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A New Family of Irrational Numbers with Curious Properties

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1. INTRODUCTION

The "*metallic means family*" (MMF) includes all the quadratic irrational numbers that are positive solutions of algebraic equations of the type

$$x^2 - nx - 1 = 0$$
$$x^2 - x - n = 0$$

where n is a natural number. The most outstanding member of the MMF is the well-known "golden number." Then we have the silver number, the bronze number, the copper number, the nickel number and many others. The golden number has been widely used by a great number of very old cultures, as a base of proportions to compose music, to create sculptures and paintings or to build temples and palaces (see the first chapter of Reference [1]). With respect to the many relatives of the golden number, a great part of them have been used in different researches that analyze the behavior of non linear dynamical systems when they proceed from a periodic régime to a chaotic one. Notwithstanding, there exist many instances of application of these numbers in quite different knowledge fields, like the one described by the mathematician Jay Kappraff [2] in his study of the old Roman proportion system of construction. This system was based on the silver number, on account of a mathematical property, which is not unique but is common to all members of the MMF, as we shall prove.

Being irrational numbers, all the members of the MMF have to be approximated by ratios of integer numbers in applications to different scientific fields. The analysis of the relation between the members of the MMF and their approximate ratios is one of the goals of this paper.

2. CONTINUED FRACTION EXPANSIONS

The expansion of a real number in continued fractions is one of the most useful tools of arithmetic. Every real number x may be expanded in continued fractions

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}$$

denoted by $x = [a_0, a_1, a_2, \ldots]$. The first coefficient can be equal to zero (whenever the real number is between 0 and 1), but the rest of the coefficients are positive integers. The sequence of coefficients is <u>finite</u> if and only if x is a rational number, like

$$\frac{18}{7} = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3}}} = [2,1,1,3].$$

If, instead, x is an irrational number, the expansion is infinite and if we take a finite number of terms like

$$\sigma_k = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\dots + \cfrac{1}{a_k}}}$$

we get a sequence of "*rational approximants*" to the number x that converge to x when $k \longrightarrow \infty$.

Some irrational numbers like π and *e* have approximants that converge very quickly. The number $\pi = [3, 7, 15, 1, 292, ...]$ converges so quickly that the third rational approximant $\sigma_3 = \frac{335}{113} = 3.1415929\ldots$ has six exact decimal figures. Tsu Chung Chi, China, 5th century, already knew this result! Instead $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 2, 2, 8, 1, ...]$ converges more slowly at the beginning, due to the existence of many "ones" in the expansion. The quadratic irrational numbers converge more slowly. "*Periodic*" continued fraction expansions are denoted with a bar over the period, and if the expansion is of the form $x = [\overline{a_0, a_1, \ldots, a_n}]$, we call it a "*purely periodic*" continued fraction. The French mathematician Joseph Louis Lagrange (1736-1813) proved that a number is quadratic irrational if and only if its continued fraction expansion is periodic (not necessarily purely periodic).

Property No. 1 of the MMF

They are all positive quadratic irrational numbers.

In fact, if we solve the quadratic equation

(2.1) $$x^2 - nx - 1 = 0$$

we find that the positive solutions are of the form $x = \frac{n + \sqrt{n^2 + 4}}{2}$. For $n = 1$, we have the well-known golden number $\phi = \frac{1 + \sqrt{5}}{2} = 1.618\ldots$. How do we get its continued fraction expansion? Simply, divide equation (2.1) by x (not zero): $x = n + \frac{1}{x}$ and replace the x of the right member iteratively by $n + 1/x$. In this way, we get after N iterations:

$$x = n + \cfrac{1}{n + \cfrac{1}{n + \cfrac{1}{\ldots + \cfrac{1}{n + \cfrac{1}{x}}}}}$$

If $N \longrightarrow \infty$, then $x = n + \cfrac{1}{n + \cfrac{1}{\ddots}} = [\overline{n}]$, that is, a purely periodic continued fraction. Obviously, the golden number has the most simple continued fraction expansion of all the metallic numbers

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}} = [\overline{1}].$$

Similarly, solving the quadratic equation $x^2 - 2x - 1 = 0$, whose positive root is the silver number, we get its purely periodic continued fraction expansion

$$\sigma_{Ag} = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}} = [\overline{2}].$$

Solving $x^2 - 3x - 1 = 0$, we get the bronze number

$$\sigma_{Ag} = \frac{3 + \sqrt{13}}{2} = [\overline{3}].$$

Summarizing, looking for positive solutions of quadratic equations of the type $x^2 - nx - 1 = 0$ (n natural number), we obtain members of the MMF whose continued fraction expansion is purely periodic $x = [\overline{n}]$. Similarly, the positive solutions of quadratic equations

(2.2) $$x^2 - x - n = 0,$$

(n natural number), are members of the MMF whose continued fraction expansion is periodic $[m, \overline{n_1, n_2, \ldots, n_n}]$. Some of these members are natural numbers of the form $[n, \overline{0}]$. This subset of metallic numbers has curious mathematical properties related to the frequency with which natural numbers appear as well as to the period length or the appearance of "*stable cycles*" (see [3]).

Of all the members of the MMF, the golden number has the most slowly convergent expansion, that is

The golden number ϕ is the most irrational of all irrational numbers.

It is easy to prove that it is sufficient to consider the positive solutions of equation (2.1) and (2.2), since in the remaining cases we have the following results:

a) $x^2 + nx - 1 = 0$. Same solutions as (2.1) but only its decimal part.
b) $x^2 + nx + 1 = 0$. No positive solutions.
c) $x^2 - nx + 1 = 0$. Positive solutions have periodic continued fractions expansions.
d) $x^2 + x - n = 0$. Positive solutions have periodic continued fractions expansions.
e) $x^2 + x + n = 0$. No positive solutions.
f) $x^2 - x + n = 0$. No positive solutions.

3. FIBONACCI SEQUENCES

A Fibonacci sequence is constructed by taking each term equal to the sum of the two precedents. Beginning with F(0) = 1; F(1) = 1, we have

(3.1) 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

where

(3.2) $F(n+1) = F(n) + F(n-1)$.

This sequence may be generalized by taking each term equal to a linear combination of the two precedents a, b, $pb + qa$, $p(pb + qa) + qb$, ... These sequences are called "*generalized Fibonacci sequences,*" (GFS) and they satisfy relations like

(3.3) $G(n+1) = pG(n) + qG(n-1)$

with p and q natural numbers. Dividing both members by G(n) we have

$$\frac{G(n+1)}{G(n)} = p + q\frac{G(n-1)}{G(n)} = p + \frac{q}{\frac{G(n)}{G(n-1)}}.$$

Taking limits and assuming that $\lim_{n\to\infty}\frac{G(n+1)}{G(n)}$ exists and is equal to a real number x (for a proof see Reference [4]), we get $x = p + \frac{q}{x}$ or $x^2 - px - q = 0$, whose positive solution is $x = \frac{p + \sqrt{p^2 + 4q}}{2}$. This implies that

(3.4) $$\lim_{n\to\infty}\frac{G(n+1)}{G(n)} = \frac{p + \sqrt{p^2 + 4q}}{2}$$

Let us now verify the following property

Property No. 2 of the MMF

They are all limits of ratios of successive terms of GFS.

Indeed, let us assume $G(0) = G(1) = 1$ and choose $p = q = 1$ in (3.4). The sequence coincides with (3.1), and, as is well known, the ratio of two successive terms of it converges to the golden number

$$x = \frac{1 + \sqrt{5}}{2} = \phi = [\bar{1}].$$

If $p = 2$, $q = 1$, the sequence is

(3.5) 1, 1, 3, 7, 17, 41, 99, 140, ...

and similarly we get the silver number,

$$\sigma_{Ag} = \lim_{n\to\infty}\frac{G(n+1)}{G(n)} = [\bar{2}]..$$

If $p = 3$, $q = 1$, the sequence

(3.6) 1, 1, 4, 13, 43, 142, 469, ...

gives the bronze number,

$$\sigma_{Br} = \lim_{n\to\infty}\frac{G(n+1)}{G(n)} = \frac{3 + \sqrt{13}}{2} = [\bar{3}]..$$

If $p = 1$, $q = 2$, the sequence

(3.7) 1, 1, 3, 5, 11, 21, 43, 85, ...

gives the copper number $\sigma_{Cu} = 2 = [2, \bar{0}]$. Finally, if $p = 1$, $q = 3$, the sequence 1, 1, 4, 7, 19, 40, 97, ... gives rise to the nickel number, $\sigma_{Ni} = \frac{1+\sqrt{13}}{2} = [2,\bar{3}]$.

4. ADDITIVE PROPERTIES

If we consider the sequence of ratios of consecutive terms of (3.1)

$$\frac{1}{1},\frac{2}{1},\frac{3}{2},\frac{5}{3},\frac{8}{5},\frac{13}{8},\frac{21}{13},\frac{34}{21},\frac{55}{34},\frac{89}{55},\ldots$$

we know that this sequence converges to the golden number ϕ. Let us now form a geometric progression of ratio ϕ: $\ldots,\frac{1}{\phi^2},\frac{1}{\phi},1,\phi,\phi^2,\phi^3,\ldots$ This geometric progression is also a Fibonacci sequence satisfying condition (3.2), as is easy to prove: $\frac{1}{\phi^2}+\frac{1}{\phi}=\frac{1+\phi}{\phi^2}=1$. The same happens with the silver number σ_{Ag}, starting with the sequence $\frac{1}{1},\frac{3}{1},\frac{7}{3},\frac{17}{7},\frac{41}{17},\frac{99}{41},\frac{140}{99},\ldots$ that converges to the silver number. Indeed the sequence $\ldots\frac{1}{\sigma_{Ag}{}^2},\frac{1}{\sigma_{Ag}},1,\sigma_{Ag},\sigma_{Ag}{}^2,\sigma_{Ag}{}^3,\ldots$ is a geometric progression of ratio σ_{Ag} that satisfies (3.4) with $p = 2$, $q =$ 1, as is easy to verify by the following equalities:

$$\frac{1}{\sigma_{Ag}{}^2}+2=\sigma_{Ag};1+2\sigma_{Ag}=\sigma_{Ag}{}^2;+2=\sigma_{Ag}{}^3;\ldots$$

The same procedure can be applied to every member of the MMF and we may assert

Property No. 3 of the MMF

They are the only positive quadratic irrational numbers that originate geometric progressions that simultaneously satisfy additive properties.

This curious property endows all the members of the MMF with interesting characteristics that become a base of different proportion systems in design.

5. QUASICRYSTALS: FORBIDDEN SYMMETRIES

Among the many physical, chemical, biological and ecological problems in which the members of the MMF appear, one of the most striking is the study of a quasicrystal structure. "*Crystals*" are the most regular, periodic and symmetric of all the real entities. On the opposite edge, there exist the most disordered or amorph configurations: the "*glasses*." How do we distinguish between a crystal and a glass? The answer is very simple: you may model a real crystal by putting an atom or a molecule at every vertex of a regular triangular, square or hexagonal lattice that enjoys symmetry of order 3, 4 and 6, respectively. In such a way, the problem of the structure of matter becomes one of pure geometry. This was the situation until 1984, when Shechtman et al. [5], [6], registering electron diffraction patterns in a rapidly cooled metal alloy, found a fivefold rotational symmetry when making projections with an angle equal to the golden number (see [4]). This new solid state of matter was called a "*quasicrystal*." As is well known, it is impossible to tile a plane using only shapes that have fivefold symmetry. But the two-dimensional analogue of quasicrystals – Penrose tilings – has this symmetry. It is interesting to note that the golden number arises in this tiling. The tiling is made of two types of parallelograms, and in an infinite Penrose tiling, the ratio between the numbers of units in these two types is the golden number.

Starting with this experimental discovery, there appeared experimentally new quasicrystals with other forbidden symmetries. For example, the silver number $\sigma_{Ag} = 1+\sqrt{2} = [\bar{2}]$, generates a quasicrystal enjoying a forbidden symmetry of order 8 (see [7], [8]), while the number $[\bar{2}] = \phi^3$ appears in another forbidden symmetry of order 12 (see [9]).

In particular, Gumbs, Ali et al., in different papers ([10],[11]), have analyzed electronic, optical, acoustic and superconducting properties of quasi-periodic systems. The research consisted of the construction of one-dimensional models of new types of quasicrystals, designed taking as a module the different members of the MMF. They were very interested in these quasicrystals, due to many important physical applications, like light transmission through a multi-layered medium. Among their most prominent experimental results, they found big differences in the behavior of metallic numbers whose continued fraction expansion was purely periodic (the golden number, the silver number and the bronze number) and those metallic numbers whose continued fraction expansion was only periodic (the copper number and the nickel number).

6. CONCLUSIONS

In analyzing, from a mathematical point of view, the similarities as well as the differences among the members of the MMF, it is obvious that these characteristics are strongly linked with the transition from periodic to quasi-periodic dynamics. But simultaneously, from the beginning of humanity, there have been philosophical, natural and aesthetic considerations that have given them primacy in the establishment of geometrical proportions based on some members of this family. Such a broad range of applications opens the road to new multi-disciplinary investigations that undoubtedly will contribute to clarifying the relations between art and technology, building a bridge that should join rational scientific thinking with aesthetical emotion. Hopefully, this new perspective could help us to confer on technology, from which we depend every day more and more for our survival, a more human character.

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