Three-Dimensional Panel

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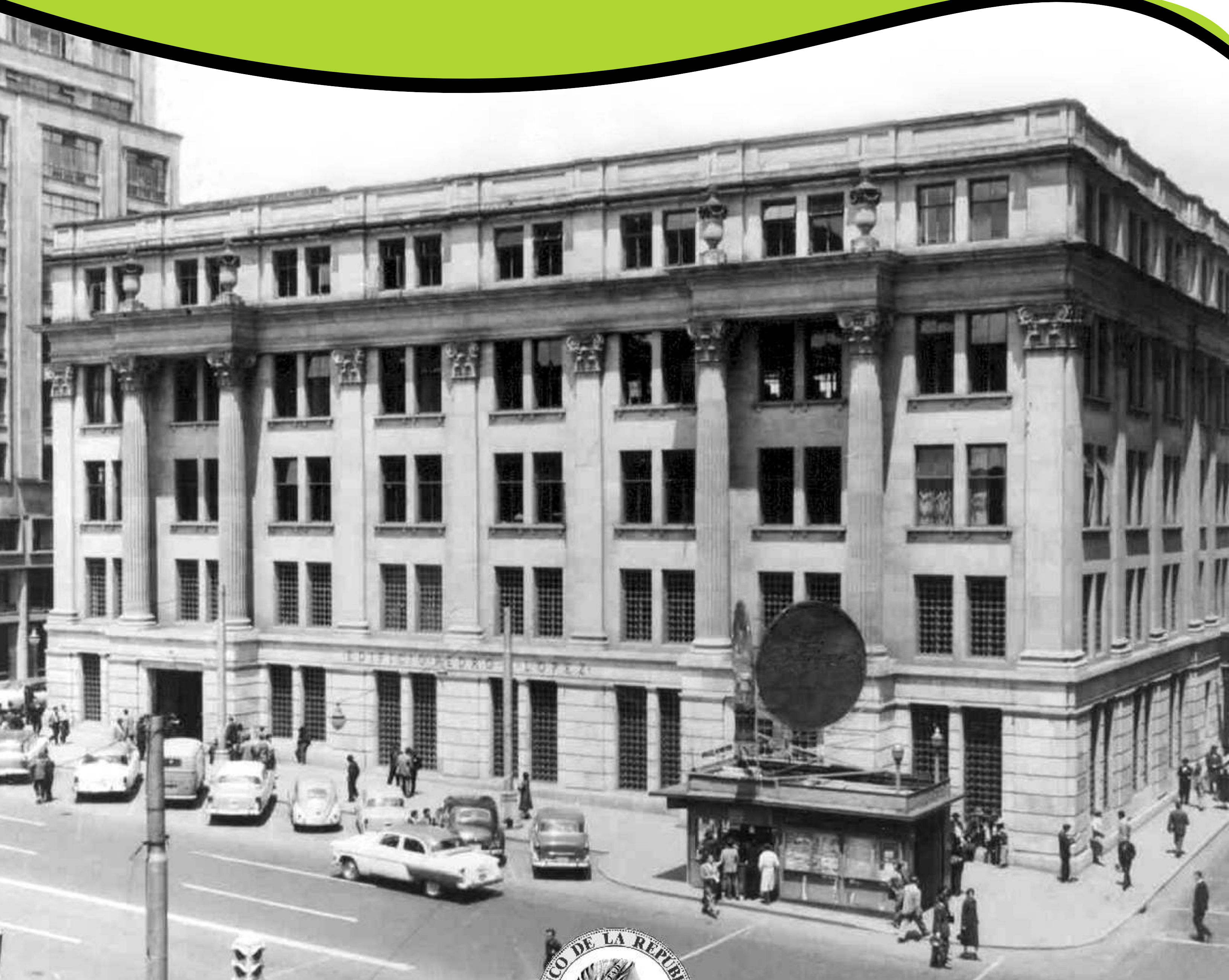

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COINTEGRATION VECTOR ESTIMATION BY DOLS FOR A THREE-DIMENSIONAL PANEL

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ABSTRACT. This paper extends the asymptotic results of the dynamic ordinary least squares (DOLS) cointegration vector estimator of Mark and Sul (2003) to a three-dimensional panel. We use a balanced panel of N and M lengths observed over T time periods. The cointegration vector is homogenous across individuals but we allow for individual heterogeneity using different short-run dynamics, individual-specific fixed effects and individual-specific time trends. Both individual effects are considered for the first two dimensions. We also model some degree of cross-sectional dependence using time-specific effects.The estimator has a Gaussian sequential limit distribution that is obtained first letting $T \to \infty$ and then letting $N \to \infty$, $M \to \infty$.

This paper was motivated by the three-dimensional panel cointegration analysis used to estimate the total factor productivity for Colombian regions and sectors during 1975-2000 by Iregui, Melo and Ramírez (2007). They used the methodology proposed by Marrocu, Paci and Pala (2000); however, hypothesis testing is not valid under this technique. The methodology we are currently proposing allows us to estimate the long-run relationship and to construct asymptotically valid test statistics in the 3D-panel context.

Key words and phrases. Cointegration, Dynamic OLS estimation, panel data in three dimensions.
JEL clasification. C13, C33.

1. INTRODUCTION

This paper proposes an extension of the dynamic ordinary least squares (DOLS) cointegration panel estimators of Mark and Sul (2003) to a three-dimensional panel. The single equation DOLS for estimating and testing hypothesis about cointegration was proposed by Phillips and Loretan (1991), Saikkonen (1991) and generalized by Stock and Watson (1993).

DOLS is a single equation cointegration technique that overcome the common problems of the static and modified OLS. The static OLS finite sample estimates of long-run relationships are potentially biased and inferences cannot be drawn using t-statistics (Banerjee et al, 1986, Kremers et al, 1992). DOLS methodology is based on an equation that includes lags and leads of right-hand side variables which eliminates the effect of the endogeneity of these variables. Therefore, it is possible to construct asymptotically valid test statistics and also to estimate the long-run relationships.

Panel DOLS (PDOLS) has been analyzed by Kao and Chiang (2000) and Mark and Sul (2003). Kao and Chiang (2000) studies the properties of panel DOLS when there are fixed effects in the cointegration regressions. Mark and Sul (2003) allow for individual heterogeneity through different short-run dynamics, individual-specific fixed effects and individual-specific time trends. They also permit a limited degree of

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cross-sectional dependence through the presence of time-specific effects.

For extending the results of Mark and Sul (2003), we use a balanced panel of three dimensions with lengths N, M and T. The cointegration vector is homogenous across individuals but we allow for individual heterogeneity using different short-run dynamics, individual-specific fixed effects and individual-specific time trends. Both individual effects are considered in the first two dimensions. As in Mark and Sul (2003), we also model some degree of cross-sectional dependence using time-specific effects. After obtaining the Panel DOLS estimator in three dimensions, PDOLS-3D, we present the sequential limit distribution of the estimator by first letting $T \to \infty$ then letting $N \to \infty$, $M \to \infty$.

The remainder of the paper is organized as follows. Section 2 describes the cointegration representation for a three-dimensional panel. Section 3 describes the PDOLS-3D estimator. Finally, the asymptotic distribution of the PDOLS-3D estimator is presented in Section 4.

2. Representation of a cointegrated model in panel data in three dimensions

Consider the following triangular representation of a cointegrated system for a panel with individuals indexed by $i = 1, ..., N$ and $j = 1, ..., M^\star$ over time periods $t = 1, ..., T$

$$(1) \qquad \begin{aligned} y_{ijt} &= \alpha_i^{(N)} + \alpha_j^{(M)} + \lambda_i^{(N)} t + \lambda_j^{(M)} t + \theta_t + \underline{\gamma}' \underline{x}_{ijt} + u_{ijt} \\ \underline{x}_{ijt} &= \underline{x}_{ijt-1} + \underline{v}_{ijt} \end{aligned}$$

where $\{y_{ijt}\}$ is the dependent variable integrated of order one, $\{\underline{x}_{ijt}\}$ is a k-dimensional vector of integrated series of order one and $\{u_{ijt}, \underline{v}'_{ijt}\}'$ is a covariance stationary error process independent across i and j but possibly dependent across t. In this case, the variables are said to be cointegrated for each member of the panel, with cointegrated vector $\underline{\gamma}$. Individual heterogeneity is considered through different short-run dynamics, individual-specific fixed effects of the first two dimensions, $\alpha_i^{(N)}$ and $\alpha_j^{(M)}$, and individual-specific time trends in those dimensions, $\lambda_i^{(N)}$ and $\lambda_j^{(M)}$. A limited degree of cross-sectional dependence is also permitted by the presence of time-specific effects, θ_t.

3. Panel DOLS estimator in three dimensions

PDOLS methodology is based on the estimation of the following equation

$$(2) \qquad y_{ijt} = \alpha_i^{(N)} + \alpha_j^{(M)} + \lambda_i^{(N)} t + \lambda_j^{(M)} t + \theta_t + \underline{\gamma}' \underline{x}_{ijt} + \underline{\delta}'_i \underline{z}_{ijt} + u_{ijt}$$

where $\underline{z}_{ijt} = (\Delta \underline{x}'_{ijt-p}, \ldots, \Delta \underline{x}'_{ijt}, \ldots, \Delta \underline{x}'_{ijt+p})'$ is a $(2p+1)k$-dimensional vector of leads and lags of the first differences of the variables $\underline{x}_{ijt}$. The inclusion of lags and leads eliminates the effect of the endogeneity of these variables. To avoid perfect collinearity, $\alpha_{M^\star}^{(M)} = \lambda_{M^\star}^{(M)} = 0$.

Equation (2) can be expressed as follows,

$$(3) \qquad y^\ddagger_{ijt} = \alpha_i^{(N)} + \alpha_j^{(M)} + \lambda_i^{(N)} t + \lambda_j^{(M)} t + \theta_t + \underline{\gamma}' \underline{x}^\ddagger_{ijt} + u_{ijt}$$

where $y^\ddagger_{ijt}$ and $\underline{x}^\ddagger_{ijt}$ represent the linear projection of the dependent variable and the variables $\underline{x}_{ijt}$ with respect to the short run components, $\underline{z}_{ijt}$.

Taking average of (3) in the time dimension gives

$$\frac{1}{T}\sum_{t=1}^{T} y^{\ddagger}_{ijt} = \alpha_i^{(N)} + \alpha_j^{(M)} + \left(\lambda_i^{(N)} + \lambda_j^{(M)}\right)\left[\frac{(T+1)}{2}\right] + \underline{\gamma}'\left[\frac{1}{T}\sum_{t=1}^{T}\underline{x}^{\ddagger}_{ijt}\right] + \frac{1}{T}\sum_{t=1}^{T}\theta_t + \frac{1}{T}\sum_{t=1}^{T}u_{ijt} \tag{4}$$

Subtracting (4) from (3) eliminates $\alpha_i^{(N)}$ and $\alpha_j^{(M)}$ and gives

(5)

$$y^{\ddagger}_{ijt} - \frac{1}{T}\sum_{s=1}^{T} y^{\ddagger}_{ijs} = \left(\lambda_i^{(N)} + \lambda_j^{(M)}\right)\left[t - \frac{(T+1)}{2}\right] + \underline{\gamma}'\left[\underline{x}^{\ddagger}_{ijt} - \frac{1}{T}\sum_{s=1}^{T}\underline{x}^{\ddagger}_{ijs}\right] + \left[\theta_t - \frac{1}{T}\sum_{s=1}^{T}\theta_s\right] + \left[u_{ijt} - \frac{1}{T}\sum_{s=1}^{T}u_{ijs}\right]$$

Taking double average of equation (5) in the first two dimensions gives the following result

$$\begin{aligned}\frac{1}{NM^\star}\sum_{i=1}^{N}\sum_{j=1}^{M^\star} y^{\ddagger}_{ijt} - \frac{1}{NM^\star T}\sum_{i=1}^{N}\sum_{j=1}^{M^\star}\sum_{s=1}^{T} y^{\ddagger}_{ijs} &= \left[t - \frac{(T+1)}{2}\right]\frac{1}{NM^\star}\sum_{i=1}^{N}\sum_{j=1}^{M^\star}\left(\lambda_i^{(N)} + \lambda_j^{(M)}\right) + \underline{\gamma}'\frac{1}{NM^\star}\sum_{i=1}^{N}\sum_{j=1}^{M^\star}\left[\underline{x}^{\ddagger}_{ijt} - \frac{1}{T}\sum_{s=1}^{T}\underline{x}^{\ddagger}_{ijs}\right] \\ &+ \left[\theta_t - \frac{1}{T}\sum_{s=1}^{T}\theta_s\right] + \frac{1}{NM^\star}\sum_{i=1}^{N}\sum_{j=1}^{M^\star}\left[u_{ijt} - \frac{1}{T}\sum_{s=1}^{T}u_{ijs}\right]\end{aligned} \tag{6}$$

Subtracting (6) from (5) eliminates the common time effects, then the model can be rewritten as

$$y^{\ddagger *}_{ijt} \;=\; \left(\widetilde{\lambda}_i^{(N)} + \widetilde{\lambda}_j^{(M)}\right)\widetilde{t} \;+\; \underline{\gamma}'\underline{x}^{\ddagger *}_{ijt} \;+\; u^*_{ijt} \tag{7}$$

Where the superscripts " $*$ " and " $\sim$ " denote the following deviations

$y^{\ddagger *}_{ijt} = \left(y^{\ddagger}_{ijt} - \frac{1}{T}\sum_{s=1}^{T} y^{\ddagger}_{ijs}\right) - \left(\frac{1}{NM^\star}\sum_{n=1}^{N}\sum_{m=1}^{M^\star} y^{\ddagger}_{nmt} - \frac{1}{NM^\star T}\sum_{n=1}^{N}\sum_{m=1}^{M^\star}\sum_{s=1}^{T} y^{\ddagger}_{nms}\right),$

$\underline{x}^{\ddagger *}_{ijt} = \left(\underline{x}^{\ddagger}_{ijt} - \frac{1}{T}\sum_{s=1}^{T} \underline{x}^{\ddagger}_{ijs}\right) - \left(\frac{1}{NM^\star}\sum_{n=1}^{N}\sum_{m=1}^{M^\star} \underline{x}^{\ddagger}_{nmt} - \frac{1}{NM^\star T}\sum_{n=1}^{N}\sum_{m=1}^{M^\star}\sum_{s=1}^{T} \underline{x}^{\ddagger}_{nms}\right),$

$u^{*}_{ijt} = \left(u_{ijt} - \frac{1}{T}\sum_{s=1}^{T} u_{ijs}\right) - \left(\frac{1}{NM^\star}\sum_{n=1}^{N}\sum_{m=1}^{M^\star} u_{nmt} - \frac{1}{NM^\star T}\sum_{n=1}^{N}\sum_{m=1}^{M^\star}\sum_{s=1}^{T} u_{nms}\right),$

$\widetilde{\lambda}_i^{(N)} = \lambda_i^{(N)} - \frac{1}{N}\sum_{n=1}^{N}\lambda_n^{(N)},$

$\widetilde{\lambda}_j^{(M)} = \lambda_j^{(M)} - \frac{1}{M^\star}\sum_{m=1}^{M^\star}\lambda_m^{(M)},$

$\widetilde{t} = t - \frac{(T+1)}{2}.$

Let us define the grand coefficient vector of the model as $\underline{\beta}' = \left(\underline{\gamma}', \underline{\widetilde{\lambda}}_N', \underline{\widetilde{\lambda}}_M'\right)$ where $\underline{\widetilde{\lambda}}_N' = \left(\widetilde{\lambda}_1^{(N)}, \widetilde{\lambda}_2^{(N)}, \ldots, \widetilde{\lambda}_N^{(N)}\right)$, $\underline{\widetilde{\lambda}}_M' = \left(\widetilde{\lambda}_1^{(M)}, \widetilde{\lambda}_2^{(M)}, \ldots, \widetilde{\lambda}_M^{(M)}\right)$, $M = M^\star - 1$, and the following matrices

$$
\begin{aligned}
\underline{q}_{11t}^{\ddagger *} &= \left(\underline{x}_{11t}^{\ddagger *\prime}, \quad \widetilde{t}, 0, \ldots, 0, \widetilde{t}, 0, \ldots, 0\right)^{\prime} \\
\underline{q}_{12t}^{\ddagger *} &= \left(\underline{x}_{12t}^{\ddagger *\prime}, \quad \widetilde{t}, 0, \ldots, 0, 0, \widetilde{t}, \ldots, 0\right)^{\prime} \\
&\vdots \qquad\qquad \vdots \\
\underline{q}_{1Mt}^{\ddagger *} &= \left(\underline{x}_{1Mt}^{\ddagger *\prime}, \quad \widetilde{t}, 0, \ldots, 0, 0, 0, \ldots, \widetilde{t}\right)^{\prime} \\
\underline{q}_{21t}^{\ddagger *} &= \left(\underline{x}_{21t}^{\ddagger *\prime}, \quad 0, \widetilde{t}, \ldots, 0, \widetilde{t}, 0, \ldots, 0\right)^{\prime} \\
&\vdots \qquad\qquad \vdots \\
\underline{q}_{2Mt}^{\ddagger *} &= \left(\underline{x}_{2Mt}^{\ddagger *\prime}, \quad 0, \widetilde{t}, \ldots, , 0, 0, 0, \ldots, \widetilde{t}\right)^{\prime} \\
&\vdots \qquad\qquad \vdots \\
\underline{q}_{NMt}^{\ddagger *} &= \left(\underline{x}_{NMt}^{\ddagger *\prime}, \quad 0, 0, \ldots, \widetilde{t}, 0, 0, \ldots, \widetilde{t}\right)^{\prime}
\end{aligned} \tag{8}
$$

Then, the model can finally be expressed as

$$
y_{ijt}^{\ddagger *} \;=\; \underline{\beta}^{\prime}\, q_{ijt}^{\ddagger *} \;+\; u_{ijt}^{\ddagger *} \tag{9}
$$

And the PDOLS-3D estimator of $\underline{\beta}$ is

$$
\underline{\hat{\beta}}_{NMT} = \left[\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{t=1}^{T} \underline{q}_{ijt}^{\ddagger *}\underline{q}_{ijt}^{\ddagger *\prime}\right]^{-1} \left[\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{t=1}^{T} \underline{q}_{ijt}^{\ddagger *} y_{ijt}^{\ddagger *}\right] \tag{10}
$$

4. Asymptotic distribution of the PDOLS-3D estimator

Taking into account that elements in $\underline{\hat{\beta}}_{NMT}$ have different rates of convergence, we can rewrite (10) as

$$
\mathbf{G}_{NMT}(\underline{\hat{\beta}}_{NMT} - \underline{\beta}) = [\mathbf{M}_{NMT}]^{-1}\, \underline{m}_{NMT}
$$

Where

$$
\mathbf{G}_{NMT} = \begin{bmatrix} \sqrt{NMT}T\,\mathbf{I}_k & \mathbf{0}^{\prime} & \mathbf{0}^{\prime} \\ \mathbf{0} & \sqrt{MT}T^{\frac{3}{2}}\mathbf{I}_N & \mathbf{0}^{\prime} \\ \mathbf{0} & \mathbf{0} & \sqrt{NT}T^{\frac{3}{2}}\mathbf{I}_M \end{bmatrix},
$$

$$
\mathbf{M}_{NMT} = \left[\mathbf{G}_{NMT}^{-1} \sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{t=1}^{T} (\underline{q}_{ijt}^{\ddagger *}\underline{q}_{ijt}^{\ddagger *\prime})\mathbf{G}_{NMT}^{-1}\right] = \begin{bmatrix} \mathbf{M}_{11,NMT} & \mathbf{M}_{21,NMT}^{\prime} & \mathbf{M}_{31,NMT}^{\prime} \\ \mathbf{M}_{21,NMT} & \mathbf{M}_{22,NMT} & \mathbf{M}_{32,NMT}^{\prime} \\ \mathbf{M}_{31,NMT} & \mathbf{M}_{32,NMT} & \mathbf{M}_{33,NMT} \end{bmatrix},
$$

$$
\mathbf{M}_{11,NMT} = \frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\left[\frac{1}{T^2}\sum_{t=1}^{T} \underline{x}_{ijt}^{\ddagger *}\underline{x}_{ijt}^{\ddagger *\prime}\right],
$$

$$\mathbf{M}'_{21,NMT} = \left[\frac{1}{\sqrt{N}MT^{\frac{5}{2}}} \sum_{j=1}^{M} \sum_{t=1}^{T} \widetilde{t}\underline{x}^{\ddagger*}_{1jt} \quad \cdots \quad \frac{1}{\sqrt{N}MT^{\frac{5}{2}}} \sum_{j=1}^{M} \sum_{t=1}^{T} \widetilde{t}\underline{x}^{\ddagger*}_{Njt} \right],$$

$$\mathbf{M}'_{31,NMT} = \left[\frac{1}{N\sqrt{M}T^{\frac{5}{2}}} \sum_{i=1}^{N} \sum_{t=1}^{T} \widetilde{t}\underline{x}^{\ddagger*}_{i1t} \quad \cdots \quad \frac{1}{N\sqrt{M}T^{\frac{5}{2}}} \sum_{i=1}^{N} \sum_{t=1}^{T} \widetilde{t}\underline{x}^{\ddagger*}_{iMt} \right],$$

$$\mathbf{M}_{22,NMT} = \left[\frac{1}{T^3} \sum_{t=1}^{T} \widetilde{t}^{\,2} \right] \mathbf{I}_N,$$

$$\mathbf{M}'_{32,NMT} = \left[\frac{1}{\sqrt{NM}T^3} \sum_{t=1}^{T} \widetilde{t}^{\,2} \right] (1)_{N\times M}; \quad \text{where } (1)_{N\times M} \text{ is a matrix of ones,}$$

$$\mathbf{M}_{33,NMT} = \left[\frac{1}{T^3} \sum_{t=1}^{T} \widetilde{t}^{\,2} \right] \mathbf{I}_M,$$

$$\underline{m}_{NMT} = \mathbf{G}^{-1}_{NMT} \sum_{i=1}^{N} \sum_{j=1}^{M} \sum_{t=1}^{T} \underline{q}^{\ddagger*}_{ijt} u^*_{ijt} = \begin{bmatrix} \underline{m}_{1,NMT} \\ \underline{m}_{2,NMT} \\ \underline{m}_{3,NMT} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{NM}} \sum_{i=1}^{N} \sum_{j=1}^{M} \left[\frac{1}{T} \sum_{t=1}^{T} \underline{x}^{\ddagger*}_{ijt} u^{\ddagger*}_{ijt} \right] \\ \frac{1}{\sqrt{M}T^{\frac{3}{2}}} \sum_{j=1}^{M} \sum_{t=1}^{T} \widetilde{t} u^*_{1jt} \\ \vdots \\ \frac{1}{\sqrt{M}T^{\frac{3}{2}}} \sum_{j=1}^{M} \sum_{t=1}^{T} \widetilde{t} u^*_{Njt} \\ \frac{1}{\sqrt{N}T^{\frac{3}{2}}} \sum_{i=1}^{N} \sum_{t=1}^{T} \widetilde{t} u^*_{i1t} \\ \vdots \\ \frac{1}{\sqrt{N}T^{\frac{3}{2}}} \sum_{i=1}^{N} \sum_{t=1}^{T} \widetilde{t} u^*_{iMt} \end{bmatrix}$$

The asymptotic distribution of the PDOLS-3D estimator is presented in the proposition 1 part (b). The following lemmas are required to prove this proposition. The proofs of the lemmas follow from simple extensions of the results of Mark and Sul (2002). However, for easiness of the explanation they are presented in the Appendixes A, B and C.

Following the results of Mark and Sul (2003), the null hypothesis $\mathbf{R}\underline{\gamma} = \underline{r}$ can be tested using a regular Wald statistics.

Lemma 1. *For each i and j as $T \to \infty$,*

a. $\frac{1}{T^2} \sum_{t=1}^{T} \underline{x}^{\ddagger*}_{ijt} \underline{x}^{\ddagger*\prime}_{ijt} - \frac{1}{T^2} \sum_{t=1}^{T} \underline{x}^{*}_{ijt} \underline{x}^{*\prime}_{ijt} \xrightarrow{p} \boldsymbol{0}.$

b. $\frac{1}{T^{5/2}} \sum_{t=1}^{T} \widetilde{t}\underline{x}^{\ddagger*}_{ijt} - \frac{1}{T^{5/2}} \sum_{t=1}^{T} t\underline{x}^{*}_{ijt} \xrightarrow{p} \boldsymbol{0}.$

c. $\frac{1}{T} \sum_{t=1}^{T} \underline{x}^{\ddagger*}_{ijt} u^*_{ijt} - \frac{1}{T} \sum_{t=1}^{T} \underline{x}^{*}_{ijt} u^*_{ijt} \xrightarrow{p} \boldsymbol{0}.$

This lemma demonstrates the equivalence in probability of the projected series in the z_{ijt} space and the series which are not projected. This gives an asymptotic justification for ignoring the fact that we are using projection errors instead of the original observations.

Lemma 2. *As $T \to \infty$ and then $N \to \infty$, $M \to \infty$,*

a. $\boldsymbol{M}_{11,NMT} - \frac{1}{6NM} \sum_{i=1}^{N} \sum_{j=1}^{M} \boldsymbol{\Omega}_{vv,ij} \xrightarrow{p} \boldsymbol{0}.$

b. $\boldsymbol{M}'_{21,NMT} \xrightarrow{p} \boldsymbol{0}$

$c.$ $\boldsymbol{M}'_{31,NMT} \xrightarrow{p} \boldsymbol{0}$
$d.$ $\boldsymbol{M}_{22,NMT} \xrightarrow{p} \frac{1}{12}\boldsymbol{I}_N$
$e.$ $\boldsymbol{M}'_{32,NMT} \xrightarrow{p} \boldsymbol{0}$
$f.$ $\boldsymbol{M}_{33,NMT} \xrightarrow{p} \frac{1}{12}\boldsymbol{I}_M$

This lemma shows the convergence of each element in $\mathbf{M}_{NMT}$ matrix.

Lemma 3.
$a.$ *For N and M fixed, with* $T \to \infty$, $\underline{m}_{1,NMT} \xrightarrow{p} \underline{m}_{1,NM}$ *where*

$$\underline{m}_{1,NM} = \left[\frac{NM-1}{NM}\right] \frac{1}{\sqrt{NM}} \sum_{i=1}^{N}\sum_{j=1}^{M} \sqrt{\Omega_{uu,ij}} \int \underline{\tilde{B}}_{vij} dW_{uij} - \frac{1}{NM} O_{NM}(1)$$

$b.$ *As* $T \to \infty$ *then* $N \to \infty$, $M \to \infty$, $\boldsymbol{V}_{NM}^{-\frac{1}{2}} \underline{m}_{1,NM} \xrightarrow{D} N(0, \boldsymbol{I})$ *where* $\boldsymbol{V}_{NM} = \frac{1}{6NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\Omega_{uu,ij}\boldsymbol{\Omega}_{vv,ij}$.
$c.$ *As* $T \to \infty$ *and* $N \to \infty$, $M \to \infty$, $\underline{m}_{1,NMT}$ *is independent of* $\underline{m}_{2,NMT}$ *and* $\underline{m}_{3,NMT}$

This lemma shows the convergence in distribution of $\underline{m}_{NM}$.

Proposition 1. *For the PDOLS-3D estimator in (2) , as* $T \to \infty$ *and then* $N \to \infty$, $M \to \infty$,
$a.$ $\sqrt{NM}T(\underline{\hat{\gamma}}_{NMT} - \underline{\gamma})$ *is independent of* $\sqrt{M}T^{\frac{3}{2}}(\underline{\hat{\lambda}}_N - \underline{\lambda}_N)$ *and* $\sqrt{N}T^{\frac{3}{2}}(\underline{\hat{\lambda}}_M - \underline{\lambda}_M)$.
$b.$ $\boldsymbol{C}_{NM}^{-\frac{1}{2}}\sqrt{NM}T(\underline{\hat{\gamma}}_{NMT} - \underline{\gamma}) \overset{A}{\sim} N(\underline{\boldsymbol{0}}, \boldsymbol{I}_K)$.

Where

$$\boldsymbol{C}_{NM} = \left(\boldsymbol{C}_{NM}^{\frac{1}{2}}\right)\left(\boldsymbol{C}_{NM}^{\frac{1}{2}}\right)' = \overline{\boldsymbol{M}}_{11,NM}^{-1}\overline{\boldsymbol{V}}_{11,NM}\overline{\boldsymbol{M}}_{11,NM}^{-1}$$
$$\overline{\boldsymbol{M}}_{11,NM} = \frac{1}{6NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\boldsymbol{\Omega}_{vv,ij}$$
$$\overline{\boldsymbol{V}}_{11,NM} = \frac{1}{6NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\Omega_{uu,ij}\boldsymbol{\Omega}_{vv,ij}$$

$c.$ $\widehat{\boldsymbol{D}}_{NMT} - \boldsymbol{C}_{NM} \xrightarrow{p} 0$.

Where

$$\widehat{\boldsymbol{D}}_{NMT} = \boldsymbol{M}_{11,NMT}^{-1}\widehat{\boldsymbol{V}}_{11,NMT}\boldsymbol{M}_{11,NMT}^{-1}$$
$$\boldsymbol{M}_{11,NMT} = \frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\left[\frac{1}{T^2}\sum_{t=1}^{T}\underline{x}_{ijt}^{\ddagger*}\underline{x}_{ijt}^{\ddagger*\prime}\right]$$
$$\widehat{\boldsymbol{V}}_{11,NMT} = \frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\sqrt{\widehat{\Omega}_{uu,ij}}\left[\frac{1}{T^2}\sum_{t=1}^{T}\underline{x}_{ijt}^{\ddagger*}\underline{x}_{ijt}^{\ddagger*\prime}\right]$$

and $\widehat{\Omega}_{uu,ij}$ *is a consistent estimator of* $\Omega_{uu,ij}$.

This proposition presents the sequential limit distribution of the PDOLS-3D estimator. The proof of part *(a)* follows from lemma 2 and lemma 3.c, the proof of part (b) follows from lemma 2 and lemma 3.b. The proof of part (c) is straightforward.

REFERENCES

[1] Banerjee, A., D. Hendry, and G. Smith, 1986, "Exploring Equilibrium Relationships in Econometrics through Static Models: Some Monte Carlo Evidence", *Oxford Bulletin of Economics and Statistics*, 52, pp 92-104.
[2] Hamilton, J., 1994, *Time series analysis*, Princeton University Press.
[3] Iregui, A.M., L.F. Melo and M.T. Ramírez, 2007, "Productividad Regional y Sectorial en Colombia: Análisis utilizando datos de panel", *Ensayos sobre Política Económica*, 53, pp 18-65, Special Issue 2006-2007, Forthcoming.

[4] Kao, C. and M.-H. Chiang, 2000, "On the estimation and inference of a cointegrated regression in panel data", in Baltagi B. H. (ed.), *Advances in Econometrics: Nonstationary Panels, Panel Cointegration and Dynamic Panels*, 15, pp. 179-222.

[5] Kremers, J., N. Ericsson, and J. Dolado, 1992, "The Power of Cointegration Tests", *Oxford Bulletin of Economics and Statistics*, 54, pp 325-349.

[6] Mark, N and D. Sul, 2002, "Appendix to Cointegration Vector Estimation by Panel DOLS and Long-Run Money Demand", Unpublished manuscript, University of Notre Dame, Available at http://www.nd.edu/ñmark/wrkpaper/PDOLS%20Appendix%2004-15-03.pdf.

[7] ——— , 2003, "Cointegration Vector Estimation by Panel DOLS and Long-run Money Demand", *Oxford Bulletin of Economics and Statistics*, 65 (5), pp 655-680.

[8] Marrocu, E., Paci, R. and R. Pala, 2000, "Estimation of total factor productivity for regions and sectors in Italy. A panel cointegration approach", Centro Ricerche Economiche Nord Sud-CRENos, *Working paper* 00/16.

[9] Phillips, P. and M. Loretan, 1991, "Estimating Long-Run Economic Equilibria", *Review of Economic Studies*, 58, pp 407-436.

[10] Saikkonen, P., 1991, "Asymptotically efficient estimation of cointegration regressions", *Econometric Theory*, 7, pp. 1-21.

[11] Stock, J. and M. Watson, 1993. "A simple estimator of cointegrating vectors in higher order integrated systems", *Econometrica*, 61, pp. 783-820.

[12] White, H., 2001, *Asymptotic Theory for Econometricians*, Revised Edition, Academic Press.

Appendix A. Proof of Lemma 1

Proof. Following previous definitions,

$$\underline{x}_{ijt}^{\ddagger *} = \left(\underline{x}_{ijt}^{\ddagger} - \frac{1}{T}\sum_{s=1}^{T}\underline{x}_{ijs}^{\ddagger} \right) - \left(\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\underline{x}_{ijt}^{\ddagger} - \frac{1}{NMT}\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{s=1}^{T}\underline{x}_{ijs}^{\ddagger} \right)$$

and $\underline{x}_{ijt}^{\ddagger} = \underline{x}_{ijt} - \Phi_{ij}\underline{z}_{ijt}$, is the linear projection of each $\underline{x}_{ijt}$ into $\underline{z}_{ijt}$, with Φ_{ij} a matrix of projections coefficients. Then

$$\begin{aligned}
\underline{x}_{ijt}^{\ddagger *} &= \left[\underline{x}_{ijt} - \Phi_{ij}\underline{z}_{ijt} - \frac{1}{T}\sum_{s=1}^{T}\left(\underline{x}_{ijs} - \Phi_{ij}\underline{z}_{ijt}\right) \right] \\
&\quad - \left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\underline{x}_{nmt} - \Phi_{nm}\underline{z}_{nmt}\right) - \frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\left(\underline{x}_{nms} - \Phi_{nm}\underline{z}_{nms}\right) \right] \\
&= \left[\underline{x}_{ijt} - \frac{1}{T}\sum_{s=1}^{T}\underline{x}_{ijs} - \frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}_{nmt} + \frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}_{nms} \right] \\
&\quad - \Phi_{ij}\left[\underline{z}_{ijt} - \frac{1}{T}\sum_{s=1}^{T}\underline{z}_{ijt} \right] + \frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\Phi_{nm}\left(\underline{z}_{nmt} - \frac{1}{T}\sum_{s=1}^{T}\underline{z}_{nms} \right) \\
&= \underline{x}_{ijt}^{*} - \left[\Phi_{ij}\underline{\widetilde{z}}_{ijt} - \frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\Phi_{nm}\underline{\widetilde{z}}_{nmt} \right] \qquad (11)
\end{aligned}$$

a. Using (11) we obtain the following expression

$$\begin{aligned}
\frac{1}{T^2}\sum_{t=1}^{T}\underline{x}^{\ddagger *}_{ijt}\underline{x}^{\ddagger *'}_{ijt} &= \frac{1}{T^2}\sum_{t=1}^{T}\left[\underline{x}^{*}_{ijt}-\left(\Phi_{ij}\widetilde{\underline{z}}_{ijt}-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\Phi_{nm}\widetilde{\underline{z}}_{nmt}\right)\right]\left[\underline{x}^{*'}_{ijt}-\left(\widetilde{\underline{z}}'_{ijt}\Phi'_{ij}-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\widetilde{\underline{z}}'_{nmt}\Phi'_{nm}\right)\right]\\
&= \frac{1}{T^2}\sum_{t=1}^{T}\underline{x}^{*}_{ijt}\underline{x}^{*'}_{ijt}-\frac{1}{T^2}\sum_{t=1}^{T}\left(\Phi_{ij}\widetilde{\underline{z}}_{ijt}-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\Phi_{nm}\widetilde{\underline{z}}_{nmt}\right)\underline{x}^{*'}_{ijt}\\
&\quad -\frac{1}{T^2}\sum_{t=1}^{T}\underline{x}^{*}_{ijt}\left(\widetilde{\underline{z}}'_{ijt}\Phi'_{ij}-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\widetilde{\underline{z}}'_{nmt}\Phi'_{nm}\right)\\
&\quad +\frac{1}{T^2}\sum_{t=1}^{T}\left(\Phi_{ij}\widetilde{\underline{z}}_{ijt}-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\Phi_{nm}\widetilde{\underline{z}}_{nmt}\right)\left(\widetilde{\underline{z}}'_{ijt}\Phi'_{ij}-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\widetilde{\underline{z}}'_{nmt}\Phi'_{nm}\right)\\
&= \frac{1}{T^2}\sum_{t=1}^{T}\underline{x}^{*}_{ijt}\underline{x}^{*'}_{ijt}-\frac{1}{T^2}O_p(T)-\frac{1}{T^2}O_p(T)+\frac{1}{T}O_p(T^{\frac{1}{2}})
\end{aligned}$$

then

$$\begin{aligned}
\frac{1}{T^2}\sum_{t=1}^{T}\underline{x}^{\ddagger *}_{ijt}\underline{x}^{\ddagger *'}_{ijt}-\frac{1}{T^2}\sum_{t=1}^{T}\underline{x}^{*}_{ijt}\underline{x}^{*'}_{ijt} &= \frac{1}{T}O_p(T^{\frac{1}{2}})-\frac{2}{T^2}O_p(T)\\
&\xrightarrow{p} 0
\end{aligned}$$

b. Based on (11) we also find

$$\begin{aligned}
\frac{1}{T^{5/2}}\sum_{t=1}^{T}\widetilde{t}\underline{x}^{\ddagger *}_{ijt} &= \frac{1}{T^{5/2}}\sum_{t=1}^{T}\widetilde{t}\left[\underline{x}^{*}_{ijt}-\left(\Phi_{ij}\widetilde{\underline{z}}_{ijt}-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\Phi_{nm}\widetilde{\underline{z}}_{nmt}\right)\right]\\
&= \frac{1}{T^{5/2}}\sum_{t=1}^{T}\widetilde{t}\underline{x}^{*}_{ijt}-\frac{1}{T^{5/2}}\sum_{t=1}^{T}O_p(T^{\frac{3}{2}})\\
&= \frac{1}{T^{5/2}}\sum_{t=1}^{T}\left(t-\frac{(T+1)}{2}\right)\underline{x}^{*}_{ijt}-\frac{1}{T^{5/2}}\sum_{t=1}^{T}O_p(T^{\frac{3}{2}})\\
&= \frac{1}{T^{5/2}}\sum_{t=1}^{T}t\underline{x}^{*}_{ijt}-\frac{(T+1)}{2T^{3/2}}\left(\frac{1}{T}\sum_{t=1}^{T}\underline{x}^{*}_{ijt}\right)-\frac{1}{T^{5/2}}\sum_{t=1}^{T}O_p(T^{\frac{3}{2}})
\end{aligned}$$

where

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T}\underline{x}^{*}_{ijt} &= \frac{1}{T}\sum_{t=1}^{T}\left(\underline{x}_{ijt}-\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{ijs}-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}_{nmt}+\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}_{nms}\right)\\
&= \frac{1}{T}\sum_{t=1}^{T}\underline{x}_{ijt}-\frac{1}{T^2}T\sum_{s=1}^{T}\underline{x}_{ijs}-\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{t=1}^{T}\underline{x}_{nmt}+\frac{1}{T}T\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}_{nms}\\
&= 0
\end{aligned}$$

Then

$$\frac{1}{T^{5/2}}\sum_{t=1}^{T}\widetilde{t}\underline{x}_{ijt}^{\ddagger *}-\frac{1}{T^{5/2}}\sum_{t=1}^{T}t\underline{x}_{ijt}^{*}=-\frac{1}{T^{5/2}}\sum_{t=1}^{T}O_p(T^{\frac{3}{2}})$$
$$=(-1)\sum_{t=1}^{T}\frac{O_p(T^{\frac{3}{2}})}{T^{5/2}}$$
$$\xrightarrow{p}(-1)\sum_{t=1}^{T}0=0$$

c. Again, from equation (11) we get

$$\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{ijt}^{\ddagger *}u_{ijt}^{*}=\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{ijt}^{*}u_{ijt}^{*}-\frac{1}{T}\sum_{t=1}^{T}\left[\Phi_{ij}\underline{\widetilde{z}}_{ijt}-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\Phi_{nm}\underline{\widetilde{z}}_{nmt}\right]u_{ijt}^{*}$$
$$=\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{ijt}^{*}u_{ijt}^{*}-\frac{1}{T}\left[\sum_{t=1}^{T}\Phi_{ij}\underline{\widetilde{z}}_{ijt}u_{ijt}^{*}-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\sum_{t=1}^{T}\Phi_{nm}\underline{\widetilde{z}}_{nmt}u_{ijt}^{*}\right)\right]$$
$$=\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{ijt}^{*}u_{ijt}^{*}-\frac{1}{T}O_p(T^{\frac{1}{2}})$$

Then

$$\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{ijt}^{\ddagger *}u_{ijt}^{*}-\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{ijt}^{*}u_{ijt}^{*}=-\frac{1}{T}O_p(T^{\frac{1}{2}})$$
$$\xrightarrow{p}0 \qquad \square$$

Appendix B. Proof of Lemma 2

Proof. a. First, we need to analyze the following term

$$\frac{1}{T^2}\sum_{t=1}^{T}\underline{x}_{ijt}^{*}\underline{x}_{ijt}^{*\prime}=\frac{1}{T^2}\sum_{t=1}^{T}\left(\underline{x}_{ijt}-\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{ijs}-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}_{nmt}+\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}_{nms}\right)$$
$$\left(\underline{x}'_{ijt}-\frac{1}{T}\sum_{s=1}^{T}\underline{x}'_{ijs}-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}'_{nmt}+\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}'_{nms}\right)$$
$$=\underbrace{\frac{1}{T^2}\sum_{t=1}^{T}\underline{x}_{ijt}\underline{x}'_{ijt}}_{(I)}+\underbrace{\frac{1}{T}\left[\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{ijs}\right]\left[\frac{1}{T}\sum_{s=1}^{T}\underline{x}'_{ijs}\right]}_{(II)}$$
$$+\underbrace{\frac{1}{T^2}\sum_{t=1}^{T}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}_{nmt}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}'_{nmt}\right]}_{(III)}$$
$$+\underbrace{\frac{1}{T}\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}_{nms}\right]\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}'_{nms}\right]}_{(IV)}$$

$$+\underbrace{\frac{1}{T^2}\sum_{t=1}^{T}\left[\underline{x}_{ijt}\left(\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}'_{nms}\right)+\left(\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}_{nms}\right)\underline{x}'_{ijt}\right]}_{(V)}$$

$$+\underbrace{\frac{1}{T^2}\sum_{t=1}^{T}\left[\left(\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{ijs}\right)\left(\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}'_{nmt}\right)+\left(\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}_{nmt}\right)\left(\frac{1}{T}\sum_{s=1}^{T}\underline{x}'_{ijs}\right)\right]}_{(VI)}$$

$$-\underbrace{\frac{1}{T^2}\sum_{t=1}^{T}\left[\underline{x}_{ijt}\left(\frac{1}{T}\sum_{s=1}^{T}\underline{x}'_{ijs}\right)+\left(\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{ijs}\right)\underline{x}'_{ijt}\right]}_{(VII)}$$

$$-\underbrace{\frac{1}{T^2}\sum_{t=1}^{T}\left[\underline{x}_{ijt}\left(\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}'_{nmt}\right)+\left(\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}_{nmt}\right)\underline{x}'_{ijt}\right]}_{(VIII)}$$

$$-\underbrace{\frac{1}{T^2}\sum_{t=1}^{T}\left[\left(\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{ijs}\right)\left(\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}'_{nms}\right)+\left(\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}_{nms}\right)\left(\frac{1}{T}\sum_{s=1}^{T}\underline{x}'_{ijs}\right)\right]}_{(IX)}$$

$$-\underbrace{\frac{1}{T^2}\sum_{t=1}^{T}\left[\left(\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}_{nmt}\right)\left(\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}'_{nms}\right)+\left(\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}_{nms}\right)\left(\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}'_{nmt}\right)\right]}_{(X)}$$

as $T\to\infty$, we have the following results for the ten factors of the previous expression

(i) $\frac{1}{T^2}\sum_{t=1}^{T}\underline{x}_{ijt}\underline{x}'_{ijt}\xrightarrow{d}\int\underline{B}_{vij}\underline{B}'_{vij}$

(ii) $\frac{1}{T}\left[\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{ijs}\right]\left[\frac{1}{T}\sum_{s=1}^{T}\underline{x}'_{ijs}\right]\xrightarrow{d}\left(\int\underline{B}_{vij}\right)\left(\int\underline{B}'_{vij}\right)$ (propositions 18.1g and 17.3f, Hamilton, 1994)

(iii) $\frac{1}{T^2}\sum_{t=1}^{T}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}_{nmt}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}'_{nmt}\right]\xrightarrow{d}\frac{1}{N^2M^2}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\underline{B}_{vnm}\underline{B}'_{vnm}$

(iv) $\frac{1}{T}\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}_{nms}\right]\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}'_{nms}\right]$
$=\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{3/2}}\sum_{s=1}^{T}\underline{x}_{nms}\right)\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{3/2}}\sum_{s=1}^{T}\underline{x}'_{nms}\right)\right]$
$\xrightarrow{d}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\underline{B}_{vnm}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\underline{B}'_{vnm}\right]$ (proposition 18.1g, Hamilton, 1994)

(v) $\frac{1}{T^2}\sum_{t=1}^{T}\left[\underline{x}_{ijt}\left(\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}'_{nms}\right)+\left(\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}_{nms}\right)\underline{x}'_{ijt}\right]$
$=\left[\frac{1}{T^{3/2}}\sum_{t=1}^{T}\underline{x}_{ijt}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{3/2}}\sum_{s=1}^{T}\underline{x}'_{nms}\right)\right]+\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{3/2}}\sum_{s=1}^{T}\underline{x}_{nms}\right)\right]\left[\frac{1}{T^{3/2}}\sum_{s=1}^{T}\underline{x}'_{ijt}\right]$
$\xrightarrow{d}\left[\int\underline{B}_{vij}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\underline{B}'_{vnm}\right]+\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\underline{B}_{vnm}\right]\left[\int\underline{B}'_{vij}\right]$ (proposition 16.1g, Hamilton, 1994)

(vi) $\frac{1}{T^2}\sum_{t=1}^{T}\left[\left(\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{ijs}\right)\left(\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}'_{nmt}\right)+\left(\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}_{nmt}\right)\left(\frac{1}{T}\sum_{s=1}^{T}\underline{x}'_{ijs}\right)\right]$
$=\left[\frac{1}{T^{3/2}}\sum_{s=1}^{T}\underline{x}_{ijs}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}\underline{x}'_{nmt}\right)\right]+\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}\underline{x}_{nmt}\right)\right]\left[\frac{1}{T^{3/2}}\sum_{s=1}^{T}\underline{x}'_{ijs}\right]$
$\xrightarrow{d}\left[\int\underline{B}_{vij}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\underline{B}'_{vnm}\right]+\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\underline{B}_{vnm}\right]\left[\int\underline{B}'_{vij}\right]$

(vii) $\frac{1}{T^2}\sum_{t=1}^{T}\left[\underline{x}_{ijt}\left(\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{ijs}'\right)+\left(\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{ijs}\right)\underline{x}_{ijt}'\right]$
$=\left[\frac{1}{T^{3/2}}\sum_{t=1}^{T}\underline{x}_{ijt}\right]\left[\frac{1}{T^{3/2}}\sum_{s=1}^{T}\underline{x}_{ijs}'\right]+\left[\frac{1}{T^{3/2}}\sum_{s=1}^{T}\underline{x}_{ijs}\right]\left[\frac{1}{T^{3/2}}\sum_{t=1}^{T}\underline{x}_{ijt}'\right]$
$\xrightarrow{d} 2\int\underline{B}_{vij}\int\underline{B}_{vij}'$

(viii) $\frac{1}{T^2}\sum_{t=1}^{T}\left[\underline{x}_{ijt}\left(\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}_{nmt}'\right)+\left(\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}_{nmt}\right)\underline{x}_{ijt}'\right]$
$=\frac{1}{NM}\left[\frac{1}{T^2}\sum_{t=1}^{T}\left(\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}_{ijt}\underline{x}_{nmt}'\right)\right]+\frac{1}{NM}\left[\frac{1}{T^2}\sum_{t=1}^{T}\left(\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}_{nmt}\underline{x}_{ijt}'\right)\right]$
$=\frac{1}{NM}\left[\frac{1}{T^2}\sum_{t=1}^{T}\underline{x}_{ijt}\underline{x}_{ijt}'\right]+\frac{1}{NM}\left[\frac{1}{T^2}\sum_{t=1}^{T}\underline{x}_{ijt}\underline{x}_{ijt}'\right]$
$\xrightarrow{d}\frac{1}{NM}\int\underline{B}_{vij}\underline{B}_{vij}'+\frac{1}{NM}\int\underline{B}_{vij}\underline{B}_{vij}'$

(ix) $\frac{1}{T^2}\sum_{t=1}^{T}\left[\left(\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{ijs}\right)\left(\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}_{nms}'\right)+\left(\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}_{nms}\right)\left(\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{ijs}'\right)\right]$
$=\left[\frac{1}{T^{3/2}}\sum_{s=1}^{T}\underline{x}_{ijs}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{3/2}}\sum_{s=1}^{T}\underline{x}_{nms}'\right)\right]+\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{3/2}}\sum_{s=1}^{T}\underline{x}_{nms}\right)\right]\left[\frac{1}{T^{3/2}}\sum_{s=1}^{T}\underline{x}_{ijs}'\right]$
$\xrightarrow{d}\left[\int\underline{B}_{vij}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\underline{B}_{vnm}'\right]+\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\underline{B}_{vnm}\right]\left[\int\underline{B}_{vij}'\right]$ (proposition 17.3f, Hamilton, 1994)

(x) $\frac{1}{T^2}\sum_{t=1}^{T}\left[\left(\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}_{nmt}\right)\left(\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}_{nms}'\right)+\left(\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}_{nms}\right)\left(\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}_{nmt}'\right)\right]$
$=\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}\underline{x}_{nmt}\right)\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{3/2}}\sum_{s=1}^{T}\underline{x}_{nms}'\right)\right]$
$+\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{3/2}}\sum_{s=1}^{T}\underline{x}_{nms}\right)\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}\underline{x}_{nmt}'\right)\right]$
$\xrightarrow{d} 2\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\underline{B}_{vnm}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\underline{B}_{vnm}'\right]$

Using the definition of $M_{11,NMT}$, the result of lemma 1 part a, and the previous results of the terms of $\frac{1}{T^2}\sum_{t=1}^{T}\underline{x}_{ijt}^{*}\underline{x}_{ijt}^{*\prime}$, for fixed N and M as $T\to\infty$,

$$M_{11,NMT}\xrightarrow{d}M_{11,NM}$$

where

$$M_{11,NMT}=\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\left(\frac{1}{T^2}\sum_{t=1}^{T}\underline{x}_{ijt}^{\ddagger *}\underline{x}_{ijt}^{\ddagger *\prime}\right)$$
$$\xrightarrow{p}\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\left[(i)+(ii)+(iii)+(iv)+(v)+(vi)-(vii)-(viii)-(ix)-(x)\right]$$
$$=\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\left\{\left[(i)+(iii)-(viii)\right]+\left[(ii)-(vii)\right]+\left[((iv)-(x))+((v)+(vi)-(ix))\right]\right\}$$
$$M_{11,NM}=\frac{NM-1}{NM}\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\int\underline{B}_{vij}\underline{B}_{vij}'-\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\int\underline{B}_{vij}\int\underline{B}_{vij}'$$
$$+\left[\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\int\underline{B}_{vij}\right]\left[\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\int\underline{B}_{vij}'\right]$$

As $N\to\infty$ and $M\to\infty$ we have the following result

- $\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\int\underline{B}_{vij}\underline{B}_{vij}'-\frac{1}{2NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\Omega_{vv,ij}\xrightarrow{p}0$

- $\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\int \underline{B}_{vij}\int \underline{B}_{vij}' - \frac{1}{3NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\Omega_{vv,ij} \xrightarrow{p} 0$
- $\left(\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\int \underline{B}_{vij}\right)\left(\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\int \underline{B}_{vij}'\right) \xrightarrow{p} 0$

then

$$M_{11,NM} - \frac{1}{6NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\Omega_{vv,ij} \xrightarrow{p} 0$$

b. Analyzing the $i-th$ column of $\mathbf{M}_{21,NMT}'$, defined in section 4, and using the result of lemma 1 part *b*

$$\frac{1}{M\sqrt{N}T^{5/2}}\sum_{j=1}^{M}\sum_{t=1}^{T}t\underline{x}_{ijt}^{*} = \frac{1}{M}\sum_{j=1}^{M}\underbrace{\left(\frac{1}{\sqrt{N}T^{5/2}}\sum_{t=1}^{T}t\underline{x}_{ijt}^{*}\right)}_{(xi)}$$

Examining (xi) for fixed j

$$\begin{aligned}
\frac{1}{\sqrt{N}T^{5/2}}\sum_{t=1}^{T}t\underline{x}_{ijt}^{*} &= \frac{1}{\sqrt{N}T^{5/2}}\sum_{t=1}^{T}t\left(\underline{x}_{ijt} - \frac{1}{T}\sum_{s=1}^{T}\underline{x}_{ijs} - \frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}_{nmt} + \frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}_{nms}\right)\\
&= \frac{1}{\sqrt{N}T^{5/2}}\sum_{t=1}^{T}t\underline{x}_{ijt} - \frac{1}{\sqrt{N}T^{7/2}}\sum_{t=1}^{T}t\sum_{s=1}^{T}\underline{x}_{ijs} - \frac{1}{MN^{3/2}T^{5/2}}\sum_{t=1}^{T}t\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}_{nmt}\\
&\quad + \frac{1}{N^{3/2}MT^{7/2}}\sum_{t=1}^{T}t\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}_{nms}\\
&= \frac{1}{\sqrt{N}}\left(\frac{1}{T^{5/2}}\sum_{t=1}^{T}t\underline{x}_{ijt}\right) - \frac{1}{\sqrt{N}}\left(\frac{1}{T^{2}}\sum_{t=1}^{T}t\right)\left(\frac{1}{T^{3/2}}\sum_{s=1}^{T}\underline{x}_{ijs}\right) - \frac{1}{MN^{3/2}}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{5/2}}\sum_{t=1}^{T}t\underline{x}_{nmt}\right)\\
&\quad + \left(\frac{1}{T^{2}}\sum_{t=1}^{T}t\right)\left(\frac{1}{N^{3/2}M}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{3/2}}\sum_{s=1}^{T}\underline{x}_{nms}\right)\right)\\
&\xrightarrow{d} \frac{1}{\sqrt{N}}\int r\underline{B}_{vij} - \frac{1}{2\sqrt{N}}\int \underline{B}_{vij} - \frac{1}{MN^{3/2}}\sum_{n=1}^{N}\sum_{m=1}^{M}\int r\underline{B}_{vnm} + \frac{1}{2}\left(\frac{1}{N^{3/2}M}\sum_{n=1}^{N}\sum_{m=1}^{M}\int \underline{B}_{vnm}\right)\\
&= \frac{1}{\sqrt{N}}\left[\int r\underline{B}_{vij} - \frac{1}{2}\int \underline{B}_{vij}\right] - \frac{1}{N^{3/2}M}\sum_{n=1}^{N}\sum_{m=1}^{M}\left[\int r\underline{B}_{vnm} - \frac{1}{2}\int \underline{B}_{vnm}\right]
\end{aligned}$$

then

$$\mathbf{M}_{21,NMT}' = \left[\frac{1}{\sqrt{N}MT^{\frac{5}{2}}}\sum_{j=1}^{M}\sum_{t=1}^{T}\widetilde{t\underline{x}}_{1jt}^{\ddagger *},\ \cdots\ ,\frac{1}{\sqrt{N}MT^{\frac{5}{2}}}\sum_{j=1}^{M}\sum_{t=1}^{T}\widetilde{t\underline{x}}_{Njt}^{\ddagger *}\right] \xrightarrow{d} \mathbf{M}_{21,NM}'$$

where $\left[\mathbf{M}_{21,NM}'\right]_i$ is the *i-th* column of the matrix $\mathbf{M}_{21,NM}'$ and

$$\left[\mathbf{M}_{21,NM}'\right]_i = \frac{1}{M}\sum_{j=1}^{M}\frac{1}{\sqrt{N}}\left[\left(\int r\underline{B}_{vij} - \frac{1}{2}\int \underline{B}_{vij}\right) - \frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\int r\underline{B}_{vnm} - \frac{1}{2}\int \underline{B}_{vnm}\right)\right]$$

As $N\to\infty$, $\left[\mathbf{M}_{21,NM}'\right]_i \xrightarrow{p} 0$ for all i and fixed M, then

$$\mathbf{M}_{21,NM}' \xrightarrow{p} 0$$

c. Similar to the proof of lemma 2 part *b*.

d. From previous definitions, $\mathbf{M}_{22,NMT} = \left(\frac{1}{T^3}\sum_{t=1}^{T}\widetilde{t}^{\,2}\right)\mathbf{I}_N$, with

$$\begin{aligned}
\frac{1}{T^3}\sum_{t=1}^{T}\widetilde{t}^{\,2} &= \frac{1}{T^3}\sum_{t=1}^{T}\left[t-\frac{(T+1)}{2}\right]^2\\
&= \frac{1}{T^3}\sum_{t=1}^{T}\left[t^2-\frac{2t(T+1)}{2}+\frac{(T+1)^2}{4}\right]\\
&= \frac{1}{T^3}\sum_{t=1}^{T}t^2-\frac{(T+1)}{T^3}\sum_{t=1}^{T}t+\frac{(T+1)^2}{4T^2}\\
&= \frac{(T+1)(2T+1)}{6T^2}-\frac{(T+1)^2}{2T^2}+\frac{(T+1)^2}{4T^2}\\
&\xrightarrow[T\to\infty]{} \frac{1}{3}-\frac{1}{2}+\frac{1}{4}=\frac{1}{12}
\end{aligned}$$

then

$$\mathbf{M}_{22,NMT} \xrightarrow{p} \frac{1}{12}\mathbf{I}_N$$

e. From section 4.

$$\begin{aligned}
\mathbf{M}'_{32,NMT} &= \left[\frac{1}{\sqrt{NM}T^3}\sum_{t=1}^{T}\widetilde{t}^{\,2}\right](1)_{N\times M}\\
&\xrightarrow[T\to\infty]{} \frac{1}{\sqrt{NM}}\frac{1}{12}(1)_{N\times M}\\
&\xrightarrow{p} 0 \quad \text{as } N\to\infty \text{ and } M\to\infty
\end{aligned}$$

f. Using lemma 2 part *d*

$$\mathbf{M}_{33,NMT} \xrightarrow{p} \frac{1}{12}\mathbf{I}_M$$

□

Appendix C. Proof of Lemma 3

Proof. *a*. First, we analyze the following term

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T}\underline{x}^*_{ijt}u^*_{ijt} =& \frac{1}{T}\sum_{t=1}^{T}\left(\underline{x}_{ijt}-\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{ijs}-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}_{nmt}+\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}_{nms}\right)\\
&\left(u_{ijt}-\frac{1}{T}\sum_{s=1}^{T}u_{ijs}-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}u_{nmt}+\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}u_{nms}\right)\\
=& \underbrace{\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{ijt}u_{ijt}}_{(i)}+\underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}_{nmt}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}u_{nmt}\right]}_{(ii)}\\
&+\underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{ijs}\right]\left[\frac{1}{T}\sum_{s=1}^{T}u_{ijs}\right]}_{(iii)}+\underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}_{nms}\right]\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}u_{nms}\right]}_{(iv)}
\end{aligned}$$

$$\underbrace{-\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{ijt}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}u_{nmt}\right]}_{(v)}\underbrace{-\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{ijt}\left[\frac{1}{T}\sum_{s=1}^{T}u_{ijs}\right]}_{(vi)}+\underbrace{\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{ijt}\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}u_{nms}\right]}_{(vii)}$$

$$\underbrace{-\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}_{nmt}\right]u_{ijt}}_{(viii)}+\underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}_{nmt}\right]\left[\frac{1}{T}\sum_{s=1}^{T}u_{ijs}\right]}_{(ix)}$$

$$\underbrace{-\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}_{nmt}\right]\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}u_{nms}\right]}_{(x)}\underbrace{-\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{ijs}\right]u_{ijt}}_{(xi)}$$

$$+\underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{ijs}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}u_{nmt}\right]}_{(xii)}\underbrace{-\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{ijs}\right]\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}u_{nms}\right]}_{(xiii)}$$

$$+\underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}_{nms}\right]u_{ijt}}_{(xiv)}\underbrace{-\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}_{nms}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}u_{nmt}\right]}_{(xv)}$$

$$\underbrace{-\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}_{nms}\right]\left[\frac{1}{T}\sum_{s=1}^{T}u_{ijs}\right]}_{(xvi)}$$

with

(i) $\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{ijt}u_{ijt}\xrightarrow{d}\sqrt{\Omega_{uu,ij}}\int\underline{B}_{vij}dW_{uij}$

(ii) $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}_{nmt}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}u_{nmt}\right]\xrightarrow{d}\frac{1}{(NM)^2}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}\int\underline{B}_{vnm}dW_{unm}$

(iii) $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{ijs}\right]\left[\frac{1}{T}\sum_{s=1}^{T}u_{ijs}\right]\xrightarrow{d}\left[\int\underline{B}_{vij}\right]\sqrt{\Omega_{uu,ij}}W_{uij}(1)$

(iv) $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}_{nms}\right]\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}u_{nms}\right]\xrightarrow{d}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\underline{B}_{vnm}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}W_{unm}(1)\right]$

(v) $\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{ijt}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}u_{nmt}\right]\xrightarrow{d}\frac{1}{NM}\sqrt{\Omega_{uu,ij}}\int\underline{B}_{vij}dW_{uij}$

(vi) $\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{ijt}\left[\frac{1}{T}\sum_{s=1}^{T}u_{ijs}\right]\xrightarrow{d}\left[\int\underline{B}_{vij}\right]\sqrt{\Omega_{uu,ij}}W_{uij}(1)$

(vii) $\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{ijt}\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}u_{nms}\right]\xrightarrow{d}\left[\int\underline{B}_{vij}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}W_{unm}(1)\right]$

(viii) $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}_{nmt}\right]u_{ijt}\xrightarrow{d}\frac{1}{NM}\sqrt{\Omega_{uu,ij}}\int\underline{B}_{vij}dW_{uij}$

(ix) $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}_{nmt}\right]\left[\frac{1}{T}\sum_{s=1}^{T}u_{ijs}\right]\xrightarrow{d}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\underline{B}_{vnm}\right]\sqrt{\Omega_{uu,ij}}W_{uij}(1)$

(x) $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\underline{x}_{nmt}\right]\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}u_{nms}\right]\xrightarrow{d}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\underline{B}_{vnm}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}W_{unm}(1)\right]$

(xi) $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{ijs}\right]u_{ijt}\xrightarrow{d}\left[\int\underline{B}_{vij}\right]\sqrt{\Omega_{uu,ij}}W_{uij}(1)$

(xii) $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{ijs}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}u_{nmt}\right]\xrightarrow{d}\left[\int\underline{B}_{vij}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}W_{unm}(1)\right]$

(xiii) $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{T}\sum_{s=1}^{T}\underline{x}_{ijs}\right]\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}u_{nms}\right]\xrightarrow{d}\left[\int\underline{B}_{vij}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}W_{unm}(1)\right]$

(xiv) $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}_{nms}\right]u_{ijt}\xrightarrow{d}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\underline{B}_{vnm}\right]\sqrt{\Omega_{uu,ij}}W_{uij}(1)$

(xv) $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}_{nms}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}u_{nmt}\right]\xrightarrow{d}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\underline{B}_{vnm}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}W_{unm}(1)\right]$

(xvi) $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\underline{x}_{nms}\right]\left[\frac{1}{T}\sum_{s=1}^{T}u_{ijs}\right]\xrightarrow{d}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\underline{B}_{vnm}\right]\sqrt{\Omega_{uu,ij}}W_{uij}(1)$

Using the previous results and lemma 1 part c

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\underline{x}_{ijt}^{\ddagger *}u_{ijt}^{*}\xrightarrow{d}&\left[\frac{NM-2}{NM}\right]\sqrt{\Omega_{uu,ij}}\int\underline{B}_{vij}dW_{uij}+\frac{1}{N^2M^2}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}\int\underline{B}_{vnm}dW_{unm}\\&-\left[\int\underline{B}_{vij}\right]\sqrt{\Omega_{uu,ij}}W_{uij}(1)-\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\underline{B}_{vnm}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}\sqrt{\Omega_{uu,nm}}W_{unm}(1)\right]\\&+\left[\int\underline{B}_{vij}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}W_{unm}(1)\right]+\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\underline{B}_{vnm}\right]\left[\sqrt{\Omega_{uu,ij}}W_{uij}(1)\right]\end{aligned}$$

Summing over i and j and dividing the result by $\sqrt{NM}$, we obtain

$$\begin{aligned}&\left[\frac{NM-2}{NM}\right]\frac{1}{\sqrt{NM}}\sum_{i=1}^{N}\sum_{j=1}^{M}\sqrt{\Omega_{uu,ij}}\int\underline{B}_{vij}dW_{uij}+\frac{1}{(NM)^{\frac{3}{2}}}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}\int\underline{B}_{vnm}dW_{unm}\\&-\frac{1}{\sqrt{NM}}\sum_{i=1}^{N}\sum_{j=1}^{M}\left[\int\underline{B}_{vij}\sqrt{\Omega_{uu,ij}}W_{uij}(1)\right]-\frac{1}{(NM)^{\frac{3}{2}}}\left[\sum_{n=1}^{N}\sum_{m=1}^{M}\int\underline{B}_{vnm}\right]\left[\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}W_{unm}(1)\right]\\&+\left[\frac{1}{\sqrt{NM}}\sum_{i=1}^{N}\sum_{j=1}^{M}\int\underline{B}_{vij}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}W_{unm}(1)\right]\\&+\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\underline{B}_{vnm}\right]\left[\frac{1}{\sqrt{NM}}\sum_{i=1}^{N}\sum_{j=1}^{M}\sqrt{\Omega_{uu,ij}}W_{uij}(1)\right]\end{aligned}$$

$$\begin{aligned}=&\left[\frac{NM-1}{NM}\right]\frac{1}{\sqrt{NM}}\sum_{i=1}^{N}\sum_{j=1}^{M}\sqrt{\Omega_{uu,ij}}\left(\int\underline{B}_{vij}dW_{uij}-\int\underline{B}_{vij}\int dW_{uij}\right)-\frac{1}{(NM)^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{M}\int B_{vij}\sqrt{\Omega_{uu,ij}}W_{uij}(1)\\&+\frac{1}{(NM)^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{M}\int B_{vij}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}W_{unm}(1)\\=&\left[\frac{NM-1}{NM}\right]\frac{1}{\sqrt{NM}}\sum_{i=1}^{N}\sum_{j=1}^{M}\sqrt{\Omega_{uu,ij}}\left(\int\underline{B}_{vij}dW_{uij}-\int\underline{B}_{vij}\int dW_{uij}\right)\\&-\frac{1}{(NM)^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{M}\int B_{vij}\left(\sqrt{\Omega_{uu,ij}}W_{uij}(1)-\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}W_{unm}(1)\right)\end{aligned}$$

Therefore

$$\underline{m}_{1,NM}=\left[\frac{NM-1}{NM}\right]\frac{1}{\sqrt{NM}}\sum_{i=1}^{N}\sum_{j=1}^{M}\sqrt{\Omega_{uu,ij}}\int\underline{\tilde{B}}_{vij}dW_{uij}-\frac{1}{NM}O_{NM}(1)$$

where $\widetilde{B}_{vij}=\underline{B}_{vij}-\int\underline{B}_{vij}$

b. First, we need to establish the asymptotic normality of $\left[\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\mathbf{U}_{ij}\right]^{-1}\left[\frac{1}{\sqrt{NM}}\sum_{i=1}^{N}\sum_{j=1}^{M}\underline{e}_{ij}\right]$, where $\mathbf{U}_{ij}=\int\underline{B}_{vij}\underline{B}'_{vij}$ and $\underline{e}_{ij}=\sqrt{\Omega_{uu,ij}}\int\underline{B}_{vij}dW_{uij}$.

Working with $\{\underline{e}_{Lij}\}_{i,j=0}^{\infty}$ and $L=NM$, we have

i. $\overline{\underline{e}}_{Lij}=\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\underline{e}_{Lij}$
ii. $\underline{\mu}_{Lij}=E(\underline{e}_{Lij})=E\{E(\underline{e}_{Lij}|\mathbf{U}_{Lij})\}=E\{\underline{0}\}=0$
iii. $\overline{\underline{\mu}}_{Lij}=\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\underline{\mu}_{Lij}=0$
iv. $\mathbf{V}_{Lij}=Var(\underline{e}_{Lij})=E(\underline{e}_{Lij}\underline{e}'_{Lij})=E\{E(\underline{e}_{Lij}\underline{e}'_{Lij}|\mathbf{U}_{Lij})\}=E\{\Omega_{uu,Lij}\mathbf{U}_{Lij}\}=\frac{1}{6}\Omega_{uu,Lij}\mathbf{\Omega}_{vv,Lij}$
v. $E\|\underline{e}_{Lij}\|^{2+\delta}=E\|\sqrt{\Omega_{uu,Lij}}\int\underline{B}_{v,Lij}dW_{u,Lij}\|^{2+\delta}$

$$=E\|\sqrt{\Omega_{uu,Lij}}\mathbf{\Omega}_{vv,Lij}^{1/2}\int\underline{W}_{v,Lij}dW_{u,Lij}\|^{2+\delta}$$
$$\leqslant\|\sqrt{\Omega_{uu,Lij}}\mathbf{\Omega}_{vv,Lij}^{1/2}\|^{2+\delta}E\|\int\underline{W}_{v,Lij}dW_{u,Lij}\|^{2+\delta}<\Delta<\infty$$

vi. $\mathbf{V}_{L}=\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\mathbf{V}_{Lij}=\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}Var(\underline{e}_{Lij})=\frac{1}{6NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\Omega_{uu,Lij}\mathbf{\Omega}_{vv,Lij}$

Since $\mathbf{\Omega}_{vv,Lij}$ is positive definite for all i,j, $\mathbf{V}_L$ is $O(1)$ and uniformly positive definite. Using theorem 5.11 of White (2001) follows that as $T\longrightarrow\infty$, $N\longrightarrow\infty$ and $M\longrightarrow\infty$

$$V_{NM}^{-\frac{1}{2}}\frac{1}{\sqrt{NM}}\sum_{i=1}^{N}\sum_{j=1}^{M}\underline{e}_{ij}\xrightarrow{d}N(0,\mathbf{I})$$

where $\mathbf{V}_{NM}=Var\left(\frac{1}{\sqrt{NM}}\sum_{i=1}^{N}\sum_{j=1}^{M}\underline{e}_{ij}\right)=\frac{1}{6NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\Omega_{uu,ij}\mathbf{\Omega}_{vv,ij}$.

c. Analyzing part of the $i-th$ element of $\underline{m}_{2,NM}$, for fixed N, M as $T\longrightarrow\infty$

$$\frac{1}{T^{3/2}}\sum_{t=1}^{T}\widetilde{t}u_{ijt}^{*}=\frac{1}{T^{3/2}}\sum_{t=1}^{T}tu_{ijt}^{*}$$
$$=\frac{1}{T^{3/2}}\sum_{t=1}^{T}tu_{ijt}-\left[\frac{1}{T^2}\sum_{t=1}^{T}t\right]\left[\frac{1}{T^{1/2}}\sum_{s=1}^{T}u_{ijs}\right]-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left[\frac{1}{T^{3/2}}\sum_{t=1}^{T}tu_{nmt}\right]$$
$$+\left[\frac{1}{T^2}\sum_{t=1}^{T}t\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{1/2}}\sum_{s=1}^{T}u_{nms}\right)\right]$$
$$=\frac{1}{T^{3/2}}\sum_{t=1}^{T}tu_{ijt}-\left(\frac{1}{2}+\frac{1}{2T}\right)\left[\frac{1}{T^{1/2}}\sum_{s=1}^{T}u_{ijs}\right]-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left[\frac{1}{T^{3/2}}\sum_{t=1}^{T}tu_{nmt}\right]$$
$$+\left(\frac{1}{2}+\frac{1}{2T}\right)\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{1/2}}\sum_{s=1}^{T}u_{nms}\right)\right]$$
$$\xrightarrow{d}\sqrt{\Omega_{uu,ij}}\left[W_{uij}(1)-\int_0^1W_{uij}(r)dr\right]-\frac{1}{2}\sqrt{\Omega_{uu,ij}}W_{uij}(1)$$
$$-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}\left[W_{unm}(1)-\int_0^1W_{unm}(r)dr\right]+\frac{1}{2}\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}W_{unm}(1)$$
$$=\sqrt{\Omega_{uu,ij}}\left(\int rdW_{uij}\right)-\frac{1}{2}\sqrt{\Omega_{uu,ij}}W_{uij}(1)$$
$$-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}\left(\int rdW_{unm}\right)+\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}\frac{1}{2}W_{unm}(1)$$

$$=\sqrt{\Omega_{uu,ij}}\left(\int rdW_{uij}-\frac{1}{2}W_{uij}(1)\right)-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}\left(\int rdW_{unm}-\frac{1}{2}W_{unm}(1)\right)$$

Then

$$\frac{1}{T^{3/2}}\sum_{t=1}^{T}\widetilde{t}u_{ijt}^{*}\xrightarrow{d}\sqrt{\Omega_{uu,ij}}\left(\int rdW_{uij}-\frac{1}{2}W_{uij}(1)\right)-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}\left(\int rdW_{unm}-\frac{1}{2}W_{unm}(1)\right)$$

from lemma 3 part a, $\underline{m}_{1,NMT}\xrightarrow{p}\underline{m}_{1,NM}$, and

$$\underline{m}_{1,NM}=\left[\frac{NM-1}{NM}\right]\frac{1}{\sqrt{NM}}\sum_{i=1}^{N}\sum_{j=1}^{M}\sqrt{\Omega_{uu,ij}}\int\underline{\tilde{B}}_{vij}dW_{uij}-\frac{1}{NM}O_{NM}(1)$$

Using these results and an extension of proposition 4 part (a) from Mark and Sul (2002), we have that as $T\longrightarrow\infty$, $N\rightarrow\infty$ and $M\rightarrow\infty$, $\underline{m}_{1,NM}$ is independent of $\underline{m}_{2,NM}$.

The proof of the asymptotic independence of $\underline{m}_{1,NM}$ and $\underline{m}_{3,NM}$ follows in a similar way.

□

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