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A two-country dynamic model of international trade and endogenous growth: multiple balanced growth paths and stability*

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Abstract

We formulate a two-country endogenous growth model which explain joint determination of long-run trade patterns and world growth rates. After providing the existence and local stability of the continuum of balanced growth paths, we show that main standard trade propositions hold under some modifications and that, subject to certain conditions concerning social and private rankings of factory intensities between production sectors, the higher is the growth rate, the smaller is the volume of international trade among balanced growth paths in the continuum. .

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1 Introduction

The two(-country) by two(-tradable good) by two(-factor) Heckscher-Ohlin (H-O) model has been a fundamental general equilibrium framework in trade theory for a long time. Innumberable articles and volumes have been published to extend the H-O model to various directions in such a way that many important realistic issues like increasing returns, externalities, market imperfections, non-traded goods, and trade policies... are incorporated into it. However, there are only few contributions that extend the H-O model to explain the pattern of trade and the long-run world growth rate jointly in an unified framework.[1]

This paper attempts to provide such a framework. We develop a two by two by two dynamic (à la Heckscher-Ohlin) trade model of international trade *and* endogenous growth to discuss what jointly determines the long-run pattern of international trade and the long-run growth rate. We also examine whether the main trade propositions in the static H-O model, i.e., the factor-price equalization theorem, the Stolper-Samuelson theorem, the Rybczynski theorem, the Heckscher-Ohlin theorem and the law of comparative advantage, continue to hold in some modified forms. We find that while the first three trade theorems basically hold, the last two theorems need to be modified: we show that, other things being equal, the international difference in the "psychic costs" in generating new human capital can be a determinant of the long-run trade pattern.[2]

On the one hand, the endogenous growth/trade model in this paper share the following property with the exogenous growth/trade model developed by Chen [6] and Shimomura [11]; there is a continuum of balanced growth paths with different patterns and volumes of international trade[3] and any equilibrium path starting from historically given initial international distribution of physical and human capitals converges to a balanced growth path in the continuum. On the other hand, while the long-run growth rate is naturally the same for any balanced growth path and hence uniquely determined in the exogenous growth models, we see that in the present endogenous growth/trade model balanced growth paths in the continuum have different pair of .long-run growth rate and the volume of trade. Thus, we can argue whether or not a balanced growth path with a higher growth rate involves a larger world trade volume.

After establishing the existence and local stability of the continuum of balanced growth paths, we examine whether or not the above main trade propositions can substantially hold for the continuum as a whole. As we already mentioned, we can show that the long-run pattern of trade is determined by the international difference in the above "psychic costs".

[1] A notable preceding contribution is, among others, Ventura [12]. .

[2] By psychic cost we mean disutily accompanying the acquisition of new skill and knowledge. It is a sort of adjustment cost when human capital is augmented.

[3] Since we deal with balanced growth path with positive growth rates, the volume of trade in the sense of the volumes of exports and imports also grow. When we say "the volume of trade is large (small)", we mean that the ratio of the volume to the world physical capital is large (small).

We then move to focus on the relationship between growth rate and trade volume among balanced growth paths in the continuum. One may usually associate globalization with rapid world growth, However, we find a theoretical possibility such that the relationship is reversed: Under certain conditions concerning social and private rankings of factor intensities between production sectors, the higher is the growth rate, the smaller is the volume of international trade among balanced growth paths in the continuum.

Formerly speaking, the anti-intuitive relationship is closely related to the issue called local indeterminacy[4] which has been extensively studied in macroeconomic theory over decade. Among exogenous growth models, Benhabib and Nishimura [3] first made clear that indeterminacy is possible in a multi-sector model of closed economy with factor-generated externality even if returns to scale are constant from the social perspective. Nishimura and Shimomura [9] extended it to a two-country dynamic Heckscher-Ohlin model of international trade and showed that the introduction of externality induces indeterminacy which may reverse the pattern of trade along the equilibrium path. Later, Nishimura and Shimomura [10] also showed that indeterminacy can be generated even without introducing externality; they presented a two-country dynamic trade model in which indeterminacy is based on the lack of international lending and borrowings. .

For endogenous growth models, Bond, Wang and Yip [4] studied a two-sector model of closed economy. It was extended to an open economy model by Bond, Trask and Wang [5]. By introducing externalities into a multi-sector endogenous growth model, Benhabib, Meng and Nishimura [2] provided a factor intensity condition under which indeterminacy may occur. We shall extend it to a two-country model.

The extension of the closed economy to a two-country framework is not straightforward in the endogenous growth model. Bond, Trask, and Wang focus only on a balanced growth path in the world economy, and no transitional dynamics is defined in their model. We introduce "psychic cost" and externality in our model so that transitional dynamics may be well-defined and indeterminacy may arise in the two-country endogenous growth model.

This paper is organized as follows. Section 2 presents the endogenous growth/trade model. Section 3 give the formal definition of balanced growth paths. Section 4 discusses under what conditions balanced growth paths exists Section 5 studies two types of local stability which have different implications for the relationship between long-run growth rate and volume of international trade in the continuum of balanced growth paths. Section 6 characterizes the two types of BGP's and obtains main trade-theoretic propositions. Section 7 concludes.

[4]A related work was done by Farmer and Lahiri [7]. They formulate a two-country endogenous growth model. Introducing externalities into the endogenous growth model, they show that the model has one symmetric and two asymmetric equilibria such that there is a continuum of switching equilibrium paths among the three equilibria.

2 The Model

2.1 Production

The trading world consists of two countries, the home country and the foreign country. Two tradable goods, an investment good and a consumption good, are produced by using physical and human capitals both of which are internationally immobile. We call the tradable goods *good* 1 and *good* 2, respectively. There is no international difference in production technology and the production function of good i is

$$Y_i = \varepsilon \overline{e}_i L_i^{\beta_{1i}} K_i^{\beta_{2i}}, \quad i = 1, 2, \tag{1}$$

where $\overline{e}_i = \bar{L}_i^{b_{1i}} \bar{K}_i^{b_{2i}}$ are factor-generated externality terms and $\varepsilon > 0$ is a shift parameter. Just for notational brevity we set ε to be unity for the time being. We assume $\beta_{1i} > 0, \beta_{2i} > 0, b_{1i} > 0, b_{2i} > 0$ and $\beta_{1i} + \beta_{2i} + b_{1i} + b_{2i} = 1$, which mean that the production function shows constant returns to scale from the social perspective but decreasing returns from the private perspective. Let $a_{1i} = L_i/Y_i$ and $a_{ji} = K_i/Y_i \;\; i = 1, 2$. Then full employment conditions are

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} L \\ K \end{pmatrix}. \tag{2}$$

Solving (2), we obtain

$$Y_1 = \frac{a_{22}L - a_{12}K}{a_{11}a_{22} - a_{21}a_{12}}, \quad Y_2 = \frac{a_{11}K - a_{21}L}{a_{11}a_{22} - a_{21}a_{12}}. \tag{3}$$

Since the endowments of the foreign country are given by L^* and K^*, outputs of the foreign country are given by

$$Y_1^* = \frac{a_{22}L^* - a_{12}K^*}{a_{11}a_{22} - a_{21}a_{12}}, \quad Y_2^* = \frac{a_{11}K^* - a_{21}L^*}{a_{11}a_{22} - a_{21}a_{12}}. \tag{4}$$

Let P_i, w and r be the prices of good i, the factor rewards of human and physical capitals. As technologies are internationally identical, the same domestic prices $(w,\, r)$ prevail in the foreign country under incomplete specialization in both countries: the factor price equalization theorem holds in our framework. Profit maximization implies

$$P_i\beta_{1i} = wa_{1i}, \quad P_i\beta_{2i} = ra_{2i} \tag{5}$$

From (3), (4) and (5) we obtain

$$Y_1 = \frac{w\beta_{22}L - r\beta_{12}K}{P_1\Delta}, \quad Y_2 = \frac{r\beta_{11}K - w\beta_{21}L}{P_2\Delta} \tag{6}$$

$$Y_1^* = \frac{w\beta_{22}L^* - r\beta_{12}K^*}{P_1\Delta}, \quad Y_2^* = \frac{r\beta_{11}K^* - w\beta_{21}L^*}{P_2\Delta} \tag{7}$$

where $\Delta = \beta_{11}\beta_{22} - \beta_{12}\beta_{21}$. If $\Delta > (<) \, 0$, the consumption good, good 2, is physical (human) capital intensive from the private perspective.

Let the investment good, good 1, be the numeraire: $P_1 = 1$. Then $P_2 = p$ is the price of good 2 in terms of good 1.

Let $\widehat{\beta}_{ji} = \beta_{ji} + b_{ji}$. From (5) and production functions (1)

$$1 = \varepsilon \left(\frac{\beta_{11}}{w} \right)^{\widehat{\beta}_{11}} \left(\frac{\beta_{21}}{r} \right)^{\widehat{\beta}_{21}}, \quad 1 = \varepsilon \left(\frac{p\beta_{12}}{w} \right)^{\widehat{\beta}_{12}} \left(\frac{p\beta_{22}}{r} \right)^{\widehat{\beta}_{22}} \tag{8}$$

By solving (8) with respect to w and r,

$$w(p) = \varepsilon \hat{w} p^{-\widehat{\beta}_{21}/\widehat{\Delta}}, \quad r(p) = \varepsilon \hat{r} p^{\widehat{\beta}_{11}/\widehat{\Delta}} \tag{9}$$

where

$$\hat{w} = \left(\beta_{12}^{\widehat{\beta}_{12}\widehat{\beta}_{21}} \beta_{11}^{-\widehat{\beta}_{11}\widehat{\beta}_{22}} \left(\beta_{22}/\beta_{21} \right)^{\widehat{\beta}_{21}\widehat{\beta}_{22}} \right)^{-1/\widehat{\Delta}}$$

and

$$\hat{r} = \left(\beta_{21}^{\widehat{\beta}_{12}\widehat{\beta}_{21}} \beta_{22}^{-\widehat{\beta}_{11}\widehat{\beta}_{22}} \left(\beta_{11}/\beta_{12} \right)^{\widehat{\beta}_{11}\widehat{\beta}_{12}} \right)^{-1/\widehat{\Delta}}.$$

Let $\widehat{\Delta} = \widehat{\beta}_{11}\widehat{\beta}_{22} - \widehat{\beta}_{12}\widehat{\beta}_{21} = \widehat{\beta}_{11} - \widehat{\beta}_{12}$. From (9), we have the Stolper-Samuelson properties;

$$\frac{pw'(p)}{w(p)} = -\frac{\widehat{\beta}_{21}}{\widehat{\Delta}}, \quad \frac{pr'(p)}{r(p)} = \frac{\widehat{\beta}_{11}}{\widehat{\Delta}} \tag{10}$$

We denote $pw'(p)/w(p)$ by θ_w, and $pr'(p)/r(p)$ by θ_r. If $\widehat{\Delta} > (<) \, 0$, the consumption good is capital (labor) intensive from the social perspective.

Finally, for the subsequent argument, we define

$$\overline{\beta}_{ij} \equiv \frac{\beta_{ij}}{\Delta}, \quad i, j = 1, 2 \tag{11}$$

2.2 Households

Next let us consider the dynamic optimization problem an home household faces. The population of each country is normalized to be one. Each household chooses C and Z to maximize the discounted sum of utility,

$$\int_0^\infty \frac{C^{1-\sigma} - Z^{1-\sigma}}{1-\sigma} e^{-\rho t} dt, 0 < \sigma < 1, \rho > 0 \tag{12}$$

subject to

$$\begin{aligned} \overset{\bullet}{K} &= Y_1 + pY_2 - pC \\ &= (wL + rK) + \Pi - pC \qquad (13) \\ \overset{\bullet}{L} &= Z^\eta L^{1-\eta}, \; 0 < \eta < 1 \qquad (14) \end{aligned}$$

where Z is disutility, or psychic cost, for accumulating human capital L; K is physical capital; ρ is the subjective discount rate; σ is the inverse of the intertemporal elasticity of substitution. Note that Π is generally positive because of decreasing returns from the private perspective.

(13) is the flow budget constraint. In this paper we follow Benhabib and Nishimura [3] and Benhabib, Meng and Nishimura [2] in assuming some kind of entry barriers that make possible positive profits. An alternative assumption would be that there are sector-specific factors with negative externalities in each industry in such a way that both social and private returns to scale are constant concerning both general and specific factors of production as a whole. Under the assumption, Π is interpreted as the sector-specific factor income and we do not need to assume entry barriers.

(14) is the human-capital formation function in this paper. We assume the followings

(i) Each household can increase her human capital, i.e., knowledge and skill, by studying and learning new knowledge and skill. In order to obtain more human capital, she has to pay larger psychological or disutility cost.

(ii) The larger is her existing stock of human, the higher the productivity of disutilty cost is.

(iii) Solving the above dynamic optimization problem, she recognizes the positive relationship between L and $\dot{L}$ which is assumed in (ii)[5].

Associated with the above optimization problem is the Hamiltonian

$$H \equiv \frac{C^{1-\sigma} - Z^{1-\sigma}}{1-\sigma} + \Lambda \left[rK + wL + \Pi - pC \right] + MZ^{\eta}L^{1-\eta} \tag{15}$$

The necessary conditions for optimality are

$$\frac{\partial H}{\partial C} = C^{-\sigma} - \Lambda p = 0 \tag{16a}$$

$$\frac{\partial H}{\partial Z} = -Z^{-\sigma} + \eta M Z^{\eta-1} L^{1-\eta} = 0 \tag{16b}$$

$$\dot{K} = wL + rK + \Pi - pC \tag{16c}$$

$$\dot{L} = Z^{\eta} L^{1-\eta} \tag{16d}$$

$$\dot{\Lambda} = \rho\Lambda - r\Lambda \tag{16e}$$

$$\dot{M} = \rho M - \Lambda w - M(1-\eta) Z^{\eta} L^{-\eta} \tag{16f}$$

[5] Let N be the home population. Since we assume that the population of each country is constant over time, we can derive the aggregate human capital formation function from individual ones in a straightforwad manner:

$$\dot{(NL)} = N\dot{L} = NZ^{\eta}L^{1-\eta} = (NZ)^{\eta}(NL)^{1-\eta}$$

and the transversality condition

$$\lim_{t\to\infty}[e^{-\rho t}K(t)\Lambda(t) + e^{-\rho t}L(t)M(t)] = 0 \quad (17)$$

The second-order condition for maximizing the Hamiltonian with respect to Z is satisfied if the following holds.

ASSUMPTION 1: $\sigma < \beta_{21} < 1 - \eta$.

We can similarly formulate the dynamic optimization problem for each foreign household. The Hamiltonian is

$$\begin{aligned} H^* \equiv & \frac{C^{*1-\sigma^*} - Z^{*1-\sigma^*}}{1-\sigma^*} + \Lambda^*[rK^* + wL^* + \Pi^* - pC^*] \\ & + M^* b Z^{*\eta^*} L^{*1-\eta^*} \end{aligned} \quad (18)$$

where an asterisk(*) is attached to each foreign variable. The parameter b denotes the international difference in ability to generate new human capital[6]. The necessary conditions for optimality are

$$\frac{\partial H^*}{\partial C^*} = (C^*)^{-\sigma^*} - \Lambda^* p = 0 \quad (19a)$$

$$\frac{\partial H^*}{\partial Z^*} = -Z^{-\sigma^*} + \eta^* M^* Z^{*\eta^*-1} L^{*1-\eta^*} = 0 \quad (19b)$$

$$\dot{K}^* = wL^* + rK^* + \Pi^* - pc^* \quad (19c)$$

$$\dot{L}^* = b\,(Z^*)^{\eta^*}\,(L^*)^{1-\eta^*} \quad (19d)$$

$$\dot{\Lambda}^* = \rho^*\Lambda^* - r\Lambda^* \quad (19e)$$

$$\dot{M}^* = \rho^* M^* - \Lambda^* w - M^* b(1-\eta^*) Z^{*\eta^*} L^{*-\eta^*} \quad (19f)$$

and the transversality condition

$$lim_{t\to\infty}[e^{-\rho^* t}K^*(t)\Lambda^*(t) + e^{-\rho^* t}L^*(t)M^*(t)] = 0 \quad (20)$$

In what follows we assume $\sigma = \sigma^*$, and $\rho = \rho^*$.

Finally, the world market-clearing condition for the consumption good is

$$C + C^* = Y_2 + Y_2^* \quad (21)$$

Equations, (16a)-(17), (19a)-(20) and (21), give the complete two-country dynamic general equilibrium (DGE) model. Note that the co-state (jump) variables are Λ, Λ^*, M, and M^*.

[6] The parameter b reflects environmental differences between two countries: Even if home and foreign people have the same ability to generate new human capital, people who live in a more harsh natutal environment have b that is lower than people in the other country.

2.3 Growth Paths under Incomplete Specialization

Let us focus on DGE paths under incomplete specialization in both countries. Combining (6), (7) and (9) together, we derive

$$Y_1 + pY_2 = \frac{r(p)K(\beta_{11} - \beta_{12}) - w(p)L(\beta_{21} - \beta_{22})}{\Delta} \tag{22a}$$

$$Y_1^* + pY_2^* = \frac{r(p)K^*(\beta_{11} - \beta_{12}) - w(p)L^*(\beta_{21} - \beta_{22})}{\Delta} \tag{22b}$$

$$Y_2 = \frac{r(p)\beta_{11}K - w(p)\beta_{21}L}{p\Delta} \tag{22c}$$

$$Y_2^* = \frac{r(p)\beta_{11}K^* - w(p)\beta_{21}L^*}{p\Delta} \tag{22d}$$

Making use of (22a)-(22d), we can write the above dynamic equilibrium system as follows:

$$\frac{p(1+m^{-\frac{1}{\sigma}})}{(p\Lambda)^{\frac{1}{\sigma}}} = \frac{r(p)\beta_{11}(K+K^*) - (L+L^*)w(p)\beta_{21}}{\Delta} \tag{23a}$$

$$\overset{\bullet}{K} = \frac{r(p)K(\beta_{11} - \beta_{12}) - w(p)L(\beta_{21} - \beta_{22})}{\Delta} - \frac{p^{1-\frac{1}{\sigma}}}{\Lambda^{\frac{1}{\sigma}}} \tag{23b}$$

$$\overset{\bullet}{K^*} = \frac{r(p)K^*(\beta_{11} - \beta_{12}) - w(p)L^*(\beta_{21} - \beta_{22})}{\Delta} - \frac{p^{1-\frac{1}{\sigma}}}{(\Lambda^*)^{\frac{1}{\sigma}}} \tag{23c}$$

$$\overset{\bullet}{L} = Z^\eta L^{1-\eta} \tag{23d}$$

$$\overset{\bullet}{L^*} = b(Z^*)^\eta (L^*)^{1-\eta} \tag{23e}$$

$$\overset{\bullet}{\Lambda} = \Lambda[\rho - r(p)] \tag{23f}$$

$$\overset{\bullet}{\Lambda^*} = \Lambda^*[\rho - r(p)] \tag{23g}$$

$$\overset{\bullet}{M} = M[\rho - (1-\eta)Z^\eta L^{-\eta}] - \Lambda w(p) \tag{23h}$$

$$\overset{\bullet}{M^*} = M^*[\rho - b(1-\eta)(Z^*)^\eta (L^*)^{-\eta}] - \Lambda^* w(p) \tag{23i}$$

$$Z^{-\sigma} = \eta M Z^{\eta-1} L^{1-\eta} \tag{23j}$$

$$Z^{-\sigma} = \eta b M^* (Z^*)^{\eta-1} (L^*)^{1-\eta} \tag{23k}$$

(23a) follows from the world market clearing condition (21), (16a), (19a), (22c) and (22d). Note that $m \equiv \Lambda^*/\Lambda$ is time-invariant from (23f) and (23g). In what follows, we assume that the two sectors have different factor intensities both from the social and private perspectives;

Assumption 2:$\Delta\hat{\Delta} \neq 0$.

If externality is absent, $\Delta = \hat{\Delta}$. Note that if there is, it is possible that $sign[\Delta] = -sign[\hat{\Delta}] \neq 0$. As was shown in Benhabib and Nishimura [3] and Benhabib, Meng and Nishimura [2], and as will be shown in this paper, this difference in private and social factor-intensity rankings is crucial for indeterminacy to occur.

Let S be the total capital stock level in the world economy; $S \equiv K + K^*$. Since $\bar{\beta}_{ij} \equiv \beta_{ij}/\Delta$, (23a)-(23c) imply

$$\dot{S} = (L + L^*)w(p)\bar{\beta}_{22} - Sr(p)\bar{\beta}_{12} \tag{24}$$

We simplify the model by reducing the numbers of both differential equations and unknowns. Let $l \equiv L/S$, $l^* \equiv L^*/S$, $\lambda \equiv \Lambda^{-1/\sigma}/S$, $\mu \equiv M^{-1/\sigma}/S$, $\mu^* \equiv M^{-1/\sigma}/S$, $k = K/S$, $k^* \equiv K^*/S$. The dynamic system can be rewritten by

$$
\begin{aligned}
0 &= \frac{(m^{\frac{1}{\sigma}}+1)p^{\frac{\sigma-1}{\sigma}}\lambda}{m^{\frac{1}{\sigma}}} - \{r(p)\overline{\beta}_{11} - (l+l^*)w(p)\overline{\beta}_{21}\} \equiv F^c(p,l,l^*,\lambda) && \text{(25a)}\\
\overset{\bullet}{l} &= l[\eta^A(\frac{l}{\mu})^{\sigma A} - R(p,l+l^*)] \equiv F^l(p,l,l^*,\lambda,\mu,\mu^*) && \text{(25b)}\\
\overset{\bullet}{l^*} &= l^*[b^{1+A}\eta^A(\frac{l^*}{\mu^*})^{\sigma A} - R(p,l+l^*)] \equiv F^{l^*}(p,l,l^*,\lambda,\mu,\mu^*) && \text{(25c)}\\
\overset{\bullet}{\lambda} &= \lambda[-\frac{\rho}{\sigma} + \frac{1}{\sigma}r(p) - R(p,l+l^*)] \equiv F^\lambda(p,l,l^*,\lambda,\mu,\mu^*) && \text{(25d)}\\
\overset{\bullet}{\mu} &= \mu[-\frac{\rho}{\sigma} + \frac{(1-\eta)\eta^A(\frac{l}{\mu})^{\sigma A}}{\sigma} + \frac{(\frac{\mu}{\lambda})^\sigma w(p)}{\sigma} - R(p,l+l^*)] && \text{(25e)}\\
&\equiv F^\mu(p,l,l^*,\lambda,\mu,\mu^*)\\
\overset{\bullet}{\mu^*} &= \mu^*[-\frac{\rho}{\sigma} + \frac{b^{1+A}(1-\eta)\eta^A(\frac{l^*}{\mu^*})^{\sigma A}}{\sigma} + \frac{m(\frac{\mu^*}{\lambda})^\sigma w(p)}{\sigma} - R(p,l+l^*)] && \text{(25f)}\\
&\equiv F^{\mu^*}(p,l,l^*,\lambda,\mu,\mu^*)\\
\overset{\bullet}{k} &= k[r(p)\bar{\beta}_{11} - (l+l^*)w(p)\bar{\beta}_{22}] - w(p)l(\bar{\beta}_{21} - \bar{\beta}_{22}) - p^{\frac{\sigma-1}{\sigma}}\lambda && \text{(25g)}\\
\overset{\bullet}{k^*} &= k^*[r(p)\bar{\beta}_{11} - (l+l^*)w(p)\bar{\beta}_{22}] - w(p)l^*(\bar{\beta}_{21} - \bar{\beta}_{22}) - \frac{p^{\frac{\sigma-1}{\sigma}}\lambda}{m^{\frac{1}{\sigma}}} && \text{(25h)}
\end{aligned}
$$

where $R(p, l+l^*) \equiv (l+l^*)w(p)\overline{\beta}_{22} - r(p)\overline{\beta}_{12}$, and $A \equiv \eta/(1-\eta-\sigma)$.This is the simplified system to be analyzed subsequently.

3 Balanced Growth Paths

We define a balanced growth path (BGP) as a solution to the above dynamic model for given initial conditions, if all but co-state variables grow at a common and time-invariant rate $g > 0$ while all the co-state variables decreases at a rate $-\sigma g < 0$.

We study conditions under which a BGP of the system (25a)-(25h) exists. Along a BGP we have,

$$0 = \frac{(m^{\frac{1}{\sigma}}+1)p^{\frac{\sigma-1}{\sigma}}\lambda}{m^{\frac{1}{\sigma}}} - \{r(p)\overline{\beta}_{11} - (l+l^*)w(p)\overline{\beta}_{21}\} \tag{26a}$$

$$0 = \eta^A(\frac{l}{\mu})^{\sigma A} - R(p, l+l^*) \tag{26b}$$

$$0 = b^{1+A}\eta^A(\frac{l^*}{\mu^*})^{\sigma A} - R(p, l+l^*) \tag{26c}$$

$$0 = -\frac{\rho}{\sigma} + \frac{1}{\sigma}r(p) - R(p, l+l^*) \tag{26d}$$

$$0 = -\frac{\rho}{\sigma} + \frac{(1-\eta)\eta^A(\frac{l}{\mu})^{\sigma A}}{\sigma} + \frac{(\frac{\mu}{\lambda})^{\sigma}}{\sigma}w(p) - R(p, l+l^*) \tag{26e}$$

$$0 = -\frac{\rho}{\sigma} + \frac{b^{1+A}(1-\eta)\eta^A(\frac{l^*}{\mu^*})^{\sigma A}}{\sigma} + \frac{m(\frac{\mu^*}{\lambda})^{\sigma}}{\sigma}w(p) - R(p, l+l^*) \tag{26f}$$

$$0 = [r(p)\bar{\beta}_{11} - (l+l^*)w(p)\bar{\beta}_{22}]k - w(p)l(\bar{\beta}_{21} - \bar{\beta}_{22}) - p^{1-\frac{1}{\sigma}}\lambda \tag{26g}$$

$$0 = [r(p)\bar{\beta}_{11} - (l+l^*)w(p)\bar{\beta}_{22}]k^* - w(p)l^*(\bar{\beta}_{21} - \bar{\beta}_{22}) - m^{-\frac{1}{\sigma}}p^{1-\frac{1}{\sigma}}\lambda \tag{26h}$$

For a given $\sigma > 0$, a system of eight equations (26a)-(26h) determines, values of $(l, l^*, \lambda, \mu, \mu^*, k, k^*) \equiv (L/S, L^*/S,\ \Lambda^{-\frac{1}{\sigma}}/S, M^{-\frac{1}{\sigma}}/S, (M^*)^{-\frac{1}{\sigma}}/S, K/S, K^*/S)$ and p.

Making some calculations, we can obtain $(l, l^*, \lambda, \mu, \mu^*, k, k^*)$ as the functions of p.

$$l(p) \equiv \Xi(r(p))w(p)^{-1/\sigma}\lambda(p) \tag{27}$$

$$l^*(p) \equiv b^{-\frac{1+A}{\sigma A}}m^{-1/\sigma}l(p)\lambda(p) \tag{28}$$

$$k(p) \equiv \frac{\sigma\{w(p)(\bar{\beta}_{21} - \bar{\beta}_{22})l(p) + p^{1-\frac{1}{\sigma}}\lambda(p)\}}{\sigma r(p)\bar{\beta}_{11} - (1+\sigma\bar{\beta}_{12})(r(p) - r_1)} \tag{29}$$

$$k^*(p) \equiv \frac{\sigma\{w(p)(\bar{\beta}_{21} - \bar{\beta}_{22})l^*(p) + p^{1-\frac{1}{\sigma}}\lambda(p)m^{-1/\sigma}\}}{\sigma r(p)\bar{\beta}_{11} - (1+\sigma\bar{\beta}_{12})(r(p) - r_1)} \tag{30}$$

$$\mu(p) \equiv w(p)^{-1/\sigma}\left(\frac{1-\eta-\sigma}{\sigma}\right)^{1/\sigma}(r_2 - r(p))^{1/\sigma}\lambda(p) \tag{31}$$

$$\mu^*(p) \equiv w(p)^{-1/\sigma}\left(\frac{1-\eta-\sigma}{\sigma}\right)^{1/\sigma}(r_2 - r(p))^{1/\sigma}m^{-1/\sigma}\lambda(p) \tag{32}$$

$$\lambda(p) \equiv \frac{(1+\sigma\bar{\beta}_{12})(r - r_1)}{\sigma\bar{\beta}_{22}w(p)^{1-\frac{1}{\sigma}}\Xi(r(p))(1 + b^{-\frac{1+A}{\sigma A}}m^{-1/\sigma})}, \tag{33}$$

where

$$\Xi(r(p)) \equiv \left(\frac{r(p) - \rho}{\sigma\eta^A}\right)^{1/\sigma A}\left(\frac{1-\eta-\sigma}{\sigma}\right)^{1/\sigma}(r_2 - r(p))^{1/\sigma}, \tag{34}$$

and

$$r_1 \equiv \frac{\rho}{1+\sigma\bar{\beta}_{12}} \tag{35}$$

$$r_2 \equiv \frac{(1-\eta)\rho}{1-\eta-\sigma} > \rho \tag{36}$$

Substituting (27), (28) and (33) into (26a), we obtain the BGP world market-clearing condition

$$F^c \equiv [\Phi(r(p)) - \Psi_\varepsilon(r(p))] + m^{-1/\sigma}[\Phi(r(p)) - b^{-\frac{1+A}{\sigma A}}\Psi_\varepsilon(r(p))] = 0 \tag{37}$$

where

$$\Phi(r(p)) \equiv \frac{\eta^{\frac{1}{\sigma}}\sigma^{\frac{1-\sigma}{\eta\sigma}}[(\Delta+\beta_{12}\sigma)r(p)-\Delta\rho]}{(1-\eta-\sigma)^{\frac{1}{\sigma}}[r_2-r(p)]^{\frac{1}{\sigma}}[r(p)-\rho]^{\frac{1-\eta-\sigma}{\sigma\eta}}} \tag{38}$$

$$\Psi_\varepsilon(r(p)) \equiv [\frac{r(p)^{\frac{\widehat{\beta}_{22}}{\widehat{\beta}_{11}}}}{B_1\varepsilon^{1+\frac{\widehat{\beta}_{22}}{\widehat{\beta}_{11}}}}]^{\frac{1-\sigma}{\sigma}}[(\sigma-\beta_{21})r(p)+\beta_{21}\rho], \text{ where} \tag{39}$$

$$B_1 \equiv \left(\frac{\widehat{\beta}_{11}}{\beta_{11}}\right)^{\widehat{\beta}_{21}\widehat{\beta}_{22}}\left(\frac{\widehat{\beta}_{12}}{\beta_{12}}\right)^{\widehat{\beta}_{12}\widehat{\beta}_{11}}\left(\frac{\widehat{\beta}_{21}}{\beta_{21}}\right)^{\widehat{\beta}_{21}\widehat{\beta}_{22}}\left(\frac{\widehat{\beta}_{22}}{\beta_{22}}\right)^{\widehat{\beta}_{11}\widehat{\beta}_{22}}$$

The first parenthesis in (37) denotes the home BGP excess demand for the consumption good (good 2) and the rest of (37) is the foreign BGP excess demand.

In order for all BGP quantities are positive, it is necessary that $\lambda(p)$ is positive. Thus, considering (33), we have to make the following assumption.

Assumption 3: $sign[\frac{1}{\beta_{22}}(1+\sigma\bar{\beta}_{12})] = sign[\Delta+\sigma\beta_{12}] > 0$

Note that (6) can be rewritten as

$$Y_1/S = w\bar{\beta}_{22}l - r\bar{\beta}_{12}k$$

$$Y_2/S = \frac{r\bar{\beta}_{11}k - w\bar{\beta}_{21}l}{p}$$

If $m = b = 1$, it follows from (26a) and the definition of R that $Y_1/S = Y_1^*/S = p^{1-\frac{1}{\sigma}}\lambda$ and $2Y_2/S = 2Y_2^*/S = R(p, 2l)$. Therefore, if $\lambda > 0$ and $r > \rho$ along a BGP, the production in both countries are incompletely specialized along it at least for m and b that are sufficiently close to 1. In the next section we study under what conditions such BGP's exist.

4 The existence of BGP's

Now, let us investigate under what conditions (37) has a solution p. First, it is clear from (9) that if there exists r that satisfies (37), we have the solution p. Second, suppose that $1/\sigma$ and $\frac{1-\eta-\sigma}{\sigma\eta}$ are such that, if a is negative, $a^{1/\sigma}$ and $a^{\frac{1-\eta-\sigma}{\sigma\eta}}$ are imaginary numbers[7]. Since imaginary numbers make no economic sense, we focus only on a solution such that

$$r_2 \equiv \frac{(1-\eta)\rho}{1-\eta-\sigma} > r > \rho.$$

Third, we have to take into account the transversality condition (17), which is satisfied along a BGP only if

$$g - \sigma g - \rho \leq 0,^{8}$$

and if

$$g - \sigma g - \rho < 0$$

Since (24) and (26d) imply that $g = R = \frac{1}{\sigma}(r-\rho)$, this strict inequality holds if and only if

$$r < \frac{\rho}{1-\sigma},$$

which is smaller than $r_2 \equiv \frac{(1-\eta)\rho}{1-\eta-\sigma}$. That is, we have to find a solution of (37) such that

$$\rho < r(p) < \frac{\rho}{1-\sigma}$$

Based on the foregoing preliminary remarks, let us first consider the graph of (38). We have the following lemma.

Lemma 1: (i) $\Phi'(\overline{r}) = 0$ *has a unique solution* $\overline{r}$ *in* (ρ, r_2). (ii) $\rho < \overline{r} < r_0 \equiv \rho/(1-\sigma) < r_2$.

Proof. See Appendix 8.1.

Thus, the graph of $\Phi(r)$ is U-shaped with $\lim_{r\to\rho+}\Phi(r) = \lim_{r\to r_2-}\Phi(r) = \infty$. Moreover, $\Phi'(r_0) > 0$. See Figure 1.

Next, let us consider the graph of (39). First, we see that

$$\Psi_\varepsilon(0) = \Psi_\varepsilon(\frac{\beta_{21}\rho}{\beta_{21}-\sigma}) = 0 \text{ and } \Psi_\varepsilon(r) > 0 \text{ for } r \in (0, \frac{\beta_{21}\rho}{\beta_{21}-\sigma})$$

[7] For example, if $1/\sigma$ is 2.5 and $a = -1$, then $a^{1/\sigma} = (-1)^2 \times (-1)^{0.5} = i$, a pure imaginary number.

[8] Recall that Λ and M grow at $-\sigma g$.

Note that Assumption 1 guarantees that

$$r_2 \equiv \frac{(1-\eta)\rho}{1-\eta-\sigma} < \frac{\beta_{21}\rho}{\beta_{21}-\sigma}$$

Second, for any particular $\bar{r} \in (0, \frac{\beta_{21}\rho}{\beta_{21}-\sigma})$ and for any positive $\bar{\Psi}$, we can choose a value of parameter ε such that

$$\Psi_\varepsilon(\bar{r}) = \bar{\Psi}$$

Therefore, we can choose a value of ε such that the graphs of $\Phi(r)$ and $\Psi_\varepsilon(r)$ intersects (at least) twice[9], say $\Phi(r_e) = \Psi_\varepsilon(r_e)$ and $\Phi(\bar{r}_e) = \Psi_\varepsilon(\bar{r}_e)$ in such a way that

$$\rho < r_e < \bar{r}_e < r_0$$

See Figure 2. It is clear from the figure that $\Phi'(r_e) - \Psi'_\varepsilon(r_e) < 0$ and $\Phi'(\bar{r}_e) - \Psi'_\varepsilon(\bar{r}_e) > 0$. Let us summarize the existence result obtained in this section.

PROPOSITION 1: *If b and m are sufficiently close to one, there is an open set of parameter values ε such that at least two BGP's exist between ρ and r_0.*

Remark 1: Both r_e and $\bar{r}_e$ are greater than ρ, and λ is positive. Therefore, as we already remarked, as long as m and b are close to one, all BGP quantities including outputs in both countries are positive.

Remark 2: It is clear that the existence result holds at least for any m closed to unity. Therefore, we now obtain the continuum of BGP's each of which corresponds to a different value of m.

5 The stability of BGP's

Now, let us study the local stability of the continuum of BGP's. For simplicity, we focus on the symmetric BGP such that $b = m = 1$. Let us denote by p_e (resp. $\bar{p}_e$) the solution to $r(p) = r_e$ (resp. $r(p) = \bar{r}_e$). Differentiating (37) with respect to p and inspecting Figure 2, we have

$$\begin{aligned} sign[\frac{d}{dp}F^c\Big|_{p=p_e}] &= sign[r'(p_e)[\Phi'(r(p_e)) - \Psi'_\varepsilon(r(p_e))]] \\ &= -sign[r'(p_e)] = -sign[\hat{\Delta}] \end{aligned} \tag{40}$$

[9] Thus, r_e is the least one and $\bar{r}_e$ is the largest one. Note that $\Psi'_\varepsilon(r) = 0$ if and only if

$$\begin{aligned} r &= \frac{\rho\beta_{21}}{(\beta_{21}-\sigma)} \frac{\hat{\beta}_{22}(1-\sigma)}{\{\sigma\hat{\beta}_{11} + (1-\sigma)\hat{\beta}_{22}\}} \\ &< \frac{\rho\beta_{21}}{(\beta_{21}-\sigma)} \end{aligned}$$

Therefore, the intersections of $\Phi(r)$ and $\Psi_\varepsilon(r)$ are exactly two if $\frac{\hat{\beta}_{22}(1-\sigma)}{\{\sigma\hat{\beta}_{11}+(1-\sigma)\hat{\beta}_{22}\}}$ takes on some appropriate value.

$$\begin{aligned} sign[\frac{d}{dp}F^c\Big|_{p=\bar{p}_e}] &= sign[r'(\bar{p}_e)[\Phi'(r(\bar{p}_e)) - \Psi'_\varepsilon(r(\bar{p}_e))]] \\ &= sign[r'(\bar{p}_e)] = sign[\hat{\Delta}] \end{aligned} \tag{41}$$

Consider the Jacobian matrix of (26a)-(26h) evaluated at the BGP.

$$\Gamma \equiv \begin{bmatrix} \Gamma_{11} & \Gamma_{12}^T \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix},$$

where

$$\begin{aligned} \Gamma_{11} &\equiv \frac{\partial F^c}{\partial p} \\ \Gamma_{12}^T &\equiv (\frac{\partial F^c}{\partial l}, \frac{\partial F^c}{\partial l^*}, \frac{\partial F^c}{\partial \lambda}, \frac{\partial F^c}{\partial \mu}, \frac{\partial F^c}{\partial \mu^*}, \frac{\partial F^c}{\partial k}) \\ \Gamma_{21}^T &\equiv (\frac{\partial F^l}{\partial p}, \frac{\partial F^{l^*}}{\partial p}, \frac{\partial F^\lambda}{\partial p}, \frac{\partial F^\mu}{\partial p}, \frac{\partial F^{\mu^*}}{\partial p}, \frac{\partial F^k}{\partial p}) \\ \Gamma_{22} &\equiv \begin{bmatrix} \frac{\partial F^l}{\partial l} & \frac{\partial F^l}{\partial l^*} & \frac{\partial F^l}{\partial \lambda} & \frac{\partial F^l}{\partial \mu} & \frac{\partial F^l}{\partial \mu^*} & \frac{\partial F^l}{\partial k} \\ \frac{\partial F^{l^*}}{\partial l} & \frac{\partial F^{l^*}}{\partial l^*} & \frac{\partial F^{l^*}}{\partial \lambda} & \frac{\partial F^{l^*}}{\partial \mu} & \frac{\partial F^{l^*}}{\partial \mu^*} & \frac{\partial F^{l^*}}{\partial k} \\ \frac{\partial F^\lambda}{\partial l} & \frac{\partial F^\lambda}{\partial l^*} & \frac{\partial F^\lambda}{\partial \lambda} & \frac{\partial F^\lambda}{\partial \mu} & \frac{\partial F^\lambda}{\partial \mu^*} & \frac{\partial F^\lambda}{\partial k} \\ \frac{\partial F^\mu}{\partial l} & \frac{\partial F^\mu}{\partial l^*} & \frac{\partial F^\mu}{\partial \lambda} & \frac{\partial F^\mu}{\partial \mu} & \frac{\partial F^\mu}{\partial \mu^*} & \frac{\partial F^\mu}{\partial k} \\ \frac{\partial F^{\mu^*}}{\partial l} & \frac{\partial F^{\mu^*}}{\partial l^*} & \frac{\partial F^{\mu^*}}{\partial \lambda} & \frac{\partial F^{\mu^*}}{\partial \mu} & \frac{\partial F^{\mu^*}}{\partial \mu^*} & \frac{\partial F^{\mu^*}}{\partial k} \\ \frac{\partial F^k}{\partial l} & \frac{\partial F^k}{\partial l^*} & \frac{\partial F^k}{\partial \lambda} & \frac{\partial F^k}{\partial \mu} & \frac{\partial F^k}{\partial \mu^*} & \frac{\partial F^k}{\partial k} \end{bmatrix}, \end{aligned}$$

where the superscript T attached to a vector means that the row (resp. column) vector is transposed to the column (reap. row) vector. The linearized dynamical system around the BGP corresponding to $r_{bgp}(\equiv r_e$ or $\bar{r}_e$. All variables with "bgp" are their BGP values) is described as

$$\begin{bmatrix} 0 \\ \overset{\bullet}{X} \end{bmatrix} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12}^T \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} \begin{bmatrix} x_p \\ X \end{bmatrix}, \tag{42}$$

where $x_p \equiv p - p_{bgp}$ and

$$\begin{aligned} X^T &\equiv \begin{bmatrix} x_l & x_{l^*} & x_\lambda & x_\mu & x_{\mu^*} & x_k \end{bmatrix} \\ &\equiv \begin{bmatrix} l - l_{bgp} & l^* - l^*_{bgp} & \lambda - \lambda_{bgp} & \mu - \mu_{bgp} & \mu^* - \mu^*_{bgp} & k - k_{bgp} \end{bmatrix} \end{aligned}$$

First, we show how $\frac{d}{dp}F^c$ can be described by Γ_{ij}. Totally differentiating the system

$$\begin{aligned} F^c &= F^c(p, X) \\ 0 &= F^X(p, X) \end{aligned}$$

with respect F^c, p and X, we get

$$\begin{bmatrix} dF^c \\ 0 \end{bmatrix} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12}^T \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} \begin{bmatrix} dp \\ dX \end{bmatrix},$$

which yields

$$\frac{d}{dp} F^c = \Gamma_{11} - \Gamma_{12}^T \Gamma_{22}^{-1} \Gamma_{21} \tag{43}$$

it follows from (40) and (41) that

$$\begin{aligned} sign[\Gamma_{11} - \Gamma_{12}^T \Gamma_{22}^{-1} \Gamma_{21}]_{p=p_e} &= -sign[\Gamma_{11} - \Gamma_{12}^T \Gamma_{22}^{-1} \Gamma_{21}]_{p=\bar{p}_e} \\ &= -sign[\hat{\Delta}] \end{aligned} \tag{44}$$

Lemma 2: *If* $\Gamma_{11} \neq 0$, *the characteristic equation is described as*

$$J(\xi) \equiv \det \begin{bmatrix} \Gamma_{11} & \Gamma_{12}^T \\ \Gamma_{21} & \Gamma_{22} - \xi I \end{bmatrix} = 0, \tag{45}$$

where I *is the* 5×5 *identity matrix.*

Proof: See Appendix 8.2.

To simplify the characteristic equation, we use the BGP equations that are obtained from (34), (35) and (36):

$$\begin{aligned} \frac{r_{bgp} - \rho}{\sigma} &= 2(p_{bgp})^{1-\frac{1}{\sigma}} \lambda_{bgp} = \eta^A (l/\mu)^{\sigma A} = R(p_{bgp}, 2l_{bgp}) = g_{bgp} && (46) \\ 0 &= (\mu/\lambda)^{\sigma} w(p_{bgp}) - \{\rho - (1 - \eta - \sigma) R(p_{bgp}, 2l_{bgp})\} && (47) \end{aligned}$$

where $R(p_{bgp}, 2l_{bgp}) = 2l_{bgp} w(p_{bgp}) \overline{\beta}_{22} - r(p_{bgp}) \overline{\beta}_{12}$.Based on the equations, $J(\xi)$ becomes[10]

$$\begin{vmatrix} \Gamma_{11} & w\overline{\beta}_{21} & w\overline{\beta}_{21} & 2p^{1-\frac{1}{\sigma}} & 0 & 0 & 0 \\ -lR_p & \sigma AR - wl\overline{\beta}_{22} - \xi & -wl\overline{\beta}_{22} & 0 & -\frac{l}{\mu}\sigma AR & 0 & 0 \\ -lR_p & -wl\overline{\beta}_{22} & \sigma AR - wl\overline{\beta}_{22} - \xi & 0 & 0 & -\frac{l}{\mu}\sigma AR & 0 \\ \lambda[\frac{r'}{\sigma} - R_p] & -w\lambda\overline{\beta}_{22} & -w\lambda\overline{\beta}_{22} & -\xi & 0 & 0 & 0 \\ \mu[\frac{Gw'}{\sigma w} - R_p] & \frac{\mu}{l}\Omega & -w\mu\overline{\beta}_{22} & -\frac{\mu}{\lambda}G & \overline{G} - \xi & 0 & 0 \\ \mu[\frac{Gw'}{\sigma w} - R_p] & -w\mu\overline{\beta}_{22} & \frac{\mu}{l}\Omega & -\frac{\mu}{\lambda}G & 0 & \overline{G} - \xi & 0 \\ \frac{\partial F^k}{\partial l} & \frac{\partial F^k}{\partial l^*} & \frac{\partial F^k}{\partial l^*} & \frac{\partial F^k}{\partial \lambda} & \frac{\partial F^k}{\partial \mu} & \frac{\partial F^k}{\partial \mu^*} & Q - \xi \end{vmatrix} \tag{48}$$

where

$$\begin{aligned} G &\equiv \rho - (1 - \eta - \sigma) R \\ \overline{G} &\equiv G - (1 - \eta) AR \\ \Omega &\equiv (1 - \eta) AR - wl\overline{\beta}_{22} \\ Q &\equiv r(p)\bar{\beta}_{11} - 2lw(p)\bar{\beta}_{22} \end{aligned}$$

[10] For notational brevity, we abbreviate"bgp" in what follows.

We shall prove under what conditions $J(\xi) = 0$ has at least two roots with negative real parts, in which case the continuum of an equilibrium path from given initial stock $(l(0), l^*(0), k(0))$ converges to a BGP in the continuum.

Suppose that $J(\xi) = 0$ has just two roots with negative real parts and four roots with positive real parts. Then, for each initial point $(l(0), l^*(0), k(0))$ in the neighborhood of the symmetric BGP there exists a two dimensional manifold such that an equilibrium path from the initial point and along the manifold converges to a BGP in the continuum which possibly differs from the symmetric BGP. Thus, as long as an initial point is in the neighborhood of the symmetric BGP, there exists an equilibrium path converging to a BGP in the continuum. We call this case **TYPE-I stability**.

Suppose that $J(\xi)$ has three roots with negative real parts. for each initial point $(l(0), l^*(0), k(0))$ in a neighborhood of the symmetric BGP there exists $(\lambda(0), \mu(0), \mu(0)^*)$ such that a dynamic equilibrium path from the initial point converges to the symmetric BGP. Even if we choose a asymmetric BGP that corresponds to a value of m sufficiently close but not equal to 1, the characteristic equation has three roots with negative real parts. It follows that for the same $(l(0), l^*(0), k(0))$ there exists another $(\lambda(0), \mu(0), \mu^*(0))$ such that an equilibrium path from the initial point converges to the continuum. There are infinitely many equilibrium paths from each initial point $(l(0), l^*(0), k(0))$. We call this case **TYPE-II stability**.[11]

Making some tedious calculations, we can factorize $J(\xi)$ as follows[12].

$$J(\xi) = -2p^{1-\frac{1}{\sigma}}\lambda(Q-\xi)J_1(\xi)J_2(\xi), \tag{49}$$

where

$$J_1(\xi) \equiv \xi^2 - \{\rho - (1-\sigma)R\}\xi + \sigma ARG \tag{50}$$

$$J_2(\xi) \equiv \begin{vmatrix} [\frac{r'}{\sigma} - R_p] + \frac{\xi\Gamma_{11}}{2\lambda p^{1-\frac{1}{\sigma}}} & -2wl\overline{\beta}_{22} + \frac{\xi wl\overline{\beta}_{21}}{\lambda p^{1-\frac{1}{\sigma}}} \\ \begin{array}{l} [\frac{Gw'}{w} - r']AR \\ +\frac{(G-\xi)\Gamma_{11}\sigma AR}{2\lambda p^{1-\frac{1}{\sigma}}} \\ -(\overline{G}-\xi)R_p \end{array} & \begin{array}{l} (G-\xi)(\sigma AR - 2wl\overline{\beta}_{22} - \xi) \\ +(1-\eta)AR(2wl\overline{\beta}_{22} + \xi) \\ +\frac{(G-\xi)wl\overline{\beta}_{21}\sigma AR}{\lambda p^{1-\frac{1}{\sigma}}} \end{array} \end{vmatrix} \tag{51}$$

Let us check the characteristic roots corresponding to each factor in (49). First, consider the factor $J_1(\xi)$. PROPOSITION 1 states that the two BGP rental rates, r_e and $\bar{r}_e$, are in between ρ and $r_0 \equiv \rho/(1-\sigma)$, and Lemma 1 ensures us that $r_2 > r_0$. It follows from ASSUMPTION 1, (26d), i.e., $R = \frac{1}{\sigma}(r - \rho)$, and

[11] As we mentioned in the introductory section, this case corresponds to local indeterminacy discussed in the macroeconomic literature.

[12] The derivation of (49) is available from the authors on request.

the definition of G that

$$\begin{aligned} \rho - (1-\sigma)R &= \frac{(1-\sigma)}{\sigma}(r_0 - r) > 0 \\ G &\equiv \rho - (1-\eta-\sigma)R \\ &= \frac{(1-\eta-\sigma)}{\sigma}(r_2 - r) > 0 \end{aligned}$$

Therefore, the definition of $J_1(\xi)$ leads us to the results that (i) $J_1(0) = \sigma ARG > 0$ and (ii) $J_1(\xi) = 0$ has two roots with positive real parts.

Second, let us turn to the factor $J_2(\xi)$. Combining (45) and (49) at $\xi = 0$,

$$\begin{aligned} J(0) &= -2p^{1-\frac{1}{\sigma}}\lambda Q J_1(0) J_2(0) \\ &= \det \begin{bmatrix} \Gamma_{11} & \Gamma_{12}^T \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} \\ &= \det[\Gamma_{22}] \det[\Gamma_{11} - \Gamma_{12}^T \Gamma_{22}^{-1} \Gamma_{21}] \end{aligned}$$

or

$$-2p^{1-\frac{1}{\sigma}}\lambda Q J_1(0) J_2(0) = \det[\Gamma_{22}] \det[\Gamma_{11} - \Gamma_{12}^T \Gamma_{22}^{-1} \Gamma_{21}]$$

It follows from $J_1(0) = \sigma ARG > 0$ that

$$sign[J_2(0)] = -sign[Q] sign[\det[\Gamma_{22}] \det[\Gamma_{11} - \Gamma_{12}^T \Gamma_{22}^{-1} \Gamma_{21}]] \tag{52}$$

Considering (44), we see that (52) can be rewritten to

$$sign[J_2(0)] = \begin{cases} \; sign[Q] sign[\det[\Gamma_{22}]] sign[\hat{\Delta}] \Big|_{p=p_e} \\ -\, sign[Q] sign[\det[\Gamma_{22}]] sign[\hat{\Delta}] \Big|_{p=\bar{p}_e} \end{cases} \tag{53}$$

Lemma 3:

$$sign[J_2(0)]_{p=p_e} = -sign[\Delta\hat{\Delta}] = -sign[J_2(0)]_{p=\bar{p}_e} \tag{54}$$

Proof. See Appendix 8.3.

5.1 CASE 1 : $r(\bar{p}_e) <$ and $\doteqdot r_0 (\equiv \frac{\rho}{1-\sigma})$

Let us choose the positive parameter ε very small so that $r(\bar{p}_e)$ is smaller than but very close to $r_0(\equiv \frac{\rho}{1-\sigma})$. First, Lemma 3 implies that if $\Delta\hat{\Delta} > 0$, then $J_2(0) > 0$ at $p = \bar{p}_e$.

Second, we can prove the following lemma.

Lemma 4: *If $\Delta\hat{\Delta} > 0$ and σ is sufficiently small, then $J_2'(0) > 0$ at $r(\bar{p}_e)$.*

Proof. See Appendix 8.4.

Third, let us check the sign of

$$\Gamma_{11} \equiv \frac{\partial F^c}{\partial p} = 2(1-\frac{1}{\sigma})p^{-\frac{1}{\sigma}}\lambda - \{r^{'}(p)\overline{\beta}_{11} - 2lw^{'}(p)\overline{\beta}_{21}\},$$

where (10) and (11) implies

$$\begin{aligned} sign[r^{'}(p)\overline{\beta}_{11} - 2lw^{'}(p)\overline{\beta}_{21}] &= \frac{sign[r^{'}(p)\beta_{11} - 2lw^{'}(p)\beta_{21}]}{sign[\Delta]} \\ &= -\frac{sign[\hat{\Delta}]}{sign[\Delta]} \\ &= -sign[\Delta\hat{\Delta}] \end{aligned}$$

It follows from $\Delta\hat{\Delta} > 0$ and $\sigma < 1$ that $\Gamma_{11} < 0$. Those three facts lead us to the lemma as follows.

Lemma 5: *Choose the positive parameter ε very small so that $r(p_e)$ is smaller than but very close to r_0. If* $\Delta\hat{\Delta} > 0$ and σ is sufficiently small, *then $J_2(\xi) = 0$ has one positive real root and two roots with negative real parts.*
Proof: See Appendix 8.5.

5.2 CASE $2 : r(p_e) >$ and $\doteqdot \rho$

Next, let us choose the positive parameter ε very small so that $r(p_e)$ is larger than but very close to ρ. Then $R \doteqdot 0$, $\bar{G} \fallingdotseq G \fallingdotseq \rho$, and $\Omega = -wl\overline{\beta}_{22}$. It follows from (51) that

$$\begin{aligned} J_2(\xi) &\doteqdot \begin{vmatrix} (\frac{r'}{\sigma} - R_p + \frac{\xi\Gamma_{11}}{2\lambda p^{1-\frac{1}{\sigma}}}) & -(2wl\overline{\beta}_{22} - \frac{\xi wl\overline{\beta}_{21}}{\lambda p^{1-\frac{1}{\sigma}}}) \\ (\xi-\rho)R_p & (\xi-\rho)(2wl\overline{\beta}_{22} + \xi) \end{vmatrix} \\ &= (\xi-\rho)[\frac{\Gamma_{11}}{2\lambda p^{1-\frac{1}{\sigma}}}\xi^2 + (\frac{r'}{\sigma} - R_p + \frac{wl\overline{\beta}_{22}\Gamma_{11} - wl\overline{\beta}_{21}R_p}{\lambda p^{1-\frac{1}{\sigma}}})\xi \\ &\quad +2\frac{r'}{\sigma}wl\overline{\beta}_{22}] \end{aligned}$$

Inspecting this equation and considering that

$$sign[\hat{\Delta}] = sign[r'] = -sign[w'] = -sign[R_p]sign[\Delta],$$

we see that if $\Gamma_{11} < 0$, $\Delta > 0$ and $\hat{\Delta} < 0$, then

$$\begin{aligned} sign[\frac{\Gamma_{11}}{2\lambda p^{1-\frac{1}{\sigma}}}] &= sign[\frac{r'}{\sigma} - R_p + \frac{wl\overline{\beta}_{22}\Gamma_{11} - wl\overline{\beta}_{21}R_p}{\lambda p^{1-\frac{1}{\sigma}}}] \\ &= sign[2\frac{r'}{\sigma}wl\overline{\beta}_{22}] < 0, \end{aligned}$$

which implies that the equation

$$\begin{aligned} 0 &= \frac{\Gamma_{11}}{2\lambda p^{1-\frac{1}{\sigma}}}\xi^2 + (\frac{r'}{\sigma} - R_p + \frac{wl\overline{\beta}_{22}\Gamma_{11} - wl\overline{\beta}_{21}R_p}{\lambda p^{1-\frac{1}{\sigma}}})\xi \\ & +2\frac{r'}{\sigma}wl\overline{\beta}_{22} \end{aligned}$$

has two roots with negative real parts. It follows that if we can show in CASE 2 that Γ_{11} is negative for a sufficiently small σ, then we arrive at the following lemma.

Lemma 6: *Choose the positive parameter ε very small so that $r(p_e)$ is larger than but very close to ρ. If $\Delta > 0$, $\hat{\Delta} < 0$* and σ is sufficiently small, *then $J_2(\xi) = 0$ has one positive real root that is close to ρ and two roots with negative real parts.*

Proof. See Appendix 8.6.

5.3 the sign of Q

What remains to investigate concerning stability is the sign of Q

$$Q \equiv r(p)\bar{\beta}_{11} - 2lw(p)\bar{\beta}_{22}$$

along a BGP. If $Q > 0$ under either of the above two cases, we have TYPE-l stability On the other hand, if $Q < 0$, we have TYPE-II stability.

First, considering $r - \rho = \sigma R(p, 2l) = \sigma\{2lw\bar{\beta}_{22} - r\bar{\beta}_{12}\}$, we have

$$\begin{aligned} Q &= r\bar{\beta}_{11} - r\bar{\beta}_{12} - \frac{1}{\sigma}(r - \rho) \\ &= \frac{1}{\sigma}[\sigma r(\bar{\beta}_{11} - \bar{\beta}_{12}) - (r - \rho)] \end{aligned}$$

Thus, suppose that there is no externality. Then $\beta_{ij} = \hat{\beta}_{ij}$ and $\Delta = \hat{\Delta} = \beta_{11} - \beta_{12}$, which implies that $\bar{\beta}_{11} - \bar{\beta}_{12} = \frac{\beta_{11} - \beta_{12}}{\Delta} = 1$. Therefore,

$$Q = \frac{(1 - \sigma)}{\sigma}[r_0 - r],$$

which has to be positive due to the transversality condition. That is, if no externality, we have TYPE-I stability at a BGP value $\bar{p}_e$ for which $r(\bar{p}_e)$ is close to r_0[13].

Second, suppose that

$$\beta_{11} - \beta_{12} < 0 \text{ and } \Delta > 0,$$

[13]Note that in the case of no externaity, we have $\Delta\hat{\Delta} = \hat{\Delta}^2 > 0$. Thus, the previous argument concerning the sign of ξ^2 means that we can say that in this case we have local determinacy for a BGP value $\bar{p}_e$ for which $r(\bar{p}_e)$ is close to r_0.

which are possible (only if there is externality) and compatible with $\hat{\Delta} > 0$. Then $Q < 0$ and we have TYPE-II stability at a BGP value p_e for which $r(p_e)$ is close to ρ[14].

Let us summarize the stability results obtained in this section.

PROPOSITION 2: (i) *If* $\Delta\hat{\Delta} > 0$ *and* σ is sufficiently small, then *there is an open interval of the parameter value of* ε *such that a BGP at* $\bar{p}_e$ *corresponding to each* ε *in the interval is TYPE-I stable and* $r(\bar{p}_e)$ *is smaller than but close to* r_0. (ii) *If* $\Delta > 0, \hat{\Delta} < 0$, $\beta_{11} - \beta_{12} < 0$, *and* σ is sufficiently small, *then there is an open interval of the parameter value* ε *such that a BGP at* p_e *corresponding to each* ε *in the interval is locally indeterminate and* $r(p_e)$ *is close to* ρ. *In the case of no externality, we never have TYPE-II stability.*

6 Characterizing the two types of BGP's

Based on the foregoing argument, we now have two types of locally stable BGP's.

(i) a TYPE-I BGP (IBGP), which is obtained if $\Delta\hat{\Delta} > 0$ and σ is sufficiently small..

(ii) a TYPE-II BGP (IIBGP), which is obtained if $\Delta > 0, \hat{\Delta} < 0$, $\beta_{11} - \beta_{12} < 0$, and σ is sufficiently small.

As we remarked at the end of the last section, in the standard case with no externality we could only have an IBGP.

Let us characterize each of the two types by using Table 1 and Figures 3 and 4. In what follows we focus on the case such that the "psychic costs" of generating new human capital is larger in the foreign country; that is, we assume $b < 1$.

[14] In the indeterminacy case,

$$Q = r\bar{\beta}_{11} - r\bar{\beta}_{12} - \frac{1}{\sigma}(r - \rho) < 0$$

Thus, the denominators of (29) and (30) are negative. One may wonder whether the numerators are also positive. Fortunatorily the answer is yes. For, using (27), we can rewrite the numerators as

$$\begin{aligned} & \sigma[(\bar{\beta}_{21} - \bar{\beta}_{22})wl + p^{1-\frac{1}{\sigma}}\lambda] \\ = \; & \sigma[(\bar{\beta}_{21} - \bar{\beta}_{22})wl + \frac{1}{2}(r\bar{\beta}_{11} - 2wl\beta_{21})] \\ = \; & \frac{\sigma}{2}[r\bar{\beta}_{11} - 2wl\bar{\beta}_{22}] \\ = \; & \frac{\sigma}{2}[\sigma r(\bar{\beta}_{11} - \bar{\beta}_{12}) - (r - \rho)], \end{aligned}$$

which implies that $k = 1/2 > 0$, as was to be obtained.

6.1 IBGP

We summarize the main results concerning IBGP in Figure 3 and the first column of Table 1. The point E is a IBGP.[15] Note that, depending on the value of m, there is a continuum of IBGP's from E^h to E^f, which are the home and foreign autarkic IBGP's, respectively. Which BGP is realized is determined by the historically given initial condition of all state variables.

6.1.1 trade pattern

It is clear from Figure 3 that $\Phi(r(\bar{p}_e)) > \Psi_\varepsilon(r(\bar{p}_e))$. That is, if $b < 1$, the home country exports Good 2, the consumption good, along any IBGP in the continuum. Intuition behind this result is that since the foreign household incurs a higher disutility than the home household in order to produce the same amount of new human capital, it wants to consume more instead of accumulating human capital. Thus, the foreign country imports the consumption good. Note that the factor-intensity ranking, from both of the private perspective (i.e., the sign of Δ) and the social perspective (i.e., the sign of $\hat{\Delta}$), does not affect the pattern of trade along any IBGP.

PROPOSITOIN 3: *The country whose psychic costs of generating new human capital is more expensive than the other country exports the investment good.*

6.1.2 the relationship between growth rates and the (relative) volume of trade

The long-run growth rate g^e is determined as

$$g^e = \frac{1}{\sigma}(r(\bar{p}_e) - \rho)$$

Thus, the higher is $r(\bar{p}_e)$ along a BGP, the higher the long-run growth rate. On the other hand, the vertical difference between $\Phi(r(\bar{p}_e))$ and $\Psi_\varepsilon(r(\bar{p}_e))$ is the BGP volume of trade of good 2 (the consumption good) divided by the total capital stock in the world economy, $S = K + K^*$, which we thereafter call the *relative* volume of trade. Hence, Figure 3 shows that the higher is $r(\bar{p}_e)$ along a BGP, the higher is the relative volume of trade. The following proposition holds.

PROPOSITION 4: *The higher is the long-run growth rate, the larger is the foreign relative volume of the consumption good.*

[15] Point E corresponds to one of the intersections of $\Phi(r)$ and $\frac{(1+m^{-1-\sigma}b^{-\frac{1+\sigma}{\sigma A}})}{1+m^{-1-\sigma}}\Psi_\varepsilon(r)$ in Figure 2, $r(\bar{p}_e)$.

6.1.3 the law of comparative advantage and the Heckscher-Ohlin relationship

According to Proposition 3, if one country can produce new human capital less efficiently than the other country, then the former country imports the consumption good. That is, the trade pattern depends on whether b is larger or smaller than one. On the other hand, it is recognized in the standard trade theory that trade pattern is closely related to (i) the international differences in (i) autarkic prices and (ii) BGP factor endowment ratios.

the law of comparative advantage If the autarkic BGP price of a tradable

good in terms of the other good is lower than the foreign autarkic price, then it is said that the home country has comparative advantage in the tradable good. The law of comparative advantage is that *each country exports a good in which the country has comparative advantage.*

Is PROPOSITION 3 compatible with the law of comparative advantage? See Figure 3, where $\bar{p}_e^h$ and $\bar{p}_e^f$ are the BGP autarkic prices of the consumption good in the home and foreign countries respectively. PROPOSITION 3 asserts that the home country exports good 2, i.e., the consumption good. Since $r(\bar{p}_e^h) < r(\bar{p}_e) < r(\bar{p}_e^f)$ as is shown in Figure 3, the law of comparative advantage, $\bar{p}_e^h < \bar{p}_e^f$, holds iff the consumption good is capital intensive from the social perspective, *i.e.*, $r'(p) > 0$.

the Heckscher-Ohlin trade pattern The international difference in the

factor endowment ratios plays an important role to determine the long-run trade pattern in the standard trade theory. The Heckscher-Ohlin trade pattern is that *each country exports a good whose production intensively uses the factor of production which is relatively abundant in the country.* Let us compare the home and foreign factor endowment ratio along an IBGP. From (27)-(30) we see that

$$\begin{aligned}\frac{k^*(\bar{p}_e)}{l^*(\bar{p}_e)} &= \frac{\sigma\{w(\bar{p}_e)(\bar{\beta}_{21}-\bar{\beta}_{22}) + b^{\frac{1+A}{\sigma A}}(\bar{p}_e)^{1-\frac{1}{\sigma}}w(\bar{p}_e)^{1/\sigma}\Xi(r(\bar{p}_e))^{-1}\}}{\sigma r(\bar{p}_e)\bar{\beta}_{11} - (1+\sigma\bar{\beta}_{12})(r(\bar{p}_e)-r_1)} \\ &< (resp. >)\frac{k(\bar{p}_e)}{l(\bar{p}_e)}, \quad \text{if } b < (resp. >)1.\end{aligned}$$

That is, $\frac{k^*(\bar{p}_e)}{l^*(\bar{p}_e)}$ is smaller (resp. larger) than $\frac{k(\bar{p}_e)}{l(\bar{p}_e)}$ if b is smaller (resp. larger) than one. Thus, we can say that under the condition that b is smaller (resp. larger) than one, the Heckscher-Ohlin trade pattern is established if the consumption good is more physical capital (resp. human capital) intensive, i.e., if $r'(\bar{p}_e) >$ (resp. $<$)0.

Summarizing the foregoing argument, we see that

- if $b <$ (resp. $>$)1 and $r'(\bar{p}_e) >$ (resp. $<$)0, then both the law of comparative advantage and the Heckscher-Ohlin trade pattern hold.
- if $b <$ (resp. $>$)1 and $r'(\bar{p}_e) <$ (resp. $>$)0, then neither the law of comparative advantage nor the Heckscher-Ohlin trade pattern holds.

We now arrive at the following proposition.

PROPOSITION 5: *The Heckscher-Ohilin trade pattern and the law of comparative advantage holds together if* $sign(r'(\bar{p}_e)) = sign(1-b)$. *Neither of them holds if* $sign(r'(\bar{p}_e)) = -sign(1-b)$.

6.1.4 factor price equalization, Stolper-Samuelson and Rybczynski

Recalling the production structure stated in Section 2, we see that factor price equalization holds. Comparing IBGP's in the above continuum, it is clear that the Stolper-Samuelson theorem holds concerning the social factor-intensity ranking (See (9)) while the Rybczynski theorem holds concerning the private factor-intensity ranking (See (6)).

6.2 IIBGP

We summarize the main results concerning IIBGP in Figure 4 and the second column of Table 1. The point F is a IIBGP. Note that, depending on the value of m, there is a continuum of IIBGP's from F^h to F^f, which are the home and foreign autarkic IBGP's, respectively. Which BGP is realized is determined by the historically given initial condition of all state variables.

6.2.1 trade pattern

However, whichever IIDBGP is realized, the same pattern of trade is realized. As is depicted in Figure 4, along any IIDBGP on F^hF^f the home country always exports the consumption good, if $b < 1$. The trade pattern is reversed if $b > 1$; PROPOSITION 3 holds whether IBGP or IIBGP.

6.2.2 the relationship between growth rates and the (relative) volume of trade

On the other hand, comparing Figures 3 and 4, we see that the relationship between growth rates and the (relative) volume of trade in the continuum of IIBGP's is opposite to that in the continuum of IBGP's. In the former continuum the higher is the BGP $r(p_e)$, i.e., the higher is the growth rate, the smaller is the relative volume of trade of good 2.

6.2.3 the law of comparative advantage and the Heckscher-Ohlin relationship

Note that in the equality,

$$\frac{k^*(p_e)}{l^*(p_e)} = \frac{\sigma\{w(p_e)(\bar{\beta}_{21} - \bar{\beta}_{22}) + b^{\frac{1+A}{\sigma A}}(p_e)^{1-\frac{1}{\sigma}} w(p_e)^{1/\sigma}\Xi(r(p_e))^{-1}\}}{\sigma r(p_e)\bar{\beta}_{11} - (1+\sigma\bar{\beta}_{12})(r(p_e) - r_1)},$$

both the denominator and the numerator are negative along a IIBGP. Therefore, $b < (resp. >)1$ means that

$$\frac{k^*(p_e)}{l^*(p_e)} > (resp. <)\frac{k(p_e)}{l(p_e)}$$

It follows that if $r'(p_e) < (resp. >)0$, then the home country exports the good for which she has comparative advantage and the trade pattern is Heckscher-Ohlin. We have the following result.

PROPOSITION 5′ : *Trade pattern is Heckscher-Ohlin and the law of comparative advantage holds together if* $sign(r'(\bar{p}_e)) = -sign(1-b)$.

6.2.4 Factor price equalization, Stolper-Samuelson and Rybczynski

There is no difference in those three issues between IBGP and IIBGP.

7 Concluding Remarks

In this paper we have tried to provide a dynamic trade model of international trade and endogenous growth that can be regarded as a development of the conventional two by two by two Heckscher-Ohlin model. While we are not the first to explore an endogenous growth trade theory, we believe that we first provide a dynamic trade model with constant-returns-to-scale technologies in which both existence and stability of balanced growth paths with incomplete specialization in both countries are established. Moreover, we show that under some conditions concerning production technologies and externalities a continuum of indeterminate balanced growth paths is possible and that any balanced growth path in the continuum has the same pattern of international trade.

In international trade theory the effects of economic growth on international trade have been an important issue for a long time. However, the papers that study multi-country dynamic general equilibrium models of international trade are still few. It may be informative to the reader to compare the present paper with some of recent works on this topic. Let us compare the present paper with Bond, Trask, and Wang (BTW) [5]and the two papers by Nishimura and Shimomura (NS) ([9], [10]).

As is shown there, BTW [5] and the present paper deal with endogenous growth models, while the two Nishimura and Shimomura papers study exogenous growth models. As to production externality, while NS (2002) and the present paper assume factor-generated externalities, BTW [5] and NS [10] assume away such distortions.

Next, let us compare the implications of those models. First, while it is possible in NS [10] that there is a continuum of equilibrium paths in a neighborhood of a steady state, BTW [5] concentrate on characterizing balanced growth paths.

Second, both NS [9] and the present paper derive the local indeterminacy.

Third, let us compare trade pattern among the four papers. BTW [5] define static and dynamic Heckscher-Ohlin trade patterns; static trade pattern means that at each point in time the relationship between the difference in factor endowment ratios and trade pattern is as suggested by the Heckscher-Ohlin theorem in the standard trade theory; dynamic trade pattern means that the relationship between the difference in factor endowment ratios *at a point in time* and trade pattern after that point in time is Heckscher-Ohlin. In the case of the BTW model, static Heckscher-Ohlin trade pattern does not necessarily hold because in the model there is an educational sector which produces a nontraded good called educational services which uses physical and human capital; thus for example, under the assumption that the production of educational services is physical capital intensive, if the physical capital abundant country produce more educational services than the other country, it is possible that the ratio of the amounts of physical and human capital stocks that can be used in the investment and consumption goods in the former country is smaller than the other country, in which case static H-O trade pattern does not hold. If static H-O pattern does not hold, neither does dynamic H-O pattern.

On the other hand, both NS [9] and NS [10] satisfy static H-O trade pattern. However, if indeterminacy takes place, dynamic H-O trade pattern may not hold. See Figure 5. In those papers there is a continuum of steady state, say ABC. Indeterminacy in those models mean that as long as the initial international distribution of physical and human capital stocks is in a neighborhood of each steady state, an equilibrium path starting from the initial distribution converges to the steady state. In a word, it is possible that an equilibrium path can cross the 45-degree line like DEB. Inspecting this equilibrium path, we see that while the home country is relatively capital abundant near the initial point D and therefore it exports a capital intensive good, after crossing the 45-degree line at E the country becomes relatively labor abundant country and starts exporting a labor intensive good. Thus, although static H-O trade pattern holds, dynamic H-O trade pattern does not hold along the equilibrium path.

Compared with the existing three papers, it is ambiguous whether static and dynamic H-O pattern holds if $b = 1$. If b is not equal to one, the BGP trade pattern is clearly obtained. For exempt, if $b < 1$, i.e., if the foreign household incur higher disutility (pain) than the home household in order to produce a given amount of new human capital, the foreign household will choose more consumption good and less psychic cost. Therefore, the home household exports the investment good.

Trade pattern is still ambiguous along a transitional path when $b = 1$. However, based on the results from the exogenous growth model in the NS [9], we may conjecture that dynamic H-O pattern do not necessarily hold when indeterminacy takes place.

In this paper we assume that the human capital is augmented by the effort of households involving "psychic cost" or disutility. Alternatively, one could assume that to accumulate human capital, the household incur a pecuniary cost or time cost. It is certainly interesting to pursue the same topic as we did in this paper and to compare the results with ours. In doing so, it may be crucial how those costs are formulated in dynamic trade models. For example, let us incorporate a pecuniary cost in such way that the flow budget constraint (13) and the dynamic equation of human capital (14) are rewritten as

$$\overset{\bullet}{K} = wL + rK + \Pi - pC - h$$

$$\overset{\bullet}{L} = h^{\eta} L^{1-\eta},$$

where h denotes the pecuniary cost measured by the investment good, and reformulate the discounted sum of utility as

$$\int_0^{\infty} \frac{C^{1-\sigma}}{1-\sigma} e^{-\rho t} dt, 0 < \sigma < 1, \rho > 0$$

Then, one may find that the dynamic general equilibrium model based on the pecuniary cost is formally almost the same as the one developed in this paper; one could then conjecture that the analytical implications are also similar.

The model presented in this paper can be thought as a benchmark model in the sense that, just like the conventional static two by two by two Heckscher-Ohlin model, there seem to be many directions of extending it. Introducing factor-generated externalities, which we have done here, is only one of them. Imperfect competition, output-generated externalities, increasing returns, trade policies, ... are other issues we may be able to incorporate into the present benchmark model,which are our future research agenda.

References

[1] Benhabib, J., Farmer, R., Indeterminacy and sunspots in macroeconomics, Chapter 6, J.B.Tayor and M.Woodford eds., *Handbook of Macroeconomics* Vol.1, 387-448, Amsterdam: North-Holland, (1999).

[2] Benhabib, J., Meng, Q., Nishimura, K.: Indeterminacy under constant returns to scale in multisector economies. *Econometrica* 68, 1541-1548 (2000).

[3] Benhabib, J., Nishimura, K.: Indeterminacy and sunspots with constant returns. *Journal Economic Theory* 81, 58-96 (1998).

[4] Bond, E., Wang, P, Yip, C., A general two-sector model of endogenous growth with human and physical capital: balanced growth paths and transitional dynamics, *Journal of Economic Theory* 68, 149-173, (1996).

[5] Bond, E., Trask, K., Wang, P.: Factor accumulation and trade pattern: dynamic comparative advantage with endogemous human capital, Vanderbilt University, *International Economic Review* 44, 1041-1060, (2002).

[6] Chen, Z.: Long-run equilibria in a dynamic Heckscher-Ohlin model, *Canadian Journal. Economics* 25, 923-943, (1992).

[7] Farmer, R., and Lahiri, A.: A two-country model of endogenous growth, *Review of Economic Dynamics* 8, 68-88, 2005.

[8] Nishimura, K., Shimomura, K.: Indeterminacy in a dynamic small open economy, Kobe University, *Journal of Economic Dynamics & Control* 27, 271-281 (2002).

[9] Nishimura, K., Shimomura, K., Trade and indeterminacy in a dynamic general equilibrium model, *Journal of Economic Theory* 105, 244-60 (2002).

[10] Nishimura, K., Shimomura, K., Indeterminacy in a dynamic two-country model, to appear in *Economic Theory* (2006).

[11] Shimomura, K: A two-sector dynamic general equilibrium model of distribution. *Dynamic Economic Models and Optimal Control*, ed. by G. Feichtinger. North-Holland (1992).

[12] Ventura, J., Growth and Interdependence, *Quarterly Journal of Economics* 112, 57-84 (1997).

8 Appendix

8.1 The proof of Lemma 1

(i) Logarithmically differentiating $\Phi(r)$ with respect to r, we see that

$$\begin{aligned}\frac{d\ln\Phi(r)}{dr} &= \frac{\Phi'(r)}{\Phi(r)} \\ &= \frac{1}{r-r_1}+\frac{1}{\sigma(r-r_2)}-\frac{1-\eta-\sigma}{\sigma\eta(r-\rho)} \\ &= \frac{\sigma\eta(r-r_2)(r-\rho)-\eta(r-r_1)(r-\rho)-(1-\eta-\sigma)(r-r_1)(r-r_2)}{\sigma\eta(r-r_1)(r-\rho)(r-r_2)} \qquad \text{(a1)}\end{aligned}$$

Since the numerator,

$$\sigma\eta(r-r_2)(r-\rho)-\eta(r-r_1)(r-\rho)-(1-\eta-\sigma)(r-r_1)(r-r_2),$$

is a quadratic equation of r, $\Phi'(r) = 0$ has at most two roots. On the other hand, $\lim_{r\to\rho+} \Phi(r) = \lim_{r\to r_2-} \Phi(r) = \infty$ implies that $\Phi'(r) = 0$ must have an odd number of roots. Therefore, $\Phi'(r) = 0$ has a unique solution in $(\rho\ , r_2)$. (ii) From (i), $\rho < \overline{r} < r_2$. It can be easily shown that $\rho < r_0 < r_2$.

Let us prove $\overline{r} < r_0$. Substituting r_0 into (a1), we have

$$\frac{1}{r_0 - r_1} + \frac{1}{\sigma(r_0 - r_2)} - \frac{1-\eta-\sigma}{\sigma\eta(r_0-\rho)} = \frac{(1-\sigma)(1+\sigma\overline{\beta}_{12})}{\rho\sigma(1+\overline{\beta}_{12})} > 0$$

Since $\Phi'(r) > 0$ for $r \in (\overline{r}, r_2)$ and $\Phi(\overline{r}) = 0$, $\overline{r} < r_0$ follows from $\Phi'(r_0) > 0$. (QED)

8.2 The Proof of Lemma 2

First, we note that (42) is rewritten as

$$\begin{aligned} 0 &= \Gamma_{11}x_p + \Gamma_{12}^T X \\ \overset{\bullet}{X} &= \Gamma_{21}x_p + \Gamma_{22}X \end{aligned}$$

or, since $\Gamma_{11} \neq 0$[16],

$$\begin{aligned} \overset{\bullet}{X} &= -\Gamma_{21}\Gamma_{11}^{-1}\Gamma_{12}^T X + \Gamma_{22}X \\ &= [\Gamma_{22} - \Gamma_{21}\Gamma_{11}^{-1}\Gamma_{12}^T]X \end{aligned}$$

Therefore, the characteristic equation is

$$\det\left[\Gamma_{22} - \Gamma_{21}\Gamma_{11}^{-1}\Gamma_{12}^T - \xi I\right] = 0 \tag{a2}$$

On the other hand, making use of an elementary theorem in linear algebra, we see that

$$\begin{aligned} &\det\begin{bmatrix} \Gamma_{11} & \Gamma_{12}^T \\ \Gamma_{21} & \Gamma_{22} - \xi I \end{bmatrix} \\ =\ &\det\begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ 0 & \Gamma_{22} - \Gamma_{21}\Gamma_{11}^{-1}\Gamma_{12}^T - \xi I \end{bmatrix} \\ =\ &\Gamma_{11}\det\left[\Gamma_{22} - \Gamma_{21}\Gamma_{11}^{-1}\Gamma_{12}^T - \xi I\right] \end{aligned} \tag{a3}$$

Since $\Gamma_{11} \neq 0$, it follows that all characteristic roots are the same between (a2) and (a3). (QED)

8.3 The Proof of Lemma 3

To prove the lemma, let us calculate $\det[\Gamma_{22}]$.

[16]Note that Γ_{11} is a scalar.

$$
\begin{aligned}
\det[\Gamma_{22}] &= \begin{vmatrix}
\sigma AR - wl\overline{\beta}_{22} & -wl\overline{\beta}_{22} & 0 & -\frac{l}{\mu}\sigma AR & 0 & 0 \\
-wl\overline{\beta}_{22} & \sigma AR - wl\overline{\beta}_{22} & 0 & 0 & -\frac{l}{\mu}\sigma AR & 0 \\
-w\lambda\overline{\beta}_{22} & -w\lambda\overline{\beta}_{22} & 0 & 0 & 0 & 0 \\
\frac{\mu}{l}\Omega & -w\mu\overline{\beta}_{22} & -\frac{\mu}{\lambda}G & \overline{G} & 0 & 0 \\
-w\mu\overline{\beta}_{22} & \frac{\mu}{l}\Omega & -\frac{\mu}{\lambda}G & 0 & \overline{G} & 0 \\
\frac{\partial F^k}{\partial l^*} & \frac{\partial F^k}{\partial l^*} & \frac{\partial F^k}{\partial \lambda} & \frac{\partial F^k}{\partial \mu} & \frac{\partial F^k}{\partial \mu^*} & Q
\end{vmatrix} \\
&= Q \\
&\quad \times \begin{vmatrix}
\sigma AR - wl\overline{\beta}_{22} & -wl\overline{\beta}_{22} & 0 & -\frac{l}{\mu}\sigma AR & 0 \\
-wl\overline{\beta}_{22} & \sigma AR - wl\overline{\beta}_{22} & 0 & 0 & -\frac{l}{\mu}\sigma AR \\
-w\lambda\overline{\beta}_{22} & -w\lambda\overline{\beta}_{22} & 0 & 0 & 0 \\
\frac{\mu}{l}\Omega & -w\mu\overline{\beta}_{22} & -\frac{\mu}{\lambda}G & \overline{G} & 0 \\
-w\mu\overline{\beta}_{22} & \frac{\mu}{l}\Omega & -\frac{\mu}{\lambda}G & 0 & \overline{G}
\end{vmatrix} \\
&= Q \\
&\quad \times \begin{vmatrix}
0 & 0 & 0 & -\frac{l}{\mu}\sigma AR & 0 \\
0 & 0 & 0 & 0 & -\frac{l}{\mu}\sigma AR \\
-w\lambda\overline{\beta}_{22} & -w\lambda\overline{\beta}_{22} & 0 & 0 & 0 \\
\frac{\mu}{l}\Omega + \frac{\mu}{l}\overline{G} & -w\mu\overline{\beta}_{22} & -\frac{\mu}{\lambda}G & \overline{G} & 0 \\
-w\mu\overline{\beta}_{22} & \frac{\mu}{l}\Omega + \frac{\mu}{l}\overline{G} & -\frac{\mu}{\lambda}G & 0 & \overline{G}
\end{vmatrix} \\
&= Q(\frac{l}{\mu}\sigma AR)^2 \\
&\quad \times \begin{vmatrix}
-w\lambda\overline{\beta}_{22} & -w\lambda\overline{\beta}_{22} & 0 \\
\frac{\mu}{l}\Omega + \frac{\mu}{l}\overline{G} & -w\mu\overline{\beta}_{22} & -\frac{\mu}{\lambda}G \\
-w\mu\overline{\beta}_{22} & \frac{\mu}{l}\Omega + \frac{\mu}{l}\overline{G} & -\frac{\mu}{\lambda}G
\end{vmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= Q(\frac{l}{\mu}\sigma AR)^2(-\frac{\mu}{\lambda}G) \\
&\quad \times \begin{vmatrix}
-w\lambda\overline{\beta}_{22} & -w\lambda\overline{\beta}_{22} & 0 \\
\frac{\mu}{l}\{-wl\overline{\beta}_{22} + \rho - (1-\eta-\sigma)R\} & -w\mu\overline{\beta}_{22} & 1 \\
-w\mu\overline{\beta}_{22} & \frac{\mu}{l}\{-wl\overline{\beta}_{22} + \rho - (1-\eta-\sigma)R\} & 1
\end{vmatrix} \\
&= Q(\frac{l}{\mu}\sigma AR)^2(\frac{\mu}{\lambda}G)(w\lambda\overline{\beta}_{22}) \\
&\quad \times \begin{vmatrix}
1 & 1 & 0 \\
\frac{\mu}{l}\{\rho - (1-\eta-\sigma)R\} & 0 & 1 \\
0 & \frac{\mu}{l}\{\rho - (1-\eta-\sigma)R\} & 1
\end{vmatrix} \\
&= -2Q\overline{\beta}_{22}wl(G\sigma AR)^2
\end{aligned}
$$

Thus, $sign[\det\Gamma_{22}] = -sign[Q]sign[\overline{\beta}_{22}]$. From (11) we see that $sign[\overline{\beta}_{22}] = sign[\Delta]$. It follows from (53) that

$$\begin{aligned}
sign[J_2(0)] &= \begin{cases} \;\; sign[Q]sign[\det[\Gamma_{22}]]sign[\hat{\Delta}]\Big|_{p=p_e} \\ -\, sign[Q]sign[\det[\Gamma_{22}]]sign[\hat{\Delta}]\Big|_{p=\bar{p}_e} \end{cases} \\
&= \begin{cases} -\left(sign[Q]\right)^2 sign[\Delta]sign[\hat{\Delta}]\Big|_{p=p_e} \\ \;\; \left(sign[Q]\right)^2 sign[\Delta]sign[\hat{\Delta}]\Big|_{p=\bar{p}_e} \end{cases} \\
&= \begin{cases} -sign[\Delta]sign[\hat{\Delta}]\Big|_{p=p_e} \\ \;\; sign[\Delta]sign[\hat{\Delta}]\Big|_{p=\bar{p}_e}, \end{cases}
\end{aligned}$$

which implies (54). (QED)

8.4 The proof of Lemma 4

Differentiating (51) with respect to ξ at 0,

$$\begin{aligned}
J_2'(0) &= \begin{vmatrix} \frac{\Gamma_{11}}{2p^{1-\frac{1}{\sigma}}} & \frac{w\overline{\beta}_{21}}{p^{1-\frac{1}{\sigma}}} \\ \mu[\frac{Gw'}{\sigma w} - R_p]\sigma AR & \frac{\mu}{l}\Omega\sigma AR + \frac{\overline{G}}{\frac{l}{\mu}}\sigma AR \\ +\frac{\frac{\mu}{\lambda}G\Gamma_{11}\sigma AR}{2p^{1-\frac{1}{\sigma}}} - \frac{\overline{G}}{\frac{l}{\mu}} lR_p & -w\mu\overline{\beta}_{22}\sigma AR + \frac{\frac{\mu}{\lambda}Gw\overline{\beta}_{21}\sigma AR}{p^{1-\frac{1}{\sigma}}} - \frac{2\overline{G}}{\frac{l}{\mu}}wl\overline{\beta}_{22} \end{vmatrix} \\
&\quad + \begin{vmatrix} \lambda[\frac{r'}{\sigma} - R_p] & -2w\lambda\overline{\beta}_{22} \\ -\mu R_p & -\frac{\mu}{l}\{\rho - (1-\sigma)R + 2wl\overline{\beta}_{22}\} \end{vmatrix} \\
&= \frac{\mu}{2p^{1-\frac{1}{\sigma}}l} \begin{vmatrix} \Gamma_{11} & 2wl\overline{\beta}_{21} \\ [(\frac{Gw'}{\sigma w} - R_p)\sigma AR & [\Omega\sigma AR + \overline{G}\sigma AR \\ -\overline{G}R_p] & -wl\overline{\beta}_{22}\sigma AR - 2wl\overline{\beta}_{22}\overline{G}] \end{vmatrix} \\
&\quad - \begin{vmatrix} [\frac{r'}{\sigma} - R_p] & 2wl\overline{\beta}_{22} \\ -\frac{r'}{\sigma} & \{\rho - (1-\sigma)R\} \end{vmatrix} \frac{\lambda\mu}{l}
\end{aligned}$$

$$\begin{aligned}
\frac{2p^{1-\frac{1}{\sigma}}l}{\mu}J_2'(0) &= \begin{vmatrix} \Gamma_{11} & 2wl\overline{\beta}_{21} \\ \frac{Gw'}{w}AR - R_p(\rho - (1-\sigma)R) & G\sigma AR \\ & -2wl\overline{\beta}_{22}(\rho - (1-\sigma)R) \end{vmatrix} \\
&\quad -2p^{1-\frac{1}{\sigma}}\lambda \begin{vmatrix} [\frac{r'}{\sigma} - R_p] & 2wl\overline{\beta}_{22} \\ -\frac{r'}{\sigma} & \{\rho - (1-\sigma)R\} \end{vmatrix}
\end{aligned}$$

Considering $r \doteqdot R \doteqdot \frac{\rho}{1-\sigma}$, we have

$$\frac{2p^{1-\frac{1}{\sigma}}l}{\mu}J_2'(0) = \begin{vmatrix} \Gamma_{11} & 2wl\overline{\beta}_{21} \\ \frac{Gw'}{w}AR & G\sigma AR \end{vmatrix} - 2p^{1-\frac{1}{\sigma}}\lambda \begin{vmatrix} [\frac{r'}{\sigma} - R_p] & 2wl\overline{\beta}_{22} \\ -\frac{r'}{\sigma} & 0 \end{vmatrix}$$

$$= GAR\{\Gamma_{11}\sigma - 2l\overline{\beta}_{21}w'\} + 4p^{1-\frac{1}{\sigma}}\lambda wl\overline{\beta}_{22}\frac{r'}{\sigma}$$

It follows that if $r'\Delta > 0$ ($w'\Delta < 0$) and $\sigma \doteqdot 0$, then $J_2'(0) > 0$.(QED)

8.5 The Proof of Lemma 5

It is clear from $J_2(0) > 0$ and $sign[J_2'''(0)] = sign[\Gamma_{11}] < 0$ that $J_2(\xi) = 0$ has at least one positive real root. Let us denote the positive real root by ξ_0 and the remaining two roots by ξ_1 and ξ_2. Then $J_2(\xi)$ can be written as

$$\begin{aligned} J_2(\xi) &= \Gamma_{11}(\xi-\xi_0)(\xi-\xi_1)(\xi-\xi_2) \\ &= \Gamma_{11}\xi^3 - \Gamma_{11}(\xi_0+\xi_1+\xi_2)\xi^2 \\ &\quad +\Gamma_{11}\{\xi_1\xi_2 + \xi_0(\xi_1+\xi_2)\}\xi \\ &\quad -\Gamma_{11}\xi_1\xi_2\xi_0 \end{aligned}$$

If $J_2(0) = -\Gamma_{11}\xi_1\xi_2\xi_0 > 0$, $\Gamma_{11} < 0$ and $\xi_0 > 0$, then $\xi_1\xi_2 > 0$. Therefore, if the coefficient of ξ, $J_2'(0) = \Gamma_{11}\{\xi_1\xi_2 + \xi_0(\xi_1+\xi_2)\}$, is positive, the real parts of both ξ_1 and ξ_2 have to be negative due to the following reason: suppose that ξ_1 and ξ_2 are complex roots, $\varsigma \pm \zeta i$, where ς and ζ are real numbers: then,

$$\xi_1\xi_2 = \varsigma^2 + \zeta^2 > 0 \text{ and } \xi_1 + \xi_2 = 2\varsigma;$$

therefore, if $\Gamma_{11}\{\xi_1\xi_2 + \xi_0(\xi_1+\xi_2)\} > 0$, then $\xi_1 + \xi_2 = 2\varsigma < 0$.($QED$)

8.6 The Proof of Lemma 6

Let us prove that, if $\Delta > 0$ and $\hat{\Delta} < 0$, Γ_{11} is negative for a sufficiently small σ. Let $r(p_e)$ be approximately equal to ρ. Then, since $R = 2wl\beta_{22} - \rho\bar{\beta}_{12} \fallingdotseq \frac{1}{\sigma}(\rho-\rho) = 0$,

$$\begin{aligned} p\Gamma_{11} &= (1-\frac{1}{\sigma})\{\rho\bar{\beta}_{11} - 2lw(p_e)\bar{\beta}_{21}\} \\ &\quad -\{\theta_r\rho\overline{\beta}_{11} - 2lw(p_e)\theta_w\overline{\beta}_{21}\} \\ &\fallingdotseq (1-\frac{1}{\sigma})\{\rho\bar{\beta}_{11} - \rho\bar{\beta}_{12}\frac{\bar{\beta}_{21}}{\bar{\beta}_{22}}\} \\ &\quad -\{\theta_r\rho\bar{\beta}_{11} - \theta_w\rho\bar{\beta}_{12}\frac{\bar{\beta}_{21}}{\bar{\beta}_{22}}\} \\ &= \frac{\rho}{\bar{\beta}_{22}}[(1-\frac{1}{\sigma})(\bar{\beta}_{11}\bar{\beta}_{22} - \bar{\beta}_{12}\bar{\beta}_{21}) \\ &\quad -\{\theta_r\bar{\beta}_{11}\bar{\beta}_{22} - \theta_w\bar{\beta}_{12}\bar{\beta}_{21}\}] \\ &= \frac{\rho}{\beta_{22}}[(1-\frac{1}{\sigma}) - \frac{1}{\Delta}\{\theta_r\beta_{11}\beta_{22} - \theta_w\beta_{12}\beta_{21}\}], \end{aligned}$$

which is negative iff

$$\frac{1}{\sigma} > 1 - \frac{1}{\Delta}\{\theta_r\beta_{11}\beta_{22} - \theta_w\beta_{12}\beta_{21}\},$$

or

$$\sigma < \frac{\Delta}{\Delta - \{\theta_r \beta_{11} \beta_{22} - \theta_w \beta_{12} \beta_{21}\}}$$

In order that this inequality makes sense, the right-hand side of it must be positive. Considering (10), $\hat{\Delta} < 0$ means $\theta_r < 0$ and $\theta_w > 0$. Therefore, $\Delta > 0$ guarantees the positiveness.

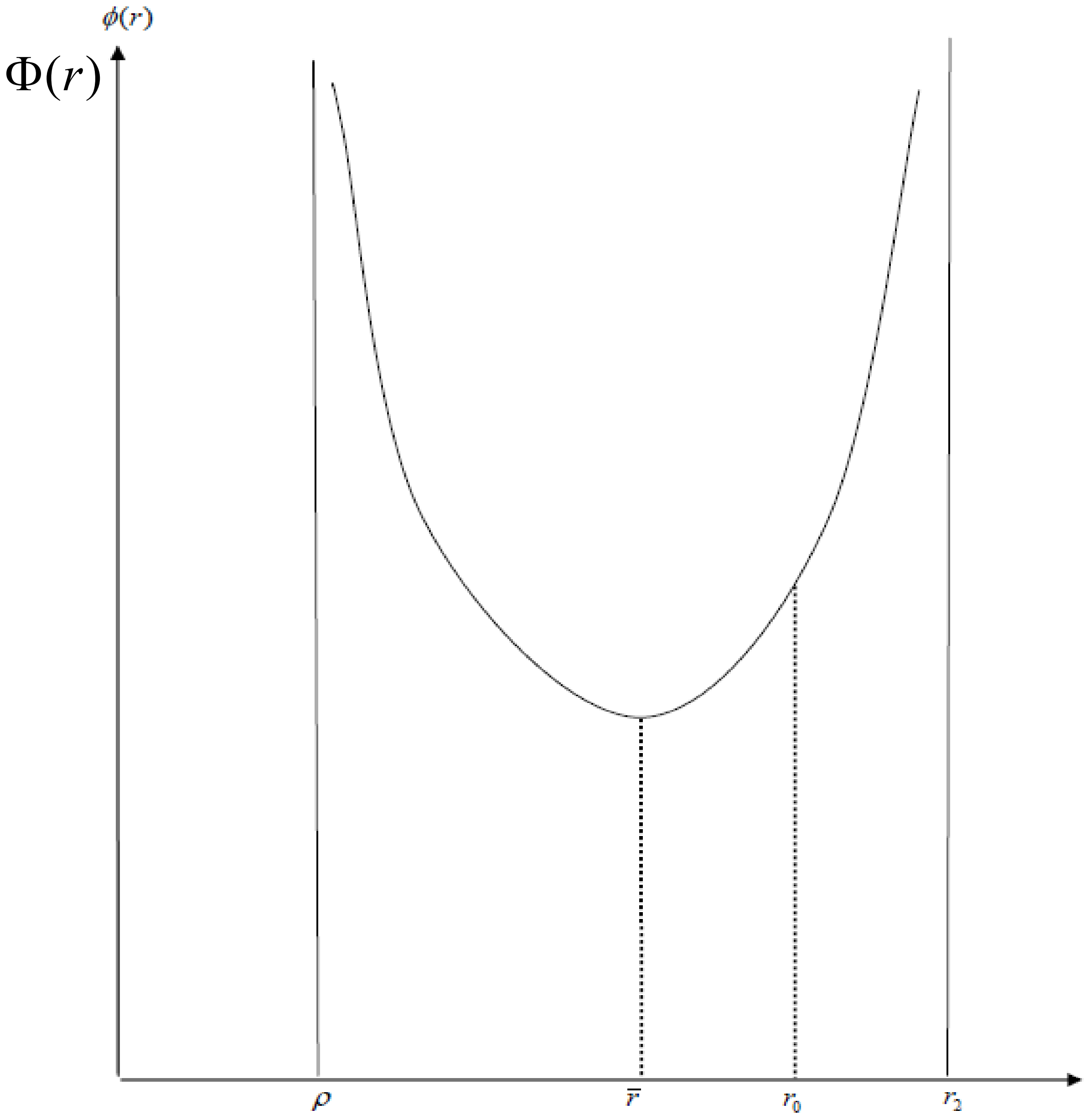

Figure 1: the graph of $\Phi(r)$

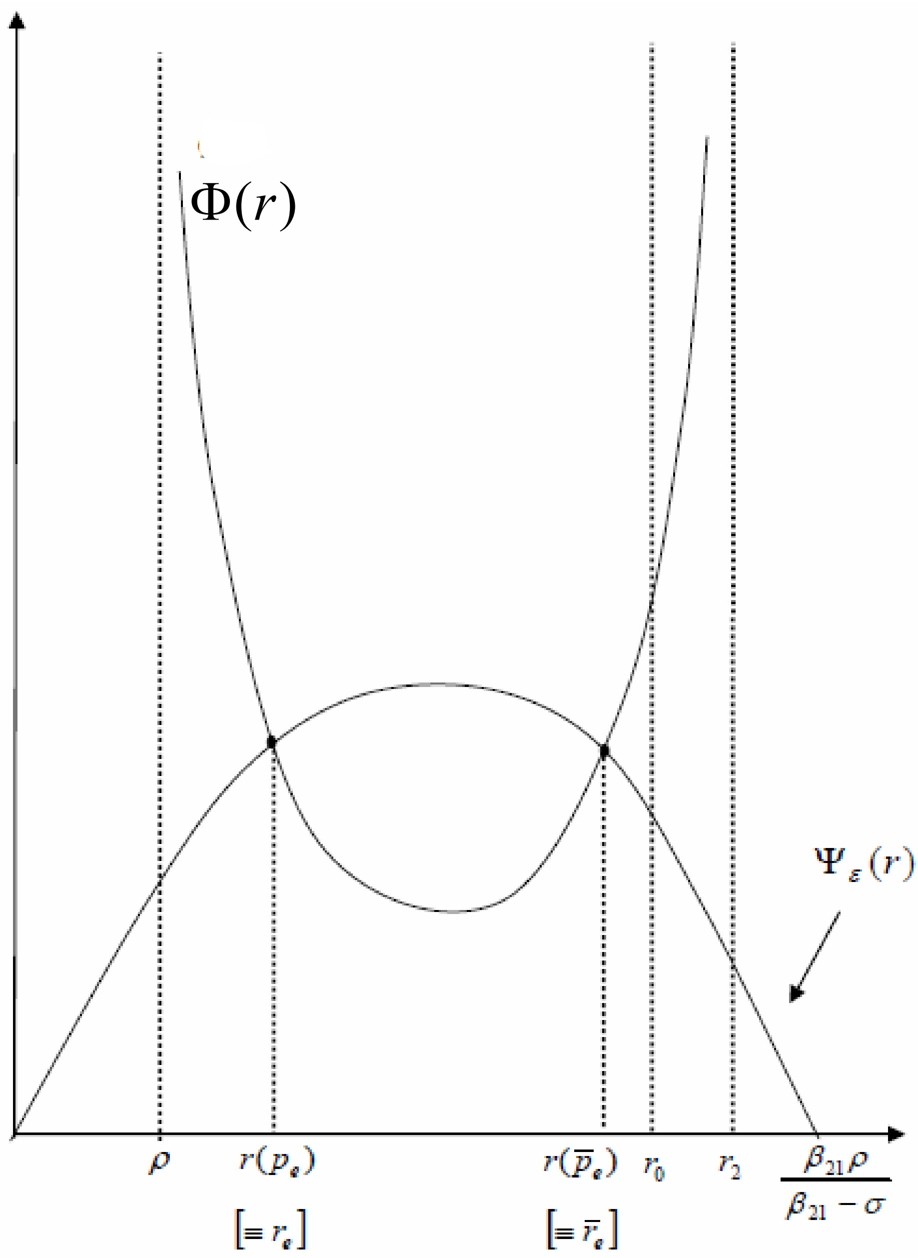

Figure 2： two possible BGP's

Table 1: Main Results of This Paper

b<1: The foreign household is less efficient than the home household in human capital investment

		IDBGP (Figure 3)	IIBGP (Figure 4)
Growth rate and trade volume in terms of good 1		Higher growth rate and larger relative trade volume	Higher growth rate and smaller relative trade volume
Trade pattern		①The home country exports the consumption good.	② The home country exports the consumption good.
BGP factor endowment ratio		③ $K^* / L^* < K / L$	④ $K^* / L^* > K / L$
Autarkic prices	$r' < 0$	⑤ $p^h > p^f$	⑥ $p^h < p^f$
	$r' > 0$	⑦ $p^h < p^f$	⑧ $p^h > p^f$
Trade pattern	$r' < 0$	Anti-H-O, due to ① and ③	Anti-H-O, due to ② and ④
	$r' > 0$	H-O, due to ① and ③	H-O, due to ② and ④
The law of comparative advantage	$r' < 0$	① and ⑤ contradict the Law.	② and ⑦ mean the Law.
	$r' > 0$	① and ⑦ mean the Law	② and ⑧ contradict the Law.

Figure 3: IBGP, b<1

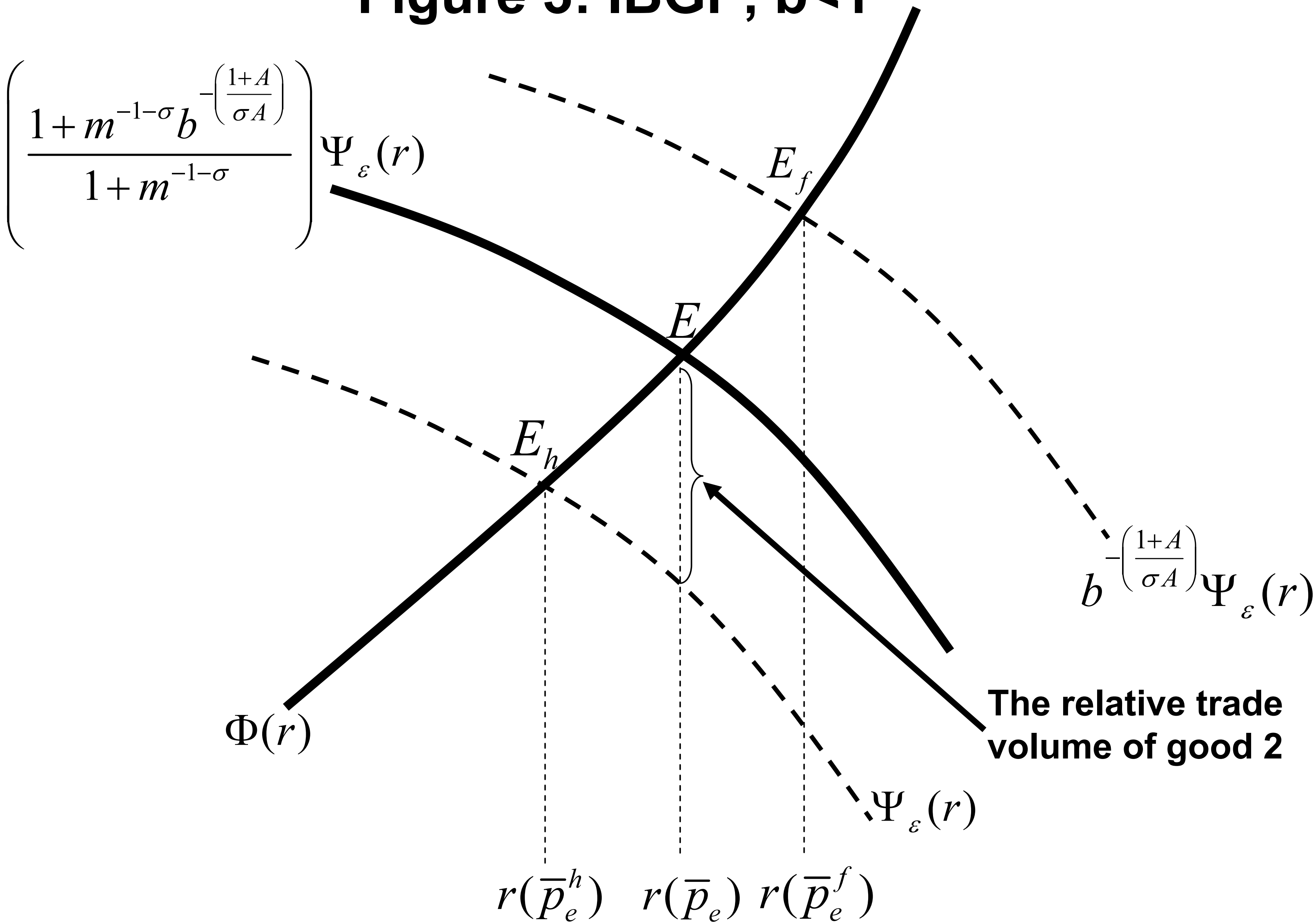

Figure 4: IIBGP, b<1

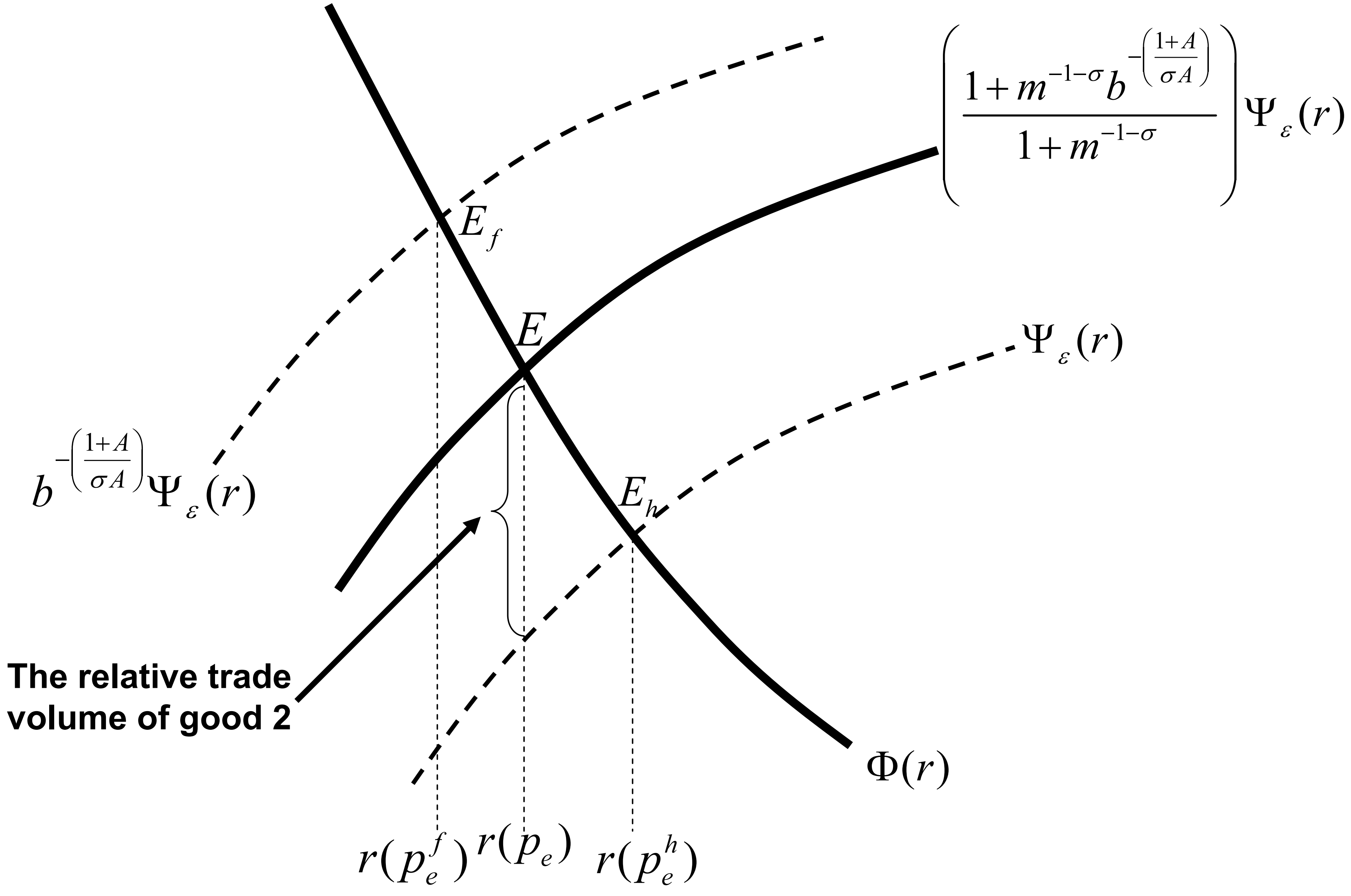

Figure 5