Preparation and toolkit learning

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Abstract: A product set of pure strategies is a prep set ('prep' is short for 'preparation') if it contains at least one best reply to any consistent belief that a player may have about the strategic behavior of his opponents. Minimal prep sets are shown to exist in a class of strategic games satisfying minor topological conditions. The concept of minimal prep sets is compared with (pure and mixed) Nash equilibria, minimal curb sets, and rationalizability. Additional dynamic motivation for the concept is provided by a model of adaptive play that is shown to settle down in minimal prep sets.

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1 Introduction

Simplicity, the bias towards strategic behavior of low complexity, is a major driving force in actual decision making and game theory. This can be due to various reasons: complex plans may be hard to implement and likely to break down, since they are more susceptible to mistakes. They are more difficult to learn, and a decision maker may not have access to extremely intricate strategies due to bounds on his rationality or cognition.

Pure Nash equilibria of strategic games have several advantages over equilibria in mixed strategies, most of which boil down to their simplicity. In the first place, they provide clear and unambiguous recommendations by avoiding complicated randomization devices. In the second place, pure strategies are directly observable, but it is hard to observe mixed strategies, unless as an aggregate of long run behavior. Moreover, mixed strategies may in some occasions not be reasonable objects of choice; see for instance the discussion in section 3.2 of Osborne and Rubinstein (1994). Finally, experimental evidence seems to point out that players relatively easily learn to play pure equilibria, while it is intuitively much more difficult to learn the exact probability measures that constitute a mixed equilibrium. Selten *et al.* (2001, p. 23), for instance, state that:

> "...under favorable conditions the game theoretic notion of a pure strategy equilibrium is strongly supported by the observed behavior of sophisticated players. Of course the tendency towards pure strategy equilibrium plays was much less pronounced in the beginning. The game theoretic equilibrium notion is not naturally present in the minds of most unexperienced subjects. But, as far as pure strategy equilibrium is concerned, it is clearly learned in repeated tournaments."

On the other hand (ibid, p. 16):

> "In the case of games without pure strategy equilibria, we do not observe a tendency towards a similarly clear answer to the question how to play such games."

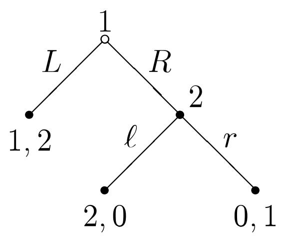
The attraction of pure equilibria induces an interest in strategic games possessing such equilibria, like the different classes of potential games studied by Monderer and Shapley (1996), Voorneveld and Norde (1997), and Voorneveld (2000). In general, however, existence of pure equilibria is not guaranteed.

This paper introduces a *set-valued extension* of the pure Nash equilibrium concept that is shown to give a nonempty set of recommendations in a large class of strategic games. Players, whether they are organizations (households, firms, boards of directors, etc.) or individual decision makers, often do not stick to playing a single action. Rather, they seek recourse to a certain 'toolkit' of strategies that is supposed to provide optimal responses to the eventualities expected by each player in the game. This insight has generated a body of literature suggesting set-valued solutions instead of the usual point-valued solutions like the Nash equilibrium concept and its refinements. Examples of such set-valued solution concepts include the product sets of minimax/maximin strategies in two-player zero-sum games (von Neumann, 1928) or rationalizable strategies (Bernheim, 1984, Pearce, 1984), persistent retracts (Kalai and Samet, 1984), curb sets (Basu and Weibull, 1991), and cyclically stable sets (Matsui, 1992).

The set-valued solution concept introduced in this paper combines a standard rationality condition, stating that the set of recommended strategies to each player must contain at least one best reply to whatever belief he may have that is consistent with the recommendations to the other players, with the players' aim at simplicity, which encourages them to maintain a set of strategies that is as small as possible. These are two opposite effects in our *minimal prep sets* ('prep' is short for 'preparation'). On one hand, each player has to be prepared: each player's toolkit must be sufficiently large, so that it contains at least one best reply against any belief he may entertain about the behavior of his opponents that is consistent with the solution. On the other hand, the decision makers' aim at simplicity motivates a set that is as small as possible. This is what discerns minimal prep sets from the minimal curb sets introduced by Basu and Weibull (1991), which are product sets of pure strategies containing not just some, but all best responses against beliefs restricted to opponents' recommendations.

Notice that the strategies played in a pure equilibrium indeed constitute a minimal prep set: each player uses one pure strategy, which is as far as one can go without violating nonemptiness, and it provides him with a best response against the unique belief on the opponents' strategy profile that is consistent with this recommendation.

In the game in Figure 1, the subgame perfect Nash equilibrium (L, r) is the unique pure Nash equilibrium. The unique minimal prep set $\{L\} \times \{r\}$ provides the same recommendation, but the curb notion has no cutting power: the only curb set is the entire strategy space. In this example, the minimal prep set provides a clearer recommendation than the curb notion, but there is always a trade-off between robustness and cutting power. The curb notion, by requiring that a curb set contains all its best replies, is a very robust concept, but — as in the example — may not be very selective. On the other hand, minimal prep sets, while typically singling out smaller sets, may exclude equivalent best replies. Thus, minimal prep sets put more stress on the simplicity of behavior, one of the major concerns in procedurally rational decision making (Rubinstein, 1998). There is substantial empirical support for this type of behavior that can be due to inertia or a status quo bias; see Kahneman *et al.* (1991) for an overview of experimental findings and Vega-Redondo (1993) for possible theoretical underpinnings. This motivates a closer study of simple and inertial behavior in a game theoretic context, aspects that minimal prep sets are meant to capture.



	ℓ	r
L	1, 2	1, 2
R	2, 0	0, 1

Figure 1: An extensive form game and its associated strategic form

Different types of learning processes have been shown to eventually settle down within the minimal curb sets of a strategic game (cf. Hurkens, 1995, Young, 1998, and Kosfeld *et al.*, 2001), thus providing dynamic support for the notion of minimal curb sets. The final part of this paper introduces a class of finite Markov processes as a model of adaptive play in finite strategic games to support the notion of minimal prep sets. Each player adjusts over time the toolkit of actions that he considers appropriate by selecting some pure strategies and discarding others. Players have a limited memory and choose best responses to beliefs supported by observed past play. This conforms with much of the literature on learning (cf. Fudenberg and Levine, 1998). Two distinctive features of the model are the following:

Status quo bias/inertia: Each time the game is played, each player first checks whether his current toolkit contains a best reply to his belief about the strategic behavior of his opponents; this conforms with a status quo bias (cf. Kahneman *et al.*, 1991).

Recent past as focal point: If the toolkit does not leave the player prepared with a best response, he adds a new pure strategy to his toolkit by backtracking and selecting one of the most recently discarded best replies (or an arbitrary best reply in case he has not played such before).

If players act in accordance with such an adjustment process, play eventually settles down in a minimal prep set.

The material is organized as follows. Section 2 contains notation and preliminaries. In Section 3, prep sets are formally defined and minimal prep sets are shown to exist subject to minor topological constraints. Moreover, the concept is compared with other solution concepts: the Nash equilibrium concept, rationalizability, and curb sets. Section 4 discusses a class of adjustment processes which in Section 5 are shown to settle down to play within minimal prep sets. Section 6 contains concluding remarks, including a discussion of consistency of minimal prep sets and experimental support.

2 Notation and preliminaries

Some set-theoretical notation: $\subseteq$ denotes weak set inclusion, $\subset$ denotes proper set inclusion. For a fixed set X, the complement of $Y \subseteq X$ (w.r.t. X) is denoted by $Y^c := X \setminus Y$.

A strategic game is a tuple $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$, where N is a nonempty, finite set of players, each player $i \in N$ has a nonempty set of pure strategies (or actions) A_i and a von Neumann-Morgenstern utility function $u_i : \times_{j \in N} A_j \to \mathbb{R}$. Write $A = \times_{i \in N} A_i$ and for each $i \in N$, $A_{-i} = \times_{j \in N \setminus \{i\}} A_j$.

Payoffs are extended to mixed strategies in the usual way. Assuming each A_i to be a topological space, $\Delta(A_i)$ denotes the set of Borel probability measures over A_i. Using a common, minor abuse of notation, α_{-i} denotes both an element of $\times_{j \in N \setminus \{i\}} \Delta(A_j)$ specifying a profile of mixed strategies of the opponents of player $i \in N$, and the probability measure it induces over the set A_{-i} of pure strategy profiles of his opponents. Beliefs of player i take the form of such a mixed strategy profile. Similarly, if $B_i \subseteq A_i$ is a Borel set, then $\Delta(B_i)$ denotes the set of Borel probability measures with support in B_i:

$$\Delta(B_i) = \{\alpha_i \in \Delta(A_i) \mid \alpha_i(B_i) = 1\}.$$

As usual, (a_i, α_{-i}) is the profile of strategies where player $i \in N$ plays $a_i \in A_i$ and his opponents play according to the mixed strategy profile $\alpha_{-i} = (\alpha_j)_{j \in N \setminus \{i\}} \in \times_{j \in N \setminus \{i\}} \Delta(A_j)$.

$\mathcal{G}$ is the class of strategic games $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ where for each player $i \in N$:

(a) A_i is a compact Hausdorff topological space;

(b) u_i is sufficiently measurable: for each $a_i \in A_i$ and each $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(A_j)$, the expected payoff $u_i(a_i, \alpha_{-i}) = \int_{A_{-i}} u_i(a_i, a_{-i}) \, \mathrm{d}\alpha_{-i}$ is well-defined and finite;

(c) u_i is upper semicontinuous (u.s.c.) on A_i, i.e., for each $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(A_j)$ and each $r \in \mathbb{R}$, the set $\{a_i \in A_i \mid u_i(a_i, \alpha_{-i}) \geq r\}$ is closed.

The set $\mathcal{G}$ contains two subclasses that are of importance in the remainder of the paper, namely the set of games where each A_i is a compact subset of a metric space and each u_i is continuous (as in Proposition 3.3), and the set of finite strategic games, i.e., the set of strategic games in which each of the players has a finite set of pure strategies (as in the examples and Sections 4 and 5).

Let $i \in N$ and let $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(A_j)$ be a belief of player i. The set

$$BR_i(\alpha_{-i}) = \{a_i \in A_i \mid \forall b_i \in A_i : u_i(a_i, \alpha_{-i}) \geq u_i(b_i, \alpha_{-i})\}$$

is the set of pure best responses of player i against α_{-i}. Since every u.s.c. function on a compact set achieves a maximum, it follows by definition of $\mathcal{G}$ that each player in a game $G \in \mathcal{G}$ always has a nonempty set of best responses against an arbitrary belief.

Theorem 3.2, the existence theorem for minimal prep sets, uses the following version of the Cantor Intersection Principle, a proof of which is included for easy reference.

Lemma 2.1 [Cantor Intersection Principle] *Let X be a compact Hausdorff topological space and $\{F_k \mid k \in I\}$ a collection of compact subsets of X with the finite intersection property:*

$$\forall J \subseteq I, J \text{ finite: } \cap_{k \in J} F_k \neq \emptyset.$$

Then $\cap_{k \in I} F_k$ is nonempty and compact.

Proof. Suppose that $\cap_{k \in I} F_k = \emptyset$. Then

$$X = (\cap_{k \in I} F_k)^c = \cup_{k \in I} F_k^c. \tag{1}$$

Since each F_k is a compact subset of the Hausdorff space X, each F_k is closed (Aliprantis and Border, 1994, Lemma 2.30. Recall that compact subsets of non-Hausdorff spaces need *not* be closed, ibid, Example 2.31), so its complement F_k^c is open. By (1), the sets $\{F_k^c \mid k \in I\}$ form an open cover of the compact set X, so there is a finite set $J \subseteq I$ with $X = \cup_{k \in J} F_k^c$. This implies that $\cap_{k \in J} F_k = \emptyset$, contradicting the finite intersection property. So $\cap_{k \in I} F_k \neq \emptyset$. Since each F_k is closed, $\cap_{k \in I} F_k$ is a closed subset of the compact set X, hence compact. □

3 Preparation

The set-valued solution concept introduced in this paper combines a standard rationality condition, stating that the set of recommended strategies to each player must contain at least one best reply to whatever belief he may have that is consistent with the recommendations to the other players, with the players' aim at simplicity, which encourages them to maintain a set of strategies that is as small as possible. Formally, (minimal) prep sets are defined as follows.

Definition 3.1 Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \mathcal{G}$. A *prep set* is a product set $X = \times_{i \in N} X_i$, where

(a) for each $i \in N$, $X_i \subseteq A_i$ is a nonempty, compact set of pure strategies;

(b) for each $i \in N$ and each belief α_{-i} of player i with support in X_{-i}, the set X_i contains at least one best response of player i against his belief:

$$\forall i \in N, \forall \alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(X_j) : \quad BR_i(\alpha_{-i}) \cap X_i \neq \emptyset.$$

A prep set X is *minimal* if no prep set is a proper subset of X. ◁

Every strategic game in $\mathcal{G}$ has a minimal prep set.

Theorem 3.2 *Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \mathcal{G}$. Then G has a minimal prep set.*

Proof. Let Q denote the set of all prep sets of G. Since u_i is u.s.c. in the i-th coordinate and A_i is compact, A_i contains a best response against any belief $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(A_j)$ that an arbitrary player $i \in N$ may have: the entire strategy space A is a prep set. So Q is nonempty and partially ordered via set inclusion. According to the Hausdorff Maximality Principle, Q contains a maximal nested subset R. For each $i \in N$, let $X_i = \cap_{Y \in R} Y_i$ be the intersection of player i's strategies in the nested set R. The product set $X = \times_{i \in N} X_i$ is shown to be a minimal prep set.

Let $i \in N$. Since Y_i is nonempty and compact for each prep set $Y \in R$ and the collection $\{Y_i \mid Y \in R\}$ is nested, Lemma 2.1 implies that X_i is nonempty and compact. Let α_{-i} be a belief of player i over $\times_{j \in N \setminus \{i\}} X_j$. To see that $X_i \cap BR_i(\alpha_{-i}) \neq \emptyset$, write

$$\begin{aligned} X_i \cap BR_i(\alpha_{-i}) &= [\cap_{Y \in R} Y_i] \cap BR_i(\alpha_{-i}) \\ &= \cap_{Y \in R} [Y_i \cap BR_i(\alpha_{-i})]. \end{aligned} \tag{2}$$

For each $Y \in R$, the set $Y_i \cap BR_i(\alpha_{-i})$ is nonempty (since Y is a prep set and $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(X_j) \subseteq \times_{j \in N \setminus \{i\}} \Delta(Y_j)$) and compact, since it is an intersection of the compact set Y_i and the set $BR_i(\alpha_{-i})$, which is closed, since it is the set of maximizers of an u.s.c. function. Moreover, the collection $\{Y_i \cap BR_i(\alpha_{-i}) \mid Y \in R\}$ is nested, since R is nested. Again applying Lemma 2.1 yields that the set in (2) is nonempty. So X is indeed a prep set. The fact that it is minimal follows directly from the fact that R is a maximal nested subset of Q. □

Having proved the existence of minimal prep sets in a wide class of games, let us compare the concept with other solution concepts. As mentioned before, if $a \in A$ is a pure strategy Nash equilibrium, then $\times_{i \in N} \{a_i\}$ is easily seen to be a minimal prep set. Consequently, the minimal prep notion can be seen as a set-valued extension of the pure Nash equilibrium concept. Every prep set of a sufficiently structured game contains the support of a Nash equilibrium.

Proposition 3.3 *Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game where for each $i \in N$:*

- *A_i is a compact subset of a metric space;*
- *u_i is a continuous von Neumann-Morgenstern utility function.*

Then every prep set of G contains the support of a Nash equilibrium in mixed strategies.

Proof. Let $X = \times_{i \in N} X_i \subseteq \times_{i \in N} A_i$ be a prep set of G. The game $\langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ obtained from G by restricting attention to the nonempty and compact strategy sets $(X_i)_{i \in N}$ meets the conditions of Glicksberg's (1952) theorem for the existence of a mixed strategy Nash equilibrium, say α. This is also a Nash equilibrium of the original game G, since X was assumed to be a prep set and consequently contains at least one best response in G to α_{-i} for each player $i \in N$: although $A_i \setminus X_i$ may contain alternative best responses to α_{-i}, there are no better options. □

Conversely, not every strategy in the support of a mixed Nash equilibrium is necessarily contained in a minimal prep set. See Example 3.4.

It is easy to see that every strategy in a minimal prep set $X = \times_{i \in N} X_i$ of a finite strategic game is rationalizable: for every $i \in N$ and $x_i \in X_i$, strategy x_i has to be a best response to some belief α_{-i} over X_{-i}, otherwise X would not be a minimal prep set, since x_i can be omitted from X_i and the resulting product set would be a smaller prep set. Example 3.4 indicates that the set of strategies included in a minimal prep set can be a proper subset of the set of rationalizable strategies.

Recall from Basu and Weibull (1991) that a curb set ('curb' is mnemonic for 'closed under rational behavior') of a strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a product set $X = \times_{i \in N} X_i$, where

(a) for each $i \in N$, $X_i \subseteq A_i$ is a nonempty, compact set of pure strategies;

(b) for each $i \in N$ and each belief α_{-i} of player i with support in X_{-i}, the set X_i contains *all* best responses of player i against his belief:

$$\forall i \in N, \forall \alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(X_j) : \quad BR_i(\alpha_{-i}) \subseteq X_i.$$

A curb set X is minimal if no curb set is a proper subset of X. Comparing this with Definition 3.1, one sees that minimal curb sets are highly robust by requiring all best replies against beliefs consistent with the solution concept to be included in the solution, while prep sets require that every player is prepared against such beliefs by including at least some best replies. Thus, minimal prep sets put more stress on the simplicity of behavior, one of the major concerns in procedurally rational decision making (Rubinstein, 1998). Recall from the example in Figure 1 that this may provide significant additional cutting power over the curb notion.

Example 3.4 The minimal prep sets and minimal curb sets of the game in Figure 2 are $\{T\} \times \{L\}$ and $\{M\} \times \{R\}$. The strategy combination $(B, \frac{1}{2}L + \frac{1}{2}R)$ is a Nash equilibrium, so B is rationalizable, but not included in any of the minimal prep or curb sets. $\triangleleft$

Every curb set of a game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \mathcal{G}$ contains a minimal prep set: if $X = \times_{i \in N} X_i \subseteq A$ is a curb set of G, the game $\langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ obtained from G

	L	R
T	$4,1$	$0,0$
M	$0,0$	$4,1$
B	$3,1$	$3,1$

Figure 2: Proper selection of rationalizable strategies

by restricting attention to the nonempty and compact strategy sets $(X_i)_{i \in N}$ is again an element of $\mathcal{G}$ and consequently contains a minimal prep set. Since X was a curb set, this is easily seen to be a minimal prep set of the original game G as well. Example 3.5 indicates that the minimal prep sets may contain a proper subset of the strategies contained in the minimal curb sets. Finally, every curb set is a prep set, so if a curb set is contained in a minimal prep set, the two sets are necessarily equal.

Example 3.5 All pure strategies in the game in Figure 3 are rationalizable. The unique (minimal) curb set is $\{T, B\} \times \{L, R\}$, the minimal prep sets are $\{T\} \times \{L\}$ and $\{B\} \times \{L\}$. Pure strategy R, which is included in the minimal curb set of the game, is not contained in any of its minimal prep sets. $\triangleleft$

	L	R
T	$1,1$	$1,0$
B	$1,0$	$0,0$

Figure 3: Proper selection of curb strategies

Another example illustrating the difference in the cutting power between curb and prep sets is the following class of games. Consider a two-player game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$, where $A_1 = A_2 = \{a_1, \ldots, a_k\}$ for some $k \in \mathbb{N}, k \geq 2$. Assume that player 1's unique best reply to an action of player 2 is to play the same action:

$$BR_1(a_\ell) = \{a_\ell\} \text{ for all } a_\ell \in A_2. \tag{3}$$

Player 2 is indifferent between his actions if player 1 chooses a_1:

$$BR_2(a_1) = A_2. \tag{4}$$

If player 1 chooses a different action, the best replies of player 2 are a nonempty subset of actions with a lower index:

$$BR_2(a_\ell) \neq \emptyset, BR_2(a_\ell) \subseteq \{a_1, \ldots, a_{\ell-1}\} \text{ if } 1 < \ell \leq k. \tag{5}$$

Let us sketch an economic examples of this class of games: The two players may — for instance — have produced a software package and decide on which of k possible dates to release its product. Firm 1, an incumbent in the market, knows that if he releases the package at the same moment as or earlier than Firm 2, a market entrant, he will get the entire market due to its good reputation. To guarantee maximal profit, Firm 1 (for instance due to the ability of further debugging) has an incentive to release at the same moment as Firm 2 does, thus motivating (3). If Firm 1 releases immediately, the market is satiated, making Firm 2 indifferent between his release dates, motivating (4); releasing ahead of Firm 1 will give Firm 2 a fixed part of the market, thus motivating a best-response correspondence of type (5).

This is a class of games in which the unique minimal prep set coincides with the unique pure Nash equilibrium of the game. The curb notion, however, has no cutting power: the unique curb set consists of the entire set of pure strategy profiles.

Proposition 3.6 *In the game G the following hold:*

(a) the unique pure Nash equilibrium equals (a_1, a_1);

(b) If $X_1 \times X_2$ is a prep set of G, then $a_1 \in X_1$;

(c) the unique minimal prep set equals $\{a_1\} \times \{a_1\}$;

(d) the unique (hence minimal) curb set equals $A_1 \times A_2$.

Proof. (a): follows easily from the best reply functions.

(b): Suppose $a_1 \notin X_1$. Let $a_\ell \in X_1$ be the element of X_1 with lowest index $\ell > 1$. Since $X_1 \times X_2$ is a prep set, (5) implies that there is an $a_m \in BR_2(a_\ell) \cap X_2 \subseteq \{a_1, \ldots, a_{\ell-1}\} \cap X_2$, which implies $m < \ell$. But then $BR_1(a_m) \cap X_1 = \{a_m\} \cap X_1 \neq \emptyset$, so that $a_m \in X_1$. Since $m < \ell$, this contradicts our assumption that a_ℓ is the element of X_1 with lowest index. Hence $a_1 \in X_1$.

(c): Since (a_1, a_1) is a pure Nash equilibrium, it follows that $\{a_1\} \times \{a_1\}$ is a minimal prep set. Suppose there exists a different minimal prep set $X = X_1 \times X_2$. Then $a_1 \in X_1$ by (b) and $a_1 \notin X_2$, otherwise $\{a_1\} \times \{a_1\} \subset X$, contradicting minimality of X. We prove by induction that $a_\ell \notin X_1$ and $a_\ell \notin X_2$ for all $\ell \in \{2, \ldots, k\}$. With $a_1 \notin X_2$ this implies that $X_2 = \emptyset$, a contradiction.

Since $a_1 \notin X_2$, and $BR_2(a_2) = \{a_1\}$ by (5), and X is a prep set, it follows that $a_2 \notin X_1$. Since $a_2 \notin X_1$, $BR_1(a_2) = \{a_2\}$, and X is a prep set, it follows that $a_2 \notin X_2$. This proves the claim for $\ell = 1$.

Assume the claim is true for all indices in $\{2, \ldots, \ell\}$ with $\ell < k$. To show: $a_{\ell+1} \notin X_1$ and $a_{\ell+1} \notin X_2$. We know by induction and (5) that $BR_2(a_{\ell+1}) \cap X_2 \subseteq \{a_1, \ldots, a_\ell\} \cap X_2 = \emptyset$, so the fact that X is a prep set implies that $a_{\ell+1} \notin X_1$. Since $a_{\ell+1} \notin X_1$, $BR_1(a_{\ell+1}) = \{a_{\ell+1}\}$, and X is a prep set, it follows that $a_{\ell+1} \notin X_2$. This finishes the inductive proof: $\{a_1\} \times \{a_1\}$ is indeed the *unique* minimal prep set.

(d): Let $X = X_1 \times X_2$ be a curb set of G. Every curb set of G is a prep set, so $a_1 \in X_1$ by (b). Since X is curb and $BR_2(a_1) = A_2$ by (4), this implies that $X_2 = A_2$. Moreover, for every $\ell \in \{1, \ldots, k\}$ it holds that $a_\ell \in X_2 = A_2$ and $BR_1(a_\ell) = \{a_\ell\}$, so $a_\ell \in X_1$. Conclude that $X_1 = A_1$, finishing the proof. □

4 Myopic adaptive play

This section presents a class of finite Markov chains as a model of adaptive play to support the notion of minimal prep sets: each player adjusts over time the toolkit of actions that he considers appropriate by selecting some pure strategies and discarding others. In line with much of the literature on learning models (cf. Young, 1998, Fudenberg and Levine, 1998), players have a limited memory and choose best responses to beliefs supported by observed past play. Two distinctive features of the learning model are the following:

Status quo bias/inertia: Each time the game is played, each player first checks whether his current toolkit contains a best reply to his belief about the strategic behavior of his opponents.

Recent past as focal point: If the toolkit does not leave the player prepared with a best response, he adds a new pure strategy to his toolkit by backtracking and selecting one of the most recently discarded best replies (or an arbitrary best reply in case he has not played such before).

Thus, the learning model aims to capture two common experimental observations: the 'unwillingness' to change strategic behavior as predicted by the literature on status quo biases and inertia (cf. Kahneman *et al.*, 1991) and the presence of focal points (cf. Schelling, 1960).

4.1 State space

A finite strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is played once every time period in discrete time. At an arbitrary time t, each player $i \in N$ is characterized by

- a nonempty 'toolkit' $X_i \subseteq A_i$ of actions that he considers appropriate at time t,
- a linear (i.e., complete, reflexive, and transitive) order R_i on the set of remaining actions $A_i \setminus X_i$, where $x R_i y$ indicates that $x \in A_i \setminus X_i$ is ranked at least as high as $y \in A_i \setminus X_i$,

and only observes a sequence with fixed length $T \in \mathbb{N}$ of past action profiles. In this section and the next, the Markov process specifies an adjustment process for a fixed finite strategic game G and histories of fixed length T. The suffixes G and T are suppressed for notational convenience.

Formally, each element s of the finite set S of *states* of the Markov process is described by a tuple

$$s = (h, (X_i)_{i \in N}, (R_i)_{i \in N}),$$

with a history $h = (a^T, a^{T-1}, \ldots, a^1) \in A^T$ of past play indicating that for each $k \in \{1, \ldots, T\}$, the profile of actions played k periods ago was $a^k = (a_i^k)_{i \in N} \in A$, a sequence $(X_i)_{i \in N}$ of toolkits, and a sequence $(R_i)_{i \in N}$ of linear orders over $(A_i \setminus X_i)_{i \in N}$.

Notice two specific features here. First of all, players adjust *sets of strategies*, rather than a single strategy. This corresponds with the models in Artificial Intelligence and psychology that consider learning as the acquisition and modification of a collection of skills or methods. In these models, computers and human subjects learn to respond to a changing environment by adopting and adjusting a set of skills or methods within the limits set by their technological and cognitional constraints (their strategy space). Moreover, it corresponds with Popper's evolutionary approach to interactive learning, where people hold a number of provisional hypotheses or responses to the current environment and apply a process of adjustment and refutal. His theory (Popper, 1979, p. 261) '...is a largely Darwinian theory...: we try to solve our problems, and to obtain, by a process of elimination, something approaching adequacy in our tentative solutions.'

Secondly, in state $s = (h, (X_i)_{i \in N}, (R_i)_{i \in N})$, the set X_i^c of actions outside i's current toolkit is assumed to be linearly ordered. This is simply an assumption that specifies how the players store the elements of their pure strategy set in their memory. The properties of the myopic adjustment process explained in the next subsection — in particular property A4 — guarantee that this will be a temporal order. This corresponds in a natural way with actual learning by both machines and humans. In artificial intelligence, the potential choices of a machine are typically ordered in a list of records; new items are often stacked on top of the list, so that a search for items satisfying certain criteria proceeds according to a last-in-first-out principle. The same ordering on basis of time is common in human learning: it is easiest to remember that which has been added most recently to the memory. Consequently, in both cases, there is a clear focality on the recent past, justifying a temporal ordering R_i.

Given a history $h = (a^T, a^{T-1}, \ldots, a^1) \in A^T$ of past play, let $r(h) = a^1 \in A$ denote the rightmost element of h, the profile of actions played in the previous period according to history h. For each $i \in N$ and each $k \in \{1, \ldots, T\}$, $p_i(h,k) = \{a_i^k, a_i^{k-1}, \ldots, a_i^1\}$ is the set of actions played by i in the previous k periods and $p_i(h) := p_i(h,T)$ is the set of actions played by i in the entire history of length T that he can remember. A *successor* of history h is any history $h' \in A^T$ obtained by deleting the leftmost element of h and attaching a new rightmost element.

4.2 Transition matrices

Having defined the set S of states, we proceed to the set of *transition matrices* $P : S \times S \to [0,1]$, where $P(s,s')$ indicates the probability of a transition from state s to state s' in one period. Let $\mathcal{P}$ be the set of transition matrices P where $P(s,s') > 0$ if and only if states $s = (h, (X_i)_{i \in N}, (R_i)_{i \in N})$ and $s' = (h', (X_i')_{i \in N}, (R_i')_{i \in N})$ satisfy conditions A1 to A4 below. Conditions A1 and A2 are standard: history h is replaced by a successor h' where the pure strategy $r_i(h') \in A_i$ chosen by each player $i \in N$ is a best response from his toolkit X_i' to some belief $\alpha_{-i}^* \in \times_{j \in N \setminus \{i\}} \Delta(p_j(h))$ over the observed past play. Condition A3 indicates that player i will check whether his current toolkit X_i contains a best reply against his belief α_{-i}^*. If so, there is no reason to expand it: $X_i' \setminus X_i = \emptyset$. Otherwise, he plays and adds to his toolkit the best response (see A2) $r_i(h')$ that is selected by checking the pure strategies in $A_i \setminus X_i$ in the order indicated by R_i and choosing the first/highest ranked best response he encounters (if there are several such highest ranked best responses, he chooses among them arbitrarily). This condition states that *addition* of actions to a toolkit only happens if the current toolkit is insufficient. This leaves space for players to actually discard certain actions from their toolkit between two periods, for instance because such actions may be considered unnecessary. As mentioned in section 4.1, condition A4 guarantees that the order on the actions outside a player's toolkit coincides with the order in which they have been discarded: the most recently discarded actions will be ranked highest. Together with condition A3, this implies that if an action is added to a toolkit, this action is a most recently discarded best response.

[A1] h' is a successor of h;

And for each player $i \in N$:

[A2: best response] $r_i(h') \in X_i' \cap BR_i(\alpha^*_{-i})$ for some belief $\alpha^*_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(p_j(h))$;

[A3: inertia and backtracking] If $BR_i(\alpha^*_{-i}) \cap X_i \neq \emptyset$, then $X_i' \setminus X_i = \emptyset$. Otherwise, $r_i(h') \in \{a_i \in BR_i(\alpha^*_{-i}) \mid a_i R_i b_i \text{ for all } b_i \in BR_i(\alpha^*_{-i})\}$ is a highest ranked best reply against α^*_{-i}, and $X_i' \setminus X_i = \{r_i(h')\}$.

[A4, Last in, first out] the ranking R_i' is obtained from R_i by stacking all recently dismissed actions on top: Let $x_i, y_i \in A_i \setminus X_i'$. Then R_i' satisfies:

- if $x_i, y_i \in A_i \setminus X_i$, then $x_i R_i' y_i \Leftrightarrow x_i R_i y_i$;
- if $x_i \in X_i$, but $y_i \notin X_i$, then $x_i R_i' y_i$, but not $y_i R_i' x_i$.

The first point requires that elements that remain outside the toolkit keep the same order. The second point indicates that newly rejected elements rank higher than elements that already were outside the toolkit in the previous period.

In these two subsections, we specified the state space and conditions on the transition matrices underlying the myopic adjustment process. The process aims to capture a number of practical aspects of learning by humans and machines: what is learned is a set of tools or skills that are meant to prepare the players with optimal reactions against beliefs they might have about opponents' behavior. These beliefs are based on observations from a limited past (the last T rounds of play). Players display a status quo bias or inertial behavior by sticking to their toolkit whenever this provides a best response against their belief. If no best response is contained in the current toolkit, a most recently played best response is added, thus stressing the focality of the recent past.

5 Convergence of adaptive play

The purpose of this section is to show that for an arbitrary finite game and a sufficiently long memory, each of the adjustment processes in the class $\mathcal{P}$ satisfying conditions A1 to A4 eventually settles down within a minimal prep set. Three lemmas will pave the road. The first (see also Hurkens, 1995, Lemma 1) indicates that a sequence $a^1, \ldots, a^K \in A$ of pure strategy profiles with the property that for each $k = 2, \ldots, K$ some player i selects a pure strategy $a_i^k \notin \{a_i^1, \ldots, a_i^{k-1}\}$, i.e., a pure strategy that he has not used earlier in the sequence, can have at most length $\sum_{i \in N} |A_i| - |N| + 1$. The proof is simple: there are $\sum_{i \in N} |A_i|$ pure strategies, $a^1 \in A$ captures $|N|$ of them, and at least one pure strategy is added in each step.

Lemma 5.1 *Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a finite strategic game and let $a^1, \ldots, a^K \in A$ be pure strategy profiles such that*

$$\forall k = 1, \ldots, K-1: \quad \times_{i \in N} \{a_i^1, \ldots, a_i^k\} \subset \times_{i \in N} \{a_i^1, \ldots, a_i^{k+1}\}.$$

Then $K \leq \sum_{i \in N} |A_i| - |N| + 1$.

The following lemma deals with the expansion of toolboxes. Suppose the process is in a state where the action of each player in the past t periods is contained in his current toolbox, but these actions do not constitute a prep set. Then there is a positive probability of moving to a state where the actions from the past $t+1$ periods are contained in the players' toolkits and the product set of these actions is strictly larger than in the previous period.

This lemma will be used to construct an increasing sequence of product sets, which according to the bound set in Lemma 5.1 must yield a prep set after at most $\sum_{i \in N} |A_i| - |N| + 1$ steps.

Lemma 5.2 *Consider a state $s^t = (h^t, (X_i^t)_{i \in N}, (R_i^t)_{i \in N})$ with*

$$\begin{aligned} h^t &= (a^{T-t}, \ldots, a^1, b^1, \ldots, b^t), \\ p_i(h^t, t) &= \{b_i^1, \ldots, b_i^t\} \subseteq X_i^t \text{ for each } i \in N, \end{aligned}$$

and suppose that $\times_{i \in N} p_i(h^t, t)$ *is not a prep set. Then the process moves with positive probability to a state* $s^{t+1} = (h^{t+1}, (X_i^{t+1})_{i \in N}, (R_i^{t+1})_{i \in N})$ *where*

$$\begin{aligned} h^{t+1} &= (a^{T-t-1}, \ldots, a^1, b^1, \ldots, b^t, b^{t+1}), \\ p_i(h^{t+1}, t+1) &= \{b_i^1, \ldots, b_i^t, b_i^{t+1}\} \subseteq X_i^{t+1} \text{ for each } i \in N, \qquad (6) \\ \times_{i \in N} p_i(h^t, t) &\subset \times_{i \in N} p_i(h^{t+1}, t+1). \qquad (7) \end{aligned}$$

Proof. Since $\times_{i \in N} p_i(h^t, t)$ is not a prep set, there is a nonempty set $T \subseteq N$ of players $i \in N$ with a belief $\alpha_{-i}^* \in \times_{j \in N \setminus \{i\}} \Delta(p_j(h^t, t))$ over the play in the past t periods such that $BR_i(\alpha_{-i}^*) \cap p_i(h^t, t) = \emptyset$. Fix such a belief α_{-i}^* for each $i \in T$, let $b_i^{t+1} \in \{b_i \in BR_i(\alpha_{-i}^*) \mid b_i R_i^t a_i \text{ for all } a_i \in BR_i(\alpha_{-i}^*)\}$ be a highest ranked best reply against α_{-i}^*, and define $X_i^{t+1} := X_i^t \cup \{b_i^{t+1}\}$. For each $i \in N \setminus T$, let $b_i^{t+1} \in p_i(h^t, t)$ be a best response against an arbitrary belief over play in the past t periods and define $X_i^{t+1} := X_i^t$.

Conditions A1 to A4 imply that the process moves with positive probability from state s^t to state $s^{t+1} = (h^{t+1}, (X_i^{t+1})_{i \in N}, (R_i^{t+1})_{i \in N})$ where $h^{t+1} = (a^{T-t-1}, \ldots, a^1, b^1, \ldots, b^t, b^{t+1})$, the set X_i^{t+1} by construction satisfies (6) for each player $i \in N$, and the linear orders R_i^{t+1} coincide with the linear orders R_i^t restricted to the subsets $A_i \setminus X_i^{t+1}$ for each $i \in N$. Moreover, (7) holds, since $p_i(h^t, t) \subseteq p_i(h^{t+1}, t+1)$ for each $i \in N$, with strict inclusion for each $i \in T$. □

A history $h = (a^T, \ldots, a^1)$ lies in $X \subseteq A$ if $a^t \in X$ for each $t = 1, \ldots, T$. The final lemma indicates that an absorbing set of the Markov chain has been reached if the process is in a state $s = (h, (X_i)_{i \in N}, (R_i)_{i \in N})$ where $X := \times_{i \in N} X_i$ is a minimal prep set and the history h of observed past play lies in X: all future action profiles and all future toolkits are contained in the minimal prep set X.

Lemma 5.3 *Consider a state* $s = (h, (X_i)_{i \in N}, (R_i)_{i \in N})$*, where* $X := \times_{i \in N} X_i$ *is a minimal prep set and the history* h *lies in* X*. Every state* $s^t = (h^t, (X_i^t)_{i \in N}, (R_i^t)_{i \in N})$ *that can be reached from* s *with positive probability in a finite number* $t \in \mathbb{N}$ *of steps satisfies:*

(i) h^t *lies in* X*,*

and for each player $i \in N$:

(ii) $X_i^t \subseteq X_i$,

(iii) if $x \in A_i \setminus X_i$ *and* $y \in X_i \setminus X_i^t$, *then* $yR_i^t x$, *but not* $xR_i^t y$.

Proof. The proof is by induction on $t \in \mathbb{N}$. Let $s^1 = (h^1, (X_i^1)_{i \in N}, (R_i^1)_{i \in N})$ be a state reached from s within one period, let $i \in N$, and let $r_i(h^1) \in A_i$ be the last action played by i. A2 implies that $r_i(h^1) \in X_i^1 \cap BR_i(\alpha_{-i}^*)$ for some $\alpha_{-i}^* \in \times_{j \in N \setminus \{i\}} \Delta(p_j(h))$. X is a minimal prep set and h lies in X, so $BR_i(\alpha_{-i}^*) \cap X_i \neq \emptyset$. A3 then implies that $X_i^1 \setminus X_i = \emptyset$, i.e., $X_i^1 \subseteq X_i$, proving *(ii)* for $t = 1$. Since h lies in X and $r_i(h^1) \in X_i^1 \subseteq X_i$ for each $i \in N$, it follows that h^1 lies in X, proving *(i)* for $t = 1$. Assumption A4 directly implies *(iii)* for $t = 1$.

Now assume the statement is true up to a certain $t \in \mathbb{N}$ and consider a state $s^{t+1} = (h^{t+1}, (X_i^{t+1})_{i \in N}, (R_i^{t+1})_{i \in N})$ that can be reached from s with positive probability in $t+1$ steps. Let $i \in N$ and let $r_i(h^{t+1}) \in A_i$ be the last action played by i. A2 implies that $r_i(h^{t+1}) \in X_i^{t+1} \cap BR_i(\alpha_{-i}^*)$ for some $\alpha_{-i}^* \in \times_{j \in N \setminus \{i\}} \Delta(p_j(h^t))$.

If $BR_i(\alpha_{-i}^*) \cap X_i^t \neq \emptyset$, A3 implies $X_i^{t+1} \setminus X_i^t = \emptyset$, so that $X_i^{t+1} \subseteq X_i^t \subseteq X_i$ by part *(ii)* of the induction hypothesis. On the other hand, if $BR_i(\alpha_{-i}^*) \cap X_i^t = \emptyset$, part *(iii)* of the induction hypothesis guarantees that all actions in $X_i \setminus X_i^t$ are ranked above actions in $A_i \setminus X_i$. Moreover, $BR_i(\alpha_{-i}^*) \cap X_i \neq \emptyset$ since X is a minimal prep set and h^t lies in X by part *(i)* of the induction hypothesis. A3 then implies that $r_i(h^{t+1})$ is a highest ranked best response against α_{-i}^*, i.e., an element of $X_i \setminus X_i^t \subseteq X_i$, and $X_i^{t+1} = X_i^t \cup \{r_i(h^{t+1})\} \subseteq X_i$. Conclude that in both cases $X_i^{t+1} \subseteq X_i$, proving *(ii)* for $t+1$. History h^t lies in X by part *(i)* of the induction hypothesis and $r_i(h_i^{t+1}) \in X_i$ for each $i \in N$, so h^{t+1} lies in X, proving *(i)* for $t+1$. A4 and part *(iii)* of the induction hypothesis immediately imply *(iii)* for $t+1$, completing the inductive argument. □

The three lemmas provide the basis for the convergence theorem, according to which players eventually learn to coordinate on play inside a minimal prep set. The proof roughly proceeds as follows. Starting from an arbitrary state, Lemma 5.2 is used to construct an increasing sequence of product sets of actions used inside the players' toolboxes. Lemma 5.1 indicates a bound for the length of this sequence, assuring that after a certain number of steps, the product set of toolboxes must contain a prep set. Choosing a minimal prep set from this collection, it is shown that the process proceeds with positive probability to a state where the players' toolboxes coincide with their components from the minimal prep set and the entire observed history of observed past play lies inside this minimal prep set. Lemma 5.3 indicates that the process has then reached an absorbing set: the process settles down inside this minimal prep set. Since this happens with positive probability for every state, it eventually happens with probability one.

Theorem 5.4 *For any finite strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$, memory length $T \geq \sum_{i \in N} |A_i| - |N| + 1$, and any adjustment process $P \in \mathcal{P}$ satisfying A1 to A4, play eventually settles down in a minimal prep set.*

Proof. The proof proceeds in three steps.

Step 1: From an arbitrary state $s = (h, (X_i)_{i \in N}, (R_i)_{i \in N})$ with $h = (a^T, a^{T-1}, \ldots, a^1)$, the process always moves to a state $s^1 = (h^1, (X_i^1)_{i \in N}, (R_i^1)_{i \in N})$ with $h^1 = (a^{T-1}, \ldots, a^1, b^1)$ by A1 and satisfying, by A2, for each $i \in N$:

$$p_i(h^1, 1) = r_i(h^1) = \{b_i^1\} \subseteq X_i^1.$$

Starting from s^1, Lemma 5.2 can be applied repeatedly. Using the bound from Lemma 5.1, there is a $K \in \mathbb{N}, K \leq \sum_{i \in N} |A_i| - |N| + 1 \leq T$ such that the process moves with positive probability in K steps to a state $s^K = (h^K, (X_i^K)_{i \in N}, (R_i^K)_{i \in N})$ where $h^K = (a^{T-K}, \ldots, a^1, b^1, \ldots, b^K)$, $p_i(h^K, K) = \{b_i^1, \ldots, b_i^K\} \subseteq X_i^K$ for each $i \in N$, and $\times_{i \in N} p_i(h^K, K)$ is a prep set.

Step 2: Let $X = \times_{i \in N} X_i \subseteq \times_{i \in N} p_i(h^K, K)$ be a *minimal* prep set. For each $i \in N$, let R_i be a linear order on $A_i \setminus X_i$ that (analogous with condition A4) coincides with R_i^K on the subset $A_i \setminus X_i^K$ and ranks the additional elements, i.e., those in $X_i^K \setminus X_i$, above the elements in $A_i \setminus X_i^K$: if $x \in X_i^K \setminus X_i$ and $y \in A_i \setminus X_i^K$, then $x R_i y$, but not $y R_i x$.

For each $t = 1, \ldots, T$, recursively define the strategy profile $c^t = (c_i^t)_{i \in N} \in A$ as follows. For each player $i \in N$, $c_i^1 \in X_i$ is a best response to an arbitrary belief $\alpha_{-i} \in$

$\times_{j \in N \setminus \{i\}} \Delta(X_j)$ over X_{-i} and from them on, c_i^{t+1} is a best response in X_i to the previous pure strategy profile c_{-i}^t:

$$\forall i \in N, \forall t = 1, \ldots, T-1: \quad c_i^{t+1} \in BR_i(c_{-i}^t) \cap X_i.$$

Since X is a prep set, the profiles $c^1, \ldots, c^T$ are well-defined.

Conditions A1 to A4 imply that the process moves with positive probability from state s^K to state

$$s^{K+1} = ((a^{T-K-1}, \ldots, a^1, b^1, \ldots, b^K, c^1), (X_i)_{i \in N}, (R_i)_{i \in N})$$

in one period, to state

$$s^{K+2} = ((a^{T-K-2}, \ldots, a^1, b^1, \ldots, b^K, c^1, c^2), (X_i)_{i \in N}, (R_i)_{i \in N})$$

in two periods, and, continuing, to state

$$s^{K+T} = ((c^1, c^2, \ldots, c^T), (X_i)_{i \in N}, (R_i)_{i \in N})$$

in T periods.

Step 3: State s^{K+T} satisfies the conditions of Lemma 5.3, so from this state onward, players only play actions from the minimal prep set X.

In conclusion, starting from state $s \in S$, there is a positive probability of proceeding to a state meeting the requirements of Lemma 5.3, after which play settles down in a minimal prep set, i.e., a positive probability of proceeding to an absorbing set of states. Since $s \in S$ was chosen arbitrarily, this eventually happens with probability one, finishing the proof. □

6 Concluding remarks

Motivated in part by the prominence of pure Nash equilibria in game theoretic experiments, minimal prep sets were introduced as a set-valued extension of the pure Nash equilibrium concept. As a consequence, minimal prep sets have the same robustness problem as pure equilibria: as opposed to minimal curb sets, equivalent best replies against beliefs that are consistent with the solution may be excluded from a minimal prep set. Such behavior, however, has substantial empirical support: human choice behavior often reflects a tendency towards some inertia by favoring the status quo, and one of the basic tenets from bounded rationality is the urge towards relatively simple behavior. Moreover, dynamic support for the concept was provided by a myopic adjustment process leading players to eventually coordinate on play within minimal prep sets. The learning process was motivated by common experimental observations: status quo bias/inertia and the presence of focal points, in this case the fact that more importance was assigned to the recent past than to older observations.

CONSISTENCY: The notion of minimal prep sets fits into the research program initiated by Peleg and Tijs (1996) and Peleg, Potters, and Tijs (1996), who concentrate on the consistency of behavior of players in strategic games. Consistency essentially requires that if a nonempty set of players commits to playing according to a certain solution, the players in the reduced game should not have an incentive to deviate from the initial solution. This appears to be a minimal property that a solution concept should satisfy (see Aumann, 1987, pp. 478-479, for a general appraisal): if others play the game according to a certain equilibrium, the solution concept should recommend you to do the same. Yet, the axiom has a rather dramatic impact: Norde *et al.* (1996) proved that the unique point-valued solution concept for the set of strategic games satisfying consistency, in combination with standard utility maximizing behavior in one-player games and nonemptiness, is the Nash equilibrium concept. This implies that none of the concepts from the equilibrium refinement literature is actually consistent. Dufwenberg *et al.* (1998) indicate that a transition to set-valued solution concepts overcomes this problem: there is a multiplicity of consistent set-valued solution concepts that satisfy nonemptiness and recommend utility maximization in one-player games. Minimal prep sets constitute one such a solution concept. To see that it is consistent, let $X = \times_{i \in N} X_i$ be a minimal prep set of a strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$. Suppose that a set $S \subset N$ of players commits to playing according to their recommendation $\times_{i \in S} X_i$ from the minimal prep set X. In the reduced game G', this implies that the set A_i of each player $i \in S$ is reduced to the set X_i, while each player $i \in N \setminus S$ is still allowed to choose freely from A_i. The set X is easily seen to be a minimal prep set of the reduced game G' as well. Consequently, the notion of minimal prep sets contributes to the research on consistent behavior in strategic games.

EXPERIMENTAL EVIDENCE: The author does not have the facilities to conduct large scale experiments to test the prominence of minimal prep sets in a controlled environment. Still, the minimal prep sets and the corresponding adjustment process were motivated by simple and inertial behavior of the players, assumptions for which ample experimental evidence is available. An experimental study of minimal prep sets is an interesting venue for further research.

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