Choosing Factors in a Multifactor Asset Pricing Model: A Bayesian Approach

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Abstract

We use Bayesian techniques to select factors in a general multifactor asset pricing model. From a given set of 15 factors we evaluate all possible pricing models by the extent to which they describe the data as given by the posterior model probabilities. Interest rates, premiums, returns on broadbased portfolios and macroeconomic variables are included in the set of factors.

Using different portfolios as the investment universe we find strong evidence that a general multifactor pricing model should include the market excess return, the size premium, the value premium and the momentum factor. In addition, we find evidence that the credit risk spread should be included as an additional factor. There are some indications that industrial production also is an important factor.

Keywords: asset pricing, factor models, Bayesian model selection.
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1 Introduction

The capital asset pricing model, CAPM, developed by Sharpe (1964), Lintner (1965) and Black (1972), predict that the expected asset return is a linear function of the risk, where the risk is measured by the covariance between its return and that of a market portfolio. The empirical evidence on the CAPM is mixed. Black, Jensen and Scholes (1972), Fama and MacBeth (1973) and Blume and Friend (1973) find support for CAPM whereas Basu (1977) and Banz (1981), Fama and French (1992, 1993), DeBondt and Thaler (1985) and Jegadeesh and Titman (1993) find evidence against the CAPM. The mixed evidence naturally leads to consideration of multifactor asset pricing models.

Multifactor pricing models was introduced by Ross (1976) through the Arbitrage Pricing Theory and by Merton (1973) through the Intertemporal CAPM. The multifactor pricing model imply that the expected return on an asset is a linear function of factor risk premiums and their associated factor sensitivities. The underlying theory is, however, not very explicit on the exact nature of these factors. The selection of an appropriate set of factors is thus largely an empirical issue.

There are two strands in the empirical literature on multifactor asset pricing models. One focusing on unobservable or latent factors, e.g. Lehmann and Modest (1988) who use factor analysis and find weak evidence in favor of a ten factor model but they also argue that the tests have little power to discriminate among models with different number of factors or Connor and Korajczyk (1988) who use principal components and find little sensitivity to increasing the number of factors beyond five. Chen, Roll and Ross (1986) and Fama and French (1993) on the other hand consider observable factors. Chen et al. (1986) find evidence of five priced macroeconomic factors. The Fama and French study use firm characteristics to form factor portfolios and result in the well known three-factor model while Carhart (1997) finds evidence for a fourth, momentum, factor. There is thus a lack of consensus about the number and the identity of the factors.

In this paper we conduct an exhaustive evaluation of multifactor asset pricing models based on observable factors. Based on a set of 15 factors we use Bayesian techniques to rank the 2^{15} possible models based on the posterior model probabilities. The priors for the model parameters are relatively uninformative, which ensures that the posterior results are dominated by the data.

The rest of the paper is organized as follows. In the next section we present a general multifactor pricing model. Section 3 describes the Bayesian model selection procedure. Section 4 and 5 contains the data and empirical results, respectively, and section 6 concludes.

2 The Model

In general, a multifactor pricing model states that the returns of different assets are explained by a set of common factors in a linear model. For the return on N assets we have the general multifactor model

$$\mathbf{r}_t = \mathbf{a} + \beta_1 \mathbf{f}_{1t} + \beta_2 \mathbf{f}_{2t} + \varepsilon_t \tag{1}$$

where $\mathbf{r}_t = [r_{1t}\ r_{2t}\ ...\ r_{Nt}]'$ is a $N \times 1$ vector of excess returns, $\mathbf{a}$ is a $N \times 1$ vector of intercepts, $\mathbf{f}_{1t}$ is a $K_1 \times 1$ vector of general economic factors with $E[\mathbf{f}_{1t}] = 0$ and $\mathbf{f}_{2t}$ is a $K_2 \times 1$ vector of asset returns on reference portfolios. The error term $\varepsilon_t = [\varepsilon_{1t}, ..., \varepsilon_{Nt}]'$ is a $N \times 1$ random vector with $E[\varepsilon_t] = 0$ and $E[\varepsilon_t \varepsilon_t'] = \mathbf{\Sigma}$. The matrices β_1 and β_2 are factor sensitivities with dimension $N \times K_1$ and $N \times K_2$, respectively. For convenience we rewrite (1) as a multivariate regression model

$$\mathbf{R} = \mathbf{XB} + \mathbf{E}, \tag{2}$$

where the rows of $\mathbf{R}$, $\mathbf{X}$ and $\mathbf{E}$ are given by $\mathbf{r}_t'$, $\begin{bmatrix} 1 & \mathbf{f}_{1t}' & \mathbf{f}_{2t}' \end{bmatrix}$ and ε_t'. Finally, $\mathbf{B} = \begin{bmatrix} \mathbf{a}' & \beta_1' & \beta_2' \end{bmatrix}'$.

Generally, asset pricing theory offers little guidance when selecting the factors. Theory suggests that assets will have to pay high average returns if they do poorly in bad times, in which investors would particularly like their investments not to perform badly, and are willing to sacrifice some expected return in order to ensure that it is so. Consumption, or more correctly marginal utility, should provide the purest measure of bad times. Investors consume less when their income are low or if they think future returns will be bad. But, the empirical evidence that relate asset returns to consumption is weak.[1] Therefore, empirical asset pricing models examine more indirect measures of good or bad times, interest rates, returns on broadbased portfolios, and growth in consumption, production and other macroeconomic variables that measure the state of the economy. Furthermore, variables that signals change in the future, such as term premiums, credit spreads, etc. are also reasonable to include.

The set of possible factors we consider is based on previous studies. Fama and French (1992,1993,1996) advocate a model with the market return, the return of small less big stocks (SMB) and the return of high less low book-to-market stocks (HML) as factors. Carhart (1997) find support of a four-factor model with the three factors of Fama and French and an additional factor that captures the momentum anomaly. Several authors have used macroeconomic variables as factors. Jagannathan and Wang (1996) and Reyfman (1997) use labour income. Chen et al. (1986) test whether innovations in several

[1]See Cochrane (2001), chapter 2 for more details.

macroeconomic variables are risks that are rewarded in the stock market. Included variables are: the spread between long and short interest rate, expected and unexpected inflation, industrial production, the spread between high and low-grade bonds, market portfolio, aggregate consumption and oil price. Other[2] empirical evidence suggests that yields and yield spreads in corporate and Treasury bond markets are important in asset pricing models. More details about the factors included in this study are given in section 4.

3 Bayesian Model Selection

The Bayesian approach to model selection offers several advantages. In particular, the Bayesian approach is conceptually the same, regardless of the number of models under consideration, and the interpretation of the Bayes factor and the posterior model probabilities are straightforward.

From a given set of K factors, we evaluate all 2^K different models by the extent to which they describe the data as given by the posterior model probabilities. Hence, we consider all possible models of the form

$$M_i : \mathbf{R} = \mathbf{X}_i \mathbf{B}_i + \mathbf{E},\ \ i = 1, ..., 2^K \tag{3}$$

where $\mathbf{X}_i$ is $T \times (q_i + 1)$, q_i is the number of factors included in the model, and the parameter matrix $\mathbf{B}_i$ is $(q_i + 1) \times N$.

Given the prior distribution,

$$\pi(\mathbf{B}_i, \mathbf{\Sigma} | M_i)$$

[2]Ferson and Harvey (1991, 1999), Schwert (1990), Kothari and Shanken (1997), Whitelaw (1997), Campbell and Shiller (1988), and Campbell (1987).

for the parameters in model i, the marginal likelihood under model M_i is obtained as

$$m(\mathbf{R}|M_i) = \int L(\mathbf{R}|\mathbf{B}_i, \mathbf{\Sigma}, M_i)\pi(\mathbf{B}_i, \mathbf{\Sigma}|M_i)d\mathbf{B}_i d\mathbf{\Sigma} \tag{4}$$

where $L(\mathbf{R}|\mathbf{B}_i, \mathbf{\Sigma}, M_i)$ is the likelihood for model M_i. The marginal likelihood meassures how well the model (and the prior) fits the data. Model comparison can be conducted through the use of Bayes factors. The Bayes factor for M_i versus M_j is given by

$$B_{ij} = \frac{m(\mathbf{R}|M_i)}{m(\mathbf{R}|M_j)} = \frac{\int L(\mathbf{R}|\mathbf{B}_i, \mathbf{\Sigma}, M_i)\pi(\mathbf{B}_i, \mathbf{\Sigma}|M_i)d\mathbf{B}_i d\mathbf{\Sigma}}{\int L(\mathbf{R}|\mathbf{B}_j, \mathbf{\Sigma}, M_j)\pi(\mathbf{B}_j, \mathbf{\Sigma}|M_j)d\mathbf{B}_j d\mathbf{\Sigma}} \tag{5}$$

and measures how much our belief in M_i relative M_j has changed after viewing the data. If prior probabilities $P(M_i)$, $i = 1, ..., 2^K$, of the models are available, the Bayes factor can be used to compute the posterior model probabilities

$$P(M_i|\mathbf{R}) = \frac{m(\mathbf{R}|M_i)P(M_i)}{\sum_{j=1}^{2^K} m(\mathbf{R}|M_j)P(M_j)} = \left[\sum_{j=1}^{2^K} \frac{P(M_j)}{P(M_i)} B_{ji}\right]^{-1}. \tag{6}$$

Finally we note that if $P(M_i) = 1/2^K$ the posterior model probabilities are given by the normalized marginal likelihoods

$$P(M_i|\mathbf{R}) = \frac{m(\mathbf{R}|M_i)}{\sum_{j=1}^{2^K} m(\mathbf{R}|M_j)} = \left[\sum_{j=1}^{2^K} B_{ji}\right]^{-1}. \tag{7}$$

There are two main difficulties with Bayesian model selection. Firstly, we have to select prior distributions for the parameters of each model. In general, these priors must be informative since improper noninformative priors yields indeterminate marginal likelihoods. Secondly, to obtain the Bayes factors and the posterior model probabilities we need to compute the integration in equation (4). To overcome these problems we use natural conjugate priors for the factor sensitivities, $\mathbf{B}$, and for the covariance matrix , $\boldsymbol{\Sigma}$, we follow Berger and Pericchi (2001) and specify a diffuse prior since $\boldsymbol{\Sigma}$ is common for all models and the indeterminate factors cancels in the Bayes factor. The prior for $\mathbf{B}_j$ given $\boldsymbol{\Sigma}$ is given by the matrix variate normal distribution[3]

$$\mathbf{B}_j|\boldsymbol{\Sigma}, M_j \sim MN_{(q_j+1)\times N}\left(\mathbf{B}_j|\bar{\mathbf{B}}_j, \boldsymbol{\Sigma}, \mathbf{Z}_j^{-1}\right) \tag{8}$$

and the improper prior for $\boldsymbol{\Sigma}$ is given by

$$\pi\left(\boldsymbol{\Sigma}\right) \propto |\boldsymbol{\Sigma}|^{-\frac{1}{2}(N+1)}. \tag{9}$$

Using the above prior settings, the marginal likelihood for model M_i can be derived analytically. Let $\hat{\mathbf{B}}_i$ be the OLS estimator of $\mathbf{B}_i$ and let $\mathbf{S}_i = (\mathbf{R} - \mathbf{X}_i\hat{\mathbf{B}}_i)'(\mathbf{R} - \mathbf{X}_i\hat{\mathbf{B}}_i)$. Then, the Bayes factor for model M_i versus M_j is

$$B_{ij} = \frac{|\mathbf{Z}_i|^{N/2}|\mathbf{A}_i|^{-N/2} C_{IW}\left(\mathbf{S}_i^*, T, N\right)}{|\mathbf{Z}_j|^{N/2}|\mathbf{A}_j|^{-N/2} C_{IW}\left(\mathbf{S}_j^*, T, N\right)} \tag{10}$$

where $\mathbf{S}_i^* = \mathbf{S}_i + (\bar{\mathbf{B}}_i - \hat{\mathbf{B}}_i)'\left[\mathbf{Z}_i^{-1} + (\mathbf{X}_i'\mathbf{X}_i)^{-1}\right]^{-1}(\bar{\mathbf{B}}_i - \hat{\mathbf{B}}_i)$, $\mathbf{A}_i = \mathbf{Z}_i + \mathbf{X}_i'\mathbf{X}_i$ and

$$C_{IW}\left(\mathbf{S}, v, q\right) = 2^{\frac{1}{2}vq}\pi^{\frac{1}{4}q(q-1)}\prod_{i=1}^{q}\Gamma\left(\frac{v+1-i}{2}\right)|\mathbf{S}|^{-\frac{1}{2}v}. \tag{11}$$

Choosing the prior hyperparameters can be difficult in the absence of prior information. Reflecting the lack of consensus in the finance literature about the identity of the factors the prior mean of $\mathbf{B}$ conditional on specific model is $\bar{\mathbf{B}}_j = 0$ and for the prior covariance matrix we follow Fernández, Ley and Steel (2001), Hall, Hwang and Satchell (2002) and

[3] That is $E(vec\mathbf{B}_j) = vec(\bar{\mathbf{B}}_j)$ and $Cov(vec\mathbf{B}_j) = \boldsymbol{\Sigma}\otimes\mathbf{Z}_j^{-1}$, where $\otimes$ denotes the Kronecker product.

Smith and Kohn (2000) and use the g-prior by Zellner (1986). Thus,

$$\mathbf{Z}_j = g\left(\mathbf{X}_j'\mathbf{X}_j\right) \tag{12}$$

where $g > 0$. The parameter g is chosen such that the prior variance is large relative to the OLS counterpart. The Bayes factor finally simplifies to

$$B_{ij} = \frac{\left(\frac{g}{g+1}\right)^{\frac{1}{2}N(q_i+1)}}{\left(\frac{g}{g+1}\right)^{\frac{1}{2}N(q_j+1)}} \left(\frac{\left| \mathbf{S}_j + \hat{\mathbf{B}}_j{}' \frac{g}{g+1} \left(\mathbf{X}_j'\mathbf{X}_j\right) \hat{\mathbf{B}}_j) \right|}{\left| \mathbf{S}_i + \hat{\mathbf{B}}_i{}' \frac{g}{g+1} \left(\mathbf{X}_i'\mathbf{X}_i\right) \hat{\mathbf{B}}_i) \right|} \right)^{\frac{1}{2}T} \tag{13}$$

and we can easily calculate the posterior model probabilities given by equation (7).

4 The Data

The data in this study is monthly observations on US stock excess returns and a set of factors over July 1963 through December 2002. Estimation and testing of multifactor asset pricing models is typically done on portfolios of assets, rather than on individual assets. The reason is that the returns must be stationary in the sense that they have approximately the same mean and covariance. Individual assets are usually very volatile, which makes it hard to obtain precise estimates and to be able to reject anything. In this study we use six sets of portfolios[4]. The first set contains the six benchmark portfolios of Fama and French sorted on size[5] and book-to-market[6], (B/M). The second set contains the 25 Fama and French (1993) portfolios formed on size and B/M. The third set contains 10 industry portfolios. The last three sets contains 10 portfolios formed on cashflow, earnings and dividends. Based on theoretical considerations and previous empirical studies, we specify the following set of candidate factors in our evaluation.

[4]The portfolios include all NYSE, AMEX, and NASDAQ stocks
[5]Market equity (size) is price times shares outstanding
[6]Book equity to market equity (BE/ME).

1. Market excess returns, the difference between value-weighted return on all NYSE, AMEX, and NASDAQ stocks and the one-month Treasury bill rate, and dividend yield on S&P'S Composite common stock. Size premium (SMB), value premium (HML) and a momentum factor.

2. Credit risk spread, the difference between yields of Moody's Baa and the yields of Moddy's Aaa rated bonds. This is a state variables that measure changes in the risk of corporate bonds. Interest rate variables, change in yield on three month Treasury bill, difference in annualized yield of ten-year and one-year Treasuries (Term spread long), and the difference between the one–year Treasuries and the Federal Funds rate (Term spread short). These variables are expected to signal changes in the future. Macroeconomic factors that capture the state of the economy are monthly and yearly growth rate in industrial production, monthly change in inflation rate, monthly growth rate in consumption and in disposable income.

In addition, we treat the intercept as a factor, resulting in $K = 15$ factors to choose from. Returns on portfolios, size premium, value premium and momentum was kindly provided by Kenneth French[7]. Data on interest rates and price variables was obtained from the Federal Reserve Board.[8] The macroeconomic factors are demeaned.

5 Empirical Results

Using a set of $K = 15$ factors we compare all $2^{15} = 32768$ possible multifactor pricing models. Equation (13) computes the Bayes factor and by allocating the prior model probabilities equally over all models equation (7) yields the posterior model probabilities. In

[7]A decsription of the data obtained from Kenneth French can be found at http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

[8]http://www.federalreserve.gov

Table 1 Probability of Inclusion, $g = 0.05$

Factor	Benchmark	Size-B/M	Industry	Cashflow	Earning	Dividend
Intercept	0.068	0.999	0.379	0.557	0.519	0.218
SMB	1.000	1.000	1.000	1.000	1.000	1.000
HML	1.000	1.000	1.000	1.000	1.000	1.000
Momentum	1.000	0.000	0.846	0.923	0.438	1.000
Market excess return	1.000	1.000	1.000	1.000	1.000	1.000
Term spread Short	0.001	0.000	0.000	0.000	0.002	0.000
Termp spread Long	0.079	0.000	0.000	0.000	0.000	0.000
Credit risk spread	0.689	0.001	0.490	0.308	0.360	0.452
Yield on TB3M	0.003	0.000	0.000	0.001	0.000	0.000
Dividend SP500	0.348	0.000	0.140	0.137	0.121	0.331
Industrial prod. (M)	0.000	0.000	0.931	0.011	0.000	0.000
Industrial prod. (Y)	0.006	0.000	0.000	0.001	0.000	0.001
Income	0.000	0.000	0.000	0.000	0.000	0.000
Consumption	0.004	0.000	0.000	0.000	0.000	0.001
Inflation	0.001	0.000	0.000	0.000	0.003	0.000

the prior settings we only need to specify the parameter g, the amount of prior information relative to the information in the data. The results presented here are based on $g = 0.05$. That is, the prior information correspond to about two years of the monthly data.

In Table 1 we report the posterior probability of inclusion for the 15 factors and the different sets of portfolios. It is computed as the total sum of the posterior probabilities of all 2^{15} models in which the particular factor is included.

Focusing on what is common among the different portfolios, Table 1 shows that size premium, value premium and market excess return all have a high probability of inclusion. This indicates that each of the factors have a high probability to appear in a weighted asset pricing model. In addition, the momentum factor have a high probability of inclusion except when we use the 25 size and book-to-market portfolios as the investment universe. The credit risk spread seems also to be important in asset pricing. None of the macroeconomic factors obtain a high probability of inclusion except for the industry portfolios where the monthly growth rate in industrial production has a probability of 0.93.

Table 2a The three models with the highest posterior model probabilities, $g = 0.05$

	Benchmark			Size-B/M			Industry		
Factor	1	2	3	1	2	3	1	2	3
Intercept	0	0	0	1	0	1	0	1	0
SMB	1	1	1	1	1	1	1	1	1
HML	1	1	1	1	1	1	1	1	1
Momentum	1	1	1	0	0	0	1	1	1
Market excess return	1	1	1	1	1	1	1	1	1
Term spread Short	0	0	0	0	0	0	0	0	0
Termp spread Long	0	0	0	0	0	0	0	0	0
Credit risk spread	1	0	1	0	1	0	1	0	0
Yield on TB3M	0	0	0	0	0	0	0	0	0
Dividend SP500	0	1	1	0	0	0	0	0	1
Industrial prod. (M)	0	0	0	0	0	0	1	1	1
Industrial prod. (Y)	0	0	0	0	0	1	0	0	0
Income	0	0	0	0	0	0	0	0	0
Consumption	0	0	0	0	0	0	0	0	0
Inflation	0	0	0	0	0	0	0	0	0
Probability	0.523	0.241	0.083	0.999	0.001	0	0.365	0.312	0.109

In Table 2a and Table 2b the best models with the highest posterior model probabilities are represented by combinations of zeros and ones, where one indicates that a specific factor is included in the model.

Starting with the 25 size-B/M portfolios as the investment universe, the best model clearly dominates with a posterior model probability of 0.99. The factor pricing model includes, the intercept, size and value premiums, and the market excess return. This is consistent with the three factor model of Fama and French (1993). For the six benchmark portfolios also constructed by sorting stocks on size and book-to-market the result differs from the 25 size-B/M case in several ways. First, we note that the model with the highest probability contains the momentum factor and the credit risk spread in addition to the three Fama and French factors. Second, the posterior model probabilities for the best model is much lower. For the second and the third model the posterior model probabilities are 0.24 and 0.08 respectively. This indicates the importance of model uncertainty in asset pricing models. An overall result for the size and book-to-market sorted portfolios is that

Table 2b The three models with the highest posterior model probabilities, $g = 0.05$

	Cashflow			Earning			Dividend		
Factor	1	2	3	1	2	3	1	2	3
Intercept	1	0	0	1	1	0	0	0	1
SMB	1	1	1	1	1	1	1	1	1
HML	1	1	1	1	1	1	1	1	1
Momentum	1	1	1	0	1	0	1	1	1
Market excess return	1	1	1	1	1	1	1	1	1
Term spread Short	0	0	0	0	0	0	0	0	0
Termp spread Long	0	0	0	0	0	0	0	0	0
Credit risk spread	0	1	0	0	0	1	1	0	0
Yield on TB3M	0	0	0	0	0	0	0	0	0
Dividend SP500	0	0	1	0	0	0	0	1	0
Industrial prod. (M)	0	0	0	0	0	0	0	0	0
Industrial prod. (Y)	0	0	0	0	0	0	0	0	0
Income	0	0	0	0	0	0	0	0	0
Consumption	0	0	0	0	0	0	0	0	0
Inflation	0	0	0	0	0	0	0	0	0
Probability	0.494	0.285	0.129	0.294	0.221	0.196	0.450	0.330	0.217

we find evidence that the three factors introduced by Fama and French and the momentum factor are important in explaining asset returns. In addition, the credit risk spread is also an important factor in asset pricing. The difference in the results between the two sets of portfolios may be explained by diversification. The 25 size-B/M contains less stocks in each portfolio and this can result in idiosyncratic effects.

In the above regressions, the dependent returns and the two explanatory returns SMB and HML are portfolios formed on size and book-to-market. Thus, it is a chance that the inclusion of these two factors is spurious. To investigate this we examine whether these factors explains returns on portfolios formed on other variables.

The last three columns in Table 2a shows the result when stocks are sorted by industry. The best model includes the three Fama and French factors and momentum, but also the credit risk spread and the monthly growth rate in industrial production. However, the posterior model probability for the best model is only 0.365, indicating substantial model uncertainty. Most of the uncertainty is over the inclusion of the intercept.

The results when stocks are sorted on cashflows, earnings and dividends are shown in Table 2b. Using portfolios formed on cashflow yields a factor model that includes the intercept, the three Fama and French factors and the momentum factors. The best model has a posterior probability equal to 0.49. The second best model with a posterior probability of 0.29 includes the credit risk spread but not a constant term. The best model when we use stocks sorted by earnings and dividends includes the three Fama and French factors and an intercept and the three Fama and French factors, the momentum factor and the credit risk spread, respectively. The three top models have a probability of inclusion of 0.29, 0.22 and 0.20 for stocks sorted by earnings and 0.45, 0.33 and 0.22 for stocks sorted by dividends. From Table 2a and Table 2b we finally note that the model uncertainty is quite substantial. It is only for the 25 size-B/M portfolios and the benchmark portfolios that the probability for the best model is larger than 0.5.

In a well specified asset pricing model, with excess returns as dependent variable and returns on zero-investment portfolios as explanatory variables, we can expect that the intercept is not included (Merton(1973)). In our case, the zero-investment portfolios have very high probabilities of inclusion but a lot of the model uncertainty is over the inclusion of the intercept. Hence, it is not clear if there exists any constant misspricing of the assets.

Remember that size, value and the momentum factor are expressed in portfolio returns. Hence, they are constructed to mimic economy wide risk factors and can be viewed as factor-mimicking portfolios. As argued by Cochrane (2001), a model with factor-mimicking portfolios will almost always outperform a model with real economic factors. By removing the factors expressed as portfolio returns we can investigate which real economic factors are important in explaining asset returns and what kind of economic risk the mimicking portfolios are proxies for. In Table 3 we present the probability

Table 3 Probability of Inclusion, $g = 0.05$

Factor	Benchmark	Size-B/M	Industry	Cashflow	Earning	Dividend
Intercept	0.019	0.857	0.005	0.076	0.056	0.047
Term spread Short	0.002	0.000	0.000	0.000	0.007	0.000
Termp spread Long	0.021	0.000	0.000	0.000	0.001	0.000
Credit risk spread	0.982	0.143	0.994	0.921	0.940	0.952
Yield on TB3M	0.998	0.002	0.493	0.520	0.035	0.845
Dividend SP500	0.051	0.000	0.007	0.011	0.016	0.005
Industrial prod. (M)	0.004	0.000	0.483	0.007	0.000	0.000
Industrial prod. (Y)	0.718	0.001	0.106	0.198	0.137	0.024
Income	0.002	0.000	0.000	0.000	0.000	0.000
Consumption	0.995	0.000	0.002	0.051	0.000	0.000
Inflation	0.005	0.000	0.000	0.000	0.000	0.000

Table 4a The three models with the highest posterior model probabilities, $g = 0.05$

	Benchmark			Size-B/M			Industry		
Factor	1	2	3	1	2	3	1	2	3
Intercept	0	0	0	1	0	1	0	0	0
Term spread Short	0	0	0	0	0	0	0	0	0
Termp spread Long	0	0	0	0	0	0	0	0	0
Credit risk spread	1	1	1	0	1	0	1	1	1
Yield on TB3M	1	1	1	0	0	1	0	1	0
Dividend SP500	0	0	1	0	0	0	0	0	0
Industrial prod. (M)	0	0	0	0	0	0	1	0	0
Industrial prod. (Y)	1	0	0	0	0	0	0	0	0
Income	0	0	0	0	0	0	0	0	0
Consumption	1	1	1	0	0	0	0	0	0
Inflation	0	0	0	0	0	0	0	0	0
Probability	0.661	0.232	0.027	0.854	0.143	0.002	0.331	0.322	0.116

of inclusion for the six portfolios and in Table 4a and Table 4b the best models with the highest posterior model probabilities are presented.

The results in Table 3 and Table 4a-4b are mixed, indicating substantial model uncertainty. As we can expect from the earlier results, monthly growth rate in industrial production is included when using portfolios formed by industry, but note that the probability of inclusion for industrial production has increased for almost all portfolios. However, the overall evidence indicates that the change in yield on three month Treasury bill and the credit risk spread are the most important factors. This indicates that the Fama

Table 4b The three models with the highest posterior model probabilities, $g = 0.05$

	Cashflow			Earning			Dividend		
Factor	1	2	3	1	2	3	1	2	3
Intercept	0	0	0	0	0	1	0	0	1
Term spread Short	0	0	0	0	0	0	0	0	0
Termp spread Long	0	0	0	0	0	0	0	0	0
Credit risk spread	1	1	1	1	1	0	1	1	0
Yield on TB3M	0	1	0	0	0	0	1	0	1
Dividend SP500	0	0	0	0	0	0	0	0	0
Industrial prod. (M)	0	0	0	0	0	0	0	0	0
Industrial prod. (Y)	0	0	1	0	1	0	0	0	0
Income	0	0	0	0	0	0	0	0	0
Consumption	0	0	0	0	0	0	0	0	0
Inflation	0	0	0	0	0	0	0	0	0
Probability	0.351	0.333	0.091	0.769	0.129	0.041	0.776	0.148	0.046

and French factors may be proxies for risk corresponding to the term structure and to some extent, factors relating to industrial production.

5.1 Sensitivity analysis

The exact results obtained are dependent on a number of choices such as the composition of the portfolios, the sample used and the prior specification. The preceding section gave some results on the sensitivity to portfolio composition. In this section we address the latter two issues.

Splitting the data into two subsamples, 196307 - 198212 and 198301 - 200212, the results (see appendix in Table A.1 to Table A.12) are similar to the full sample results with the exception of the intercept, which has a higher inclusion probability in the later sub period.

Addressing the issue of prior sensitivity we first consider the prior for the innovation variance, Σ. Specifying a proper inverse Wishart prior, $\Sigma \sim iW(I, v)$ with $v = N + 2$ corresponding to a prior mean[9] for the variances of $I/(v - N - 1) = I$, instead of the

[9]Weak prior information requires a small value of v. The value $v = N + 2$ is a convenient small value

Table 5 Probability of Inclusion, $g = 1/T$

Factor	Benchmark	Size-B/M	Industry	Cashflow	Earning	Dividend
Intercept	0.955	1.000	0.000	0.997	0.861	0.149
SMB	1.000	1.000	1.000	1.000	1.000	1.000
HML	1.000	1.000	1.000	1.000	1.000	1.000
Momentum	1.000	0.000	0.955	0.000	0.000	0.454
Market excess return	1.000	1.000	1.000	1.000	1.000	1.000
Term spread Short	0.000	0.000	0.000	0.000	0.000	0.000
Termp spread Long	0.364	0.000	0.000	0.000	0.000	0.000
Credit risk spread	0.608	0.000	0.959	0.001	0.136	0.313
Yield on TB3M	0.000	0.000	0.000	0.000	0.000	0.000
Dividend SP500	0.034	0.000	0.041	0.002	0.003	0.538
Industrial prod. (M)	0.000	0.000	0.000	0.000	0.000	0.000
Industrial prod. (Y)	0.605	0.000	0.000	0.000	0.000	0.000
Income	0.000	0.000	0.000	0.000	0.000	0.000
Consumption	0.000	0.000	0.000	0.000	0.000	0.000
Inflation	0.000	0.000	0.000	0.000	0.000	0.000

improper Jeffreys prior leads to a well defined marginal likelihood and might thus be preferable. The results (see appendix in Table B.1 to Table B.6) are, however, not affected in any substantial way by this change in the prior specification.

Next we consider the choice of g, measuring the tightness or information content of the prior. Letting g vary between $1/T$ (the information in one observation) and 0.05 (5% of the sample) we find some sensitivity to g. The inclusion probabilities for $g = 1/T$ are given in Table 5 and a full set of results are in appendix (Table C.1 to Table C.11). As g decreases, the prior is made less informative, the inclusion probability for the constant increases for the benchmark, cashflow and earnings portfolios and the inclusion probability for the momentum factor decreases for the cashflow, earnings and dividend portfolios. In addition to these broader trends we note that the support for industrial production disappears in the industry portfolio but increases in the benchmark portfolio.

since it is the smallest integer value such that the expectation of Σ exists.

5.2 Classical results

In a classical setting the choice of model is often based on information criteria. While this frequently gives reasonable results and provide information on the merits of different models it is much less informative than the Bayesian procedure proposed here. In particular information criteria can not be used to quantify model uncertainty and only provide a relative measure of the merits of a model. In this sense information criteria are similar to Bayes factors. By committing to a set of prior model probabilities absolute measures of the merit of different models and measures of model uncertainty are available in the form of posterior model probabilities.

Still, a comparison with classical methods is of some interest. In Table 6a we report which factors are included in the best model according to the Akaikes information criterion (AIC) and in Table 6b we report the results for the Bayesian information criterion (BIC). Note that using BIC is asymptotically the same as using Bayes factors for model selection. The AIC always selects one or two more factors than when we use posterior model probabilities. Especially, the growth rate in industrial production shows up more frequently. Using the BIC yields the same results or one less factor than the posterior model probabilities. Notably, is that the momentum factor is excluded for the last three portfolios using the BIC criterion.

6 Conclusions

In this paper we use Bayesian techniques to select the factors in a general multifactor asset pricing model. From a given set of 15 factors we evaluate and rank all $2^{15} = 32768$ different pricing models by their posterior model probabilities. Interest rates, premiums, returns on broadbased portfolios and macroeconomic variables are included in the set of considered factors.

Table 6a The best models when we use Akaike's information criterion.

Factor	Benchmark	Size-B/M	Industry	Cashflow	Earning	Dividend
Intercept	1	1	1	1	1	1
SMB	1	1	1	1	1	1
HML	1	1	1	1	1	1
Momentum	1	1	1	1	1	1
Market excess return	1	1	1	1	1	1
Term spread Short	0	0	0	0	0	0
Termp spread Long	1	0	1	1	0	0
Credit risk spread	1	0	1	1	0	1
Yield on TB3M	1	0	0	0	0	0
Dividend SP500	0	0	0	0	0	0
Industrial prod. (M)	0	0	1	0	0	0
Industrial prod. (Y)	1	1	0	0	1	1
Income	0	0	0	0	0	0
Consumption	0	0	0	0	0	0
Inflation	0	0	0	1	1	0

Table 6b The best models when we use Bayesian information criterion.

Factor	Benchmark	Size-B/M	Industry	Cashflow	Earning	Dividend
Intercept	1	1	0	1	1	0
SMB	1	1	1	1	1	1
HML	1	1	1	1	1	1
Momentum	1	0	1	0	0	0
Market excess return	1	1	1	1	1	1
Term spread Short	0	0	0	0	0	0
Termp spread Long	0	0	0	0	0	0
Credit risk spread	1	0	1	0	0	0
Yield on TB3M	0	0	0	0	0	0
Dividend SP500	0	0	0	0	0	1
Industrial prod. (M)	0	0	0	0	0	0
Industrial prod. (Y)	1	0	0	0	0	0
Income	0	0	0	0	0	0
Consumption	0	0	0	0	0	0
Inflation	0	0	0	0	0	0

Using different portfolios as the investment universe we find strong evidence that a general multifactor pricing model should include the market excess return, the size premium, and the value premium. The evidence in favor of the momentum factor is more sensitive to the sample used and the prior specification. In addition, we find evidence that the credit risk spread should be included as an additional factor. There are some indications that industrial production also is an important factor. Furthermore, when only using real economic factors, risk factors related to the term structure are important factors when explaining asset returns.

A large part of the model uncertainty is over the inclusion of the intercept. The intercept is included more frequently over the period 198301-200212. The results obtained here and the Bayesian approach, accounting for model uncertainty, should be useful in several areas of application. Examples are selecting portfolios, evaluating portfolio performance and estimating the cost of capital.

The interpretation of the momentum and the three factors of Fama and French as risk factors have caused a large debate in the finance literature. Lo and MacKinlay (1990) and MacKinlay (1995) argue that CAPM anomalies may be the result of data-snooping or of selection bias. Our results indicates that these factors may be proxies for risk corresponding to the term structure.

A Appendix

A.1 Subsample analysis

Period: 196307 - 198212.

Table A.1 Probability of Inclusion, $g = 0.05$. Period: 196307 - 198212.

Factor	Benchmark	Size-B/M	Industry	Cashflow	Earning	Dividend
Intercept	0.014	0.499	0.021	0.025	0.071	0.057
SMB	1.000	1.000	1.000	1.000	1.000	0.247
HML	1.000	1.000	1.000	1.000	1.000	1.000
Momentum	0.529	0.028	1.000	0.975	0.000	1.000
Market excess return	1.000	1.000	1.000	1.000	1.000	1.000
Term spread Short	0.000	0.000	0.001	0.001	0.035	0.000
Termp spread Long	0.005	0.000	0.000	0.001	0.000	0.000
Credit risk spread	0.811	0.133	0.227	0.787	0.118	0.361
Yield on TB3M	0.048	0.000	0.014	0.007	0.000	0.000
Dividend SP500	0.184	0.312	0.753	0.197	0.811	0.583
Industrial prod. (M)	0.001	0.000	0.000	0.000	0.000	0.000
Industrial prod. (Y)	0.002	0.000	0.000	0.000	0.001	0.016
Income	0.001	0.000	0.000	0.000	0.000	0.000
Consumption	0.003	0.000	0.001	0.000	0.000	0.002
Inflation	0.006	0.000	0.000	0.004	0.001	0.000

Table A.2 The three models with the highest posterior model probabilities, $g = 0.05$. Period: 196307 - 198212.

	Benchmark			Size-B/M			Industry		
Factor	1	2	3	1	2	3	1	2	3
Intercept	0	0	0	1	0	0	0	0	1
SMB	1	1	1	1	1	1	1	1	1
HML	1	1	1	1	1	1	1	1	1
Momentum	1	0	0	0	0	0	1	1	1
Market excess return	1	1	1	1	1	1	1	1	1
Term spread Short	0	0	0	0	0	0	0	0	0
Termp spread Long	0	0	0	0	0	0	0	0	0
Credit risk spread	1	1	0	0	0	1	0	1	0
Yield on TB3M	0	0	0	0	0	0	0	0	0
Dividend SP500	0	0	1	0	1	0	1	0	0
Industrial prod. (M)	0	0	0	0	0	0	0	0	0
Industrial prod. (Y)	0	0	0	0	0	0	0	0	0
Income	0	0	0	0	0	0	0	0	0
Consumption	0	0	0	0	0	0	0	0	0
Inflation	0	0	0	0	0	0	0	0	0
Probability	0.417	0.332	0.091	0.495	0.309	0.120	0.740	0.224	0.020

Table A.3 The three models with the highest posterior model probabilities, $g = 0.05$. Period: 196307 - 198212.

	Cashflow			Earning			Dividend		
Factor	1	2	3	1	2	3	1	2	3
Intercept	0	0	1	0	0	1	0	0	0
SMB	1	1	1	1	1	1	0	0	1
HML	1	1	1	1	1	1	1	1	1
Momentum	1	1	1	0	0	0	1	1	1
Market excess return	1	1	1	1	1	1	1	1	1
Term spread Short	0	0	0	0	0	0	0	0	0
Termp spread Long	0	0	0	0	0	0	0	0	0
Credit risk spread	1	0	0	0	1	0	0	1	0
Yield on TB3M	0	0	0	0	0	0	0	0	0
Dividend SP500	0	1	0	1	0	0	1	0	1
Industrial prod. (M)	0	0	0	0	0	0	0	0	0
Industrial prod. (Y)	0	0	0	0	0	0	0	0	0
Income	0	0	0	0	0	0	0	0	0
Consumption	0	0	0	0	0	0	0	0	0
Inflation	0	0	0	0	0	0	0	0	0
Probability	0.757	0.184	0.014	0.783	0.114	0.065	0.424	0.272	0.151

Table A.4 Probability of Inclusion, $g = 0.05$. Period: 196307 - 198212.

Factor	Benchmark	Size-B/M	Industry	Cashflow	Earning	Dividend
Intercept	0.333	0.000	0.000	0.026	0.003	0.001
Term spread Short	0.014	0.000	0.003	0.000	0.005	0.000
Termp spread Long	0.003	0.000	0.004	0.026	0.075	0.003
Credit risk spread	0.468	0.000	0.015	0.063	0.008	0.010
Yield on TB3M	1.000	0.000	0.493	0.664	0.032	0.273
Dividend SP500	0.088	0.000	0.001	0.006	0.002	0.000
Industrial prod. (M)	0.005	0.000	0.004	0.002	0.000	0.000
Industrial prod. (Y)	0.103	0.000	0.003	0.001	0.012	0.004
Income	0.006	0.000	0.001	0.001	0.000	0.001
Consumption	0.860	0.000	0.007	0.004	0.003	0.207
Inflation	0.008	0.000	0.000	0.001	0.000	0.000

Table A.5 The three models with the highest posterior model probabilities, $g = 0.05$. Period: 196307 - 198212.

	Benchmark			Size-B/M			Industry		
Factor	1	2	3	1	2	3	1	2	3
Intercept	0	1	0	0	0	0	0	0	0
Term spread Short	0	0	0	0	0	0	0	0	0
Termp spread Long	0	0	0	0	0	0	0	0	0
Credit risk spread	1	0	0	0	0	0	0	0	1
Yield on TB3M	1	1	1	0	1	0	0	1	1
Dividend SP500	0	0	0	0	0	0	0	0	0
Industrial prod. (M)	0	0	0	0	0	0	0	0	0
Industrial prod. (Y)	0	0	0	0	0	1	0	0	0
Income	0	0	0	0	0	0	0	0	0
Consumption	1	1	1	0	0	0	0	0	0
Inflation	0	0	0	0	0	0	0	0	0
Probability	0.289	0.254	0.144	1.000	0.000	0.000	0.488	0.474	0.007

Table A.6 The three models with the highest posterior model probabilities, $g = 0.05$. Period: 196307 - 198212.

	Cashflow			Earning			Dividend		
Factor	1	2	3	1	2	3	1	2	3
Intercept	0	0	0	0	0	0	0	0	0
Term spread Short	0	0	0	0	0	0	0	0	0
Termp spread Long	0	0	0	0	1	0	0	0	0
Credit risk spread	0	0	1	0	0	0	0	0	0
Yield on TB3M	1	0	1	0	0	1	0	1	1
Dividend SP500	0	0	0	0	0	0	0	0	0
Industrial prod. (M)	0	0	0	0	0	0	0	0	0
Industrial prod. (Y)	0	0	0	0	0	0	0	0	0
Income	0	0	0	0	0	0	0	0	0
Consumption	0	0	0	0	0	0	0	0	1
Inflation	0	0	0	0	0	0	0	0	0
Probability	0.591	0.285	0.037	0.863	0.075	0.030	0.612	0.163	0.105

Period: 198312 - 200212.

Table A.7 Probability of Inclusion, $g = 0.05$. Period: 198301 - 200212.

Factor	Benchmark	Size-B/M	Industry	Cashflow	Earning	Dividend
Intercept	0.144	0.998	0.223	0.370	0.370	0.238
SMB	1.000	1.000	1.000	1.000	1.000	1.000
HML	1.000	1.000	1.000	1.000	1.000	1.000
Momentum	1.000	0.003	0.987	0.170	0.944	0.443
Market excess return	1.000	1.000	1.000	1.000	1.000	1.000
Term spread Short	0.015	0.000	0.000	0.000	0.000	0.000
Termp spread Long	0.004	0.000	0.000	0.000	0.000	0.000
Credit risk spread	0.608	0.001	0.381	0.606	0.601	0.656
Yield on TB3M	0.000	0.000	0.000	0.000	0.000	0.000
Dividend SP500	0.249	0.000	0.396	0.020	0.028	0.066
Industrial prod. (M)	0.003	0.000	0.153	0.001	0.000	0.001
Industrial prod. (Y)	0.004	0.000	0.003	0.000	0.069	0.017
Income	0.005	0.000	0.000	0.000	0.000	0.000
Consumption	0.000	0.000	0.000	0.001	0.001	0.000
Inflation	0.001	0.000	0.002	0.002	0.000	0.003

Table A.8 The three models with the highest posterior model probabilities, $g = 0.05$. Period: 198301 - 200212.

	Benchmark			Size-B/M			Industry		
Factor	1	2	3	1	2	3	1	2	3
Intercept	0	0	1	1	1	0	0	0	1
SMB	1	1	1	1	1	1	1	1	1
HML	1	1	1	1	1	1	1	1	1
Momentum	1	1	1	0	1	0	1	1	1
Market excess return	1	1	1	1	1	1	1	1	1
Term spread Short	0	0	0	0	0	0	0	0	0
Termp spread Long	0	0	0	0	0	0	0	0	0
Credit risk spread	1	0	0	0	0	1	0	1	0
Yield on TB3M	0	0	0	0	0	0	0	0	0
Dividend SP500	0	1	0	0	0	0	1	0	0
Industrial prod. (M)	0	0	0	0	0	0	0	0	0
Industrial prod. (Y)	0	0	0	0	0	0	0	0	0
Income	0	0	0	0	0	0	0	0	0
Consumption	0	0	0	0	0	0	0	0	0
Inflation	0	0	0	0	0	0	0	0	0
Probability	0.591	0.235	0.139	0.996	0.003	0.001	0.330	0.318	0.184

Table A.9 The three models with the highest posterior model probabilities, $g = 0.05$. Period: 198301 - 200212.

	Cashflow			Earning			Dividend		
Factor	1	2	3	1	2	3	1	2	3
Intercept	0	1	0	0	1	0	0	0	1
SMB	1	1	1	1	1	1	1	1	1
HML	1	1	1	1	1	1	1	1	1
Momentum	0	0	1	1	1	1	0	1	0
Market excess return	1	1	1	1	1	1	1	1	1
Term spread Short	0	0	0	0	0	0	0	0	0
Termp spread Long	0	0	0	0	0	0	0	0	0
Credit risk spread	1	0	1	1	0	1	1	1	0
Yield on TB3M	0	0	0	0	0	0	0	0	0
Dividend SP500	0	0	0	0	0	0	0	0	0
Industrial prod. (M)	0	0	0	0	0	0	0	0	0
Industrial prod. (Y)	0	0	0	0	0	1	0	0	0
Income	0	0	0	0	0	0	0	0	0
Consumption	0	0	0	0	0	0	0	0	0
Inflation	0	0	0	0	0	0	0	0	0
Probability	0.502	0.306	0.102	0.528	0.325	0.039	0.343	0.299	0.129

Table A.10 Probability of Inclusion, $g = 0.05$. Period: 198301 - 200212.

Factor	Benchmark	Size-B/M	Industry	Cashflow	Earning	Dividend
Intercept	0.019	0.993	0.133	0.165	0.228	0.170
Term spread Short	0.001	0.000	0.000	0.000	0.000	0.000
Termp spread Long	0.010	0.000	0.000	0.000	0.000	0.000
Credit risk spread	0.965	0.007	0.732	0.796	0.733	0.738
Yield on TB3M	0.001	0.000	0.000	0.000	0.000	0.000
Dividend SP500	0.042	0.000	0.123	0.016	0.033	0.054
Industrial prod. (M)	0.000	0.000	0.278	0.000	0.000	0.000
Industrial prod. (Y)	0.012	0.000	0.062	0.000	0.035	0.215
Income	0.002	0.000	0.000	0.000	0.000	0.000
Consumption	0.119	0.000	0.005	0.068	0.014	0.002
Inflation	0.003	0.000	0.002	0.003	0.000	0.001

Table A.11 The three models with the highest posterior model probabilities, $g = 0.05$. Period: 198301 - 200212.

	Benchmark			Size-B/M			Industry		
Factor	1	2	3	1	2	3	1	2	3
Intercept	0	0	0	1	0	0	0	0	1
Term spread Short	0	0	0	0	0	0	0	0	0
Termp spread Long	0	0	0	0	0	0	0	0	0
Credit risk spread	1	1	1	0	1	0	1	1	0
Yield on TB3M	0	0	0	0	0	0	0	0	0
Dividend SP500	0	0	1	0	0	1	0	0	0
Industrial prod. (M)	0	0	0	0	0	0	0	1	0
Industrial prod. (Y)	0	0	0	0	0	0	0	0	0
Income	0	0	0	0	0	0	0	0	0
Consumption	0	1	0	0	0	0	0	0	0
Inflation	0	0	0	0	0	0	0	0	0
Probability	0.806	0.109	0.023	0.993	0.007	0	0.490	0.191	0.080

Table A.12 The three models with the highest posterior model probabilities, $g = 0.05$. Period: 198301 - 200212.

	Cashflow			Earning			Dividend		
Factor	1	2	3	1	2	3	1	2	3
Intercept	0	1	0	0	1	0	0	0	1
Term spread Short	0	0	0	0	0	0	0	0	0
Termp spread Long	0	0	0	0	0	0	0	0	0
Credit risk spread	1	0	1	1	0	0	1	1	0
Yield on TB3M	0	0	0	0	0	0	0	0	0
Dividend SP500	0	0	0	0	0	1	0	0	0
Industrial prod. (M)	0	0	0	0	0	0	0	0	0
Industrial prod. (Y)	0	0	0	0	0	0	0	1	0
Income	0	0	0	0	0	0	0	0	0
Consumption	0	0	1	0	0	0	0	0	0
Inflation	0	0	0	0	0	0	0	0	0
Probability	0.736	0.158	0.055	0.697	0.218	0.032	0.578	0.158	0.133

A.2 Proper prior on Σ

Table B.1 Probability of Inclusion, g = 0.05

Factor	Benchmark	Size-B/M	Industry	Cashflow	Earning	Dividend
Intercept	0.067	1.000	0.388	0.559	0.522	0.216
SMB	1.000	1.000	1.000	1.000	1.000	1.000
HML	1.000	1.000	1.000	1.000	1.000	1.000
Momentum	1.000	0.000	0.896	0.949	0.529	1.000
Market excess return	1.000	1.000	1.000	1.000	1.000	1.000
Term spread Short	0.001	0.000	0.000	0.000	0.003	0.000
Termp spread Long	0.087	0.000	0.000	0.000	0.000	0.000
Credit risk spread	0.696	0.000	0.487	0.308	0.361	0.455
Yield on TB3M	0.003	0.000	0.000	0.001	0.000	0.000
Dividend SP500	0.352	0.000	0.136	0.135	0.117	0.331
Industrial prod. (M)	0.001	0.000	0.956	0.015	0.000	0.000
Industrial prod. (Y)	0.007	0.000	0.000	0.001	0.000	0.001
Income	0.000	0.000	0.000	0.000	0.000	0.000
Consumption	0.004	0.000	0.000	0.000	0.000	0.001
Inflation	0.001	0.000	0.000	0.000	0.003	0.000

Table B.2 The three models with the highest posterior model probabilities, $g = 0.05$

	Benchmark			Size-B/M			Industry		
Factor	1	2	3	1	2	3	1	2	3
Intercept	0	0	0	1	0	1	0	1	0
SMB	1	1	1	1	1	1	1	1	1
HML	1	1	1	1	1	1	1	1	1
Momentum	1	1	1	0	0	0	1	1	1
Market excess return	1	1	1	1	1	1	1	1	1
Term spread Short	0	0	0	0	0	0	0	0	0
Termp spread Long	0	0	0	0	0	0	0	0	0
Credit risk spread	1	0	1	0	1	0	1	0	0
Yield on TB3M	0	0	0	0	0	0	0	0	0
Dividend SP500	0	1	1	0	0	0	0	0	1
Industrial prod. (M)	0	0	0	0	0	0	1	1	1
Industrial prod. (Y)	0	0	0	0	0	1	0	0	0
Income	0	0	0	0	0	0	0	0	0
Consumption	0	0	0	0	0	0	0	0	0
Inflation	0	0	0	0	0	0	0	0	0
Probability	0.514	0.235	0.092	0.999	0.000	0.000	0.397	0.338	0.115

Table B.3 The three models with the highest posterior model probabilities, $g = 0.05$

	Cashflow			Earning			Dividend		
Factor	1	2	3	1	2	3	1	2	3
Intercept	1	0	0	1	1	0	0	0	1
SMB	1	1	1	1	1	1	1	1	1
HML	1	1	1	1	1	1	1	1	1
Momentum	1	1	1	1	0	1	1	1	1
Market excess return	1	1	1	1	1	1	1	1	1
Term spread Short	0	0	0	0	0	0	0	0	0
Termp spread Long	0	0	0	0	0	0	0	0	0
Credit risk spread	0	1	0	0	0	1	1	0	0
Yield on TB3M	0	0	0	0	0	0	0	0	0
Dividend SP500	0	0	1	0	0	0	0	1	0
Industrial prod. (M)	0	0	0	0	0	0	0	0	0
Industrial prod. (Y)	0	0	0	0	0	0	0	0	0
Income	0	0	0	0	0	0	0	0	0
Consumption	0	0	0	0	0	0	0	0	0
Inflation	0	0	0	0	0	0	0	0	0
Probability	0.510	0.291	0.128	0.269	0.248	0.195	0.453	0.330	0.214

Table B.4 Probability of Inclusion, g = 0.05

Factor	Benchmark	Size-B/M	Industry	Cashflow	Earning	Dividend
Intercept	0.018	0.868	0.005	0.081	0.055	0.046
Term spread Short	0.002	0.000	0.000	0.000	0.009	0.000
Termp spread Long	0.023	0.000	0.000	0.000	0.000	0.000
Credit risk spread	0.983	0.132	0.995	0.916	0.942	0.953
Yield on TB3M	0.998	0.012	0.541	0.613	0.046	0.893
Dividend SP500	0.053	0.000	0.008	0.012	0.019	0.004
Industrial prod. (M)	0.004	0.000	0.524	0.011	0.000	0.000
Industrial prod. (Y)	0.750	0.008	0.136	0.259	0.183	0.033
Income	0.002	0.000	0.000	0.000	0.000	0.000
Consumption	0.996	0.000	0.003	0.068	0.000	0.000
Inflation	0.006	0.000	0.000	0.000	0.000	0.000

Table B.5 The three models with the highest posterior model probabilities, $g = 0.05$

	Benchmark			Size-B/M			Industry		
Factor	1	2	3	1	2	3	1	2	3
Intercept	0	0	0	1	0	1	0	0	0
Term spread Short	0	0	0	0	0	0	0	0	0
Termp spread Long	0	0	0	0	0	0	0	0	0
Credit risk spread	1	1	1	0	1	0	1	1	1
Yield on TB3M	1	1	1	0	0	1	0	1	1
Dividend SP500	0	0	1	0	0	0	0	0	0
Industrial prod. (M)	0	0	0	0	0	0	1	0	1
Industrial prod. (Y)	1	0	0	0	0	0	0	0	0
Income	0	0	0	0	0	0	0	0	0
Consumption	1	1	1	0	0	0	0	0	0
Inflation	0	0	0	0	0	0	0	0	0
Probability	0.688	0.204	0.027	0.851	0.129	0.012	0.317	0.308	0.151

Table B.6 The three models with the highest posterior model probabilities, $g = 0.05$

	Cashflow			Earning			Dividend		
Factor	1	2	3	1	2	3	1	2	3
Intercept	0	0	0	0	0	1	0	0	1
Term spread Short	0	0	0	0	0	0	0	0	0
Termp spread Long	0	0	0	0	0	0	0	0	0
Credit risk spread	1	1	1	1	1	0	1	1	0
Yield on TB3M	1	0	1	0	0	0	1	0	1
Dividend SP500	0	0	0	0	0	0	0	0	0
Industrial prod. (M)	0	0	0	0	0	0	0	0	0
Industrial prod. (Y)	0	0	1	0	1	0	0	0	0
Income	0	0	0	0	0	0	0	0	0
Consumption	0	0	0	0	0	0	0	0	0
Inflation	0	0	0	0	0	0	0	0	0
Probability	0.357	0.255	0.131	0.717	0.170	0.035	0.816	0.101	0.045

A.3 Results for $g = 1/T$, improper prior on Σ

Table C.1 The three models with the highest posterior model probabilities, g = 1/T.

	Benchmark			Size-B/M			Industry		
Factor	1	2	3	1	2	3	1	2	3
Intercept	1	1	1	1	0	0	0	0	0
SMB	1	1	1	1	1	1	1	1	1
HML	1	1	1	1	1	1	1	1	1
Momentum	1	1	1	0	0	0	1	0	1
Market excess return	1	1	1	1	1	1	1	1	1
Term spread Short	0	0	0	0	0	0	0	0	0
Termp spread Long	1	0	0	0	0	0	0	0	0
Credit risk spread	1	0	1	0	1	0	1	1	0
Yield on TB3M	0	0	0	0	0	0	0	0	0
Dividend SP500	0	0	0	0	0	1	0	0	1
Industrial prod. (M)	0	0	0	0	0	0	0	0	0
Industrial prod. (Y)	0	1	1	0	0	0	0	0	0
Income	0	0	0	0	0	0	0	0	0
Consumption	0	0	0	0	0	0	0	0	0
Inflation	0	0	0	0	0	0	0	0	0
Probability	0.360	0.358	0.235	1.000	0.000	0.000	0.914	0.045	0.041

Table C.2 The three models with the highest posterior model probabilities, $g = 1/T$

	Cashflow			Earning			Dividend		
Factor	1	2	3	1	2	3	1	2	3
Intercept	1	0	0	1	0	0	0	0	0
SMB	1	1	1	1	1	1	1	1	1
HML	1	1	1	1	1	1	1	1	1
Momentum	0	0	0	0	0	0	0	1	1
Market excess return	1	1	1	1	1	1	1	1	1
Term spread Short	0	0	0	0	0	0	0	0	0
Termp spread Long	0	0	0	0	0	0	0	0	0
Credit risk spread	0	0	1	0	1	0	0	0	1
Yield on TB3M	0	0	0	0	0	0	0	0	0
Dividend SP500	0	1	0	0	0	1	1	1	0
Industrial prod. (M)	0	0	0	0	0	0	0	0	0
Industrial prod. (Y)	0	0	0	0	0	0	0	0	0
Income	0	0	0	0	0	0	0	0	0
Consumption	0	0	0	0	0	0	0	0	0
Inflation	0	0	0	0	0	0	0	0	0
Probability	0.997	0.002	0.001	0.861	0.136	0.003	0.302	0.236	0.173

Table C.3 Probability of Inclusion, $g = 1/T$

Factor	Benchmark	Size-B/M	Industry	Cashflow	Earning	Dividend
Intercept	0.002	0.000	0.000	0.001	0.003	0.000
Term spread Short	0.000	0.000	0.000	0.000	0.000	0.000
Termp spread Long	0.001	0.000	0.000	0.000	0.000	0.000
Credit risk spread	0.997	0.000	0.042	0.096	0.071	0.057
Yield on TB3M	0.066	0.000	0.000	0.000	0.000	0.000
Dividend SP500	0.000	0.000	0.000	0.000	0.000	0.000
Industrial prod. (M)	0.000	0.000	0.000	0.000	0.000	0.000
Industrial prod. (Y)	0.001	0.000	0.000	0.000	0.000	0.000
Income	0.000	0.000	0.000	0.000	0.000	0.000
Consumption	0.026	0.000	0.000	0.000	0.000	0.000
Inflation	0.000	0.000	0.000	0.000	0.000	0.000

Table C.4 The three models with the highest posterior model probabilities, g = 1/T.

	Benchmark			Size-B/M			Industry		
Factor	1	2	3	1	2	3	1	2	3
Intercept	0	0	0	0	1	0	0	0	1
Term spread Short	0	0	0	0	0	0	0	0	0
Termp spread Long	0	0	0	0	0	0	0	0	0
Credit risk spread	1	1	1	0	0	1	0	1	0
Yield on TB3M	0	1	0	0	0	0	0	0	0
Dividend SP500	0	0	0	0	0	0	0	0	0
Industrial prod. (M)	0	0	0	0	0	0	0	0	0
Industrial prod. (Y)	0	0	0	0	0	0	0	0	0
Income	0	0	0	0	0	0	0	0	0
Consumption	0	0	1	0	0	0	0	0	0
Inflation	0	0	0	0	0	0	0	0	0
Probability	0.910	0.061	0.022	1	0.000	0.000	0.958	0.042	0.000

Table C.5 The three models with the highest posterior model probabilities, $g = 1/T$

	Cashflow			Earning			Dividend		
Factor	1	2	3	1	2	3	1	2	3
Intercept	0	0	1	0	0	1	0	0	1
Term spread Short	0	0	0	0	0	0	0	0	0
Termp spread Long	0	0	0	0	0	0	0	0	0
Credit risk spread	0	1	0	0	1	0	0	1	0
Yield on TB3M	0	0	0	0	0	0	0	0	0
Dividend SP500	0	0	0	0	0	0	0	0	0
Industrial prod. (M)	0	0	0	0	0	0	0	0	0
Industrial prod. (Y)	0	0	0	0	0	0	0	0	0
Income	0	0	0	0	0	0	0	0	0
Consumption	0	0	0	0	0	0	0	0	0
Inflation	0	0	0	0	0	0	0	0	0
Probability	0.902	0.096	0.001	0.926	0.071	0.003	0.943	0.057	0.000

Table C.6 Probability of Inclusion, $g = 1/T$. Period: 196307 - 198212.

Factor	Benchmark	Size-B/M	Industry	Cashflow	Earning	Dividend
Intercept	0.000	0.001	0.000	0.000	0.000	0.001
SMB	1.000	1.000	1.000	1.000	1.000	0.000
HML	1.000	1.000	1.000	1.000	1.000	1.000
Momentum	0.002	0.000	1.000	0.001	0.000	1.000
Market excess return	1.000	1.000	1.000	1.000	1.000	1.000
Term spread Short	0.000	0.000	0.000	0.000	0.000	0.036
Termp spread Long	0.131	0.000	0.000	0.000	0.000	0.000
Credit risk spread	0.952	0.078	0.001	0.083	0.004	0.001
Yield on TB3M	0.002	0.000	0.000	0.000	0.000	0.000
Dividend SP500	0.050	0.921	0.999	0.917	0.996	0.998
Industrial prod. (M)	0.000	0.000	0.000	0.000	0.000	0.000
Industrial prod. (Y)	0.001	0.000	0.000	0.000	0.000	0.136
Income	0.000	0.000	0.000	0.000	0.000	0.000
Consumption	0.000	0.000	0.000	0.000	0.000	0.000
Inflation	0.000	0.000	0.000	0.000	0.000	0.000

Table C.7 The three models with the highest posterior model probabilities, $g = 1/T$. Period: 196307 - 198212.

	Benchmark			Size-B/M			Industry		
Factor	1	2	3	1	2	3	1	2	3
Intercept	0	0	0	0	0	1	0	0	0
SMB	1	1	1	1	1	1	1	1	1
HML	1	1	1	1	1	1	1	1	1
Momentum	0	0	0	0	0	0	1	1	1
Market excess return	1	1	1	1	1	1	1	1	1
Term spread Short	0	0	0	0	0	0	0	0	1
Termp spread Long	0	1	0	0	0	0	0	0	0
Credit risk spread	1	1	0	0	1	0	0	1	0
Yield on TB3M	0	0	0	0	0	0	0	0	0
Dividend SP500	0	0	1	1	0	0	1	0	1
Industrial prod. (M)	0	0	0	0	0	0	0	0	0
Industrial prod. (Y)	0	0	0	0	0	0	0	0	0
Income	0	0	0	0	0	0	0	0	0
Consumption	0	0	0	0	0	0	0	0	0
Inflation	0	0	0	0	0	0	0	0	0
Probability	0.814	0.131	0.048	0.921	0.078	0.001	0.999	0.001	0.000

Table C.8 The three models with the highest posterior model probabilities, $g = 1/T$. Period: 196307 - 198212.

	Cashflow			Earning			Dividend		
Factor	1	2	3	1	2	3	1	2	3
Intercept	0	0	0	0	0	0	0	0	0
SMB	1	1	1	1	1	1	0	0	0
HML	1	1	1	1	1	1	1	1	1
Momentum	0	0	1	0	0	0	1	1	1
Market excess return	1	1	1	1	1	1	1	1	1
Term spread Short	0	0	0	0	0	1	0	0	1
Termp spread Long	0	0	0	0	0	0	0	0	0
Credit risk spread	0	1	0	0	1	0	0	0	0
Yield on TB3M	0	0	0	0	0	0	0	0	0
Dividend SP500	1	0	1	1	0	1	1	1	1
Industrial prod. (M)	0	0	0	0	0	0	0	0	0
Industrial prod. (Y)	0	0	0	0	0	0	0	1	0
Income	0	0	0	0	0	0	0	0	0
Consumption	0	0	0	0	0	0	0	0	0
Inflation	0	0	0	0	0	0	0	0	0
Probability	0.917	0.083	0.001	0.996	0.004	0	0.827	0.135	0.036

Table C.9 Probability of Inclusion, $g = 1/T$. Period: 198301 - 200212.

Factor	Benchmark	Size-B/M	Industry	Cashflow	Earning	Dividend
Intercept	0.710	0.999	0.029	0.720	0.631	0.204
SMB	1.000	1.000	1.000	0.615	1.000	0.976
HML	1.000	1.000	1.000	1.000	1.000	1.000
Momentum	1.000	0.000	0.481	0.000	0.002	0.000
Market excess return	1.000	1.000	1.000	1.000	1.000	1.000
Term spread Short	0.000	0.000	0.000	0.000	0.000	0.000
Termp spread Long	0.000	0.000	0.000	0.000	0.000	0.000
Credit risk spread	0.199	0.001	0.929	0.279	0.368	0.700
Yield on TB3M	0.000	0.000	0.000	0.000	0.000	0.000
Dividend SP500	0.091	0.000	0.042	0.000	0.000	0.007
Industrial prod. (M)	0.000	0.000	0.000	0.000	0.000	0.000
Industrial prod. (Y)	0.003	0.000	0.000	0.000	0.000	0.000
Income	0.000	0.000	0.000	0.000	0.000	0.000
Consumption	0.000	0.000	0.000	0.000	0.000	0.000
Inflation	0.000	0.000	0.000	0.000	0.000	0.000

Table C.10 The three models with the highest posterior model probabilities, $g = 1/T$. Period: 198301 - 200212.

	Benchmark			Size-B/M			Industry		
Factor	1	2	3	1	2	3	1	2	3
Intercept	1	0	0	1	0	0	0	0	1
SMB	1	1	1	1	1	1	1	1	1
HML	1	1	1	1	1	1	1	1	1
Momentum	1	1	1	0	0	0	0	1	1
Market excess return	1	1	1	1	1	1	1	1	1
Term spread Short	0	0	0	0	0	0	0	0	0
Termp spread Long	0	0	0	0	0	0	0	0	0
Credit risk spread	0	1	0	0	1	0	1	1	0
Yield on TB3M	0	0	0	0	0	0	0	0	0
Dividend SP500	0	0	1	0	0	1	0	0	0
Industrial prod. (M)	0	0	0	0	0	0	0	0	0
Industrial prod. (Y)	0	0	0	0	0	0	0	0	0
Income	0	0	0	0	0	0	0	0	0
Consumption	0	0	0	0	0	0	0	0	0
Inflation	0	0	0	0	0	0	0	0	0
Probability	0.708	0.198	0.090	0.999	0.001	0	0.492	0.437	0.025

Table C.11 The three models with the highest posterior model probabilities, $g = 1/T$. Period: 198301 - 200212.

	Cashflow			Earning			Dividend		
Factor	1	2	3	1	2	3	1	2	3
Intercept	1	1	0	1	0	1	0	1	0
SMB	1	0	1	1	1	1	1	1	1
HML	1	1	1	1	1	1	1	1	1
Momentum	0	0	0	0	0	1	0	0	0
Market excess return	1	1	1	1	1	1	1	1	1
Term spread Short	0	0	0	0	0	0	0	0	0
Termp spread Long	0	0	0	0	0	0	0	0	0
Credit risk spread	0	0	1	0	1	0	1	0	0
Yield on TB3M	0	0	0	0	0	0	0	0	0
Dividend SP500	0	0	0	0	0	0	0	0	0
Industrial prod. (M)	0	0	0	0	0	0	0	0	0
Industrial prod. (Y)	0	0	0	0	0	0	0	0	0
Income	0	0	0	0	0	0	0	0	0
Consumption	0	0	0	0	0	0	0	0	0
Inflation	0	0	0	0	0	0	0	0	0
Probability	0.397	0.323	0.218	0.630	0.368	0.002	0.700	0.203	0.066

A.4 Description of the Data

The Portfolios

The 25 size-BE/ME Portfolios

The portfolios, which are constructed at the end of each June, are the intersections of 5 portfolios formed on size (market equity, ME) and 5 portfolios formed on the ratio of book equity to market equity (BE/ME). The size breakpoints for year t are the NYSE market equity quantiles at the end of June of t. BE/ME for June of year t is the book equity for the last fiscal year end in t-1 divided by ME for December of t-1. The BE/ME breakpoints are NYSE quantiles.

The portfolios for July of year t to June of t+1 include all NYSE, AMEX, and NASDAQ stocks for which we have market equity data for December of t-1 and June of t, and (positive) book equity data for t-1.

The Benchmark Portfolios

The benchmark portfolios are rebalanced quarterly using two independent sorts, on size and book-to-market. See the description of the 25 size-BE/ME for details.

The Industry Portfolios

Each NYSE, AMEX, and NASDAQ stock is sorted into an industry portfolio at the end of June of year t. The industries are consumer nondurables,consumer durables, oil, manufacturing, telecom, utilities, shops, Finance and other.

The Cashflow/P Portfolios

Portfolios are formed on CF/P at the end of each June using NYSE breakpoints. The cashflow used in June of year t is total earnings before extraordinary items, plus equity's share of depreciation, plus deferred taxes (if available) for the last fiscal year end in t-1. P is price times shares outstanding at the end of December of t-1.

The Earnings/P Portfolios

Portfolios are formed on E/P at the end of each June using NYSE breakpoints. The earnings used in June of year t are total earnings before extraordinary items for the last fiscal year end in t-1.

The Dividends/P Portfolios

Portfolios are formed on D/P at the end of each June using NYSE breakpoints. The dividend yield use to form portfolios in June of year t is the total dividends paid from July of t-1 to June of t per dollar of equity in June of t.

The Factors

The factors can be divided into three parts, (i) returns on broadbased portfolios, (ii) factors that measure the state of the economy and, (ii) factors that signals change in the future. The first part contains the Fama and French factors and the momentum factor and the other two contains macroeconomic variables and interest rates.

Returns on broadbased portfolios and Stock prices

The Fama-French factors are constructed using the 6 value-weighted portfolios formed on size and book-to-market.

- Size premium, SMB (Small Minus Big), is the average return on the three small portfolios minus the average return on the three big portfolios.
- Value premium, HML (High Minus Low), is the average return on the two value portfolios minus the average return on the two growth portfolios.

- The momentum factor (UMD) is constructed using six value-weight portfolios formed on size and the past 2 to 12 month returns. The portfolios, which are formed monthly, are the intersections of two portfolios formed on size and three portfolios formed on prior return. The monthly size breakpoint is the median NYSE market equity. The monthly prior return breakpoints are the 30th and 70th NYSE percentiles. Then UMD (Up Minus Down) is the average return on the two high prior return portfolios minus the average return on the two low prior return portfolios
- Market excess return is the value-weighted return on all NYSE, AMEX, and NASDAQ stocks minus the one-month Treasury bill rate.
- S&P'S Composite common stock: Dividend yield (% PER ANNUM)

Interest rates and Macroeconomic Factors[10]

Interest Rates

- Bond Yield: Moody's Aaa corporate*.
- Bond Yield: Moody's Baa corporate*.
- Interest rate: Federal Funds Rate*.
- Interest rate: Three-month U.S.Treasury Bills*
- Interest rate: One-year U.S.Treasury Bills*.
- Interest rate: Ten-year U.S.Treasury*
- Credit risk spread: Difference between the yield on Moody's Baa rated bonds and the yield on Moody's Aaa rated bonds
- Term spread (Short): Difference between the yield on one-year Treasuries and the Federal Funds rate.

[10] *Indicates that the series is used to calculate a factor.

- Term spread (Long): Difference between the yield on ten-year and one-year Treasuries.
- Change in yield on three-month Treasury Bills.

Macroeconomic Factors

- Industrial production*: Total Index (1992=100,SA)
- Monthly growth rate in industrial production: First difference of the log series.
- Yearly growth rate in industrial production: Twelfth difference of the log series.
- Producer Price Index*: Finished goods (1982=100,SA).
- Change in inflation: Second difference of the log producer price index.
- Personal income* BIL 92$,SAAR.
- Monthly growth in personal income: First difference of the log series.
- Personal consumption* (expend) BIL 92$,SAAR.
- Monthly growth in personal consumption: First difference of the log series.

A.5 Derivation of the marginal likelihood

Consider the linear model

$$\mathbf{R} = \mathbf{X}\mathbf{B} + \mathbf{e} \tag{14}$$

where $\mathbf{R} = (\mathbf{r}_1, ..., \mathbf{r}_N)$ is the $T \times N$ matrix of returns for N assets, $\mathbf{X}$ is the $T \times (q+1)$ matrix of a vector of ones and the n factors, $\mathbf{B} = (\beta_1, ..., \beta_N)$ is the $(q+1) \times N$ parameter matrix and $\mathbf{e}$ is a $T \times N$ random matrix assumed to be matrix variate normal $MN_{T\times N}(\mathbf{0}, \mathbf{\Sigma}, \mathbf{I}_T)$ where $\mathbf{\Sigma}$ is a positive definite $N \times N$ matrix.

The likelihood is

$$\begin{aligned} P(\mathbf{R}|\mathbf{B}_i, \mathbf{\Sigma}) &= (2\pi)^{-\frac{TN}{2}} |\mathbf{\Sigma}|^{-\frac{T}{2}} \exp\left[-\frac{1}{2}tr\left[\mathbf{\Sigma}^{-1}(\mathbf{R}-\mathbf{X}\mathbf{B})'(\mathbf{R}-\mathbf{X}\mathbf{B})\right]\right] \\ &= (2\pi)^{-\frac{TN}{2}} |\mathbf{\Sigma}|^{-\frac{T}{2}} \exp\left[-\frac{1}{2}tr\left[\mathbf{\Sigma}^{-1}\left(\mathbf{S} + \left(\mathbf{B}-\widehat{\mathbf{B}}\right)'\mathbf{X}'\mathbf{X}\left(\mathbf{B}-\widehat{\mathbf{B}}\right)\right)\right]\right] \end{aligned} \tag{15}$$

where $\mathbf{S} = (\mathbf{R}-\mathbf{X}\hat{\mathbf{B}})'(\mathbf{R}-\mathbf{X}\hat{\mathbf{B}})$ and $\hat{\mathbf{B}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{R}$. Prior densities are given by

$$\pi(\mathbf{B}|\mathbf{\Sigma}) = (2\pi)^{-\frac{(q+1)N}{2}} \left|\mathbf{Z}_0^{-1}\right|^{-\frac{N}{2}} |\mathbf{\Sigma}|^{-\frac{(q+1)}{2}} \exp\left[-\frac{1}{2}tr\left[\mathbf{\Sigma}^{-1}(\mathbf{B}-\bar{\mathbf{B}})'\mathbf{Z}_0(\mathbf{B}-\bar{\mathbf{B}})\right]\right] \tag{16}$$

and

$$\pi(\mathbf{\Sigma}) \propto |\mathbf{\Sigma}|^{-\frac{1}{2}(N+1)}. \tag{17}$$

Combining the likelihood and the priors yields

$$\begin{aligned}
P(\mathbf{R}|\mathbf{B},\mathbf{\Sigma})\pi(\mathbf{B}|\mathbf{\Sigma})\pi(\mathbf{\Sigma}) &\propto (2\pi)^{-\frac{TN+(q+1)N}{2}} \left|\mathbf{Z}_0^{-1}\right|^{-\frac{N}{2}} |\mathbf{\Sigma}|^{-\frac{T+q+1+N+1}{2}} \\
&\times \exp\left[-\frac{1}{2}tr\left[\mathbf{\Sigma}^{-1}\left(\mathbf{S}+\left(\mathbf{B}-\widehat{\mathbf{B}}\right)'\mathbf{X}'\mathbf{X}\left(\mathbf{B}-\widehat{\mathbf{B}}\right)\right)\right]\right] \\
&\times \exp\left[-\frac{1}{2}tr\left[\mathbf{\Sigma}^{-1}(\mathbf{B}-\bar{\mathbf{B}})'\mathbf{Z}_0(\mathbf{B}-\bar{\mathbf{B}})\right]\right] \qquad (18) \\
&= (2\pi)^{-\frac{TN+(q+1)N}{2}} \left|\mathbf{Z}_0^{-1}\right|^{-\frac{N}{2}} |\mathbf{\Sigma}|^{-\frac{T+q+1+N+1}{2}} \\
&\times \exp\left\{-\frac{1}{2}tr\left(\mathbf{\Sigma}^{-1}\mathbf{S}_1\right)\right\}\exp\left\{-\frac{1}{2}tr\left[\mathbf{\Sigma}^{-1}\left(\mathbf{B}-\mathbf{B}_1\right)'\mathbf{A}_1\left(\mathbf{B}-\mathbf{B}_1\right)\right]\right\}
\end{aligned}$$

where $\mathbf{S}_1 = \mathbf{S}+\left(\bar{\mathbf{B}}-\widehat{\mathbf{B}}\right)'\left[\mathbf{Z}_0^{-1}+(\mathbf{X}'\mathbf{X})^{-1}\right]^{-1}\left(\bar{\mathbf{B}}-\widehat{\mathbf{B}}\right)$, $\mathbf{A}_1 = \mathbf{Z}_0+\mathbf{X}'\mathbf{X}$ and $\mathbf{B}_1 = \mathbf{Z}_1^{-1}\left(\mathbf{Z}_0\bar{\mathbf{B}}+\mathbf{X}'\mathbf{X}\widehat{\mathbf{B}}\right)$.

To obtain the marginal likelihood we integrate (18) with respect to $\mathbf{B}$ and $\mathbf{\Sigma}$. We first note that the second exp term in (18) is the kernel of a matrix variate normal density and integrating with respect to $\mathbf{B}$ yields

$$(2\pi)^{-\frac{TN}{2}} |\mathbf{Z}_0|^{\frac{N}{2}} |\mathbf{A}_1|^{-\frac{N}{2}} |\mathbf{\Sigma}|^{-\frac{T+N+1}{2}} \exp\left\{-\frac{1}{2}tr\left(\mathbf{\Sigma}^{-1}\mathbf{S}_1\right)\right\}. \qquad (19)$$

To integrate (19) with respect to $\mathbf{\Sigma}$ we note that $|\mathbf{\Sigma}|^{-\frac{T+N+1}{2}}\exp\left\{-\frac{1}{2}tr\left(\mathbf{\Sigma}^{-1}\mathbf{S}_1\right)\right\}$ is the kernel of an inverted Wishart density. Hence, the marginal likelihood is

$$m(\mathbf{R}) = (2\pi)^{-\frac{TN}{2}} |\mathbf{Z}_0|^{\frac{N}{2}} |\mathbf{A}_1|^{-\frac{N}{2}} C_{IW}(\mathbf{S}^*, T; N)k. \qquad (20)$$

where k is the proportionality constant relating to $\pi(\Sigma)$. Taking the ratio of two marginal likelihoods k cancels and (10) obtains. Inserting the g-prior specification yields (13).

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