Global Dynamics in Infinitely Repeated Games with Additively Separable Continuous Payoffs*

Takashi Kamihigashi† and Taiji Furusawa‡

November 13, 2007

Abstract

This paper studies a class of infinitely repeated games with two players in which the action space of each player is an interval, and the one-shot payoff of each player is additively separable in their actions. We define an immediately reactive equilibrium (IRE) as a pure-strategy subgame perfect equilibrium such that each player's action in each period is a stationary function of the other player's last action. We completely characterize IREs and their dynamics in terms of certain indifference curves. In a special case we establish a folk-type theorem using only IREs that are continuous and punish deviations in a minimal way. Our results are used to show that in a prisoners' dilemma game with observable mixed strategies, gradual cooperation occurs when the players are sufficiently patient, and that in a certain duopoly game, kinked demand curves emerge naturally.

Keywords: Immediately reactive equilibria; additively separable payoffs; kinked demand; gradual cooperation; prisoners' dilemma

---

*This is a revised version of the paper entitled "Immediately Reactive Equilibria in Infinitely Repeated Games with Additively Separable Continuous Payoffs." Earlier versions were presented in seminars at Kyoto University, University of Venice, University of Paris 1, and GREQAM. We would like to thank Atsushi Kajii, Tomoyuki Nakajima, Tadashi Sekiguchi, Olivier Tercieux, Julio Davila, Sergio Currarini, Piero Gottardi, and anonymous referees for helpful comments and suggestions.

†Corresponding author. RIEB, Kobe University, Rokkodai, Nada, Kobe 657-8501 JAPAN. Email: tkamihig@rieb.kobe-u.ac.jp. Tel/Fax: +81-78-803-7015.

‡Graduate School of Economics, Hitotsubashi University, 2-1 Naka, Kunitachi, Tokyo 186-8601 JAPAN. Email: furusawa@econ.hit-u.ac.jp. Tel/Fax: +81-42-580-8866.

1 Introduction

In infinitely repeated games with a prisoners' dilemma-like stage game, Nash reversion trigger strategies (Friedman, 1971) are often used to show that cooperation (or collusion) can be sustained by the threat to revert to a noncooperative Nash equilibrium. In such equilibria, each player continues to cooperate as long as all the other players cooperate, but will choose to behave selfishly once anyone defects.

While Nash reversion equilibria are simple and intuitive, they seem to have two disadvantages. First, small deviations are punished as harshly as large deviations.[1] Second, there are no nontrivial transition dynamics between the cooperative state and the noncooperative state. The first disadvantage can be avoided by considering continuous strategies, which are insensitive to small deviations. This approach has been adopted by Samuelson (1987), Friedman and Samuelson (1990, 1994a, 1994b), and Langlois and Sachs (1993), who established the existence of nontrivial continuous equilibria. As for nontrivial transition dynamics, however, very few general results seem to be available in the literature.

In this paper we study a class of infinitely repeated two-person games in which the action space of each player is an interval. We follow Friedman (1968, 1973, 1976) in focusing on strategies such that the action chosen in each period is a stationary function of the other player's last action. We call such strategies *immediately reactive*, and say that a subgame perfect equilibrium is an *immediately reactive equilibrium (IRE)* if each player chooses an immediately reactive strategy.[2] IREs are the natural choice in alternating move games (e.g., Maskin and Tirole, 1988a; Bhaskar and Vega-Redondo, 2002). Since they can be continuous or discontinuous, IREs extend Nash reversion equilibria, avoiding their first disadvantage mentioned above. Furthermore, the global dynamics of IREs are typically nontrivial and can completely be characterized graphically.

In our framework, interesting dynamics arise naturally. For example,

[1]This also applies to more complex trigger strategy equilibria (Green and Porter 1984; Abreu, 1986, 1988).

[2]Friedman (1968, 1973, 1976) called IREs "reaction function equilibria." We avoid his terminology since it has been used in a broader sense in subsequent literature. The concept of IRE is related to a few other ones (Friedman and Samuelson, 1994a; Kalai, Samet, and Stanford, 1988; Maskin and Tirole, 1988a, 1988b). Detailed discussions are given in Subsections 2.4 and 2.5.

cooperation is achieved gradually in a repeated prisoners' dilemma with observable mixed strategies if the players are sufficiently patient.[3] In a duopoly game, kinked demand curves emerge naturally.[4] In a collusive steady state, each firm cuts its price if the other firm does so, but neither firm responds if the other firm raises its price. In "inefficient" IREs, the collusive steady state is unstable: after a small price cut by either firm, price war starts gradually, accelerates, and leads to noncollusive prices in the long run. In "efficient" IREs, however, the collusive steady state is stable: a price cut by either firm is matched by a smaller price cut, and the steady state is restored in the long run. All of these dynamic phenomena are properties of *IREs in regular form,* which we define as IREs that are continuous and punish deviations in a minimal way.

Unfortunately, these results come at a cost. In particular, we assume that the one-shot payoff of each player is additively separable in their actions. This assumption is necessary since, as shown by Stanford (1986) and Robson (1986), the only possible IRE is a trivial one in certain duopoly games with additively non-separable payoffs. On the other hand, the assumption of additive separability holds in various games, including a prisoner's dilemma with observable mixed strategies and a duopoly game in which each firm's one-shot payoff is given by the logarithm of its profit.[5] Another example is a two-country model of international trade in which each country sets its tariff rate to maximize the sum of tariff revenue, consumer surplus, and producer surplus.[6] These and other examples are discussed in Subsection 2.2. In addition to additive separability, we assume that each player's one-shot payoff is continuous, monotone in the other player's action, and monotone or unimodal in his own action.

[3]Gradual cooperation is known to arise in certain partnership games (e.g., Kranton, 1996; Watson, 1999, 2002; Furusawa and Kawakami, 2006). Our example shows a simplest mechanism of gradual cooperation.

[4]Although there are game-theoretic models of kinked demand in the literature (e.g., Maskin and Tiroel, 1988b; Bhaskar, Machin, and Reid, 1991; Radner, 2003; Sen, 2004), they typically require rather specific assumptions. Though our example also requires specific assumptions, it allows one to derive and visualize kinked demand curves in an extremely simple manner.

[5]This can be justified by assuming that the owner of each firm is "risk averse" or, more precisely, prefers stable profit streams to unstable ones.

[6]Furusawa and Kamihigashi (2006) study such a model, focusing on issues specific to international trade. A preliminary version of Furusawa and Kamihigashi (2006) contained some of the arguments in this paper, which now appear exclusively in this paper.

Given a stage game satisfying the assumptions mentioned above, we show that the set of IREs in the simultaneous move game is identical to that in the alternating move game.[7] In both games, we completely characterize IREs in terms of indifference curves associated with what we call *effective payoffs.* The effective payoff of a player is the part of his discounted sum of payoffs that is directly affected by his current action. By additive separability, the effective payoff consists of only two functions. This structure substantially simplifies each player's dynamic maximization problem.[8]

We show that in any IRE, any equilibrium path stays on the associated indifference curves except for the initial period. By this result, equilibrium dynamics are always characterized by two indifference curves. Our main result characterizes what pairs of indifference curves can be supported as IREs: given a pair of indifference curves, there is an associated IRE if and only if the following two graphical conditions are met. First, the intersection of the areas on or above the indifference curves must be nonempty. Second, the lowest point of each indifference curve must not be too low relative to the other indifference curve.

In a special case in which each player's payoff depends monotonically on his own action, we provide a necessary and sufficient condition for an IRE to be *effective efficient,* i.e., Pareto optimal among IREs in terms of effective payoffs. The necessary and sufficient condition is that the intersection of the areas strictly above the indifference curves be empty. Effective efficiency has interesting dynamic implications, such as gradual cooperation in a prisoners' dilemma with observable mixed strategies, as well as globally stable collusive prices in a duopoly game, as mentioned above. In the same special case, we also obtain the following folk-type theorem: if both players are sufficiently patient, any "strictly individually rational" action profile—or any pair of actions in which each player's payoff is strictly greater than his minimax payoff—can be supported as a steady state of an IRE in regular form.

Our folk-type theorem is similar in spirit to those shown by Friedman and Samuelson (1994a, 1994b). Their results show that the main idea of

[7]Section 4 discusses the relationship of this result to Lagunoff and Matsui's (1997) anti-folk theorem for alternating move games of pure coordination. See Haller and Lagunoff (2000) and Yoon (2001) for further results on alternating move games.

[8]Effective payoffs are similar to what Kamihigashi and Roy (2006) call partial gains in an optimal growth model with linear utility. Equations (2.19)–(2.21) in this paper are similar to (3.7)–(3.9) in Kamihigashi and Roy (2006), but essentially this is the only similarity in analysis between the two papers.

the standard folk theorem (Fudenberg and Maskin, 1986) is valid even if one confines oneself to continuous equilibria. In our case the equilibria that we consider are stationary, continuous, and immediately reactive. In addition, the IREs used in our folk-type theorem punish deviations in a minimal way.

The rest of the paper is organized as follows. Section 2 describes the one-shot game and our assumptions, discusses several examples, and introduces the simultaneous and the alternating move games. Section 3 characterizes the best responses of a player given the other player's strategy, developing and utilizing various graphical tools. The main result of Section 3 has some immediate implications on IREs, which are shown in Section 4. Section 5 discusses the dynamics induced by IREs. Section 6 gives a complete characterization of IREs. Section 7 characterizes effectively efficient IREs in a special case and shows a folk-type theorem. Section 8 applies our results to a prisoner's dilemma game and a duopoly game. Section 9 concludes the paper. The appendix contains the proof of our characterization result.

2 The Games

2.1 The One-Shot Game

Before introducing repeated games, let us describe the one-shot game. There are two players, 1 and 2. Define

$$Q = \{(1, 2), (2, 1)\}. \tag{2.1}$$

For $(i, j) \in Q$, let S_i denote player i's action space, $\pi_i : S_i \times S_j \to \mathbb{R}$ player i's payoff. The following assumptions are maintained throughout.

Assumption 2.1. For $i = 1, 2$, $S_i \subset \mathbb{R}$ is an interval with nonempty interior.

Assumption 2.2. For $(i, j) \in Q$, there exist $u_i : S_i \to [-\infty, \infty)$ and $v_i : S_j \to \mathbb{R}$ such that

$$\forall (s_i, s_j) \in S_i \times S_j, \quad \pi_i(s_i, s_j) = u_i(s_i) + v_i(s_j). \tag{2.2}$$

Assumption 2.3. v_1 and v_2 are continuous. Either both are strictly increasing or both are strictly decreasing.

Assumption 2.4. For $i = 1, 2$, u_i is continuous, and there exists $\hat{s}_i \in S_i$ such that u_i is strictly increasing on $S_i \cap (-\infty, \hat{s}_i)$ provided $S_i \cap (-\infty, \hat{s}_i) \neq \emptyset$, and strictly decreasing on $S_i \cap (\hat{s}_i, \infty)$ provided $S_i \cap (\hat{s}_i, \infty) \neq \emptyset$.[9]

Assumption 2.5. For $i = 1, 2$, u_i is bounded above, and v_i is bounded.

Assumption 2.2 is our key assumption. Assumptions 2.4 and 2.5 imply that $\hat{s}_i$ is player i's strictly dominant strategy and that $(\hat{s}_1, \hat{s}_2)$ is the unique static Nash equilibrium. Assumption 2.2 allows u_i to be unbounded below because such cases are common in economic models.

2.2 Examples

Though Assumption 2.2 may appear rather strong as a restriction on general games with two players, it is satisfied in various games. We provide specific examples below. Our intention here is not to claim that our assumptions are general, but to suggest that our framework is useful in analyzing certain types of games as well as special cases of more general games.

2.2.1 Tariff War

Consider a two-country world in which the payoff of each country is given by the sum of its tariff revenue, consumer surplus, and producer surplus. Each country is better off if the other country reduces its tariff rate, while each country has an incentive to choose the tariff rate that maximizes the sum of its tariff revenue and consumer surplus. To be more specific, let $\hat{s}_i$ be this maximizing tariff rate, and s_i be country i's tariff rate imposed on imports from country j. Under standard assumptions, country i's producer surplus is strictly decreasing in s_j, while the sum of its tariff revenue and consumer surplus is strictly increasing in s_i for $s_i \leq \hat{s}_i$ and strictly decreasing for $s_i \geq \hat{s}_i$. This game satisfies our assumptions, and is analyzed in detail in Furusawa and Kamihigashi (2006).

2.2.2 Aggregative Games

Consider a game in which the payoff of player i can be written as a function of s_i and $s_i + s_j$, i.e., $\pi_i(s_i, s_j) = \tilde{\pi}_i(s_i, s_i + s_j)$ for some $\tilde{\pi}_i$. This type of

[9] We follow the convention that if $u_i(r) = -\infty$ for some $r \in S_i$, then u_i is continuous at r if $\lim_{s_i \to r} u_i(s_i) = -\infty$. Such r can only be $\min S_i$ or $\max S_i$ by Assumption 2.4.

game is called an aggregative game (Corchon, 1994). For example, s_i can be player i's contribution to a public good, or his pollution emission. If $\tilde{\pi}$ is additively separable and depends linearly on $s_i + s_j$, then there are various cases in which our assumptions are satisfied.

2.2.3 Bertrand Competition

Consider a game played by two firms, each producing a differentiated product with a constant marginal cost c_i and with no fixed cost. Firm i faces a demand function $D_i(p_i, p_j)$ that depends on the prices p_i and p_j chosen by the two firms. Firm i's profit is $D_i(p_i, p_j)(p_i - c_i)$. Suppose D_i is multiplicatively separable: $D_i(p_i, p_j) = d_i^i(p_i)d_i^j(p_j)$ for some functions d_i^i and d_i^j. Then the profit maximization problem of firm i is equivalent to maximizing $u_i(p_i) + v_i(p_j)$, where

$$u_i(p_i) = \ln d_i^i(p_i) + \ln(p_i - c_i), \quad v_i(p_j) = \ln d_i^j(p_j). \tag{2.3}$$

This transformation is innocuous in the one-shot game, and our assumptions are satisfied under reasonable assumptions on d_i^i and d_i^j. In repeated games the above transformation may be justified by assuming that the owners of the firms are "risk averse" or, more precisely, prefer stable profit streams to unstable ones.

2.2.4 Prisoner's Dilemma

Though the action spaces are assumed to be intervals in this paper, our framework applies to 2×2 games with mixed strategies. A case in point is the prisoner's dilemma game in Figure 1 (with $a, c > 0$), which is a parametrized version of the game discussed by Fudenberg and Tirole (1991, p. 10, p. 111). For $i = 1, 2$, let s_i be player i's probability of choosing action C. Let $\pi_i(s_i, s_j)$ be player i's expected payoff:

$$\pi_i(s_i, s_j) = s_i s_j c + s_i(1 - s_j)(-a) + (1 - s_i)s_j(c + a) \tag{2.4}$$

$$= -as_i + (c + a)s_j. \tag{2.5}$$

Let $S_1 = S_2 = [0, 1]$. Then all our assumptions are clearly satisfied with $\hat{s}_1 = \hat{s}_2 = 0$.[10]

[10]Furusawa and Kawakami (2006) use a payoff function similar to (2.4) to analyze perfect Bayesian equilibria in a model with stochastic outside options.

		Player 2	
		C	D
Player 1	C	c, c	$-a, c+a$
	D	$c+a, -a$	$0, 0$

Figure 1: Prisoner's dilemma

		Player 2	
		1	2
Player 1	1	p_1^{11}, p_2^{11}	p_1^{12}, p_2^{12}
	2	p_1^{21}, p_2^{21}	p_1^{22}, p_2^{22}

Figure 2: General 2×2 game

2.2.5 General 2×2 Games

The preceding example suggests that our framework applies to more general 2×2 games. To see this, consider the 2×2 game in Figure 2. For $i = 1, 2$, let s_i be player i's probability of choosing action 1. Let $\pi_i(s_i, s_j)$ be player i's expected payoff:

$$\begin{aligned}
&\pi_i(s_i, s_j) && (2.6)\\
&= s_i s_j p_i^{11} + s_i(1-s_j)p_i^{12} + (1-s_i)s_j p_i^{21} + (1-s_i)(1-s_j)p_i^{22} && (2.7)\\
&= (p_i^{12} - p_i^{22})s_i + (p_i^{21} - p_i^{22})s_j + (p_i^{11} - p_i^{12} - p_i^{21} + p_i^{22})s_i s_j + p_i^{22}. && (2.8)
\end{aligned}$$

It is easy to see that all our assumptions hold if and only if $p_i^{12} \neq p_i^{22}$, $(p_1^{21} - p_1^{22})(p_2^{21} - p_2^{22}) > 0$, and $p_i^{11} - p_i^{12} - p_i^{21} + p_i^{22} = 0$. The last condition suggests some form of additive separability. For example, it can be written as $p_i^{11} - p_i^{21} = p_i^{12} - p_i^{22}$, i.e., player i's choice has the same effect on his payoff independently of player j's action. Alternatively, it can be written as $p_i^{11} - p_i^{12} = p_i^{21} - p_i^{22}$, i.e., player j's choice has the same effect on player i's payoff independently of player i's action.

2.3 Normalizing Assumptions

To simplify the exposition, we introduce some assumptions that can be made without loss of generality.

Assumption 2.6. For $i = 1, 2, \inf S_i = 0$ and $\sup S_i = 1$.

This can be assumed without loss of generality since none of our assumptions is affected by strictly increasing, continuous transformations of S_i. If $0 \notin S_i$ and/or $1 \notin S_i$, we extend u_i and v_i to 0 and/or 1 as follows:

$$u_i(0) = \lim_{s \downarrow 0} u_i(s), \quad u_i(1) = \lim_{s \uparrow 1} u_i(s), \tag{2.9}$$

$$v_i(0) = \lim_{s \downarrow 0} v_i(s), \quad v_i(1) = \lim_{s \uparrow 1} v_i(s). \tag{2.10}$$

By the above and Assumption 2.5, the following can be assumed without loss of generality.

Assumption 2.7. For $i = 1, 2$, $u_i : [0, 1] \to [-\infty, \infty)$ and $v_i : [0, 1] \to \mathbb{R}$ are continuous, and $u_i((0, 1]), v_i([0, 1]) \subset \mathbb{R}$.

Strictly speaking, the next assumption is not a normalization, but it is innocuous and is made merely for notational simplicity.[11]

Assumption 2.8. $S_1 = S_2 = [0, 1]$.

The following is our last normalizing assumption.

Assumption 2.9. For $i = 1, 2$, v_i is strictly increasing.

To see that this is a normalization, suppose v_1 and v_2 are both strictly decreasing (recall Assumption 2.3). For $(i, j) \in Q$, define $\tilde{S}_i = [0, 1], \tilde{s}_i = 1 - s_i, \tilde{u}_i(\tilde{s}_j) = u_i(1 - \tilde{s}_j)$, and $\tilde{v}_i(\tilde{s}_i) = v_i(1 - \tilde{s}_i)$. Then $\tilde{v}_1$ and $\tilde{v}_2$ are strictly increasing, and $\tilde{u}_1, \tilde{v}_1, \tilde{S}_1, \tilde{u}_2, \tilde{v}_2$, and $\tilde{S}_2$ satisfy all the other assumptions.

2.4 The Repeated Game with Simultaneous Moves

Consider the infinitely repeated game in which the stage game is given by the one-shot game described above. For $i = 1, 2$, let $\delta_i \in (0, 1)$ be player i's discount factor. We restrict ourselves to pure-strategy subgame perfect

[11] It is innocuous because removing 0 and/or 1 from S_i does not affect our analysis.

equilibria in which player i's action in period t, $s_{i,t}$, is a stationary function f_i of player j's action in period $t-1$, $s_{j,t-1}$. We call such strategies *immediately reactive.*

Friedman (1968) called such strategies reaction functions. Immediately reactive strategies are a special case of single-period-recall strategies (Friedman and Samuelson, 1994a) and reactive strategies (Kalai et al., 1988). Single-period-recall strategies depend only on both players' last actions, and reactive strategies depend only on the other player's past actions. We focus on stationary strategies that depend only on the other player's last action. This feature is shared by well-known strategies such as grim trigger and tit-for-tat.

Let F be the set of all functions from $[0,1]$ to $[0,1]$. Let $(i,j) \in Q$. Taking player j's strategy $f_j \in F$ as given, player i faces the following problem:

$$\max_{\{s_{i,t}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \delta_i^{t-1}[u_i(s_{i,t}) + v_i(s_{j,t})] \tag{2.11}$$

$$\text{s.t.} \quad \forall t \in \mathbb{N}, \quad s_{j,t} = f_j(s_{i,t-1}), \tag{2.12}$$

$$\forall t \in \mathbb{N}, \quad s_{i,t} \in [0,1]. \tag{2.13}$$

We say that $f_i \in F$ is a *best response to* f_j if for any $(s_{i,0}, s_{j,0}) \in [0,1]^2$, the above maximization problem has a solution $\{s_{i,t}\}_{t=1}^{\infty}$ such that $s_{i,t} = f_i(s_{j,t-1})$ for all $t \in \mathbb{N}$. We call a strategy profile $(f_1, f_2) \in F^2$ an *immediately reactive equilibrium (IRE)* if f_1 is a best response to f_2 and vice versa. Note that f_1 and f_2 are not required to be continuous or even measurable, but the maximization problem (2.11)–(2.13) is required to be well defined given f_j.[12]

2.5 The Repeated Game with Alternating Moves

Now consider the case of alternating moves. Player 1 updates his action in odd periods, while player 2 updates his action in even periods.[13] Define

$$T_1 = \{1, 3, 5, \cdots\}, \quad T_2 = \{2, 4, 6, \cdots\}. \tag{2.14}$$

[12]Our results are unaffected even if f_1 and f_2 are required to be continuous or upper semi-continuous. The same remark applies to the alternating move game.

[13]In alternating move games, it is often assumed that the players play simultaneously in the initial period and take turns afterwards. Such an assumption does not affect our analysis, which is concerned only with stationary subgame perfect equilibria.

As in the simultaneous move case, we restrict ourselves to subgame perfect equilibria in which each player chooses an immediately reactive strategy, i.e, in each period $t \in T_i$, player i chooses an action $s_{i,t}$ according to a stationary function f_i of player j's last (or equivalently current) action $s_{j,t-1}$.

Let $(i, j) \in Q$. Given player j's strategy $f_j \in F$, player i faces the following problem:

$$\max_{\{s_{i,t}\}_{t=1}^{\infty}} \sum_{t=i}^{\infty} \delta_i^{t-i}[u_i(s_{i,t}) + v_i(s_{j,t})] \tag{2.15}$$

$$\text{s.t.} \quad \forall t \in T_j, \quad s_{j,t} = f_j(s_{i,t-1}),\ s_{i,t} = s_{i,t-1}, \tag{2.16}$$

$$\forall t \in T_i, \quad s_{i,t} \in [0,1],\ s_{j,t} = s_{j,t-1}. \tag{2.17}$$

We say that $f_i \in F$ is a *best response to* f_j if for any $s_{j,i-1} \in [0,1]$,[14] the above maximization problem has a solution $\{s_{i,t}\}_{t=1}^{\infty}$ such that $s_{i,t} = f_i(s_{j,t-1})$ for all $t \in T_i$. We call a strategy profile $(f_1, f_2) \in F^2$ an *immediately reactive equilibrium (IRE)* if f_1 is a best response to f_2 and vice versa. This equilibrium concept is consistent with one definition of Markov perfect equilibrium (Maskin and Tirole, 1988b, Section 2), but distinct from another (Maskin and Tirole, 2001) due to additive separability of payoffs.

2.6 Effective Payoffs

We now introduce a function that plays a central role in our analysis. For $(i, j) \in Q$, define $w_i : [0,1]^2 \to \mathbb{R}_+$ by

$$w_i(s_i, s_j) = u_i(s_i) + \delta_i v_i(s_j). \tag{2.18}$$

We call this function player i's *effective payoff* since in both repeated games, player i in effect seeks to maximize the discounted sum of effective payoffs. Indeed, in both games, player i's discounted sum of payoffs from period 1

[14]Notice that for $i = 1, 2$, the first period in which player i plays is period i.

onward is written as

$$\sum_{t=1}^{\infty} \delta_i^{t-1}[v_i(s_{j,t}) + u_i(s_{i,t})] \tag{2.19}$$

$$= v_i(s_{j,1}) + \sum_{t=1}^{\infty} \delta_i^{t-1}[u_i(s_{i,t}) + \delta_i v_i(s_{j,t+1})] \tag{2.20}$$

$$= v_i(s_{j,1}) + \sum_{t=1}^{\infty} \delta_i^{t-1} w_i(s_{i,t}, s_{j,t+1}). \tag{2.21}$$

In both games, player i has no influence on $s_{j,1}$, so that player i's problem is equivalent to maximizing the discounted sum of effective payoffs.

3 Characterizing Best Responses

Let $(i, j) \in Q$. This section takes player j's strategy $f_j \in F$ as given, and studies player i's best responses. We show first a simple result that characterizes them. The purpose of this section is to reexpress the result in terms of indifference curves associated with effective payoffs so as to obtain a graphical understanding of player i's problem. The following result characterizes player i's best responses in both the simultaneous and alternating move games.

Proposition 3.1. *In both the simultaneous and the alternating move games, $f_i \in F$ is a best response to f_j if and only if*

$$\forall s_j \in [0, 1], \quad f_i(s_j) \in \operatorname*{argmax}_{s_i \in [0,1]} w_i(s_i, f_j(s_i)) \equiv M_i(f_j). \tag{3.1}$$

Proof. Consider the simultaneous move game. From (2.19)–(2.21) and (2.12), player i's discounted sum of payoffs is written as

$$\sum_{t=1}^{\infty} \delta_i^{t-1}[u_i(s_{i,t}) + v_i(s_{j,t})] = v_i(s_{j,1}) + \sum_{t=1}^{\infty} \delta_i^{t-1} w_i(s_{i,t}, f_j(s_{i,t})). \tag{3.2}$$

Thus the maximization problem (2.11)–(2.13) is equivalent to maximizing the right-hand side of (3.2), which is maximized if and only if $s_{i,t} \in M_i(f_j)$ for all $t \in \mathbb{N}$. Therefore, if $f_i \in F$ is a best response, then $f_i(s_{j,0}) \in M_i(f_j)$ for all $s_{j,0} \in [0, 1]$; thus (3.1) holds. Conversely, if $f_i \in F$ satisfies (3.1), then it is a best response since $s_{i,t} = f_i(s_{j,t-1}) \in M_i(f_j)$ for all $t \in \mathbb{N}$.

Now consider the alternating move game. From (2.19)–(2.21), (2.16), and (2.17), player i's discounted sum of payoffs from period i onward is written as

$$\sum_{t=i}^{\infty} \delta_i^{t-i}[u_i(s_{i,t}) + v_i(s_{j,t})] \tag{3.3}$$

$$= v_i(s_{j,i}) + \sum_{t \in T_i} \delta_i^{t-i}(1+\delta_i) w_i(s_{i,t}, s_{j,t+1}) \tag{3.4}$$

$$= v_i(s_{j,i}) + (1+\delta_i) \sum_{t \in T_i} \delta_i^{t-1} w_i(s_{i,t}, f_j(s_{i,t})). \tag{3.5}$$

Thus the maximization problem (2.15)–(2.17) is equivalent to maximizing the right-hand side of (3.5), which is maximized if and only if $s_{i,t} \in M_i(f_j)$ for all $t \in T_i$. Hence the proposition follows as in the simultaneous move case. □

To translate the above result into more usable forms, define

$$R(f_i) = \{f_i(s_j) \,|\, s_j \in [0,1]\}. \tag{3.6}$$

Note that $R(f_i)$ is the range of f_i. The following is a simple restatement of Proposition 3.1.

Corollary 3.1. *$f_i \in F$ is a best response to f_j if and only if $R(f_i) \subset M_i(f_j)$.*

Let us now introduce indifference curves associated with effective payoffs. Those indifference curves are closely connected with both players' best responses, and crucial in understanding IREs and their global dynamics.

Since v_i is strictly increasing by Assumption 2.9, each indifference curve $w_i(s_i, s_j) = \omega$ can be expressed as the graph of a function from s_i to s_j. We denote this function by g_j^ω, i.e.,

$$\omega = w_i(s_i, g_j^\omega(s_i)) = u_i(s_i) + \delta_i v_i(g_j^\omega(s_i)). \tag{3.7}$$

Depending on s_i and ω, however, $g_j^\omega(s_i)$ may or may not be defined. We specify the domain of g_j^ω, denoted $D(g_j^\omega)$, as follows:

$$D(g_j^\omega) = \{s_i \in [0,1] \,|\, \exists s_j \in [0,1], u_i(s_i) + \delta_i v_i(s_j) = \omega\} \tag{3.8}$$

$$= \{s_i \in [0,1] \,|\, \omega - \delta_i v_i(1) \le u_i(s_i) \le \omega - \delta_i v_i(0)\}. \tag{3.9}$$

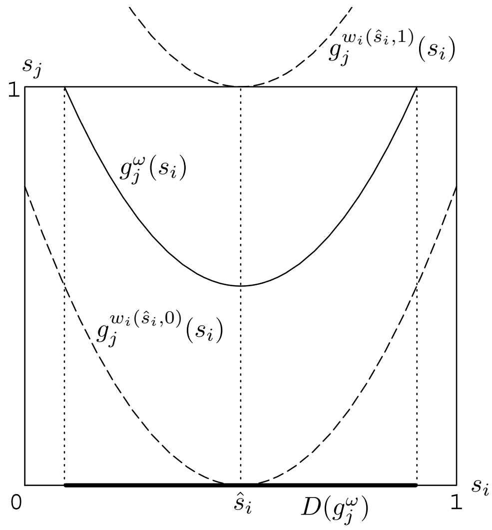


Figure 3: Indifference curve g_j^{ω} and $D(g_j^{\omega})$

See Figure 3. It follows from (3.7) that

$$\forall s_i \in D(g_j^{\omega}), \quad g_j^{\omega}(s_i) = v_i^{-1}\left(\frac{\omega - u_i(s_i)}{\delta_i}\right). \tag{3.10}$$

The following lemma collects useful observations on g_j^{ω}.

Lemma 3.1. *Let* $\Omega = [w_i(\hat{s}_i, 0), w_i(\hat{s}_i, 1)]$. (i) *For* $\omega \in \Omega, g_j^{\omega}(\cdot)$ *is continuous on* $D(g_j^{\omega})$, $\hat{s}_i \in D(g_j^{\omega})$, *and* $D(g_j^{\omega})$ *is a nonempty closed interval.* (ii) *If* $\omega, \omega' \in \Omega$ *with* $\omega < \omega'$, *then* $D(g_j^{\omega}) \subset D(g_j^{\omega'})$ *and*

$$\forall s_i \in D(g_j^{\omega'}), \quad g_j^{\omega}(s_i) < g_j^{\omega'}(s_i). \tag{3.11}$$

(iii) *Let* $\omega \in [w_i(\hat{s}_i, 0), w_i(\hat{s}_i, 1))$. *Then* $D(g_j^{\omega})$ *is a closed interval with nonempty interior. Furthermore,* $g_j^{\omega}(\cdot)$ *is strictly decreasing on* $D(g_j^{\omega}) \cap [0, \hat{s}_i]$ *provided* $\hat{s}_i > 0$, *and strictly increasing on* $D(g_j^{\omega}) \cap [\hat{s}_i, 1]$ *provided* $\hat{s}_i < 1$.

Proof. Let $\omega \in \Omega$. The continuity of g_j^{ω} is obvious. Both inequalities in (3.9) hold with $s_i = \hat{s}_i$ since

$$\omega \leq w_i(\hat{s}_i, 1) = u_i(\hat{s}_i) + \delta_i v_i(1), \tag{3.12}$$
$$\omega \geq w_i(\hat{s}_i, 0) = u_i(\hat{s}_i) + \delta_i v_i(0). \tag{3.13}$$

Hence $\hat{s}_i \in D(g_j^\omega)$. Note from Assumption 2.4 and (3.13) that

$$\forall s_i \in [0,1], \quad u_i(s_i) \le u_i(\hat{s}_i) \le \omega - \delta_i v_i(0). \tag{3.14}$$

Thus by (3.9),

$$D(g_j^\omega) = \{s_i \in [0,1] \,|\, \omega - \delta_i v_i(1) \le u_i(s_i)\}. \tag{3.15}$$

It follows by Assumption 2.4 and (3.12) that $D(g_j^\omega)$ is a nonempty closed interval. We have verified (i).

To see (ii), note that the inequality in (3.11) is immediate from (3.10) for $s_i \in D(g_j^\omega) \cap D(g_j^{\omega'})$. Thus it suffices to show $D(g_j^{\omega'}) \subset D(g_j^\omega)$. Let $s_i \in D(g_j^{\omega'})$. Then the inequality in (3.15) holds with $\omega = \omega'$, so it holds for any $\omega \le \omega'$. It follows that $D(g_j^{\omega'}) \subset D(g_j^\omega)$.

To see (iii), let $\omega \in [w_i(\hat{s}_i, 0), w_i(\hat{s}_i, 1))$. By (i), $D(g_j^\omega)$ is a nonempty closed interval. Since $\omega < w_i(\hat{s}_i, 1) = u_i(\hat{s}_i) + \delta_i v_i(1)$, i.e., $\omega - \delta_i v_i(1) < u_i(\hat{s}_i)$, it follows by (3.15) that $s_i \in D(g_j^\omega)$ for s_i sufficiently close to $\hat{s}_i$. Thus the first conclusion in (i) holds. The second conclusion is immediate from (3.10) and Assumptions 2.9 and 2.4. □

To understand Corollary 3.1 in terms of indifference curves g_j^ω, we define

$$w_i^*(f_j) = \sup_{s_i \in [0,1]} w_i(s_i, f_j(s_i)). \tag{3.16}$$

Since v_i is strictly increasing by Assumption 2.9,

$$w_i(\hat{s}_i, 0) \le w_i^*(f_j) \le w_i(\hat{s}_i, 1). \tag{3.17}$$

By Lemma 3.1(i), a higher indifference curve is associated with a higher effective payoff. Thus by (3.17),

$$g_j^{w_i(\hat{s}_i,0)}(\hat{s}_i) \le g_j^{w_i^*(f_j)}(\hat{s}_i) \le g_j^{w_i(\hat{s}_i,1)}(\hat{s}_i). \tag{3.18}$$

See Figure 3 (with $\omega = w_i^*(f_j)$).

Now consider the maximization problem associated with (3.1) (or (3.16)), which can equivalently be expressed as

$$\max_{s_i, s_j \in [0,1]} w_i(s_i, s_j) \quad \text{s.t. } s_j = f_j(s_i). \tag{3.19}$$

The graph $s_j = f_j(s_i)$ represents the set of feasible pairs (s_i, s_j) for player i, who takes player j's reaction as a constraint. Since the highest feasible indifference curve is given by $s_j = g_j^{w_i^*(f_j)}(s_i)$,

$$\forall s_i \in D(g_j^{w_i^*(f_j)}), \quad f_j(s_i) \leq g_j^{w_i^*(f_j)}(s_i). \tag{3.20}$$

See Figure 4, which shows two ad hoc examples (recall that f_j is arbitrary here). It follows that the solution to (3.19) is to choose any pair (s_i, s_j) satisfying $s_j = f_j(s_i)$ and $s_j = g_j^{w_i^*(f_j)}(s_i)$. More precisely,

$$M_i(f_j) = \{s_i \in D(g_j^{w_i^*(f_j)}) \mid f_j(s_i) = g_j^{w_i^*(f_j)}(s_i)\}. \tag{3.21}$$

See Figure 4 again. By Corollary 3.1 and (3.21),

$$R(f_i) \subset M_i(f_j) \subset D(g_j^{w_i^*(f_j)}), \tag{3.22}$$

whenever f_i is a best response to f_j. We are ready to restate Corollary 3.1 in terms of indifference curves g_j^ω.

Proposition 3.2. *$f_i \in F$ is a best response to f_j if and only if*

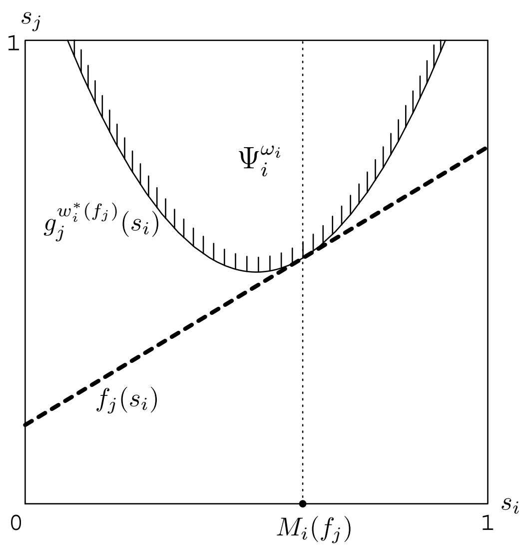
$$\forall s_i \in R(f_i), \quad s_i \in D(g_j^{w_i^*(f_j)}),\ f_j(s_i) = g_j^{w_i^*(f_j)}(s_i). \tag{3.23}$$

Proof. This holds since (3.23) is equivalent to $R(f_i) \subset M_i(f_j)$ by (3.21). □

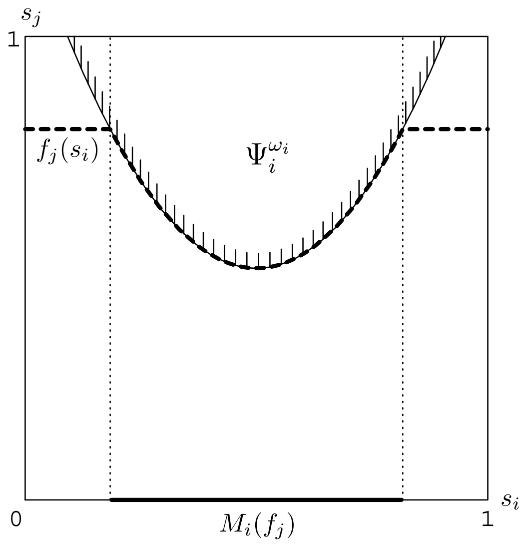
In Figure 4(a), $M_i(f_j)$ is a singleton, so by Corollary 3.1, there is a unique best response, which is the constant function from s_j to s_i corresponding to the dotted vertical line. This function trivially satisfies (3.23). In Figure 4(b), $M_i(f_j)$ is an interval, so all functions from s_j to s_i whose ranges are confined to that interval are best responses. Notice that all those functions satisfy (3.23).

4 Immediate Implications on IREs

The following result is immediate from Corollary 3.1, Proposition 3.2, and the definitions of IRE in Subsections 2.4 and 2.5.



(a)



(b)

Figure 4: $M_i(f_j)$ and $\Psi_i^{\omega_i}$ (defined in (6.1)) with $\omega_i = w_i^*(f_j)$

Theorem 4.1. *In both the simultaneous and the alternating move games, a strategy profile* $(f_1, f_2) \in F^2$ *is an IRE if and only if*

$$\forall (i,j) \in Q, \quad R(f_i) \subset M_i(f_j), \tag{4.1}$$

or, equivalently,

$$\forall (i,j) \in Q, \forall s_i \in R(f_i), \quad s_i \in D(g_j^{w_i^*(f_j)}),\ f_j(s_i) = g_j^{w_i^*(f_j)}(s_i). \tag{4.2}$$

This result implies that the simultaneous and the alternating move games are equivalent as far as IREs are concerned. This would appear in sharp contrast to the anti-folk theorem of Lagunoff and Matsui (1997) for alternating move games of pure coordination. They showed that there is a considerable difference between the simultaneous and the alternating move games in the case of pure coordination. If $u_i(s_i) = v_j(s_i)$ and $v_i(s_j) = u_j(s_j)$ for all $s_i, s_j \in [0,1]$ and $(i,j) \in Q$, then the one-shot game described in Subsection 2.1 becomes a pure coordination game. Theorem 4.1 of course applies to this case (which is consistent with our assumptions), but does not contradict Lagunoff and Matsui's result. This is because their result deals with all subgame perfect equilibria, while Theorem 4.1 deals only with IREs.[15]

For the remainder of the paper, we do not distinguish between the two games except when we explicitly consider dynamics. The differences in dynamics between the two games are discussed in Section 5.

To illustrate Theorem 4.1, let (f_1, f_2) be given by $f_i(s_j) = \hat{s}_i$ for $s_j \in [0,1]$ and $(i,j) \in Q$. See Figure 5. This strategy profile corresponds to the static Nash equilibrium. One can easily see from Figure 5 that f_2 and $g_2^{w_1^*(f_2)}$ coincide on $R(f_1) = \{\hat{s}_1\}$, and f_1 and $g_1^{w_2^*(f_1)}$ coincide on $R(f_2) = \{\hat{s}_2\}$. Thus (f_1, f_2) satisfies (4.2), so it is an IRE by Theorem 4.1.

As another example, let (f_1, f_2) be such that $f_i(s_j) = \overline{s}_i$ if $s_j = \overline{s}_j$, and $f_i(s_j) = \hat{s}_i$ otherwise, where $\overline{s}_1$ and $\overline{s}_2$ are as in Figure 6. In words, each player "cooperates" as long as the other player does so, but reverts to the static Nash equilibrium if the other player deviates in any direction. One can

[15]Theorem 4.1 suggests that the concept of IRE has some resemblance to that of conjectural variation equilibrium. The main difference between the two concepts is that while a conjectural variation equilibrium is a static equilibrium that consists of an equilibrium point and supporting conjectures that are typically required to satisfy certain local properties, an IRE is a fully dynamic equilibrium that consists of two functions that represent the players' actual reactions. See Sabourian (1992) and Tidball et al. (2000) for detailed discussions on the relation between conjectural variation equilibria and repeated games.

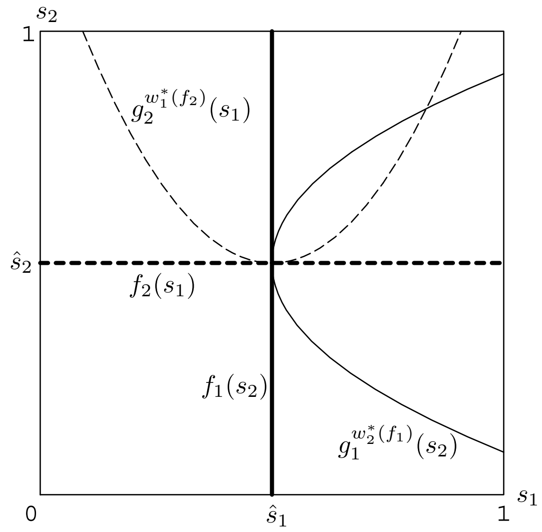


Figure 5: Static Nash equilibrium

see from Figure 6 that f_2 and $g_2^{w_1^*(f_2)}$ coincide on $R(f_1) = \{\hat{s}_1, \overline{s}_1\}$, and f_1 and $g_1^{w_2^*(f_1)}$ coincide on $R(f_2) = \{\hat{s}_2, \overline{s}_2\}$. Thus this strategy profile satisfies (4.2), so it is an IRE.

The main purpose of this paper is to characterize the entire set of IREs, many of which induce more interesting dynamics. To this end we define an *IRE associated with* $(\omega_1, \omega_2) \in \mathbb{R}^2$ as an IRE (f_1, f_2) such that

$$\forall (i,j) \in Q, \quad \omega_i = w_i^*(f_j). \tag{4.3}$$

Notice that any IRE (f_1, f_2) is associated with $(w_1^*(f_2), w_2^*(f_1))$. The following result shows one way to construct an IRE with nontrivial dynamics.

Proposition 4.1. *Let* $\omega_1, \omega_2 \in \mathbb{R}$. *Suppose*

$$\forall (i,j) \in Q, \quad D(g_j^{\omega_i}) = [0,1]. \tag{4.4}$$

Then there exists an IRE associated with (ω_1, ω_2). *In particular,* $(g_1^{\omega_2}, g_2^{\omega_1})$ *is such an IRE.*

Proof. Let $f_j = g_j^{\omega_i}$ for $(i,j) \in Q$. For $(i,j) \in Q$ and $s_i \in [0,1]$, we have $w_i(s_i, f_j(s_i)) = w_i(s_i, g_j^{\omega_i}(s_i)) = \omega_i$, so $\omega_i = w_i^*(f_j)$. To verify (4.2), let $(i,j) \in Q$ and $s_i \in R(f_j)$. Then $s_i \in [0,1] = D(g_j^{w_i^*(f_j)})$ and $f_j(s_i) = g_j^{w_i^*(f_j)}(s_i)$. Thus (4.2) holds, so (f_1, f_2) is an IRE by Theorem 4.1. $\square$

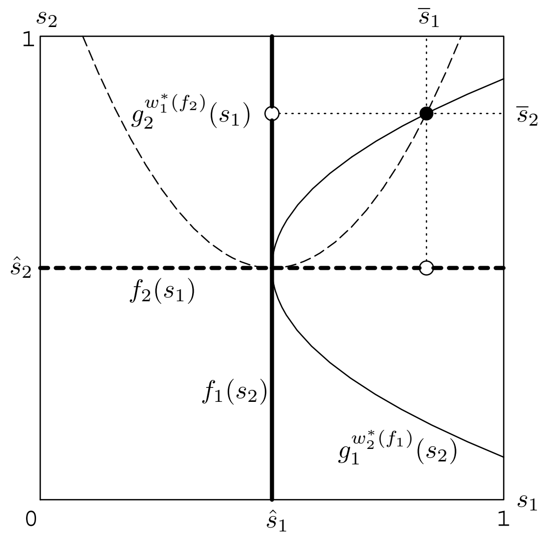


Figure 6: Nash reversion

See Figure 7 for an example of an IRE satisfying (4.4) and (4.3). Since $f_1 = g_1^{w_2^*(f_1)}$ and $f_2 = g_2^{w_1^*(f_2)}$, the example trivially satisfies (4.2).

5 Dynamics

Before we turn to a detailed characterization of IREs, it is useful to have a basic understanding of their dynamics. This section takes an IRE $(f_1, f_2) \in F^2$ as given and studies its dynamic properties.

Consider first the alternating move game. Recall that in each period $t \in \mathbb{N}$, player i with $t \in T_i$ updates his action as a function of player j's last (or current) action. So the "state variable" in each period $t \in T_i$ is player j's last action $s_{j,t-1} \in [0,1]$. Given an initial condition $s_{2,0} \in [0,1]$, the entire path $\{s_{1,t}, s_{2,t}\}_{t=1}^{\infty}$ of the game is uniquely determined by

$$\forall (i,j) \in Q, \forall t \in T_i, \quad s_{i,t+1} = s_{i,t} = f_i(s_{j,t-1}). \tag{5.1}$$

In step-by-step form,

$$s_{1,1} = f_1(s_{2,0}),\ s_{2,2} = f_2(s_{1,1}),\ s_{1,3} = f_1(s_{2,2}),\ \cdots . \tag{5.2}$$

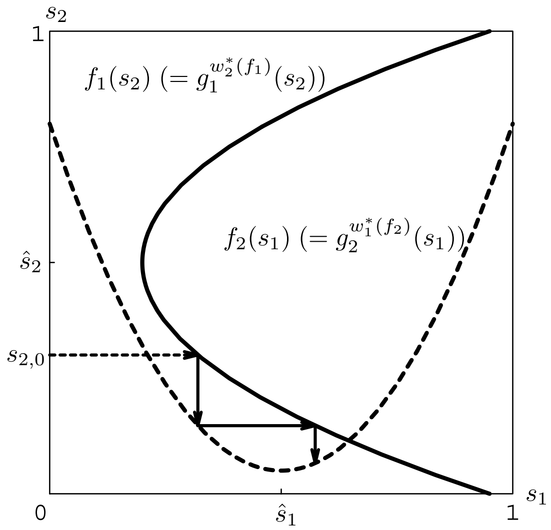


Figure 7: Example of IRE satisfying (4.4) and (4.3).

For the alternating move game, we define an *IRE path associated with* (f_1, f_2) as a sequence $\{s_{1,t}, s_{2,t}\}_{t=0}^{\infty}$ satisfying (5.1). See Figure 7 for an example an IRE path.

Now consider the simultaneous move game. The state variable in each period $t \in \mathbb{N}$ is the pair of both players' last actions $(s_{1,t-1}, s_{2,t-1}) \in [0,1]^2$. Given an initial condition $(s_{1,0}, s_{2,0}) \in [0,1]^2$, the entire path $\{s_{1,t}, s_{2,t}\}_{t=1}^{\infty}$ of the game is uniquely determined by

$$\forall (i,j) \in Q, \forall t \in \mathbb{N}, \quad s_{i,t} = f_i(s_{j,t-1}). \tag{5.3}$$

For the simultaneous move game, we define an *IRE path associated with* (f_1, f_2) as a sequence $\{s_{1,t}, s_{2,t}\}_{t=0}^{\infty}$ satisfying (5.3). Any IRE path can be decoupled into two sequences, one originating from $s_{2,0}$, the other from $s_{1,0}$:

$$s_{1,1} = f_1(s_{2,0}),\ s_{2,2} = f_2(s_{1,1}),\ s_{1,3} = f_1(s_{2,2}),\ \cdots, \tag{5.4}$$

$$s_{2,1} = f_2(s_{1,0}),\ s_{1,2} = f_1(s_{2,1}),\ s_{2,3} = f_2(s_{1,2}),\ \cdots. \tag{5.5}$$

Obviously, given $s_{2,0} \in [0,1]$, the sequences given by (5.2) and (5.4) are identical. The sequence given by (5.5) can be viewed as an IRE path for the alternating move game in which player 2 moves first. Hence an IRE path for the simultaneous move game is equivalent to a pair of IRE paths for the two alternating move games in one of which player 1 moves first and in the other of which player 2 moves first.

The following result is a simple consequence of Theorem 4.1.

Theorem 5.1. *Any IRE path* $\{s_{1,t}, s_{2,t}\}_{t=0}^{\infty}$ *associated with* (f_1, f_2) *for the simultaneous move game satisfies*

$$\forall t \geq 2, \forall (i,j) \in Q, \quad s_{i,t} = g_i^{w_j^*(f_i)}(s_{j,t-1}). \tag{5.6}$$

Furthermore, any IRE path $\{s_{1,t}, s_{2,t}\}_{t=0}^{\infty}$ *associated with* (f_1, f_2) *for the alternating move game satisfies*

$$\forall t \geq 2, \forall (i,j) \in Q, \quad t \in T_i \;\Rightarrow\; s_{i,t} = g_i^{w_j^*(f_i)}(s_{j,t-1}). \tag{5.7}$$

Proof. Consider the simultaneous move game. Let $\{s_{1,t}, s_{2,t}\}_{t=0}^{\infty}$ be an IRE path associated with (f_1, f_2). Let $(i,j) \in Q$ and $t \geq 2$. Then $s_{j,t-1} \in R(f_j)$. Hence $s_{i,t} = g_i^{w_j^*(f_i)}(s_{j,t-1})$ by (5.3) and (4.2). Thus (5.6) follows. The proof for the alternating move game is similar. $\square$

The above result shows that any IRE path is characterized by the corresponding pair of indifference curves $(g_1^{w_2^*(f_1)}, g_2^{w_1^*(f_2)})$ except for the initial period. To better understand this result, consider the alternating move game. The initial period must be excluded in (5.7) because $s_{2,0}$ is an arbitrary initial condition that need not be optimal for player 2 given f_1, i.e., it need not satisfy $s_{1,1} = g_1^{w_2^*(f_1)}(s_{2,0})$. Since all subsequent actions must be individually optimal, they must be on the optimal indifference curves. In Figure 7, any IRE path satisfies the equality in (5.7) for all $t \geq 1$. In Figure 5, by contrast, an IRE path (not shown in the figure) violates the equality for $t = 1$ unless $s_{2,0} = \hat{s}_2$, but trivially satisfies it for $t \geq 2$.

Theorem 5.1 also shows that in both cases the dynamics of an IRE associated with $(\omega_1, \omega_2) \in \mathbb{R}^2$ are essentially characterized by the same dynamical system:

$$\forall t \in T_1, \quad s_{1,t+2} = g_1^{\omega_2}(g_2^{\omega_1}(s_{1,t})). \tag{5.8}$$

To be precise, the simultaneous move game has another equation, $s_{2,t+2} = g_2^{\omega_1}(g_1^{\omega_2}(s_{2,t}))$ for $t \in T_2$, but this system is equivalent to (5.8) in terms of dynamics. Hence one can obtain conditions for dynamic properties such as monotonicity and chaos by applying numerous results available on one-dimensional dynamical systems (e.g., Devaney, 1989).[16]

[16]See Rand (1978) for an early example of complex dynamics in an "adaptive" dynamic

6 Characterizing IREs

Theorem 5.1 shows that the dynamics of an IRE are characterized by the associated pair of indifference curves. The remaining question then is what pairs of indifference curves are supported in IREs. To answer this question, we need additional notation. For $(i,j) \in Q$ and $\omega_i, \omega_j \in \mathbb{R}$, define

$$\Psi_i^{\omega_i} = \{(s_i, s_j) \in [0,1]^2 \,|\, w_i(s_i, s_j) \geq \omega_i\} \tag{6.1}$$
$$= \{(s_i, s_j) \in [0,1]^2 \,|\, s_i \in D(g_j^{\omega_i}), s_j \geq g_j^{\omega_i}(s_i)\}. \tag{6.2}$$

The set $\Psi_i^{\omega_i}$ is the collection of all pairs (s_i, s_j) with player i's effective payoff at least as large as ω_i. In the (s_i, s_j) space, it is the area on or above the graph $s_j = g_j^{\omega_i}(s_i)$; see Figure 4 in Section 3.

Provided $\Psi_i^{\omega_i} \cap \Psi_j^{\omega_j} \neq \emptyset$,[17] define

$$\overline{s}_i^{(\omega_i,\omega_j)} = \max\{s_i \in [0,1] \,|\, \exists s_j \in [0,1], (s_i, s_j) \in \Psi_i^{\omega_i} \cap \Psi_j^{\omega_j}\}, \tag{6.3}$$
$$\underline{s}_i^{(\omega_i,\omega_j)} = \min\{s_i \in D(g_j^{\omega_i}) \,|\, g_j^{\omega_i}(s_i) \leq \overline{s}_j^{(\omega_j,\omega_i)}\}. \tag{6.4}$$

By (6.2) and continuity, $g_j^{\omega_i}(\overline{s}_i^{(\omega_i,\omega_j)}) \leq \overline{s}_j^{(\omega_j,\omega_i)}$. Hence $\underline{s}_i^{(\omega_i,\omega_j)}$ exists as long as $\Psi_i^{\omega_i} \cap \Psi_j^{\omega_j} \neq \emptyset$. See Figure 8. In the case of Figure 7, $\overline{s}_1^{(\omega_1,\omega_2)} = \overline{s}_2^{(\omega_2,\omega_1)} = 1$ and $\underline{s}_1^{(\omega_1,\omega_2)} = \underline{s}_2^{(\omega_2,\omega_1)} = 0$. It follows from Lemma 3.1(iii) that

$$\forall (i,j) \in Q, \quad \underline{s}_i^{(\omega_i,\omega_j)} \leq \hat{s}_i \leq \overline{s}_i^{(\omega_i,\omega_j)}. \tag{6.5}$$

See Figure 8 again.[18] The following result characterizes all IREs in terms of effective payoffs.

model that has a structure similar to Figure 7. See Rosser (2002) for a recent survey of adaptive duopoly/oligopoly models that generate complex dynamics. This paper does not consider complex dynamics, which should be left to more specialized studies.

[17] Here it is understood that the coordinates of $\Psi_j^{\omega_j}$ (or $\Psi_i^{\omega_i}$) are interchanged so that $\Psi_i^{\omega_i}$ and $\Psi_j^{\omega_j}$ have the same order of the coordinates. Similar comments apply to similar expressions below.

[18] The first inequality in (6.5) is immediate from (6.4) and Lemma 3.1(iii). To formally verify the second inequality, let $(i,j) \in Q$ and suppose $\overline{s}_i < \hat{s}_i$. (We omit superscripts here.) By (6.2) and (6.3), $\overline{s}_j \geq g_j(\overline{s}_i)$ and $\overline{s}_i \geq g_i(\overline{s}_j)$. Since g_j is strictly decreasing at $\overline{s}_i$ by Lemma 3.1(iii), both inequalities continue to hold even if $\overline{s}_i$ is slightly increased, contradicting (6.3).

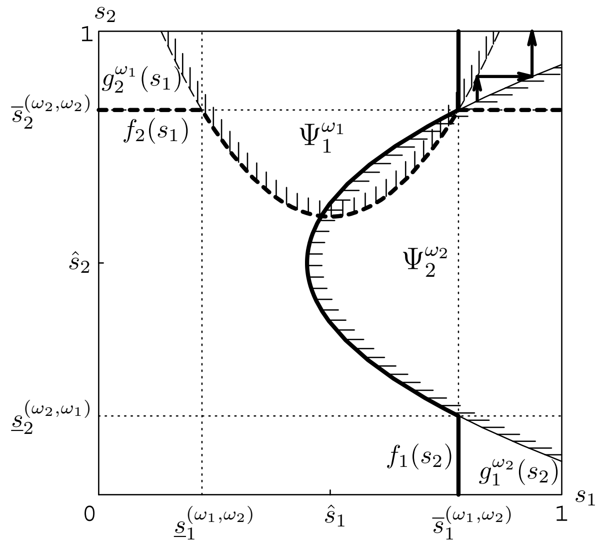


Figure 8: $\underline{s}_i$, $\overline{s}_i$, and IRE in regular form

Theorem 6.1. *There exists an IRE associated with* $(\omega_1, \omega_2) \in \mathbb{R}^2$ *if and only if*

$$\Psi_1^{\omega_1} \cap \Psi_2^{\omega_2} \neq \emptyset, \tag{6.6}$$

$$\forall (i,j) \in Q, \quad \hat{s}_j \in D(g_i^{\omega_j}),\ \underline{s}_i^{(\omega_i,\omega_j)} \leq g_i^{\omega_j}(\hat{s}_j). \tag{6.7}$$

In particular, under (6.6) and (6.7), $(f_1, f_2) \in F^2$ *is an IRE associated with* (ω_1, ω_2) *if for* $(i,j) \in Q$,

$$f_i(s_j) = \begin{cases} \min\{g_i^{\omega_j}(s_j), \overline{s}_i^{(\omega_i,\omega_j)}\} & \text{if } s_j \in D(g_i^{\omega_j}), \\ \overline{s}_i^{(\omega_i,\omega_j)} & \text{otherwise.} \end{cases} \tag{6.8}$$

Proof. See Appendix A. □

We say that an IRE satisfying (6.8) is *in regular form.* See Figure 8 for an example of an IRE in regular form. One can easily see that the example satisfies (4.2) and thus is an IRE. We call (6.6) the *nonemptiness condition*, and (6.7) the *no-sticking-out condition.* The nonemptiness condition says that the intersection of the two sets $\Psi_1^{\omega_1}$ and $\Psi_2^{\omega_2}$ must be nonempty. The no-sticking-out condition says that the graph of $g_i^{\omega_j}$ must not "stick out" of the straight line $s_i = \underline{s}_i$.

These conditions can be better understood by considering examples in which they are violated. In Figure 9(a), the nonemptiness condition (6.6) is violated. In this case, if an IRE exists, any IRE path for the alternating move game must behave like the path depicted in the figure (except for the initial period) by Theorem 5.1. But since such a path cannot stay on the indifference curves forever, it cannot be an IRE path, a contradiction. In Figure 9(b), the no-sticking-out condition (6.7) is violated for $(i, j) = (1, 2)$. In this case, if an IRE exists, there is $s_{2,0}$ such that $f_1(s_{2,0}) \leq g_1^{\omega_2}(s_{2,0}) < \underline{s}_1^{(\omega_1,\omega_2)}$.[19] As shown in the figure, the IRE path from such $s_{2,0}$ cannot stay on the indifference curves forever, contradicting Theorem 5.1.

We should mention that the IRE in regular form associated with $(\omega_1, \omega_2) \in \mathbb{R}^2$ is not the only IRE associated with (ω_1, ω_2). In fact, $f_i(s_j)$ is arbitrary for $s_j \notin [\underline{s}_j^{(\omega_j,\omega_i)}, \overline{s}_j^{(\omega_j,\omega_i)}]$ as long as it does not affect $R(f_i)$. However, any IRE satisfies one restriction:

Proposition 6.1. *Let (f_1, f_2) be an IRE associated with $(\omega_1, \omega_2) \in \mathbb{R}^2$. Then*

$$\forall (i, j) \in Q, \forall s_j \in [0, 1], \quad f_i(s_j) \leq \overline{s}_i^{(\omega_i,\omega_j)}. \tag{6.9}$$

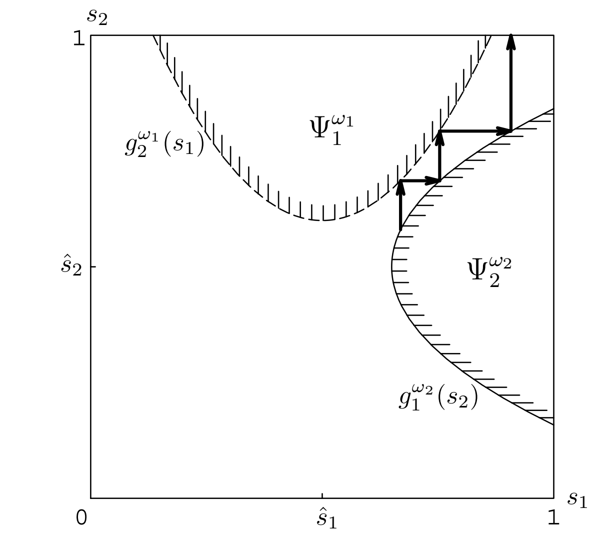
Proof. Immediate from (A.2), (A.12), and (6.3). □

To see the idea of this result, suppose the inequality in (6.9) is violated for $(i, j) = (1, 2)$. Consider the alternating move game. Then for some $s_{2,0}$, $s_{1,1} = f_1(s_{2,0}) > \overline{s}_1^{(\omega_1,\omega_2)}$. If this path is continued, it behaves like the one depicted in Figure 8 by Theorem 5.1. But such a path cannot be an IRE path since it cannot stay on the indifference curves forever.

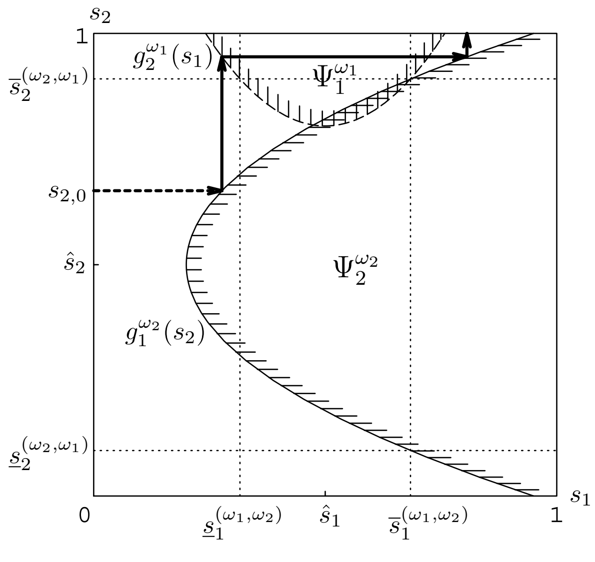
Proposition 6.1 along with (6.8) implies that if (f_1, f_2) is an IRE associated with $(\omega_1, \omega_2) \in \mathbb{R}^2$, and if $(\overline{f}_1, \overline{f}_2)$ is the IRE in regular form associated with (ω_1, ω_2), then $f_i \leq \overline{f}_i$ for $i = 1, 2$. In other words, in the IRE $(\overline{f}_1, \overline{f}_2)$, each player gives the other player the highest possible effective payoff consistent with an IRE associated with (ω_1, ω_2) in response to any action by the other player. This becomes important when we discuss our folk-type theorem in the next section.

In what follows, we say that an IRE (f_1, f_2) is *effectively efficient* if there is no IRE $(\tilde{f}_1, \tilde{f}_2)$ such that $w_1^*(f_2) \leq w_1^*(\tilde{f}_2)$ and $w_2^*(f_1) \leq w_2^*(\tilde{f}_1)$ with at least one of them holding strictly. That is, (f_1, f_2) is effectively efficient if it

[19] The first inequality holds by (3.20). In Figure 9(b), $f_1(s_{2,0}) = g_1^{\omega_2}(s_{2,0})$.



(a)



(b)

Figure 9: Examples with no IRE

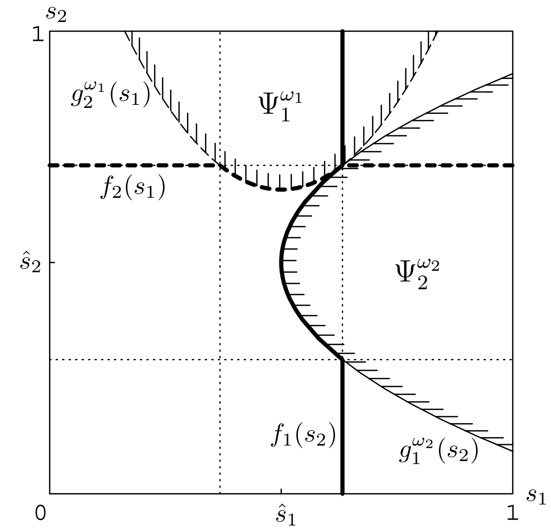


Figure 10: Effectively efficient IRE

is not Pareto dominated by any other IRE in terms of effective payoffs. As shown in Section 8, effective efficiency has important dynamic implications.

For $(i, j) \in Q$ and $\omega_i, \omega_j \in \mathbb{R}$, define

$$\tilde{\Psi}_i^{\omega_i} = \{(s_i, s_j) \in [0,1]^2 \mid w_i(s_i, s_j) > \omega_i\}. \tag{6.10}$$

It is clear from Theorem 6.1 and Lemma 3.1(ii) that an IRE associated with $(\omega_1, \omega_2) \in \mathbb{R}^2$ is effectively efficient if

$$\tilde{\Psi}_1^{\omega_1} \cap \tilde{\Psi}_2^{\omega_2} = \emptyset. \tag{6.11}$$

See Figures 10 and 9(a).

One might conjecture that (6.11) is also necessary for effective efficiency. Unfortunately it is not the case. This is because the no-sticking-out condition (6.7), a necessary condition for an IRE, is not stable under small perturbations to (ω_i, ω_j). In other words, even when (6.11) does not hold, (6.7) can be violated if either ω_i or ω_j is increased. For example, when (6.7) holds with equality for $(i, j) = (1, 2)$, it can be violated after ω_2 is slightly increased, depending on how fast the two sides of the inequality in (6.7) vary with ω_2.

Even if (6.7) holds with strict inequality, (6.7) can be violated after small perturbations to (ω_i, ω_j), since $\underline{s}_i^{(\omega_i, \omega_j)}$ need not be continuous in (ω_i, ω_j).

Figure 11 illustrates this point. There is an IRE in Figure 11(a), but there is no IRE in Figure 11(b) due to violation of (6.7). Note that both $\underline{s}_1^{(\omega_1,\omega_2)}$ and $\underline{s}_2^{(\omega_2,\omega_1)}$ are discontinuous in this example.[20]

7 Effective Efficiency and a Folk-Type Theorem: A Special Case

The anomaly in Figure 11 is largely due to the fact that the indifferent curves are unimodal there. The purpose of this section to characterize effective efficiency and to show a folk-type theorem under the assumption that both indifference curves are "upward sloping." More precisely, we focus on the case in which the following assumption holds.

Assumption 7.1. For $i = 1, 2$, $\hat{s}_i = 0$ or, equivalently, u_i is strictly decreasing.[21]

This assumption holds, for example, in the prisonner's dilemma game in Subsection 2.2.4. More generally, it holds whenever an increase in s_i is costly to player i but beneficial to player j. Assumption 7.1 is maintained throughout this section. The following result simplifies Theorem 6.1 and facilitates subsequent analysis.

Lemma 7.1. *There exists an IRE (in regular form)*[22] *associated with* $(\omega_1, \omega_2) \in \mathbb{R}^2$ *if and only if the nonemptiness condition (6.6) holds and*

$$\forall (i, j) \in Q, \quad 0 \in D(g_i^{\omega_j}). \tag{7.1}$$

Proof. By Theorem 6.1, it suffices to show that (7.1) is equivalent to the no-sticking-out condition (6.7) under (6.6). By Assumption 7.1, (6.7) implies (7.1). Conversely, assume (6.6) and (7.1). Let $(i, j) \in Q$. By Assumption

[20]Though in fact Figure 11 shows that the IRE in (a) is only "locally" effectively efficient, it should be clear that one can easily construct a fully specified example of an effectively efficient IRE that violates (6.11).

[21]If one chooses to normalize v_i in the opposite direction, i.e., if one chooses to assume that v_i is strictly decreasing for $i = 1, 2$ in Assumption 2.9, then the case considered here corresponds to the case $\hat{s}_i = 1$ for $i = 1, 2$.

[22]All the results stated in this section hold true with or without "in regular form." We include "(in regular form)" if the version with this qualification is not an immediate consequence of the version without it.

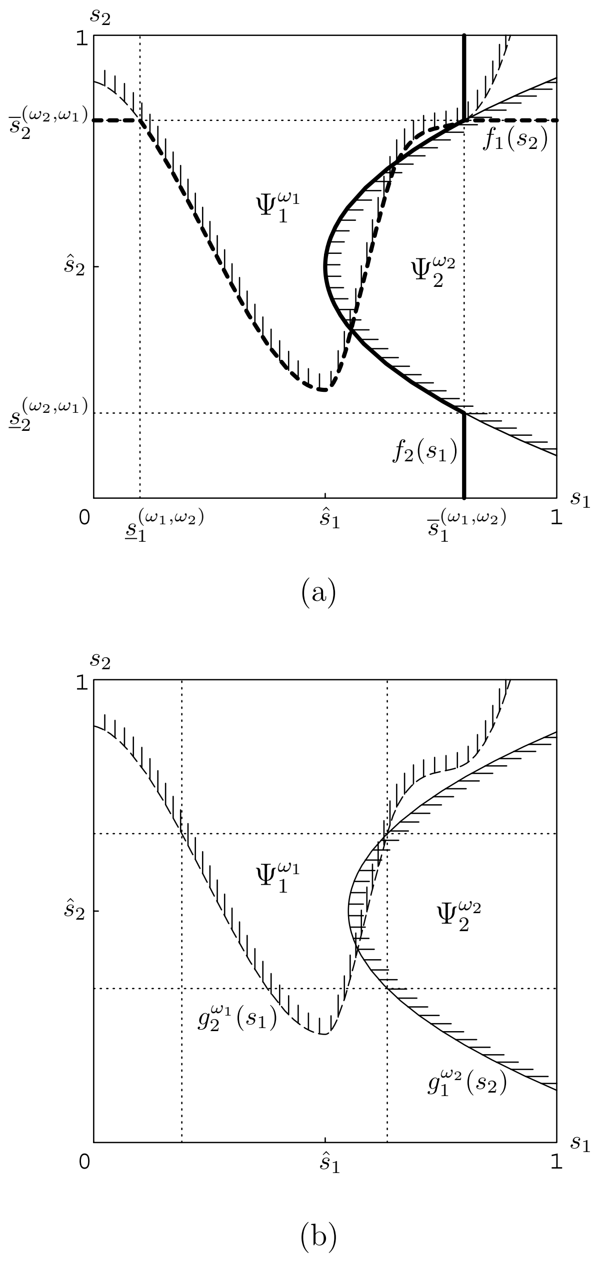


Figure 11: Effectively efficient IRE violating (6.11)

7.1 and (6.5), $\underline{s}_i^{(\omega_i,\omega_j)} = 0$. Since $0 \in D(g_i^{\omega_j})$, we have $\underline{s}_i^{(\omega_i,\omega_j)} = 0 \le g_i^{\omega_j}(0) = g_i^{\omega_j}(\hat{s}_j)$ by Assumption 7.1. Now (6.7) follows. $\square$

The above proof shows that under (6.6) and (7.1), the inequality in the no-sticking-out condition (6.7) automatically holds. This implies that if an IRE exists such that $\tilde{\Psi}_1^{\omega_1} \cap \tilde{\Psi}_2^{\omega_2} \neq \emptyset$, then an IRE continues to exist when both indifference curves are slightly shifted upward. Therefore an IRE cannot be effectively efficient if $\tilde{\Psi}_1^{\omega_1} \cap \tilde{\Psi}_2^{\omega_2} \neq \emptyset$. This is the idea of the following result.

Theorem 7.1. *Suppose an IRE associated with $(\omega_1, \omega_2) \in \mathbb{R}^2$ exists. Then it is effectively efficient if and only if (6.11) holds, i.e., $\tilde{\Psi}_1^{\omega_1} \cap \tilde{\Psi}_2^{\omega_2} = \emptyset$.*

Proof. The "if" part is obvious, as mentioned earlier. To see the "only if" part, suppose there is an IRE associated with $(\omega_1, \omega_2) \in \mathbb{R}^2$ that is effectively efficient. Suppose (6.11) does not hold, i.e.,

$$\tilde{\Psi}_1^{\omega_1} \cap \tilde{\Psi}_2^{\omega_2} \neq \emptyset. \tag{7.2}$$

Since for $(i, j) \in Q$, $g_j^{\omega_i}(\cdot)$ is strictly increasing by (3.10) and Assumption 7.1, (7.1) and (7.2) imply $0 \le g_j^{\omega_i}(0) < 1$ for $(i, j) \in Q$. Since $g_j^{\omega_i}$ is continuous and strictly increasing in ω_i by (3.10), it follows that there is $(\tilde{\omega}_1, \tilde{\omega}_2) \gg (\omega_1, \omega_2)$ such that $0 < g_j^{\tilde{\omega}_i}(0) < 1$ and $\tilde{\Psi}_1^{\tilde{\omega}_1} \cap \tilde{\Psi}_2^{\tilde{\omega}_2} \neq \emptyset$ for $(i, j) \in Q$. Hence (6.6) and (7.1) hold with $\tilde{\omega}_1$ and $\tilde{\omega}_2$ replacing ω_1 and ω_2. But this implies that the given IRE cannot be effectively efficient, a contradiction. $\square$

Let us now turn to the development of our folk-type theorem. The following result gives an alternative characterization of IREs that proves useful.

Lemma 7.2. *There exists an IRE (in regular form) associated with $(\omega_1, \omega_2) \in \mathbb{R}^2$ if and only if there exists $(s_1, s_2) \in [0, 1]^2$ such that*

$$\forall (i, j) \in Q, \quad \omega_i = w_i(s_i, s_j), \tag{7.3}$$

$$(s_1, s_2) \in \Psi_1^{w_1(0,0)} \cap \Psi_2^{w_2(0,0)}. \tag{7.4}$$

Proof. First we observe from (3.7), (6.1), and Lemma 3.1(ii) that

$$(7.3) \quad \Leftrightarrow \quad \forall (i, j) \in Q, \;\; s_i \in D(g_j^{\omega_i}), s_j = g_j^{\omega_i}(s_i), \tag{7.5}$$

$$(7.3) \quad \Rightarrow \quad (s_1, s_2) \in \Psi_1^{\omega_1} \cap \Psi_2^{\omega_2}. \tag{7.6}$$

If: Let $(\omega_1, \omega_2) \in \mathbb{R}^2$. Suppose there exists $(s_1, s_2) \in [0, 1]^2$ satisfying (7.3) and (7.4). Then by (7.6), the nonemptiness condition (6.6) holds. By

Lemma 7.1, it suffices to verify (7.1). Let $(i, j) \in Q$. By (7.4) and (6.1), $w_i(s_i, s_j) \geq w_i(0, 0)$. Since u_i is strictly decreasing by Assumption 7.1, $\omega_i = w_i(s_i, s_j) \leq w_i(0, 1)$. It follows that

$$u_i(0) + \delta_i v_i(0) \leq \omega_i \leq u_i(0) + \delta_i v_i(1), \tag{7.7}$$

which can be written as

$$\omega_i - \delta_i v_i(1) \leq u_i(0) \leq \omega_i - \delta_i v_i(0). \tag{7.8}$$

Hence (7.1) holds by (3.9).

Only if: Let there be an IRE associated with $(\omega_1, \omega_2) \in \mathbb{R}^2$. By (7.1), $g_j^{\omega_i}(0) \geq 0$ for $(i, j) \in Q$. Thus if the graphs of $g_2^{\omega_1}$ and $g_1^{\omega_2}$ have no intersection, then the nonemptiness condition (6.6) does not hold, a contradiction. Hence the graphs of $g_2^{\omega_1}$ and $g_1^{\omega_2}$ have an intersection $(s_1, s_2) \in [0, 1]^2$, i.e., $s_j = g_j^{\omega_i}(s_i)$ for $(i, j) \in Q$. Thus (7.3) holds by (7.5). We have (7.4) by (7.3), (7.6), (4.3), (3.17), Assumption 7.1, and (6.1). □

See Figure 12 for an illustration of $\Psi_1^{w_1(0,0)} \cap \Psi_2^{w_2(0,0)}$. Note that both indifference curves $g_1^{w_2(0,0)}$ and $g_2^{w_1(0,0)}$ emanate from the origin because they correspond to the effective payoffs associated with the action profile $(0, 0)$.

Given an IRE (f_1, f_2), we say that $(s_1, s_2) \in [0, 1]^2$ is a *steady state* if $s_1 = f_1(s_2)$ and $s_2 = f_2(s_1)$. In other words, any intersection of f_1 and f_2 is a steady state. Needless to say, the IRE path starting from a steady state remains there forever. Thus by Theorem 5.1, any steady state is an intersection of the associated indifference curves.

Proposition 7.1. *There exists an IRE (in regular form) such that* $(s_1, s_2) \in [0, 1]^2$ *is a steady state if and only if (7.4) holds.*

Proof. Let there be an IRE such that $(s_1, s_2) \in [0, 1]^2$ is a steady state. Define ω_1 and ω_2 by (7.3). Then this IRE is associated with (ω_1, ω_2). Thus (7.4) holds by Lemma 7.2. Conversely, let $(s_1, s_2) \in \mathbb{R}^2$ satisfy (7.4). Define ω_1 and ω_2 by (7.3). By Lemma 7.2, an IRE (f_1, f_2) in regular form associated with (ω_1, ω_2) exists. Thus (6.6) and (6.7) hold by Theorem 6.1. By (7.5), (a) $s_j = g_j^{\omega_i}(s_i)$ for $(i, j) \in Q$. Since $(s_1, s_2) \in \Psi_1^{\omega_1} \cap \Psi_2^{\omega_2}$ by (7.6), (b) $s_i \leq \overline{s}_i^{(\omega_i, \omega_j)}$ for $(i, j) \in Q$. Hence by (a) and (b), for $(i, j) \in Q$, $g_j^{\omega_i}(s_i) = s_j \leq \overline{s}_j^{(\omega_j, \omega_i)}$, so $f_j(s_i) = g_j^{\omega_i}(s_i)$ by (6.8). This together with (a) shows that (s_1, s_2) is a steady state. □

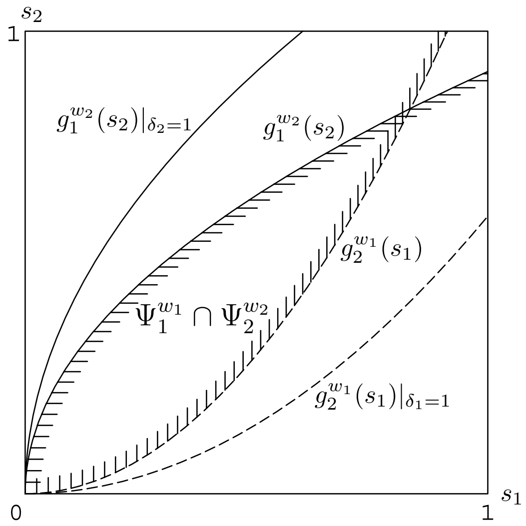


Figure 12: $\Psi_1^{w_1(0,0)} \cap \Psi_2^{w_2(0,0)}$ and lower bounds of $\tilde{\Psi}_i^*$
($w_i = w_i(0,0)$ in this figure)

By this result, the set $\Psi_1^{w_1(0,0)} \cap \Psi_2^{w_2(0,0)}$ can be viewed as the collection of all steady states supported by IREs. For $(i,j) \in Q$, define

$$\tilde{\Psi}_i^* = \{(s_i, s_j) \in [0,1]^2 \mid u_i(s_i) + v_i(s_j) > u_i(0) + v_i(0)\}. \tag{7.9}$$

Since $u_i(0) + v_i(0)$ is player i's minimax payoff in the one-shot game, $\tilde{\Psi}_1^* \cap \tilde{\Psi}_2^*$ may be called the set of "strictly individually rational" action profiles without randomization.

Note that the definition of $\tilde{\Psi}_i^{\omega_i}$ in (6.10) becomes identical to (7.9) if $\delta_i = 1$ and $\omega_i = u_i(0) + v_i(0)$. It follows from (3.10) that for $(i,j) \in Q$,

$$\forall s_i \in D(g_j^{w_i(0,0)}), \quad g_j^{w_i(0,0)}(s_i) = v_i^{-1}\left(v_i(0) + \frac{u_i(0) - u_i(s_i)}{\delta_i}\right). \tag{7.10}$$

Thus the graph of $g_j^{w_i(0,0)}$ shifts downward as δ_i increases; see Figure 12. The idea of our folk-type theorem is that as $\delta_i \uparrow 1$, $\tilde{\Psi}_1^{w_1(0,0)} \cap \tilde{\Psi}_2^{w_2(0,0)}$ "converges" to $\tilde{\Psi}_1^* \cap \tilde{\Psi}_2^*$, so by Proposition 7.1, any point in $\tilde{\Psi}_1^* \cap \tilde{\Psi}_2^*$ can be supported as a steady state of an IRE for (δ_1, δ_2) sufficiently close to $(1,1)$.

Theorem 7.2. *Let*

$$(s_1, s_2) \in \tilde{\Psi}_1^* \cap \tilde{\Psi}_2^*. \tag{7.11}$$

Then for $(\delta_1, \delta_2) \ll (1,1)$ sufficiently close to $(1,1)$, there exists an IRE (in regular form) such that (s_1, s_2) is a steady state.

Proof. Assume (7.11). By (7.9), for (δ_1, δ_2) close enough to $(1,1)$,

$$\forall (i,j) \in Q, \quad u_i(s_i) + \delta_i v_i(s_j) > u_i(0) + \delta_i v_i(0), \tag{7.12}$$

so (7.4) holds by (6.1). Now the theorem follows by Proposition 7.1. $\square$

Theorem 7.2 of course holds true without "in regular form," but in that case, it can easily be shown by using an IRE based on Nash reversion;[23] recall Figure 6. Part of the significance of Theorem 7.2 lies in the fact that IREs in regular form are continuous. In this regard the result is similar in spirit to the folk-type theorems by Friedman and Samuelson (1994a, 1994b), which show that the main idea of the standard folk theorem (Fudenberg and Maskin, 1986) is valid even if one confines oneself to continuous equilibria. Our result shows that any $(s_1, s_2) \in \tilde{\Psi}_1^* \cap \tilde{\Psi}_2^*$ is supported as a steady state of a continuous IRE for (δ_1, δ_2) sufficiently close to $(1,1)$.

In addition, it should be noted that a deviation is punished in a minimal way in IREs in regular form. Indeed, after the initial period, player i is indifferent between conforming to the current IRE path and choosing any $s_i \in [0, \overline{s}_i^{(\omega_i,\omega_j)}]$ (recall that $\underline{s}_i^{(\omega_i,\omega_j)} = 0$ here by the proof of Lemma 7.1). This suggests that sever punishment may not be necessary for maintaining a subgame perfect equilibrium or even for establishing a folk-type theorem.

8 Applications

8.1 Prisoner's Dilemma

Consider the alternating move game associated with the prisoner's dilemma game in Subsection 2.2.4.[24] For simplicity, we assume directly that the one-shot payoff of player i is given by (2.5),[25] and that both players have the

[23] We owe this observation to an anonymous referee.

[24] The simultaneous move game can be analyzed similarly; recall Proposition 6.1 and Section 5.

[25] Alternatively one may assume that player i's mixed action in period $t \in T_i$ is observable to player j at the beginning of period $t+1$. In this case, the expected one-shot payoff of player i in period t is $-as_{i,t} + (c+a)r_{j,t-1}$, where $s_{i,t}$ is player i's probability of choosing C, and $r_{j,t-1}$ is player j's realized action in period $t-1$. Since $r_{j,t-1}$ does not affect player

same discount factor: $\delta_1 = \delta_2 = \delta \in (0, 1)$. The effective payoff of player i is given by

$$w_i(s_i, s_j) = -as_i + \delta e s_j, \tag{8.1}$$

where $e = c + a$. Replacing $w_i(s_i, s_j)$ with ω_i and solving for s_j, we see that the indifference curve associated with $\omega^i \in \mathbb{R}$, or $g_j^{\omega_i}$, is linear:

$$g_j^{\omega_i}(s_i) = \frac{\omega_i}{\delta e} + \frac{a}{\delta e} s_i. \tag{8.2}$$

Since Assumption 7.1 holds here, all the results in Section 7 apply.

We consider three cases separately. First suppose $\delta < a/e$, i.e., the slope of $g_j^{\omega_i}$ is strictly greater than one. By (7.1), $g_i^{\omega_j}(0) \geq 0$ for $(i, j) \in Q$ in any IRE. Thus if $g_i^{\omega_j}(0) > 0$ for either $(i, j) \in Q$, the nonempiness condition (6.6) will be violated; see Figure 13. Hence in any IRE, $g_i^{\omega_j}(0) = 0$ for $(i, j) \in Q$. It follows that there is a unique IRE, which corresponds to the static Nash equilibrium, i.e., $f_i(s_j) = 0$ for all $s_j \in [0, 1]$ and $(i, j) \in Q$. This is because by Proposition 6.1, $f_i(s_j) \leq \overline{s}_i^{(0,0)} = 0$ for all $s_j \in [0, 1]$ and $(i, j) \in Q$. See Figure 13 again. This IRE is effectively efficient by Theorem 7.1 (or simply by uniqueness).

Now suppose $\delta = a/e$, i.e., the slope of $g_j^{\omega_i}$ is equal to one. In this knife edge case, the two indifference curves emanating from the origin coincide. The above argument still shows $g_i^{\omega_j}(0) = 0$ for $(i, j) \in Q$. Though, as in the previous case, there is an IRE corresponding to the static Nash equilibrium, there are many other IREs here. Figure 14 shows one example.

Finally, suppose $\delta > a/e$, i.e., the slope of $g_j^{\omega_i}$ is strictly less than one. In this case, there are many pairs of effective payoffs supported by IREs. A "typical" case is depicted in Figure 15, where there is a unique and globally stable steady state. The existence of a unique and globally stable steady state is a general property of this case by (5.8) and (8.2).

Figure 16 shows a symmetric IRE that is effectively efficient. In this case, gradual cooperation occurs, and full cooperation is achieved in the long run.[26] Figure 17 shows an effectively efficient IRE in which uneven gradual cooperation occurs: only player 2 fully cooperates in the long run, and player 1 enjoys the highest possible effective payoff.

i's preferences over his actions from period t onward, all our results hold in this case. This argument is unnecessary for the simultaneous move game, where $r_{j,t-1}$ must be replaced by $s_{j,t}$.

[26] Gradual cooperation is known to arise in certain partnership games; see Furusawa and Kawakami (2006) and the references therein.

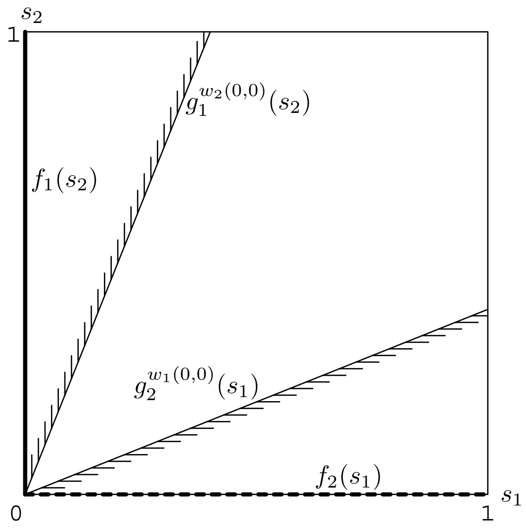


Figure 13: Prisoner's dilemma with $\delta < a/e$

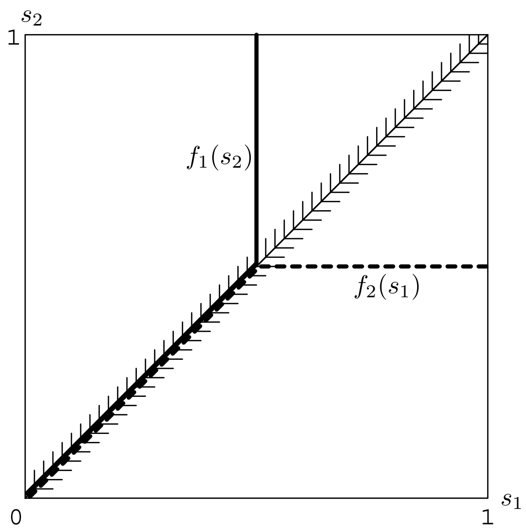


Figure 14: Prisoner's dilemma with $\delta = a/e$

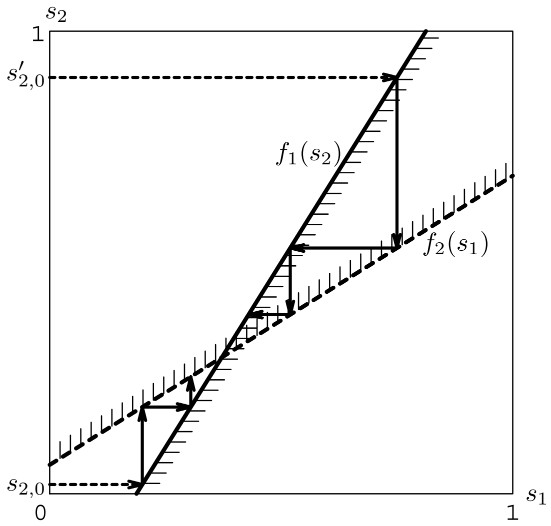


Figure 15: Prisoner's dilemma with $\delta > a/e$: "typical" case

By Theorem 7.2, any strictly individually rational action profile (s_1, s_2), which by definition satisfies $-as_i + es_j > 0$ for $(i, j) \in Q$, is supported as a steady state of an IRE for δ sufficiently close to one. Notice that the set of payoff profiles supported by strictly individually rational action profiles is convex here, so that this set coincides with the set of "strictly individually rational" payoff profiles (Fudenberge and Maskin, 1986). Thus our folk-type theorem coincides with the standard folk theorem in this example.[27]

8.2 Duopoly

Consider the alternating move game associated with the duopoly game of Subsection 2.2.3.[28] For simplicity we assume that the firms are symmetric. Let c and δ denote their common marginal cost and discount factor. Recall that firm i's one-shot profit is given by $D_i(p_i, p_j)(p_i - c)$. We parametrize D_i as follows:

$$\forall (i, j) \in Q, \quad D_i(p_i, p_j) = (\overline{p} - p_i)p_j, \tag{8.3}$$

[27]See Stahl (1991) for a characterization of subgame perfect correlated equilibria of a more general repeated prisoners' dilemma game.

[28]Once again, the dynamics of the simultaneous move game can be analyzed similarly.

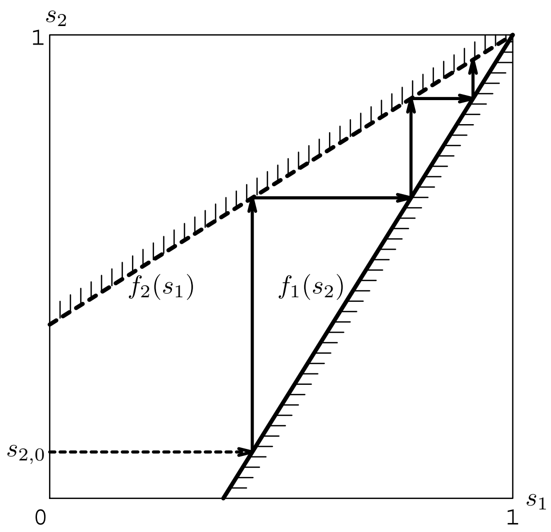


Figure 16: Gradual cooperation

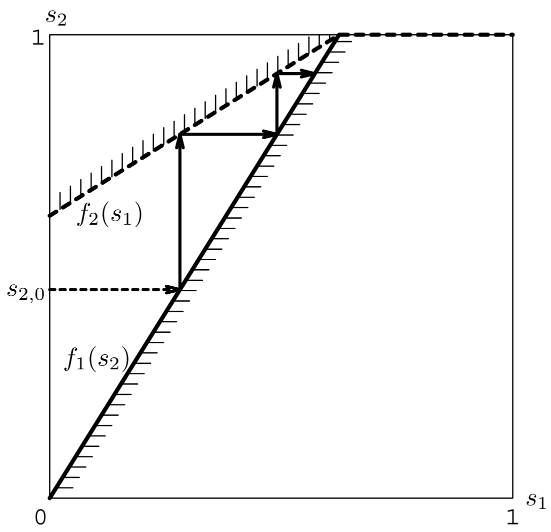


Figure 17: Uneven gradual cooperation

where $\overline{p} > c$. Let $(i, j) \in Q$. Recalling (2.3), we see that the effective payoff of firm i is given by

$$w_i(p_i, p_j) = \ln(\overline{p} - p_i) + \ln(p_i - c) + \delta \ln p_j. \tag{8.4}$$

Replacing $w_i(p_i, p_j)$ with ω_i and solving for p_j, we obtain

$$g_j^{\omega_i}(p_i) = \exp[\{\omega_i - \ln(\overline{p} - p_i) - \ln(p_i - c)\}/\delta]. \tag{8.5}$$

Note that $g_j^{\omega_i}(c) = g_j^{\omega_i}(\overline{p}) = \infty$. Direct calculation of the second derivative shows that $g_j^{\omega_i}$ is strictly convex. It is easy to see that given p_j, firm i's one-shot profit, as well as its effective payoff, is maximized at $p_i = \hat{p} \equiv (c + \overline{p})/2$. This is the price charged by both firms in the unique static Nash equilibrium.

Figure 18 illustrates a symmetric IRE in which both firms receive the effective payoff corresponding to the static Nash equilibrium. The indifference curves in this figure are similar to those in Figure 5, which shows the IRE corresponding to the static Nash equilibrium. Figure 18 shows an alternative IRE (which is in regular form). In this IRE, there is a steady state in which both firms charge the static Nash price, as in Figure 5. In Figure 18, however, there is another steady state with a higher symmetric price. At this steady state, each firm faces a "kinked demand curve." If one of the firms raises its price, the other does not follow. Proposition 6.1 implies that this kinked feature is a rather general property in the sense that in any IRE, the firms never charge prices higher than those given by the highest intersection of the two indifference curves. On the other hand, if one of the firms lowers its price, this triggers price war, and the prices converge to the lower steady state. Figure 18 shows an example of an IRE path after a small price cut by firm 2 in period 0 (which is taken as the initial condition of the model).

Clearly the above properties of the two steady states continue to hold even if the firms receive higher effective payoffs, as long as there are two steady states. It is easy to see that there can be at most two steady states by strict concavity of $g_j^{\omega_i}$, provided that the firms receive effective payoffs at least as large as the level associated with the static Nash equilibrium. Note that effective payoffs higher than the static Nash level correspond to indifference curves higher than those depicted in Figure 18.

If there is only one steady state, then the IRE is effectively efficient by (6.11). Figure 19 illustrates a symmetric, effectively efficient IRE in regular form. At the unique steady state, each firm faces a kinked demand curve once again. This steady state, however, is globally stable. If one of the firms raises

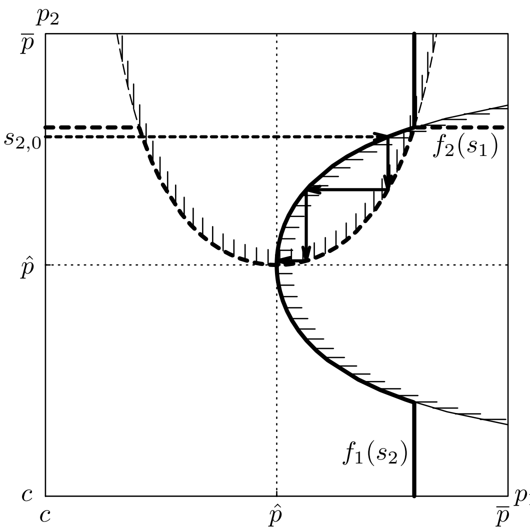


Figure 18: Kinked demand curves with unstable collusion

its price, the other does not follow, as in Figure 18. If one of them lowers its price, the other lowers its price too but by a smaller degree. Eventually the prices return to the initial high levels. This process is shown in Figure 19 assuming that firm 2 cuts its price to the static Nash level in period 0. It follows from Theorem 5.1 that the global stability of the unique steady state is a general property of any effectively efficient IRE here.

9 Concluding Comments

This paper offers a complete and graphical characterization of immediately reactive equilibria (IREs) and their global dynamics for infinitely repeated games with two players in which the action space of each player is an interval, and the one-shot payoff of each player consists of two continuous functions, one unimodal in his own action, the other strictly monotone in the other player's action. IREs extend Nash reversion equilibria by allowing for continuous strategies and nontrivial dynamics. Characterized by two indifference curves, the global dynamics of an IRE can be analyzed graphically. Though IREs constitute only a small subset of the subgame perfect equilibria, the structure of IREs is rich enough to allow us to show a folk-type theorem in a special case.

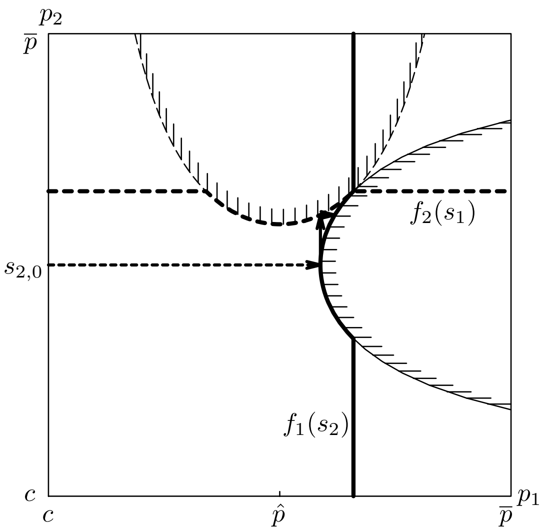


Figure 19: Kinked demand curves with stable collusion

Although additive separability, which is crucial to our analysis, is rather restrictive, there are various interesting games that satisfy it. We have analyzed two such games and characterized their IREs by applying our general results. We have shown among other things that gradual cooperation arises in an effectively efficient IRE of a prisoners' dilemma game, and that kinked demand curves with stable collusion emerge in an effectively efficient IRE of a duopoly game.

We believe that our results are useful not only in analyzing games that satisfy our assumptions, but also in constructing completely tractable special cases of more general games. Such special cases, whose dynamics can be analyzed explicitly, would enhance understanding of various interesting problems.

Appendix A Proof of Theorem 6.1

Throughout the proof, we omit the superscripts $\omega_i, \omega_j, (\omega_i, \omega_j)$, and (ω_j, ω_i).

A.1 Sufficiency

The "if" part of the proposition follows from the following.

Lemma A.1. *Let $\omega_1, \omega_2 \in \mathbb{R}$ satisfy (6.6) and (6.7). Then the strategy profile (f_1, f_2) given by (6.8) is an IRE associated with (ω_1, ω_2).*

Proof. It suffices to show $R(f_i) \subset M_i(f_j)$ for $(i, j) \in Q$ by Theorem 4.1. Let $(i, j) \in Q$. By (3.21) and (6.8),

$$M_i(f_j) = \{s_i \in D(g_j) \mid g_j(s_i) \le \overline{s}_j\}. \tag{A.1}$$

By (6.3) and (6.2), $g_j(\overline{s}_i) \le \overline{s}_j$, so $\overline{s}_i \in M_i(f_j)$. By (6.4), $g_j(\underline{s}_i) \le \overline{s}_j$, so $\underline{s}_i \in M_i(f_j)$. Let $s_i \in R(f_i)$. By (6.7), Lemma 3.1(iii), and (6.8), $\underline{s}_i \le g_i(\hat{s}_j) \le s_i \le \overline{s}_i$. Thus $s_i \in M_i(f_j)$ since $\underline{s}_i, \overline{s}_i \in M_i(f_j)$ and $M_i(f_j)$ is an interval by (A.1) and Lemma 3.1(iii). It follows that $R(f_i) \subset M_i(f_j)$. □

A.2 Necessity

We show the "only if" part in a few steps. Throughout we take as given an IRE (f_1, f_2) associated with $(\omega_1, \omega_2) \in \mathbb{R}^2$. For $(i, j) \in Q$, define

$$\underline{r}_i = \inf R(f_i), \quad \overline{r}_i = \sup R(f_i), \tag{A.2}$$

$$\tilde{r}_i = \min\{s_i \in D(g_j) \mid g_j(s_i) \le \overline{r}_j\}. \tag{A.3}$$

Lemma A.2. *For $(i, j) \in Q$,*

$$\underline{r}_i, \overline{r}_i \in D(g_j), \tag{A.4}$$

$$g_j(\underline{r}_i), g_j(\overline{r}_i) \in D(g_i), \tag{A.5}$$

$$\underline{r}_i \le g_i(g_j(\underline{r}_i)), \quad \overline{r}_i \ge g_i(g_j(\overline{r}_j)). \tag{A.6}$$

Proof. Let $(i, j) \in Q$. Recall from (3.22) that

$$R(f_i) \subset M_i(f_j) \subset D(g_j). \tag{A.7}$$

Since $D(g_j)$ is closed by (3.17) and Lemma 3.1(i), (A.4) follows from (A.7) and (A.2). Note from (4.2) and (A.7) that

$$\forall s_j \in R(f_j), \quad f_i(s_j) = g_i(s_j). \tag{A.8}$$

Hence

$$\underline{r}_i \le \inf_{s_j \in R(f_j)} g_i(s_j), \quad \overline{r}_i \ge \sup_{s_j \in R(f_j)} g_i(s_j). \tag{A.9}$$

By (A.8) and (A.7) (with i and j interchanged),

$$\forall s_i \in R(f_i), \quad g_j(s_i) = f_j(s_i) \in R(f_j) \subset D(g_i). \tag{A.10}$$

Thus (A.5) follows by continuity of g_j and closeness of $D(g_i)$.

To see (A.6), let $s_i \in R(f_i)$. By (A.10), $g_j(s_i) \in R(f_j)$. Hence by (A.9), $\underline{r}_i \le g_i(g_j(s_i))$ and $\overline{r}_i \ge g_i(g_j(s_i))$. Thus (A.6) follows by continuity of g_i and g_j. □

When $(i, j) \in Q$ is given, we interchange the coordinates in Ψ_j so that Ψ_i and Ψ_j have the same coordinates, i.e., we redefine

$$\Psi_j = \{(s_i, s_j) \in [0,1]^2 \mid s_j \in D(g_i), s_i \ge g_i(s_j)\}. \tag{A.11}$$

This is identical to (6.2) with i and j interchanged, except for the order of the coordinates; recall footnote 17.

Lemma A.3. *For* $(i, j) \in Q$,

$$(\overline{r}_i, g_j(\overline{r}_i)) \in \Psi_i \cap \Psi_j, \tag{A.12}$$

$$\text{(a)}\ \hat{s}_j \in D(g_i), \quad \text{(b)}\ \tilde{r}_i \le g_i(\hat{s}_j). \tag{A.13}$$

Proof. Let $(i, j) \in Q$. Since $\overline{r}_i \in D(g_j)$ by (A.4), we have $(\overline{r}_i, g_j(\overline{r}_i)) \in \Psi_i$ by (6.2) (with $s_j = g_j(\overline{r}_i)$ and $s_i = \overline{r}_i$). By (A.5) and (A.6), $g_j(\overline{r}_i) \in D(g_i)$ and $\overline{r}_i \ge g_i(g_j(\overline{r}_i))$. So by (A.11), $(\overline{r}_i, g_j(\overline{r}_i)) \in \Psi_j$. Thus (A.12) follows.[29]

It remains to show (A.13). Note that (A.13)(a) is immediate from (3.17) and Lemma 3.1(i). By (A.2), $f_j(s_i) \le \overline{r}_j$ for $s_i \in [0, 1]$. Thus by (3.21),

$$\forall s_i \in M_i(f_j), \quad g_j(s_i) = f_j(s_i) \le \overline{r}_j. \tag{A.14}$$

Hence $M_i(f_j) \subset \{s_i \in D(g_j) \mid g_j(s_i) \le \overline{r}_j\} \equiv B$. Thus by (3.20) and (4.1),

$$g_i(\hat{s}_j) \ge f_i(\hat{s}_j) \in M_i(f_j) \subset B. \tag{A.15}$$

Since $\tilde{r}_i = \min B$ by (A.3), (A.13)(b) follows. □

Let us now complete the "only if" part of the proof. We have (6.6) by (A.12). Let $(i, j) \in Q$. By (A.12) and (6.3), $\overline{r}_j \le \overline{s}_j$. Thus the set in (6.4) includes the set in (A.3), so $\underline{s}_j \le \tilde{r}_j$. This together with (A.13) shows (6.7).

[29]The corresponding result for $\underline{r}_i$ does not hold since $\overline{r}_i$ cannot be replaced by $\underline{r}_i$ in $(\overline{r}_i, g_j(\overline{r}_i)) \in \Psi_j$ unless $\underline{r}_i = g_i(g_j(\underline{r}_i))$; recall (A.6).

References

Abreu, D., 1986, Extremal equilibria of oligopolistic supergames, Journal of Economic Theory 39, 191–225.

Abreu, D., 1988, On the theory of infinitely repeated games with discounting, Econometrica 56, 383–396.

Bhaskar, V., F., Vega-Redondo, 2002 , Asynchronous choice and Markov equilibria, Journal of Economic Theory 103, 334–350.

Bhaskar, V., S. Machin, G. Reid, 1991, Testing a model of the kinked demand curve, Journal of Industrial Economics 39, 241–154.

Corchon, L., 1994, Comparative statics for aggregative games: the strong concavity case, Mathematical Social Sciences 28, 151–165.

Devaney, R.L., 1989, An Introduction to Chaotic Dynamical Systems, 2nd ed. Redwood City: Addison-Wesley.

Friedman, J.W., 1968, Reaction functions and the theory of duopoly, Review of Economic Studies 35, 257–272.

Friedman, J.W., 1971, A non-cooperative equilibrium for supergames, Review of Economic Studies 38, 1–12.

Friedman, J.W., 1973, On reaction function equilibria, International Economic Review 14, 721–734.

Friedman, J.W., 1976, Reaction functions as Nash equilibria, Review of Economic Studies 43, 83–90.

Friedman, J.W., L. Samuelson, 1990, Subgame perfect equilibria with continuous reaction functions, Games and Economic Behavior 2, 304–324.

Friedman, J.W., L. Samuelson, 1994a, Continuous reaction functions in duopolies, Games and Economic Behavior 6, 55–82.

Friedman, J.W., L. Samuelson, 1994b, An extension of the "folk theorem" with continuous reaction functions, Games and Economic Behavior 6, 83–96.

Fudenberg, D., E. Maskin, 1986, The folk theorem in repeated games with discounting or with incomplete information, Econometrica 54, 533–554.

Fudenberg, D., J. Tirole, 1991, *Game Theory.* Cambridge, MA: MIT Press.

Furusawa, T., T. Kamihigashi, 2006, Threats and concessions in tariff setting, work in progress.

Furusawa, T., T. Kawakami, 2006, Gradual cooperation in the existence of outside options, Journal of Economic Behavior and Organization, forthcoming.

Green, E.J., R.H. Porter, 1984, Noncooperative collusion under imperfect price information, Econometrica 52, 87–100.

Haller, H., R. Lagunoff, 2000, Markov perfect equilibria in repeated asynchronous choice games, working paper, Georgetown University.

Kalai, E., D. Samet, W. Standford, 1988, A note on reactive equilibria in the discounted prisoner's dilemma and associated games, International Journal of Game Theory 17, 177–186.

Kamihigashi, K., S. Roy, 2006, Dynamic optimization with a nonsmooth, nonconvex technology: the case of a linear objective function, Economic Theory 29, 325–340.

Kranton, R.E., 1996, The formation of cooperative relationships, Journal of Law, Economics, and Organization 12, 214–233.

Langlois, J.-P., J.A. Sachs, 1993, Existence and local stability of Pareto superior reaction function equilibria in discounted supergames, Journal of Mathematical Economics 22, 199–221.

Lagunoff, R., A. Matsui, 1997, Asynchronous choice in repeated coordination games, Econometrica 65, 1467–1477.

Maskin, E., J. Tirole, 1987, A theory of dynamic oligopoly, III: Cournot competition, European Economic Review 31, 947–968.

Maskin, E., J. Tirole, 1988a, A theory of dynamic oligopoly, I: overview and quantity competition with large fixed costs, Econometrica 56, 549–569.

Maskin, E., J. Tirole, 1988b, A theory of dynamic oligopoly, II: price competition, kinked demand curves, and edgeworth cycles, Econometrica 56, 571–599.

Maskin, E., J. Tirole, 2001, Markov perfect equilibrium I: observable actions, Journal of Economic Theory 100, 191–291.

Radner, R., 2003, Viscous demand, Journal of Economic Theory 112, 189–231.

Rand, D., 1978, Exotic phenomena in games and duopoly models, Journal of Mathematical Economics 5, 173–184.

Robson, A., 1986, The existence of Nash equilibria in reaction functions for dynamic models of oligopoly, International Economic Review 27, 539–544.

Rosser, J.B., Jr., 2002, The development of complex oligopoly dynamics theory. In: T. Puu, I. Sushko, (Eds.), Oligopoly Dynamics: Models and Tools, Springer-Verlag, Berlin, pp. 15–29.

Sabourian, H., 1992, Rational conjectural equilibrium and repeated games. In: P. Dasgupta, D. Gale, O. Hart, E. Maskin, (Eds.), Economic Analysis of Markets and Games: Essays in Honor of Frank Hahn, pp. 228–257.

Samuelson, L., Non-trivial subgame perfect duopoly equilibria can be supported by continuous reaction functions, Economics Letters 24, 207–211.

Sen, D., 2004, The kinked demand curve revisited, Economics Letters 84, 99–105.

Stahl, D.O., II, 1991, The graph of prisoners' dilemma supergame payoffs as a function of the discount factor, Games and Economic Behavior 3, 368–384.

Stanford, W.G., 1986, Subgame perfect reaction function equilibria in discounted duopoly supergames are trivial, Journal of Economic Theory 39, 226–232.

Tidball, M., A. Jean-Marie, N. Querou, 2000, Conjectural variation equilibrium, reaction functions, and indifference curves: a survey, working paper 2002-7, LAMETA.

Yoon, K., 2001, A folk theorem for asynchronously repeated games, Econometrica 69, 191–200.

Watson, J., 1999, Starting small and renegotiation, Journal of Economic Theory 85, 52–90.

Watson, J., 2002, Starting small and commitment, Games and Economic Behavior 38, 176–199.