Strategic equivalence and bounded rationality in extensive form games

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Abstract

In a large family of solution concepts for boundedly rational players — allowing players to be imperfect optimizers, but requiring that "better" responses are chosen with probabilities at least as high as those of "worse" responses — most of Thompson's "inessential" transformations for the strategic equivalence of extensive form games become far from inconsequential. Only two of the usual elementary transformations remain truly inessential: the interchange of moves, and replacing a final move by nature by simply taking expected payoffs.

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1. Introduction

In a classic paper — a semi-official status, since it was reprinted in Kuhn's (1997) "Classics in Game Theory" — Thompson (1952) shows that if two extensive form games have the same strategic form (up to duplicated pure strategies), then one may be transformed into the other via repeated application of what Kohlberg and Mertens (1986, p. 1008) refer to as "completely inessential" transformations of the game tree: these transformations leave the strategic features of the game unaffected. They continue to argue that such elementary transformations

> "...are irrelevant for correct decision making: after all, the transformed tree is merely a different presentation of the same decision problem, and decision theory should not be misled by presentation effects. We contend that to hold the opposite point of view is to admit that decision theory is useless in real-life applications..." (Kohlberg and Mertens, 1986, p. 1011).

This is a perfectly sound basis for the study of the stability and refinement of Nash equilibria under the assumption of rational behavior of the players. On the other hand, one of the major lessons from behavioral game theory is that different ways to tell the same story *do* matter if players are less rational. See for instance Tversky and Kahneman (1986) on framing.

So do these transformations really remain 'inessential' if explicit models of bounded rationality are considered? The current note investigates the effect of these transformations on one of the most commonly used classes of solution concepts for boundedly rational players in noncooperative games: the family of quantal response equilibria (QRE), where the assumption of expected utility maximization is replaced by probabilistic choice models. In these models, players are imperfect optimizers, but at least play "better" responses with probabilities not lower than "worse" responses.[2] Two properties of the family of quantal response equilibria that are of importance in our analysis are:

[QRE1] Every action is chosen with positive probability.[3]

[QRE2] In every information set, the choice probabilities are defined in terms of and are weakly increasing in conditional expected payoffs.

These properties are formalized in Section 2.2; Goeree et al. (2006) impose somewhat stronger conditions in their definition of regular QRE. The most frequently used QRE apply the logit

[2] Rosenthal (1989) and Voorneveld (2005) study properties of one of the first concepts within this literature; the variant using the logit model was introduced in McKelvey and Palfrey (1995, 1998). Numerous experimental studies (cf. Camerer, 2003, Goeree and Holt, 2001) indicate that QRE have substantial descriptive power.

[3] This assumption can be relaxed. See Section 4.

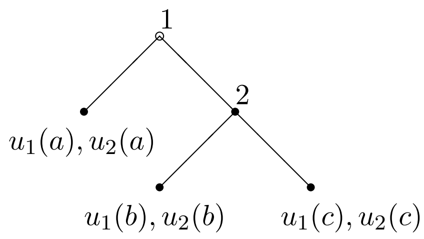

Figure 1: A simple extensive form game

choice model, well-known from pioneering contributions of McFadden (1974), where choice probabilities are proportional to a simple exponential function of the associated expected payoffs. For instance, in the two-player game in Figure 1, conditional on being in his information set, player 2 receives payoff $u_2(b)$ from moving left and $u_2(c)$ from moving right. Thus, in a logit QRE, his probability $q \in [0, 1]$ of moving left needs to satisfy

$$q = \frac{e^{\lambda u_2(b)}}{e^{\lambda u_2(b)} + e^{\lambda u_2(c)}},$$

and similarly, player 1's probability $p \in [0, 1]$ of moving left needs to satisfy

$$p = \frac{e^{\lambda u_1(a)}}{e^{\lambda u_1(a)} + e^{\lambda (q u_1(b) + (1-q) u_1(c))}},$$

with parameter $\lambda \geq 0$. This parameter is used in experiments to fit the data and provides a rough indication of 'rationality': at $\lambda = 0$, players randomize uniformly over their options without taking payoffs into account, whereas the quantal response equilibria select a subset of the game's Nash equilibria as the parameter λ goes to infinity (McKelvey and Palfrey, 1998, Thm. 4). Logit QRE are used throughout the note for the purpose of illustration.

Incidental observations on the violation of strategic equivalence in the context of logit QRE have been made before; see McKelvey and Palfrey (1998, pp. 18-19), discussed in greater detail in Section 3.1 below. Yet, to our knowledge, this note is the first[4] to provide *(a)* a full treatment of strategic equivalence by systematically going through all of the Thompson transformations (and two additional ones by Kohlberg and Mertens, 1986), and *(b)* to do so for an entire family of solution concepts for boundedly rational players, assuming only [QRE1] and [QRE2].

The remainder is set up as follows. Although the main points of this note require little or no formal definitions, standard notation for extensive form games and quantal response equilibria

[4]For instance, the books on behavioral economics by Kagel and Roth (1995) and Camerer (2003) do not mention Thompson (1952) at all.

is contained in Section 2. Section 3 shows by means of simple examples that most of the transformations for the strategic equivalence of extensive form games do affect quantal response equilibria in a nontrivial way and — in the logit case — may select decidedly different Nash equilibria in the limiting case as the parameter λ goes to infinity. Only two transformations of Thompson (1952) and Kohlberg and Mertens (1986) remain truly inessential: the interchange of moves — roughly stating that if two consecutive players move in ignorance of one another's choice, the order of their choices is irrelevant — and replacing a final move by nature by simply taking expected payoffs. Section 4 concludes. Proofs are in the Appendix.

2. Notation

2.1. Extensive form games

The notation follows, with minor changes, Osborne and Rubinstein (1994, section 11.1). A (finite) **extensive form game** Γ has the following components:

- A nonempty, finite set of **players** N.
- A finite set H of sequences satisfying:
  - the empty sequence $\emptyset$, interpreted as the starting point of the game, lies in H;
  - if $(a^1, \ldots, a^K) \in H$ and $L < K$, then $(a^1, \ldots, a^L) \in H$.

  Elements of H are **histories**; each component of a history is an action taken by a player. History $(a^1, \ldots, a^K) \in H$ is terminal if there is no a^{K+1} such that $(a^1, \ldots, a^K, a^{K+1}) \in H$. The set of **terminal histories** is denoted by $Z \subset H$. The set of **actions** available after a nonterminal history h is denoted by $A(h) = \{a : (h, a) \in H\}$.
- A **player function** $P : H \setminus Z \to N \cup \{0\}$ assigning to each nonterminal history $h \in H \setminus Z$ the player $P(h) \in N \cup \{0\}$ who takes an action after history h. If $P(h) = 0$, chance/nature determines the action.
- For each history $h \in H \setminus Z$ with $P(h) = 0$ a probability distribution $f_0(\cdot \mid h)$ over $A(h)$ specifying for each $a \in A(h)$ the probability $f_0(a \mid h)$ that nature chooses a.
- For each player $i \in N$ a partition $\mathcal{I}_i$ of $\{h \in H \setminus Z : P(h) = i\}$ with $A(h) = A(h')$ if h and h' lie in the same member of the partition. An element $I_i \in \mathcal{I}_i$ is an **information set** of player i. Since i has the same actions in all $h \in I_i$, the set of actions $A(I_i)$ in information set I_i is well-defined.

- For each player $i \in N$ a von Neumann-Morgenstern **utility function** $u_i : Z \to \mathbb{R}$.

We assume **perfect recall**: if histories h, h' lie in the same information set of player i, the sequence of player i's information sets encountered along these histories and the actions that i took there are identical. Informally: in an information set, player i remembers exactly which of his own information sets he passed, and what he did there.

A **behavioral strategy** β_i of player $i \in N$ is a collection of (independent) probability distributions $(\beta_i(I_i))_{I_i \in \mathcal{I}_i}$ assigning to each of his information sets $I_i \in \mathcal{I}_i$ a probability distribution over his available actions $A(I_i)$.

Let $h = (a^1, \ldots, a^K) \in H$ and let $\beta = (\beta_i)_{i \in N}$ be a behavioral strategy profile. The **probability of reaching** h using β, i.e., the product of the probabilities that the initial player $P(\emptyset)$ chooses a^1 and that for each $k = 1, \ldots, K-1$, player $P(a^1, \ldots, a^k)$ chooses action a^{k+1}, is denoted by $\Pr(h \mid \beta)$.

If h is not a terminal history, there are **continuations** of the form $c = (c^1, \ldots, c^L)$ (with $L \in \mathbb{N}$) from h to a terminal history $(h, c) = (a^1, \ldots, a^K, c^1, \ldots, c^L) \in Z$. Denote the set of continuations from h to a terminal history by

$$C(h) = \{c : (h, c) \in Z\}.$$

Conditional upon being in h, the **probability of continuation** $c = (c^1, \ldots, c^L) \in C(h)$ using β, i.e., the product of the probabilities that player $P(h)$ chooses c^1 and that for each $\ell = 1, \ldots, L-1$, player $P(h, c^1, \ldots, c^\ell)$ chooses action $c^{\ell+1}$, is denoted by $\Pr(c \mid \beta, h)$. For notational convenience, if h is a terminal history itself, we define its unique continuation to be the empty sequence $\emptyset$ with corresponding probability one:

$$h \in Z \quad \Rightarrow \quad C(h) := \{\emptyset\} \text{ and } \Pr(\emptyset \mid \beta, h) := 1.$$

2.2. Quantal response equilibria

We start by formalizing properties [QRE1] and [QRE2] from the introduction.

[QRE1] For each QRE β, each $i \in N$, each $I_i \in \mathcal{I}_i$, each $a \in A(I_i)$: $\beta_i(I_i)(a) > 0$.

This implies that all information sets are reached with positive probability. Hence, if $i \in N$ believes to be in information set $I_i \in \mathcal{I}_i$, he can determine the probability of being in any of its histories $h \in I_i$ via Bayes' law. The conditional expected payoff from choosing $a \in A(I_i)$ is

$$\mathbb{E}[u_i \mid I_i, \beta, a] = \sum_{h \in I_i} \frac{\Pr(h \mid \beta)}{\sum_{\tilde{h} \in I_i} \Pr(\tilde{h} \mid \beta)} \sum_{c \in C(h,a)} \Pr(c \mid \beta, (h,a))\, u_i(h, a, c).$$

The requirement that the choice probabilities in information sets are defined in terms of and are weakly increasing in conditional expected payoffs can be formalized as follows:

[QRE2] Fix two games Γ, Γ' with player set N. Suppose there is:

(i) for each $i \in N$ a bijection $\varphi : \mathcal{I}_i \to \mathcal{I}'_i$ between i's information sets and

(ii) for each $I_i \in \mathcal{I}_i$ a bijection $\psi : A(I_i) \to A'(\varphi(I_i))$ between i's feasible actions in Γ and Γ', respectively. Then every completely mixed behavioral strategy profile β in Γ yields an isomorphic strategy profile β' in Γ' defined by:

$$\forall i \in N, \forall I_i \in \mathcal{I}_i, \forall a \in A(I_i) : \qquad \beta'_i(\varphi(I_i))(\psi(a)) := \beta_i(I_i)(a).$$

If these give identical conditional expected payoffs in all information sets, i.e.,

$$\forall i \in N, \forall I_i \in \mathcal{I}_i, \forall a \in A(I_i) : \qquad \mathbb{E}[u_i \mid I_i, \beta, a] = \mathbb{E}[u'_i \mid \varphi(I_i), \beta', \psi(a)],$$

then Γ and Γ' have identical QRE, up to isomorphism:

$$\beta \text{ is a QRE of } \Gamma \quad \Leftrightarrow \quad \beta' \text{ is a QRE of } \Gamma'.$$

Moreover, if β is a QRE of Γ, then, for all $a, b \in A(I_i)$:

$$\mathbb{E}[u_i \mid I_i, \beta, a] \geq \mathbb{E}[u_i \mid I_i, \beta, b] \qquad \Rightarrow \qquad \beta_i(I_i)(a) \geq \beta_i(I_i)(b).$$

The most familiar member of the family of QRE uses logit choice probabilities (McFadden, 1974). Consider a game Γ and a parameter $\lambda \in [0, \infty)$. A **logit quantal response equilibrium** of Γ is a completely mixed behavioral strategy profile β with choice probabilities defined as follows:

$$\forall i \in N, \forall I_i \in \mathcal{I}_i, \forall a \in A(I_i) : \quad \beta_i(I_i)(a) = \frac{\exp \lambda \, \mathbb{E}[u_i \mid I_i, \beta, a]}{\sum_{b \in A(I_i)} \exp \lambda \, \mathbb{E}[u_i \mid I_i, \beta, b]}.$$

3. Transformations

3.1. Addition of a superfluous move

The second game in Figure 2 is obtained from the first by applying Thompson's 'addition of a superfluous move'. Player 2 is given an additional decision point in his information set, but the move there is irrelevant: it does not affect the outcome (no matter what 2 does, the game ends with outcome a). The games' QRE, however, will typically be affected in a nontrivial way: by [QRE2], the choice probability in player 2's information set is defined in terms of his expected payoff there. Conditional on being in his information set, 2's expected payoff of moving left before the transformation is $u_2(b)$, whereas after the transformation, it will be a convex combination of $u_2(a)$ and $u_2(b)$.

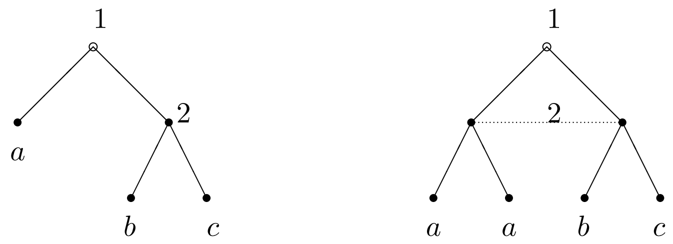

Figure 2: Addition of a superfluous move

Assigning payoffs $u_1(a) = u_2(a) = 2, u_1(b) = u_2(b) = 0, u_1(c) = 3, u_2(c) = 1$, McKelvey and Palfrey (1998, pp. 18-19) show that the logit QRE of these two games yield qualitatively different results: the logit QRE in the first game is unique (for every $\lambda \geq 0$), whereas the second game allows multiple QRE for given parameter values λ. Indeed, in the latter case, for large values of λ, there is an additional component of QRE converging to a subgame imperfect equilibrium. Nevertheless, the principal branch of the logit QRE manifold in both representations eventually selects the subgame perfect equilibrium where 1 moves left and 2 moves right.

But the differences can be even more dramatic. Suppose, for instance, that

$$u_1(a) = u_1(b) > u_1(c) \qquad \text{and} \qquad u_2(b) > u_2(c). \tag{1}$$

Let $\lambda \geq 0$. If $p(\lambda), q(\lambda) \in [0,1]$ denote the probability of player 1 and 2 moving left, respectively, the conditions for a logit QRE of the first game are:

$$\begin{aligned} q(\lambda) &= \frac{e^{\lambda u_2(b)}}{e^{\lambda u_2(b)} + e^{\lambda u_2(c)}} \\ &= \frac{1}{1 + e^{\lambda(u_2(c) - u_2(b))}}, \end{aligned} \tag{2}$$

$$\begin{aligned} p(\lambda) &= \frac{e^{\lambda u_1(a)}}{e^{\lambda u_1(a)} + e^{\lambda[q(\lambda)u_1(b) + (1-q(\lambda))u_1(c)]}} \\ &= \frac{1}{1 + e^{\lambda(1-q(\lambda))(u_1(c) - u_1(a))}}. \end{aligned} \tag{3}$$

For every $\lambda \geq 0$, (2) fixes $q(\lambda)$, which in its turn via (3) fixes $p(\lambda)$: the logit QRE given λ is unique. Moreover, given (1), we have

$$\lim_{\lambda \to \infty} (p(\lambda), q(\lambda)) = (1/2, 1), \tag{4}$$

i.e., as $\lambda \to \infty$, the principal branch of the logit QRE manifold selects the Nash equilibrium where player 1 mixes uniformly and player 2 moves left (proof in Appendix). The logit QRE of the second game, however, behave considerably more capriciously. We consider two examples.

Figure 3 contains the graph of the choice probabilities $p(\lambda)$ in the game's logit QRE if $u_i(a) = u_i(b) = 1, u_i(c) = 0$ for both $i = 1, 2$. Plot discontinuities are due to approximations.

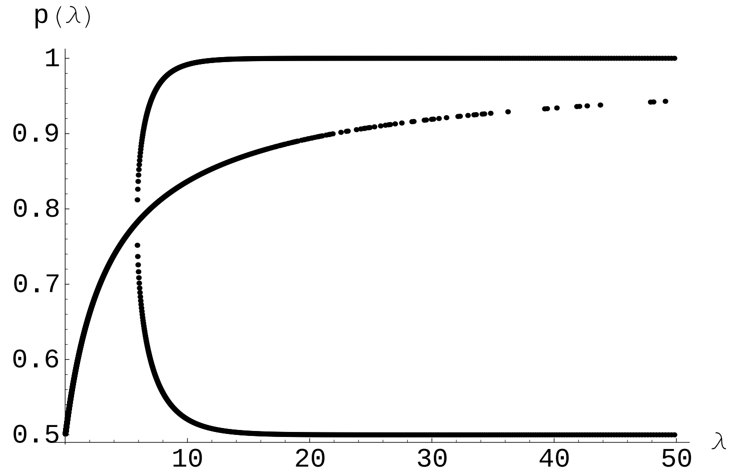

Figure 3: The choice probability $p(\lambda)$ in logit QRE of a symmetric game.

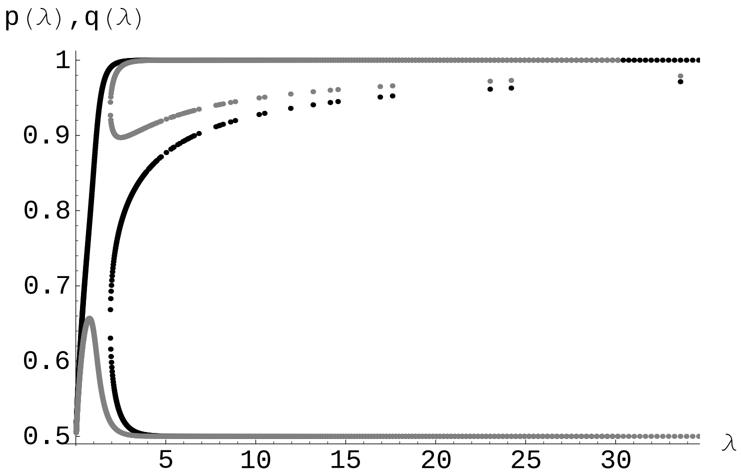

Figure 4: The choice probabilities $p(\lambda)$ (black) and $q(\lambda)$ (grey) in logit QRE.

By symmetry, the graph of $q(\lambda)$ is similar. This shows not only that there are multiple logit QRE for large values of λ, but also that there is a bifurcation: there is no principal branch. Parts of the QRE correspondence converge to different Nash equilibria: $(1,1)$, $(1/2,1)$, and $(1,1/2)$.

Figure 4 contains the graphs of the choice probabilities $p(\lambda)$ (black) and $q(\lambda)$ (grey) in the game's logit QRE if $u_1(a) = u_1(b) = 5, u_1(c) = u_2(c) = 0, u_2(a) = 2, u_2(b) = 4$. Again, there are multiple logit QRE for large values of λ, but now the principal branch converges to the Nash equilibrium $(1, 1/2)$, different from the limiting equilibrium (4) before the transformation.

3.2. Coalescing of information sets

Consider the two games in Figure 5. In the first game, player 2 moves twice in a row if he initially goes right. In the second game, these consecutive moves are contracted to a single one by giving the second player two new actions that can be interpreted as 'first right, then left' and 'twice right', respectively. This is an example of the transformation 'coalescing of information sets'. Player 2 is indifferent between his actions. By weak monotonicity [QRE2], in a quantal response equilibrium, player 2 chooses each of his actions with equal probability. Consequently, in the first game, player 1 receives payoff $u_1(a)$ by moving left and expected payoff

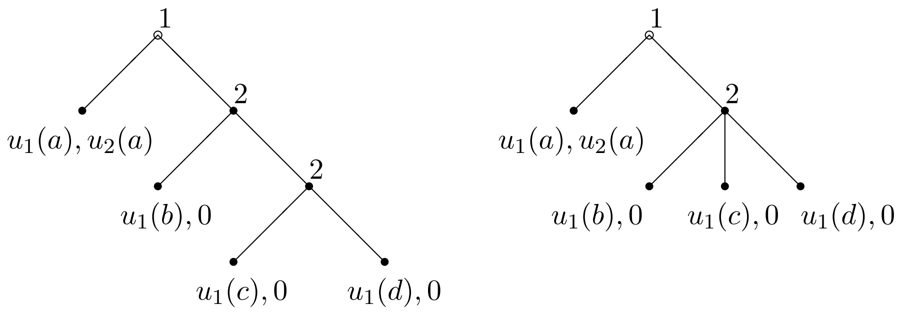

Figure 5: Coalescing information sets

$\frac{1}{2}u_1(b) + \frac{1}{4}u_1(c) + \frac{1}{4}u_1(d)$ by moving right. In the second game, 1's expected payoff from moving right is $\frac{1}{3}u_1(b) + \frac{1}{3}u_1(c) + \frac{1}{3}u_1(d)$. If player 1's payoffs are such that

$$\frac{1}{2}u_1(b) + \frac{1}{4}u_1(c) + \frac{1}{4}u_1(d) < u_1(a) < \frac{1}{3}u_1(b) + \frac{1}{3}u_1(c) + \frac{1}{3}u_1(d),$$

he prefers to move left in the first game, but right in the second. By weak monotonicity [QRE2], the probability of moving left is at least 1/2 in the first and at most 1/2 in the second game. In particular, in a logit QRE, player 1's probability of moving left is

$$p(\lambda) = \frac{e^{\lambda u_1(a)}}{e^{\lambda u_1(a)} + e^{\lambda\left(\frac{1}{2}u_1(b)+\frac{1}{4}u_1(c)+\frac{1}{4}u_1(d)\right)}}.$$

in the first and

$$\hat{p}(\lambda) = \frac{e^{\lambda u_1(a)}}{e^{\lambda u_1(a)} + e^{\lambda\left(\frac{1}{3}u_1(b)+\frac{1}{3}u_1(c)+\frac{1}{3}u_1(d)\right)}}.$$

in the second game. These probabilities move in different directions: although both equal $\frac{1}{2}$ if $\lambda = 0$, $p(\lambda)$ increases to one, whereas $\hat{p}(\lambda)$ decreases to zero as $\lambda \to \infty$.

3.3. Interchange of moves

The idea behind the interchange of moves is that if one player has no information about the other player's action when making his choice, the order of play is irrelevant. The standard example is that a two-player, simultaneous-move game can be represented in extensive form in two equivalent ways: either as a game where player 1 moves first and player 2 moves next, unaware of what 1 did, or conversely, as a game where player 2 moves first and player 1 moves next, unaware of what 2 did. Figure 6 illustrates this for a 2×2 "Stag-Hunt" game.

	L	R
T	$2, 2$	$3, 0$
B	$0, 3$	$4, 4$

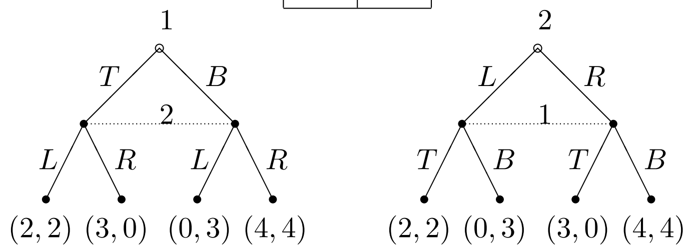

Figure 6: Interchange of moves: representations of a "Stag-Hunt" game

The interchange of moves defined in the Appendix allows slightly more general cases, but the idea remains the same: since two consecutive players act in total ignorance of one another's choices, they act on the same information in both cases. Expected payoffs are not affected and therefore, by [QRE2], the transformation is truly inessential:

Theorem 3.1 *The interchange of moves leaves the quantal response equilibria of an extensive form game unaffected.*

The proof is in the Appendix.

3.4. Three more transformations

Elmes and Reny (1994) show that Thompson's final transformation, inflation/deflation, can be dispensed with under our assumption of perfect recall. Kohlberg and Mertens (1986, pp. 1008-1009) suggest two more transformations. Firstly:

> "...whenever a move by nature leads only to terminal nodes, it is equivalent to a terminal node with the corresponding expected payoffs."

This transformation leaves the expected payoffs of a player's actions in a given information set unchanged and therefore, by [QRE2], does not affect the quantal response equilibria. Secondly:

> "...in an information set that is followed by no other information set, one may add or delete moves that lead in effect to a lottery between other moves."

This transformation is far from innocent. Consider the second game in Figure 5 and suppose that $u_1(b) < u_1(d) < u_1(c)$: player 2's third action is equivalent to a suitably chosen lottery between the first two. Removing the third action gives Figure 7. By [QRE2], since player 2 is indifferent between his actions, he will choose them with equal probability in a QRE. If player 1's payoffs are such that

$$\frac{1}{2}u_1(b) + \frac{1}{2}u_1(c) < u_1(a) < \frac{1}{3}u_1(b) + \frac{1}{3}u_1(c) + \frac{1}{3}u_1(d),$$

this means that player 1 prefers to move right in the game before the transformation and to move left after the transformation. In particular, in the logit QRE, the probabilities of moving left in the initial history move in opposite directions: both equal $\frac{1}{2}$ if $\lambda = 0$, but as $\lambda \to \infty$, the probability decreases to zero in the initial game and increases to one in the transformed game.

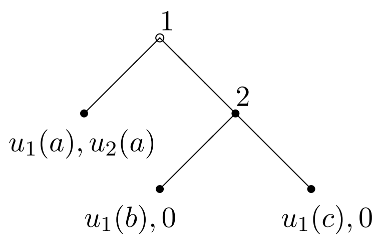

Figure 7: Removing convex combinations

4. Concluding remarks

Under weak assumptions [QRE1] and [QRE2] on a large family of solution concepts for boundedly rational players, we have shown that only two of the usual 'inessential' transformations of Thompson (1952) and Kohlberg and Mertens (1986) remain truly inessential: the interchange of moves, and replacing a final move by nature by simply taking expected payoffs. The essential assumption is [QRE2]: in an information set, only expected payoffs matter. And "better" responses are chosen with probabilities not lower than "worse" responses. Condition [QRE1], requiring the behavioral strategies to be completely mixed, guarantees that all information sets are reached with positive probability and that Bayes' law can be applied to compute the conditional probabilities of the different histories in an information set. In all our counterexamples to show that transformations were far from inessential, however, information sets were either singletons or reached with probability one, so requirement [QRE1] was not necessary. This means that our counterexamples also serve to show the relevance of such transformations in members of the QRE family where actions may be chosen with probability zero, like in Rosenthal (1989) and Voorneveld (2005).

The effect of certain transformations on choice probabilities in our probabilistic choice models has a counterpart in the discrete choice models from microeconometrics by which the QRE are inspired. The distortional effect of coalescing of moves is one of the main motivations for considering nested/hierarchical discrete choice models (cf. Ben-Akiva and Lerman, 1985, Ch. 10, Train, 2003, Ch. 4). The distortional effect of taking away/adding convex combinations of moves and in particular replicas of existing options is an example of Debreu's (1960) well-known red-bus-blue-bus paradox.

Appendix

PROOF OF (4): Since $u_2(c) - u_2(b) < 0$, it follows from (2) that $\lim_{\lambda\to\infty} q(\lambda) = 1$. Since $u_1(c) - u_1(b) = u_1(c) - u_1(a) < 0$, the power series expansion of the exponential function gives:

$$\begin{aligned}
0 &\geq \lambda(1-q(\lambda))(u_1(c)-u_1(a)) \\
&= \frac{\lambda(u_1(c)-u_1(a))e^{\lambda u_2(c)}}{e^{\lambda u_2(b)}+e^{\lambda u_2(c)}} \\
&= \frac{\lambda(u_1(c)-u_1(a))}{e^{\lambda(u_2(b)-u_2(c))}+1} \\
&= \frac{\lambda(u_1(c)-u_1(a))}{\sum_{n=0}^{\infty}\left(\frac{\lambda^n(u_2(b)-u_2(c))^n}{n!}\right)+1} \\
&\geq \frac{\lambda(u_1(c)-u_1(a))}{\left(\frac{\lambda^2(u_2(b)-u_2(c))^2}{2!}\right)} \\
&= \frac{2(u_1(c)-u_1(a))}{\lambda(u_2(b)-u_2(c))^2} \\
&\to 0 \text{ as } \lambda\to\infty.
\end{aligned}$$

Since $\lim_{\lambda\to\infty} \lambda(1-q(\lambda))(u_1(c)-u_1(a)) = 0$, it follows from (3) that $\lim_{\lambda\to\infty} p(\lambda) = \frac{1}{2}$.

INTERCHANGE OF MOVES: We follow Osborne and Rubinstein (1994, p. 208).[5] Let Γ be an extensive form game. Suppose there are players $i, j \in N$ and information sets $I_i \in \mathcal{I}_i, I_j \in \mathcal{I}_j$ capturing the following: there is a nonempty set $H' \subseteq I_i$ of histories in i's information set such that the next player to move is player j, who is completely unaware of what i did:

$$H' = \{h' \in I_i : (h', a) \in I_j \text{ for all } a \in A(h')\}.$$

The set of histories in I_j reached in this way is

$$H'' = \{(h', a) \in I_j : h' \in H', a \in A(h')\}.$$

[5]Thompson (1952) and Elmes and Reny (1994) define the transformation only for cases where both players have two-alternative moves; Osborne and Rubinstein (1994) do not impose this restriction.

Then Γ is equivalent to the extensive form game where:

- The set of histories is changed as follows. For every $h' \in H'$, $a \in A(h)$, $b \in A(h', a)$, $c \in C(h', a, b)$:
  - histories of the form (h', a) are replaced by histories of the form (h', b),
  - histories of the form (h', a, b) are replaced by histories of the form (h', b, a),
  - histories of the form (h', a, b, c) are replaced by histories of the form (h', b, a, c).
- The information set I_i of player i is replaced by the information set

$$I_i' = (I_i \setminus H') \cup \{(h', b) : h' \in H, b \in A(I_j)\}, \tag{5}$$

- The information set I_j of player j is replaced by the information set

$$I_j' = (I_j \setminus H'') \cup H'.$$

- Of course, the corresponding changes in information sets, player function, etc., are mostly cosmetic: the only relevant change is that now player j moves in his new information set I_j' before player i does so in his new information set I_i'.

Proof of Thm. 3.1: Consider an extensive form game Γ and suppose that moves of players i and j were interchanged in the way described above. Since the only information sets that were changed in a non-trivial way are I_i and I_j, it suffices to prove that the conditions for quantal response equilibrium at these information sets are unaffected. Moreover, since one gets back the original game by once again interchanging the moves of i and j, the player roles are symmetric, so it suffices to prove this for the information set of one of these players only, say player i. Finally, by [QRE2], it suffices to show that the expected payoff for every action $a^* \in A(I_i)$ is the same in both games. So let $a^* \in A(I_i)$.

Partition I_i into H' and the (possibly empty) set of remaining histories $R := I_i \setminus H'$. Let $A_j := A(I_j)$ be the set of actions available to player j in information set I_j and consequently also in I_j'. Let $\beta = (\beta_i)_{i \in N}$ be a profile of completely mixed behavioral strategies in Γ. With a minor abuse of notation, this is also a profile of behavioral strategies in the game after the interchange transformation, where player $k \in \{i, j\}$ applies probability distribution $\beta_k(I_k)$ to the actions in the new information set I_k'.

Player i's expected payoff from a^* in the games before and after the interchange of moves is:

Case 1, conditional upon being in I_i: Conditioning on the histories in I_i and computing the expected payoff from all relevant continuations (divided into the first move by j and possible consecutive moves), we find:

$$\sum_{h\in R} \frac{\Pr(h|\beta)}{\sum_{\tilde{h}\in I_i}\Pr(\tilde{h}|\beta)} \sum_{c\in C(h,a^*)} \Pr(c \mid \beta, (h,a^*))\, u_i(h,a^*,c) +$$

$$\sum_{h'\in H'} \frac{\Pr(h'|\beta)}{\sum_{\tilde{h}\in I_i}\Pr(\tilde{h}|\beta)} \sum_{b\in A_j} \sum_{c\in C(h',a^*,b)} \beta_j(I_j)(b)\, \Pr(c \mid \beta, (h',a^*,b))\, u_i(h',a^*,b,c).$$

Case 2, conditional upon being in I_i': By (5), the probability of reaching I_i' is

$$\sum_{\tilde{h}\in R} \Pr(\tilde{h} \mid \beta) \quad + \quad \sum_{b\in A_j} \sum_{h'\in H'} \beta_j(I_j)(b) \Pr(h' \mid \beta) \quad =$$

$$\sum_{\tilde{h}\in R} \Pr(\tilde{h} \mid \beta) \quad + \quad \underbrace{\sum_{b\in A_j} \beta_j(I_j)(b)}_{=1} \sum_{h'\in H'} \Pr(h' \mid \beta) \quad =$$

$$\sum_{\tilde{h}\in I_i} \Pr(\tilde{h} \mid \beta),$$

just like in Case 1. By (5), conditioning on histories in I_i' and computing the expected payoff from all relevant continuations, we find:

$$\sum_{h\in R} \frac{\Pr(h|\beta)}{\sum_{\tilde{h}\in I_i}\Pr(\tilde{h}|\beta)} \sum_{c\in C(h,a^*)} \Pr(c \mid \beta, (h,a^*))\, u_i(h,a^*,c) +$$

$$\sum_{h'\in H'} \sum_{b\in A_j} \frac{\beta_j(I_j)(b)\Pr(h'|\beta)}{\sum_{\tilde{h}\in I_i}\Pr(\tilde{h}|\beta)} \sum_{c\in C(h',a^*,b)} \Pr(c \mid \beta, (h',a^*,b))\, u_i(h',a^*,b,c),$$

which is — after rearranging terms — identical to the expression in Case 1.

References

Ben-Akiva, M., Lerman, S.R., 1985. Discrete choice analysis. Cambridge, MA: MIT Press.

Camerer, C., 2003. Behavioral game theory; Experiments in strategic interaction. Princeton NJ: Princeton University Press.

Debreu, G., 1960. Review of R. D. Luce, Individual Choice Behavior. Amer. Econ. Rev. 50, 186-188.

Elmes, S., Reny, P.J., 1994. On the strategic equivalence of extensive form games. J. Econ. Theory 62, 1-23.

Goeree, J.K., Holt, C.A., 2001. Ten little treasures of game theory and ten intuitive contradictions. Amer. Econ. Rev. 91, 1402-1422.

Goeree, J.K., Holt, C.A., Palfrey, T.R., 2006. Regular quantal response equilibrium. Experimental Economics, to appear.

Kagel, J.H., Roth, A.E. (editors), 1995. The handbook of experimental economics. Princeton, NJ: Princeton University Press.

Kohlberg, E., Mertens, J.-F., 1986. On the strategic stability of equilibria. Econometrica 54, 1003-1037.

Kuhn, H.W. (editor), 1997. Classics in game theory. Princeton, NJ: Princeton University Press.

McFadden, D., 1974. Conditional logit analysis of qualitative choice behavior. In: Zarembka, P. (Ed.), Frontiers in Econometrics. New York: Academic Press, pp. 105-142.

McKelvey, R.D., Palfrey, T.R., 1995. Quantal response equilibria for normal form games. Games Econ. Behav. 10, 6-38.

McKelvey, R.D., Palfrey, T.R., 1998. Quantal response equilibria for extensive form games. Exper. Econ. 1, 9-41.

Osborne, M.J., Rubinstein, A., 1994. A course in game theory. Cambridge, MA: MIT Press.

Rosenthal, R.W., 1989. A bounded rationality approach to the study of noncooperative games. Int. J. Game Theory 18, 273-292.

Thompson, F.B., 1952. Equivalence of games in extensive form. Rand Corporation, Research Memorandum RM-759.

Train, K.E., 2003. Discrete choice models with simulation. Cambridge: Cambridge University Press.

Tversky, A., Kahneman, D., 1986. Rational choice and the framing of decisions. J. Business 59, S251-S278.

Voorneveld, M., 2005. Probabilistic choice in games: Properties of Rosenthal's t-solutions. Int. J. Game Theory, to appear.