_t\mathbf{P}_t) = \Sigma\mathbf{1}_N \Rightarrow \\
&r_t\mathbf{P}_t = \mathbf{g} + \frac{W_{2,t}}{W_t}\delta - \frac{1}{W_t}\Sigma\mathbf{1}_N.
\end{aligned} \tag{33}$$

This is the stated equation for the price process.

Premultiplying the price equation with $\mathbf{1}'_N$ and using $\mathbf{1}'_N\mathbf{P}_t = W_t$ leads to the equation for the interest rate (11). Finally, plugging in the portfolio choices (32) into the true equation for the wealth process equation (7), leads to the wealth processes:

$$\frac{dW_{k,t}}{W_{k,t}} = (-\rho + r_t)dt + (\mathbf{g}^k - r_t\mathbf{P}_t)'\Sigma^{-1}\Big((\mathbf{g}^1 - r_t\mathbf{P}_t)dt + \Lambda\mathbf{S}d\mathbf{B}_t\Big). \tag{34}$$

From the pricing equation, (33), it follows that

$$\mathbf{g}^1 - r_t\mathbf{P}_t = -\frac{W_{2,t}}{W_t}\delta + \frac{1}{W_t}\Sigma\mathbf{1}_N, \tag{35}$$

$$\mathbf{g}^2 - r_t\mathbf{P}_t = \frac{W_{1,t}}{W_t}\delta + \frac{1}{W_t}\Sigma\mathbf{1}_N. \tag{36}$$

Plugging these into (34), together with interest rate formula (11) leads to the derived wealth dynamics (9-10).[22]
∎

Proof of Proposition 3:

We define $z_t \stackrel{\text{def}}{=} W_{1t}/W_{2t}$. Clearly, $z_t = f_t/(1-f_t)$. Itô calculus implies that

$$d\log(z_t) = d(\log(W_{1,t}) - \log(W_{2,t})) = dW_{1,t}/W_{1,t} - dW_{2,t}/W_{2,t} - \Big((dW_{1,t})^2/W_{1,t}^2 - (dW_{2,t})^2/W_{2,t}^2\Big)/2. \tag{37}$$

[22] This derivation shows that the equilibrium exists and is unique except for one special case. When $r_t = 0$, it breaks down. In this case, there is either no equilibrium, or multiple price vectors that all provide equilibria. Similar to the argument about very low levels of production in Section 2.1, a modification of the value generation process to keep interest rates positive can be made to avoid issues about existence of equilibria. Such low aggregate wealth levels are extremely rare events, occurring on average less than once in 10,000 years in our calibrations, so they have few implications for the speed of natural selection.

Equations (9-10), provide expressions for the first-order terms, and through Itô calculus imply that

$$(dW_{1,t})^2 = \frac{W_{1,t}^2}{W_t^2}\Big(dZ_1 - W_{2,t}dZ_2^2\Big)^2 = \frac{W_{1,t}^2}{W_t^2}\Big(D - 2BW_{2,t} + CW_{2,t}^2\Big)dt,$$
$$(dW_{2,t})^2 = \frac{W_{2,t}^2}{W_t^2}\Big(dZ_1 + W_{1,t}dZ_2^2\Big)^2 = \frac{W_{2,t}^2}{W_t^2}\Big(D + 2BW_{1,t} + CW_{1,t}^2\Big)dt.$$

Equation (37) together with equations (9-10) then implies that

$$\begin{aligned} d\log(z_t) = \\ \frac{dt}{W_t^2}\Big(- BW_{2,t} + CW_{2,t}^2 - BW_{1,t} + CW_{1,t}W_{2,t} - (D - 2BW_{2,t} + CW_{2,t}^2 - D - 2BW_{1,t} - CW_{1,t}^2)/2\Big) \\ + \frac{1}{W_t}\Big(dZ_{1,t} - W_{2,t}dZ_{2,t} - dZ_{1,t} - W_{1,t}dZ_{2,t}\Big) \\ = \frac{dt}{2W_t^2}C(W_{2,t}^2 + 2W_{1,t}W_{2,t} + W_{1,t}^2) - \frac{W_{1,t} + W_{2,t}}{W_t}dZ_{2,t} \\ = \frac{W_t^2}{2W_t^2}C\,dt - \frac{W_t}{W_t}dZ_{2,t} \\ = \frac{C}{2}dt + \sqrt{C}d\tilde{B}, \end{aligned}$$

where $\tilde{B}$ is a standard Brownian motion. This is, of course, in line with the literature of growth-rate optimal portfolios, which ensures that a rational investor with log-utility will dominate the market in the long run (Hakansson, 1971). The initial condition is $\log(z_0) = \log(f_0/(1 - f_0))$ and the first passage distribution of the time it takes for $\log(z_t)$ to reach $\log(f/(1 - f))$, is therefore (Ingersoll, 1987)

$$\tau_f = \frac{\log(f/(1 - f)) - \log(f_0/(1 - f_0))}{(2\pi Ct^3)^{1/2}}e^{-(\log(f/(1-f))-\log(f_0/(1-f_0))-Ct/2)^2/(2Ct)}.$$

The expected time is (Ingersoll, 1987)

$$E(\tau_f) = \frac{2}{C}\Big(\log(f/(1 - f)) - \log(f_0/(1 - f_0))\Big)$$

and the variance is

$$Var(\tau_f) = \frac{C(\log(f/(1 - f)) - \log(f_0/(1 - f_0)))}{2(C/2)^3} = \frac{4}{C^2}\Big(\log(f/(1 - f)) - \log(f_0/(1 - f_0))\Big).$$

∎

Portfolio held by investors: By (32), and (35-36), the positions held by the investors are

$$\alpha_{1,t}W_{1,t} = \frac{W_{1,t}}{W_t}\mathbf{P} - \frac{W_{1,t}W_{2,t}}{W_t}\Lambda\Sigma^{-1}\delta, \tag{38}$$
$$\alpha_{2,t}W_{2,t} = \frac{W_{2,t}}{W_t}\mathbf{P} + \frac{W_{1,t}W_{2,t}}{W_t}\Lambda\Sigma^{-1}\delta.$$

The first part of these positions correspond to the market hedging part, which is the only component if there are no sentiments. The second part is the speculative part, where investor 1 and 2 take opposite positions. Under the assumptions of parameters of Section 2.3, the second part for investor 1 becomes

$$\frac{qW_{1,t}W_{2,t}(N+1)}{W_t}\, diag(\text{sign}(\delta))\mathbf{P},$$

where the sign-operation is taken element-wise on the δ vector. This thus corresponds to a long-short portfolio, scaled up linearly in N.

Sharpe ratios: The Sharpe ratio for investor k is

$$S_{kt} = \frac{\alpha'_{kt}\Lambda^{-1}(\mathbf{g} - r_t\mathbf{P})}{\sqrt{\alpha'_{kt}\Lambda^{-1}\Sigma\Lambda^{-1}\alpha_{kt}}}. \tag{39}$$

For investor 1, this leads to

$$S_{1,t} = \frac{(\mathbf{P} - W_{2,t}\Lambda\Sigma^{-1}\delta)'\Lambda^{-1}(-W_{2,t}\delta + \Sigma\mathbf{1}_N)/W_t}{\sqrt{(\mathbf{P} - W_{2,t}\Lambda\Sigma^{-1}\delta)'\Lambda^{-1}\Sigma\Lambda^{-1}(\mathbf{P} - W_{2,t}\Lambda\Sigma^{-1}\delta)}} = \frac{\sqrt{D - 2W_{2,t}B + W_{2,t}^2C}}{W_t}.$$

Similarly, for investor 2, we get:

$$S_{2,t} = \frac{(\mathbf{P} + W_{1,t}\Lambda\Sigma^{-1}\delta)'\Lambda^{-1}(-W_{2,t}\delta + \Sigma\mathbf{1}_N)/W_t}{\sqrt{(\mathbf{P} + W_{1,t}\Lambda\Sigma^{-1}\delta)'\Lambda^{-1}\Sigma\Lambda^{-1}(\mathbf{P} + W_{1,t}\Lambda\Sigma^{-1}\delta)}} = \frac{D + (W_{1,t} - W_{2,t})B - W_{1,t}W_{2,t}C}{W_t\sqrt{D + 2W_{1,t}B + W_{1,t}^2C}}.$$

In the case of study, when $B = 0$, $C = q^2(N+1)$, $D = 1$, this reduces to

$$S_{1,t} = \frac{1}{W_t}\sqrt{1 + W_{2,t}^2q^2(N+1)}, \qquad S_{2,t} = \frac{1 - W_{1,t}W_{2,t}q^2(N+1)}{W_t\sqrt{1 + W_{1,t}^2q^2(N+1)}},$$

so for large N, the Sharpe ratio of investor 1 is basically proportional to $q\sqrt{N}$, whereas it is proportional to $-q\sqrt{N}$ for investor 2. It also follows that the market Sharpe ratio is

$$S_t = \frac{\sqrt{D}}{W_t}\Big(1 - \frac{W_{2,t}B}{D}\Big),$$

which under the assumptions of Section 2.3, with $B = 0$, reduces to exactly the same Sharpe ratio at each point in time as in the case with only one investor.

Asymptotic arbitrage: High-speed natural selection in an asymptotically well-behaved sequence of markets, $\mathcal{M}$, implies that the conditions for an asymptotic arbitrage (Ross, 1976) are satisfied. Let us define $C_N = \delta'_N\Sigma_N^{-1}\delta_N$. An arbitrage (i.e., self financed) portfolio is constructed by borrowing $\mathbf{P}'_t\Sigma^{-1}(\mathbf{g} - r_t\mathbf{P}_t)/C_N$ and investing it in the portfolio $\alpha \stackrel{\text{def}}{=} \Lambda\Sigma^{-1}(\mathbf{g} - r_t\mathbf{P}_t)/C_N$.

The instantaneous expected excess return of this portfolio is:

$$\begin{aligned}\frac{1}{C_N}(\mathbf{g} - r\mathbf{P})'\Sigma^{-1}(\mathbf{g} - r\mathbf{P}) &= \frac{1}{C_N}\left(-\frac{W_{2,t}}{W_t}\delta + \frac{1}{W_t}\Sigma\mathbf{1}_N\right)'\Sigma^{-1}\left(-\frac{W_{2,t}}{W_t}\delta + \frac{1}{W_t}\Sigma\mathbf{1}_N\right)\\ &= \frac{1}{C_N}\left(\frac{W_{2,t}^2}{W_t^2}C_N + \frac{D}{W_t^2} - 2\frac{W_{2,t}B}{W_t^2}\right) \geq \frac{W_{2t}}{W_t} - \epsilon,\end{aligned}$$

for any $\epsilon > 0$, for large N. The instantaneous variance is

$$\frac{1}{C_N^2}(\mathbf{g} - r\mathbf{P})'\Sigma^{-1}\Sigma\Sigma^{-1}(\mathbf{g} - r\mathbf{P}) = \frac{1}{C_N^2}\left(-\frac{W_{2,t}}{W_t}\delta + \frac{1}{W_t}\Sigma\mathbf{1}_N\right)'\Sigma^{-1}\left(-\frac{W_{2,t}}{W_t}\delta + \frac{1}{W_t}\Sigma\mathbf{1}_N\right)$$
$$= \frac{1}{C_N^2}\left(\frac{W_{2,t}^2}{W_t^2}C_N + \frac{D}{W_t^2} - 2\frac{W_{2,t}B}{W_t^2}\right) \leq \epsilon,$$

for any $\epsilon > 0$ for large N. Thus, the conditions for asymptotic arbitrage in $\mathcal{M}$ are satisfied (Ross, 1976).

We stress that, contrary to the partial equilibrium approach of general arbitrage arguments, in our general equilibrium approach, this is not a *scalable* asymptotic arbitrage opportunity, as the pricing system is not linear. For any fixed economy, if the demand for the portfolio is scaled up in an unbounded fashion, the pricing adjusts and the opportunity to earn abnormal returns diminishes in relative terms.

Proof of Proposition 4: The proof is a straightforward application of spectral decomposition. The spectral theorem ensures that for each N, there is a real orthogonal transformation of Σ_N into a diagonal matrix with strictly positive elements, $\Sigma_N = R_N'\Lambda_N R_N$, $\Lambda_N = diag(\rho_1, \ldots, \rho_N)$ and $R_N^{-1} = R_N'$. W.l.o.g., we can assume that the ρ's are ordered increasingly, so the spectral radius of Σ_N is ρ_N. Standard matrix-norm theory implies that

$$\min_{\delta_N' \neq \mathbf{0}_N} \frac{\delta_N'\Sigma_N^{-1}\delta_N}{\delta_N'\delta_N} = \frac{1}{\rho_N},$$

and by our assumptions, $\rho_N \leq c_2 N^{-\nu} f(N)$, $\delta_N'\delta_N \geq c_1 f(N)$, so

$$\delta_N'\Sigma_N^{-1}\delta_N \geq \frac{N^\nu}{c_2 f(N)} \times c_1 f(N) = \frac{c_1}{c_2}N^\nu.$$

∎

Proof of Proposition 5: As in the proof of the previous proposition, the spectral theorem ensures that for each N, there is a real orthogonal transformation of Σ_N into a diagonal matrix with strictly positive elements, $\Sigma_N = R_N'\Lambda_N R_N$, $\Lambda = diag(\rho_1, \ldots, \rho_N)$ and $R_N^{-1} = R_N'$. Moreover, the first assumption ensures that there is an eigenvalue, ρ_i, with corresponding eigenvector $\mathbf{1}_N$. We define $\rho^* = \rho(Q_{\mathbf{1}_N}'\Sigma_N^{-1}Q_{\mathbf{1}_N})$. Also, let us denote by P_N, the projection operator onto the one-dimensional subspace spanned by $\mathbf{1}_N$, so $Q_{\mathbf{1}_N} \perp P_N$. Clearly $P_N\delta_N = \frac{\mathbf{1}_N'\delta_N}{N}\mathbf{1}_N$. We can decompose

$$\delta_N'\Sigma_N^{-1}\delta_N = (\delta_N - P_N\delta_N + P_N\delta_N)'\Sigma^{-1}(\delta_N - P_N\delta_N + P_N\delta_N) =$$
$$(\delta_N - P_N\delta_N)'Q_{\mathbf{1}_N}'\Sigma_N^{-1}Q_{\mathbf{1}_N}(\delta_N - P_N\delta_N) + \frac{(P_N\delta_N)'(P_N\delta_N)}{\rho_i} \geq$$
$$(\delta_N - P_N\delta_N)'Q_{\mathbf{1}_N}'\Sigma_N^{-1}Q_{\mathbf{1}_N}(\delta_N - P_N\delta_N) \geq$$
$$\frac{(\delta_N - P_N\delta_N)'(\delta_N - P_N\delta_N)}{\rho^*} =$$
$$\frac{\delta_N'\delta_N - \frac{(\mathbf{1}_N'\delta_N)^2}{N}}{\rho^*}.$$

By the assumptions of the Proposition, we therefore have (for large enough N):

$$\delta_N'\Sigma_N^{-1}\delta_N \geq \frac{c_1 f(N)}{c_2 N^{-\nu} f(N)} = \frac{c_1}{c_2}N^\nu.$$

∎

Proof of Proposition 6: i) Asymptotically well-behaved markets (a.s.): For the Nth market, the market Sharpe ratio is $\overline{S}_N = \rho\sqrt{D_N}/A_N$, where ρ is the intertemporal substitution factor, and A_N and D_N of $\mathcal{M}_N$ are defined in Proposition 2. The strong law of large numbers immediately imply that $A_N \to_{a.s.} \bar{p} > 0$, $\Delta_N \to_{a.s.} E(|\tilde{q}|) \in (0, \infty)$. Furthermore,

$$D_N = \sum_i \frac{(\alpha_N^i)^2}{N^2} + \left(\frac{\sum_i \beta_N^i}{N}\right)^2,$$

and $\sum_i (\alpha_i^N)^2/N \to_{a.s.} E[(\alpha_1^1)^2] \in (0, \infty)$, and $\sum_i \tilde{\beta}_i^N/N \to_{a.s.} \bar{\beta}$, so $D \to_{a.s.} 0 + \bar{\beta}^2 \in (0, \infty)$. Thus, $\overline{S} \sim 1$ a.s., $A_N \sim 1$ a.s., and $\Delta_N \sim 1$ a.s., so the conditions for an asymptotically well-behaved sequence of markets are a.s. satisfied.

ii) High-speed natural selection of order one (a.s.): Define $\Lambda_{\alpha,N} = diag(\alpha_1^N, \ldots, \alpha_N^N)$ and $\lambda_N = N\delta_N$. We use the inversion formula $(I + \mathbf{x}\mathbf{x}')^{-1} = I - \frac{1}{1+\mathbf{x}'\mathbf{x}}\mathbf{x}\mathbf{x}'$ for an arbitrary vector $\mathbf{x}$ to get

$$\begin{aligned}
\frac{C_N}{N} &= \frac{1}{N}\delta_N'\Sigma^{-1}\delta_N = \frac{1}{N}\lambda_N'(\Lambda_{\alpha,N}^2 + \mathbf{b}_N\mathbf{b}_N')^{-1}\lambda_N \\
&= \frac{1}{N}\left(\Lambda_{\alpha,N}^{-1}\lambda_N\right)'\left(I_N + \Lambda_{\alpha,N}^{-1}(\mathbf{b}_N\mathbf{b}_N')\Lambda_{\alpha,N}^{-1}\right)^{-1}\left(\Lambda_{\alpha,N}^{-1}\lambda_N\right) \\
&= \frac{1}{N}\left(\Lambda_{\alpha,N}^{-1}\lambda_N\right)'\left(I_N - \frac{1}{1 + \mathbf{b}_N'\Lambda_{\alpha,N}^{-2}\mathbf{b}_N}\Lambda_{\alpha,N}^{-1}(\mathbf{b}_N\mathbf{b}_N')\Lambda_{\alpha,N}^{-1}\right)\left(\Lambda_{\alpha,N}^{-1}\lambda_N\right) \\
&= \frac{\lambda_N'\Lambda_{\alpha,N}^{-2}\lambda_N}{N} - \frac{1}{\frac{1}{N} + \frac{\mathbf{b}_N'\Lambda_{\alpha,N}^{-2}\mathbf{b}_N}{N}}\left(\frac{\lambda_N'\Lambda_{\alpha,N}^{-2}\mathbf{b}_N}{N}\right)^2.
\end{aligned}$$

The independence of these variables, together with the strong law of large numbers, implies that

$$\begin{aligned}
\frac{C_N}{N} &\to_{a.s.} E[(\tilde{q}_1^1)^2]E[(\alpha_1^1)^{-2}] - \frac{1}{E[(\beta_1^1)^2]E[(\alpha_1^1)^{-2}]}\left(E[\tilde{q}_1^1]E[\beta_1^1]E[(\alpha_1^1)^{-2}]\right)^2 \\
&= E[(\alpha_1^1)^{-2}]\left(E[(\tilde{q}_1^1)^2] - \frac{E[\tilde{q}_1^1]^2E[\beta_1^1]^2}{E[(\beta_1^1)^2]}\right) \\
&= \frac{E[(\alpha_1^1)^{-2}]}{E[(\beta_1^1)^2]}\left(E[(\tilde{q}_1^1)^2]E[(\beta_1^1)^2] - E[\tilde{q}_1^1]^2E[\beta_1^1]^2\right) \\
&= \frac{E[(\alpha_1^1)^{-2}]}{E[(\beta_1^1)^2]}\left((\sigma_q^2 + \bar{q}^2)(\sigma_\beta^2 + \bar{\beta}^2) - \bar{q}^2\bar{\beta}^2\right) \\
&= \frac{E[(\alpha_1^1)^{-2}]}{\sigma_\beta^2 + \bar{\beta}^2}\left(\sigma_q^2\sigma_\beta^2 + \bar{q}^2\sigma_\beta^2 + \sigma_q^2\bar{\beta}^2\right) = k \in (0, \infty].
\end{aligned}$$

The strict positivity of k is ensured, as $\sigma_\beta > 0$, $\sigma_q > 0$, and Jensen's inequality ensures that $E[(\alpha_1^1)^{-2}] \geq \frac{1}{\sigma_\alpha^2 + \bar{\alpha}^2}$. Thus, C_N grows like kN a.s. as N becomes large. This completes the proof. If $E[(\alpha_1^1)^{-2}] < \infty$ (which is not guaranteed by our assumptions) then $k < \infty$, so in this case the order of the natural selection process is exactly one. Otherwise it can be faster.

We note that the argument is easy to generalize to more general random structures. For example, a similar result can be derived for K-factor models, $K > 1$, using the same argument as above, but with the inversion rule $(I_N + \mathbf{X}\mathbf{X}')^{-1} = I_N - \mathbf{X}(I_K + \mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$. Here, $\mathbf{X}$ is an $N \times K$ random matrix, representing the factor loadings of the N stocks on K factors, I_N is the $N \times N$ identity matrix and I_K is the $K \times K$ identity matrix. ∎

Short-sales constraints: In the market model of Section 2.2-2.3, high-speed natural selection will not survive the imposition of nonnegativity constraints for investors' portfolios. imposing such constraints basically divides the market into two: one with a subset of stocks in which only the rational investors trade, and the complement in which only the irrational investors trade. Both groups view the other's set of stocks as overpriced and avoid them. The only way to get high-speed natural selection in this case would be through the risk-free asset, i.e., if the rational investors would be lending to the irrational ones at very high interest rates. However, an analysis of the constrained optimization problems does not show such behavior, at least not for the market model of Section 2.2-2.3. In fact, for asymptotically well-behaved sequences of markets, the interest rate will be bounded, regardless of the number of assets, so high-speed natural selection will not take place.

Thus, with short-sale constraints present, it is not straightforward to achieve high-speed natural selection. However, if the sequence of markets is not asymptotically well-behaved, it is still possible to achieve high-speed natural selection. For example, in a growing sequence of markets, the selection process may still be fast. Consider the following sequence of markets,

$$\mathbf{g}_N = \mathbf{1}_N, \qquad \Sigma_N = I_N + \mathbf{1}_N\mathbf{1}_N', \qquad (\delta_N)_{2i} = 0.1, \ (\delta_N)_{2i+1} = -0.1, \tag{40}$$

with $W_{10} = W_{20} = 2$. For $\rho = 10\%$, and large N, this market has a wealth far below the steady state wealth of $\bar{W}_N = A_N/\rho = N/\rho$, so consumption and wealth will grow initially. It is straightforward to check, using equation (38), that the short-selling constraint will not be binding in this case. Moreover, Proposition 5 implies that high-speed natural selection occurs. Thus, high-speed natural selection survives imposition of short-sale constraints in this special case. An interpretation of this result is that in rapidly expanding markets, high-speed natural selection can occur even with short-sale constraints, as in such markets the payoff of finding the winners is higher.

Proof of Proposition 7: Identical to the proof of Proposition 3. ∎

The return-based measure: We have $\mu^F = \Lambda^{-1}\mathbf{g}$ and $\Sigma^F = \Lambda^{-1}\Sigma\Lambda^{-1}$, which, using (35-36), leads to (24). By expanding (24), we get

$$\begin{aligned}(\Sigma^F\mathbf{s} - \mu_e)'(\Sigma^F)^{-1}(\Sigma^F\mathbf{s} - \mu_e) = \mathbf{s}'\Sigma^F\mathbf{s} - 2\mu_e'\mathbf{s} + \mu_e'(\Sigma^F)^{-1}\mu_e &= \sigma_m^2 - 2\mu_m + \mu_e'(\Sigma^F)^{-1}\mu_e \\ &\le \sigma_m^2 - 2\mu_m + N\mu_m^2/R,\end{aligned}$$

i.e., (25).

The volume-based measure: From equations (35-36), we have $\mathbf{z}_t = ((W_{1t} - W_{2t})\mathbf{1}_N - 2W_{1t}W_{2t}\Sigma^{-1}\delta)/(2W_t)$, via Itô calculus leading to

$$d\mathbf{z}_t = -W_{1t}W_{2t}W_t^{-2}dZ_{2t}\mathbf{1}_N + W_{1t}W_{2t}W_t^{-2}((W_{1t} - W_{2t})dZ_{2t} - dZ_{1t})\Sigma^{-1}\delta + o(dt^{1/2}).$$

For convenience, we define the vector $\mathbf{a} = \Sigma^{-1}\delta$. For a general $N \times N$ matrix $\mathbf{K}$, we then have

$$d\mathbf{z}_t'\mathbf{K}d\mathbf{z}_t = dt\left(\frac{W_{1t}W_{2t}}{W_t^2}\right)^2(\mathbf{1}_N dZ_{2t} + ((W_{1t} - W_{2t})dZ_{2t} - dZ_{1t})\mathbf{a})'\,\mathbf{K}\,(\mathbf{1}_N dZ_{2t} + ((W_{1t} - W_{2t})dZ_{2t} - dZ_{1t})\mathbf{a}) + o(dt).$$

We choose $\mathbf{K} = \Lambda\mathbf{S}^2\Lambda$, and assume that $\mathbf{K}$ is a good preconditioner for Σ^{-1}, i.e., that $\mathbf{K}\Sigma^{-1} \approx I_N$ (see Golub and van Loan 1991). This is a natural assumption if the idiosyncratic risk of individual stocks are not small. Using the definitions of A, B, C, and D in Proposition 2, we arrive at

$$d\mathbf{z}_t'\mathbf{K}d\mathbf{z}_t \approx dt\left(\frac{W_{1t}W_{2t}}{W_t^2}\right)^2\left(\mathbf{1}_N'\mathbf{K}\mathbf{1}_N C + (W_{1t}-W_{2t})^2C^2 + DC - 2B^2\right) + o(dt),$$

which, by the assumptions that $W_{1t} = W_{2t}$ and $B << \Delta$, reduces to

$$d\mathbf{z}_t'\mathbf{K}d\mathbf{z}_t \approx dt\frac{1}{4}\left(\mathbf{P}_t\mathbf{S}^2\mathbf{P}_t + D\right)C + o(dt),$$

justifying the approximation (27).

For (28), we note that for small Δt, $v_t \stackrel{\text{def}}{=} (\Delta t)^{-1}\int_{t-\Delta t}^{t}(d\mathbf{z}_t)_i$ is approximately $Normal(0, h^2)$ distributed for some $h > 0$, so $(\overline{q})_i = E(|v_t|) \approx h \times \pi/2$, and as $\int_t^{t+\Delta t}(d\mathbf{z})_i^2 = h^2\Delta t + o(\Delta t)$, (28) follows.

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