The target projection dynamic*

Elias Tsakas[a,†], Mark Voorneveld[b,c,‡]

[c]*Göteborg University, Department of Economics, Sweden*

[b]*Tilburg University, Department of Econometrics and Operations Research, The Netherlands*

[c]*Stockholm School of Economics, Department of Economics, Sweden*

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Abstract

This paper studies the target projection dynamic, which is a model of myopic adjustment for population games. We put it into the standard microeconomic framework of utility maximization with control costs. We also show that it is well-behaved, since it satisfies the desirable properties: Nash stationarity, positive correlation, and existence, uniqueness, and continuity of solutions. We also show that, similarly to other well-behaved dynamics, a general result for elimination of strictly dominated strategies cannot be established. Instead we rule out survival of strictly dominated strategies in certain classes of games. We relate it to the projection dynamic, by showing that the two dynamics coincide in a subset of the strategy space. We

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†Tel:+46-31-786 4106; Fax:+46-31-786 4154; E-mail: Elias.Tsakas@economics.gu.se

‡Tel:+46-8-736 9217; Fax:+46-8-31 32 07; E-mail: Mark.Voorneveld@hhs.se

show that strict equilibria, and evolutionarily stable strategies in 2×2 games are asymptotically stable under the target projection dynamic. Finally, we show that the stability results that hold under the projection dynamic for stable games, hold under the target projection dynamic too, for interior Nash equilibria.

1 Introduction

The traditional concept in the theory of strategic form games is the Nash equilibrium, which by definition embodies the notions of correct beliefs and rationality. However this approach does not say much about how players reach the point of actually implementing the equilibrium strategy. This question led to the development of a whole branch of game theory, which on the contrary to the traditional rational models is based on individual behavior that dynamically changes according to set of myopic rules. The usual questions that are addressed in this kind of evolutionary models focus on the behavioral properties of the adjustment rules, and on the limiting behavior of a population of agents who behave according to these myopic rules.

The main dynamic processes in the theory of strategic form games are the replicator dynamic (Taylor and Jonker, 1978), the best response dynamic (Gilboa and Matsui, 1991), and the Brown-Nash-von Neumann (BNN) dynamic (Brown and von Neumann, 1950). Sandholm (2005) introduced a definition for well-behaved evolutionary dynamics through a number of desiderata: *existence, uniqueness and continuity of solutions* (EUC), *Nash stationarity* (NS), and *positive correlation* (PC). He showed that unlike the replicator and the best-response dynamics, the whole family of BNN dynamics – which are known as excess payoff – are well behaved.

In the present paper we analyze the target projection dynamic that was introduced in a game-theoretic framework in the same paper by Sandholm (2005), and which – as we prove – satisfies the previous properties. Initially we go beyond the geometric structure that made this model intuitively appealing, and we present the microeconomic foundations that motivate its use. More specifically we show that it is based on a model of rational behavior under the constraint of control costs (Mattsson and Weibull, 2002). We also show that on certain subsets of the strategy space it coincides with the projection dynamic (Sandholm et al., 2006).

Following the analysis of Berger and Hofbauer (2006), and Hofbauer and Sandholm (2006) we show that there are games where strictly dominated strategies survive under the target projection dynamic. Though the original analysis was built upon the construction of pathological examples, it still prevents us from establishing a general result. We instead show that strictly dominated strategies are always eliminated in certain classes of games. Namely in the existence of two pure strategies the dominated one never survives. The same result holds if the gap between the dominated and another strategy is at least 2.

The second, and probably more challenging task is to investigate the limiting properties of the target projection. A large number of papers have been written about different dynamics and different classes of games, but most of them seem to agree that an ideal dynamic would be one that converges to a point or a set that satisfy some traditional game theoretic equilibrium concept, for a large subset of initial values. Of course no global result has been established till now (Hart and Mas Collel, 2002).

In the present paper we present a number of partial stability results for some equilibrium refinements, and different classes of games under the target projection dynamic. We show that every strict equilibrium is asymptotically stable. A similar result has been proven for evolutionarily stable strategies (ESS) under the replicator dynamic in single population random matching two-player games (Taylor and Jonker, 1978). In the target projection dynamic, we show that ESS in 2×2 games are asymptotically stable.

Finally we show that the stability results proven by Sandholm et al. (2006) for stable games under the projection dynamic hold for completely mixed equilibria under the target projection dynamic too. This is quite interesting since large classes of games, such as zero-sum, and games with interior ESS, belong to the family of stable games.

The present paper is structured as follows: Section 2 includes the notation and the basic framework of population games. Section 3 presents the target projection dynamic, its fundamental properties, and some general results about domination, strict equilibria, and ESS. Section 4 investigates the population behavior in equilibrium refinements, and special classes of games. Section 5 concludes.

2 Preliminaries

2.1 Population games

We follow the traditional framework of population games; see Sandholm (2005, 2006). Let $N = \{1, ..., n\}$, with $n \geq 1$, denote the set of populations. Agents in an arbitrary population $i \in N$ form a mass $m_i > 0$. For simplicity we take $m_i = 1$ for every $i \in N$. Let $A_i = \{a_i^1, ..., a_i^{J_i}\}$ be set of the available actions (pure strategies) to agents that belong to the i-th population, with $a_i \in A_i$ denoting the typical element. Every agent selects an action, so that the distribution of actions over a population $i \in N$ belongs to $\Delta(A_i) = \{\alpha_i \in \mathbb{R}_+^{J_i} : \sum_{j=1}^{J_i} \alpha_i^j = 1\}$. A vector $\alpha \in \Delta(A) = \times_{i=1}^n \Delta(A_i)$ is called ***(population) state***, and describes the behavior across the superpopulation N. A payoff function $U : \Delta(A) \to \mathbb{R}^J$, with $J = \sum_{i=1}^n J_i$, is a Lipschitz continuous mapping, that assigns a unique real number to every available action $a_i \in A_i$ of every population $i \in N$, when $\alpha \in \Delta(A)$ is played. The payoff assigned to action a_i^j is denoted by $U_i^j(\alpha)$ and we denote by $U_i(\alpha) = (U_i^1(\alpha), \ldots, U_i^{J_i}(\alpha))$ the vector of payoffs to the different actions available to population i.

The framework of population games allows for considerable flexibility, including the play of a single population symmetric game, as is usual in evolutionary game theory (Weibull, 1995; Maynard Smith, 1982), or the play of finite strategic games (Fudenberg and Levine, 1998).

Definition 2.1. A state α is a ***Nash equilibrium*** of the population game if each strategy that is used in α is a best response to α:

$$\text{For all } i \in N, j \in \{1, \ldots, J_i\}, \text{ if } \alpha_i^j > 0, \text{ then } U_i^j(\alpha) \geq U_i^k(\alpha) \text{ for all } k \in \{1, \ldots, J_i\}.$$

Equivalently, it is a Nash equilibrium if for each $i \in N$:

$$\alpha_i \text{ solves } \max_{\beta_i \in \Delta(A_i)} \beta_i' U_i(\alpha). \tag{1}$$

2.2 Projections

This subsection contains some results on projections. In particular, Proposition 2.1 establishes a link between maximizing a linear function and certain projection problems.

Proposition 2.2 gives a simple expression for projection onto the unit simplex.

For vectors $x, y \in \mathbb{R}^n$, let $x'y := \sum_{i=1}^{n} x_i y_i$ denote the usual inner product, and for $z \in \mathbb{R}$, let $[z]_+ := \max\{z, 0\}$. An arbitrary norm on $\mathbb{R}^n$ is denoted by $|\cdot|$, and the standard Euclidean norm by $\|\cdot\|$, i.e., $\|x\| = \sqrt{x'x} = \sqrt{\sum_{i=1}^{n} x_i^2}$.

Proposition 2.1. *Let $|\cdot|$ be a norm on $\mathbb{R}^n$. Let $C \subseteq \mathbb{R}^n$ be nonempty, convex, $a \in \mathbb{R}^n, c \in C$.*

(i) The following two claims are equivalent:

(*a*) *c solves* $\max_{x \in C} a'x$

(*b*) *c solves* $\max_{x \in C} a'x - |x - c|^2$.

(ii) If the norm $|\cdot|$ is generated by an inner product $\langle \cdot, \cdot \rangle$, let vectors $b_1, \ldots, b_n$ constitute an orthonormal basis of $\mathbb{R}^n$ (i.e., $\langle b_i, b_j \rangle$ equals one if $i = j$ and zero otherwise). Define $v : \mathbb{R}^n \to \mathbb{R}$ by taking, $v(a) := \frac{1}{2} \sum_{i=1}^{n} (a'b_i) b_i$, for each $a \in \mathbb{R}^n$. The problem in (b) then reduces to a projection problem:

$$\arg\max_{x \in C} a'x - |x - c|^2 = \arg\min_{x \in C} |x - c - v(a)|^2. \tag{2}$$

Proof. (*i*) [(*a*) $\Rightarrow$ (*b*)] Assume (*a*) holds. Since $|c - c| = 0$, it follows, for each $x \in C$, that

$$a'c - |c - c|^2 = a'c \geq a'x \geq a'x - |x - c|^2,$$

so (*b*) holds.
[(*b*) $\Rightarrow$ (*a*)] Assume (*b*) holds. Let $x \in C$ and $\lambda \in (0, 1)$. By convexity, $\lambda x + (1 - \lambda)c \in C$. Since $|c - c| = 0$, it follows that

$$\begin{aligned} a'c &\geq a'(\lambda x + (1 - \lambda)c) - |(\lambda x + (1 - \lambda)c) - c|^2 \\ &= \lambda(a'x) + (1 - \lambda)(a'c) - \lambda^2 |x - c|^2. \end{aligned}$$

Rearrange terms and divide by $\lambda > 0$ to obtain that $a'c \geq a'x - \lambda|x - c|^2$. Since $\lambda \in (0, 1)$ is arbitrary, let λ approach zero to establish (*a*).
(*ii*) Maximizing the function $x \mapsto a'x - |x - c|^2$ is equivalent with minimizing $x \mapsto |x - c|^2 - a'x$. It therefore suffices to show that the latter function is identical to the

function $x \mapsto |x - c - v(a)|^2$, up to an additive constant. Using the linearity and symmetry properties of the inner product, we find

$$\begin{aligned} |x - c - v(a)|^2 &= \langle x - c - v(a), x - c - v(a) \rangle \\ &= \langle x - c, x - c \rangle - 2\langle x, v(a) \rangle + 2\langle c, v(a) \rangle + \langle v(a), v(a) \rangle \\ &= |x - c|^2 - 2\langle x, v(a) \rangle + 2\langle c, v(a) \rangle + \langle v(a), v(a) \rangle. \end{aligned}$$

The final two terms are independent of x, so it remains to show that the two linear functions $x \mapsto a'x$ and $x \mapsto 2\langle x, v(a) \rangle$ are identical. We do so by establishing that they coincide on the orthonormal basis $b_1, \ldots, b_n$. By orthonormality:

$$2\langle b_i, v(a) \rangle = 2 \left\langle b_i, \frac{1}{2} \sum_{j=1}^{n} (a'b_j) b_j \right\rangle = \sum_{j=1}^{n} (a'b_j) \langle b_i, b_j \rangle = a'b_i,$$

for each basis vector b_i, finishing the proof. □

Remark 2.1. The usual inner product on $\mathbb{R}^n$ induces the Euclidean norm $\| \cdot \|$. If we apply Proposition 2.1 to the norm $|\cdot|$ induced by the rescaled inner product $\langle x, y \rangle := \frac{1}{2} x'y$, then an orthonormal basis is given by $\sqrt{2}e_1, \ldots, \sqrt{2}e_n$, where $e_i \in \mathbb{R}^n$ is the i-th standard basis vector with i-th coordinate equal to one and all others equal to zero. Moreover,

$$|x - y|^2 = \frac{1}{2}(x - y)'(x - y) = \frac{1}{2}\|x - y\|^2 \text{ and } v(y) = \frac{1}{2} \sum_{i=1}^{n} (y'(\sqrt{2}e_i))\sqrt{2}e_i = y,$$

so v is the identity function. ◁

The following proposition characterizes projection on a unit simplex.

Proposition 2.2. *Consider $n \in \mathbb{N}$, and let $\Delta_n = \{x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = 1\}$ be the $(n-1)$-dimensional unit simplex. Let $P : \mathbb{R}^n \to \Delta_n$ denote the projection on Δ_n w.r.t. the standard Euclidean distance.*

(i) P is Lipschitz continuous.

For every $x \in \mathbb{R}^n$, there is a unique $\lambda(x) \in \mathbb{R}$ such that

(ii) $\sum_{i=1}^n [x_i + \lambda(x)]_+ = 1$, and

(iii) $P(x) = ([x_1 + \lambda(x)]_+, \ldots, [x_n + \lambda(x)]_+)$.

Proof. (i) Recall from the Projection Theorem (see, for instance, Luenberger, 1969, p. 69) that for every $z \in \mathbb{R}^n$, $P(z)$ is characterized by $\big(z - P(z) \mid w - P(z)\big) \le 0$ for all $w \in \Delta_n$. In particular, for all $x, y \in \mathbb{R}^n$:

$$\big(x - P(x) \mid P(y) - P(x)\big) \le 0 \text{ and } \big(y - P(y) \mid P(x) - P(y)\big) \le 0.$$

Write $\big(y - P(y) \mid P(x) - P(y)\big) = \big(P(y) - y \mid P(y) - P(x)\big)$, add the two inequalities, and use Cauchy-Schwarz to establish

$$\begin{aligned} 0 &\ge \big(x - P(x) + P(y) - y \mid P(y) - P(x)\big) \\ &= \|P(y) - P(x)\|^2 - \big(y - x \mid P(y) - P(x)\big) \\ &\ge \|P(y) - P(x)\|^2 - \|y - x\| \, \|P(y) - P(x)\|. \end{aligned}$$

Conclude that $\|P(y) - P(x)\| \le \|y - x\|$, i.e., P is Lipschitz continuous with expansion factor 1.

(ii) Let $x \in \mathbb{R}^n$. The function $T : \mathbb{R} \to \mathbb{R}$ defined for each $\lambda \in \mathbb{R}$ by $T(\lambda) = \sum_{i=1}^n [x_i + \lambda]_+$ is the composition of continuous functions and therefore continuous itself. Let $m = \max\{x_1, \ldots, x_n\}$. Then $T(\lambda) = 0$ for all $\lambda \in (-\infty, -m]$ and T is strictly increasing on $[-m, \infty)$, with $T(\lambda) \to \infty$ as $\lambda \to \infty$. By the intermediate value theorem, there is a unique $\lambda(x) \in [-m, \infty)$ such that $T(\lambda(x)) = 1$.

(iii) By definition, $P(x)$ is the unique solution to $\min_{y \in \Delta_n} \frac{1}{2} \sum_{i=1}^n (y_i - x_i)^2$. This is a convex quadratic optimization problem with linear constraints, so the Karush-Kuhn-Tucker conditions are necessary and sufficient to characterize the minimum location: $y^* \in \Delta_n$ solves the problem if and only if there exist Lagrange multipliers $\mu_i \ge 0$ associated with the inequality constraints $y_i \ge 0$ and $\nu \in \mathbb{R}$ associated with the equality constraint $\sum_{i=1}^n y_i = 1$ such that for each $i = 1, \ldots, n$:

$$y_i^* - x_i - \mu_i + \nu = 0, \tag{3}$$

$$\mu_i y_i^* = 0. \tag{4}$$

Condition (3) is the first order condition obtained from differentiating the Lagrangian

$$(y, \mu_1, \ldots, \mu_n, \nu) \mapsto \frac{1}{2} \sum_{i=1}^n (y_i - x_i)^2 - \sum_{i=1}^n \mu_i y_i + \nu \left(\sum_{i=1}^n y_i - 1 \right)$$

with respect to y_i and condition (4) is the complementary slackness condition. It is now easy to see that $y^* := ([x_1 + \lambda(x)]_+, \ldots, [x_n + \lambda(x)]_+)$ solves the minimization problem: set $\mu_i = 0$ if $[x_i + \lambda(x)]_+ > 0$, $\mu_i = -x_i - \lambda(x) \geq 0$ if $[x_i + \lambda(x)]_+ \leq 0$, and $\nu = -\lambda(x)$. Substitution in (3) and (4) shows that these necessary and sufficient conditions are satisfied. □

Remark 2.2. Proposition 2.2*(iii)* immediately implies that for all $x \in \mathbb{R}^n$ and $i, j \in \{1, \ldots, n\}$: if $x_i - x_j \geq 1$, then $P_j(x) = 0$. ◁

3 The target projection dynamic

The target projection dynamic, the dynamic process governing the adjustment of population states that we study in this paper, was mentioned briefly in the concluding section of Sandholm (2005, pp. 166-167). It was originally defined for the somewhat different framework of congestion networks by Friesz et al. (1994). The formal definition is as follows:

Definition 3.1. Let $(N, (A_i)_{i \in N}, U)$ be a population game. The ***target projection dynamic (TPD)*** is defined, for each $i \in N$, by the differential equation

$$\dot{\alpha}_i = P_{\Delta(A_i)}[\alpha_i + U_i(\alpha)] - \alpha_i. \tag{5}$$

Here, $P_{\Delta(A_i)}$ denotes projection on $\Delta(A_i)$ w.r.t. the usual Euclidean distance.

The basic idea is simple and standard for most dynamic processes in game theory. The payoffs associated with the different actions determine the direction in which their weights are changed by reinforcing the better actions and decreasing the weight of worse ones. Of course, simply running in the direction of the payoff vector $U_i(\alpha)$ might take you outside the strategy simplex, but Proposition 2.2(ii) assures that projection onto the strategy simplex does not affect the order of the coordinates.

The next proposition indicates that the target projection dynamic is essentially a best-response dynamic (Gilboa and Matsui, 1991), albeit under two bounded rationality assumptions. The first is a myopia assumption: by (1), the search for Nash equilibria involves solving linear optimization problems of the form

$$\max_{\beta_i \in \Delta(A_i)} \beta_i' U_i(\alpha), \tag{6}$$

which overlooks the fact that a change by population i from α_i to β_i does not keep the payoff vector $U_i(\alpha)$ unaffected: it changes to $U_i(\beta_i, \alpha_{-i})$. The second bounded rationality assumption involves the introduction of a certain status-quo bias. We follow the control cost approach, which since its introduction by Eric van Damme (1991) in the study of equilibrium refinements has proved to be a versatile way of providing microeconomic foundations for a variety of models of strategic behavior (Hofbauer and Sandholm, 2002; Mattsson and Weibull, 2002; Voorneveld, 2006). It does so by showing that such behavior is rational for decision makers who have to make some effort (incur costs) to implement their strategic choices. One intuitive way of modeling status-quo bias by population i could be as follows. Suppose that deviation from the current α_i is costly/requires effort in the sense that by switching to a strategy β_i, population i incurs a cost of $\frac{1}{2}||\beta_i - \alpha_i||^2$, i.e., staying at the current mixed strategy is costless, whereas large deviations, i.e., strategies further away from the current one in terms of Euclidean distance, incur larger costs. Taking such costs into account changes the optimization problem in (6) to

$$\max_{\beta_i \in \Delta(A_i)} \beta_i' U_i(\alpha) - \frac{1}{2}||\beta_i - \alpha_i||^2. \tag{7}$$

Let $B_i(\alpha) \in \Delta(A_i)$ denote population i's (unique due to strict concavity of the goal function) best response against α, i.e., the unique solution to this problem. Subject to these two bounded rationality assumptions, we can now formulate the target projection dynamic as a best response dynamic:

Proposition 3.1. *Let $(N, (A_i)_{i\in N}, U)$ be a population game. The target projection dynamic is the best response dynamic for the control cost problem in (7), i.e., for each $i \in N$:*

$$\dot{\alpha}_i = P_{\Delta(A_i)}[\alpha_i + U_i(\alpha_{-i})] - \alpha_i = B_i(\alpha) - \alpha_i. \tag{8}$$

Proof. By definition,

$$P_{\Delta(A_i)}[\alpha_i + U_i(\alpha)] = \arg \min_{\beta_i \in \Delta(A_i)} \|\beta_i - \alpha_i - U_i(\alpha)\|^2, \tag{9}$$

so we need to establish that

$$\arg \min_{\beta_i \in \Delta(A_i)} \|\beta_i - \alpha_i - U_i(\alpha)\|^2 = \arg \max_{\beta_i \in \Delta(A_i)} \beta_i' U_i(\alpha) - \frac{1}{2}\|\beta_i - \alpha_i\|^2.$$

Using the multilinearity properties of the inner product, the goal function on the left can be rewritten as

$$\begin{aligned}
\|\beta_i - \alpha_i - U_i(\alpha)\|^2 &= \langle \beta_i - \alpha_i - U_i(\alpha), \beta_i - \alpha_i - U_i(\alpha)\rangle \\
&= \langle \beta_i - \alpha_i, \beta_i - \alpha_i\rangle - 2\langle \beta_i, U_i(\alpha)\rangle \\
&+ 2\langle \alpha_i, U_i(\alpha)\rangle + \langle U_i(\alpha), U_i(\alpha)\rangle \\
&= \|\beta_i - \alpha_i\|^2 - 2\langle \beta_i, U_i(\alpha)\rangle \\
&+ 2\langle \alpha_i, U_i(\alpha)\rangle + \|U_i(\alpha)\|^2.
\end{aligned}$$

As the final two terms in the sum are independent of β_i, division by -2 gives that maximizing this expression over β_i is equivalent with minimizing

$$\langle \beta_i, U_i(\alpha)\rangle - \frac{1}{2}\|\beta_i - \alpha_i\|^2. \qquad \square$$

The previous proposition indicates that the target projection dynamic can be interpreted in terms of boundedly rational populations striving for best responses. It does not, however, imply that the dynamic is susceptible to the extensive literature on (perturbed) best response dynamics (Gilboa and Matsui, 1991; Hofbauer and Sandholm, 2002; Fudenberg and Levine, 1998). Our status-quo bias models control costs arising due to deviations from the current state. The usual approaches use control cost functions which:

- are independent of the current state: they define costs in terms of deviations from a fixed strategy, often uniform randomization (close your eyes and pick an action) as in Mattsson and Weibull (2002), and Voorneveld (2006),
- are usually required to be steep near the boundary of the strategy space, as in Hofbauer and Sandholm (2002).

3.1 General properties

Theorem 3.1 states that the target projection dynamic satisfies a number of desirable properties of "nice" evolutionary dynamics. Indeed, Sandholm (2005) calls a dynamic ***well-behaved*** if it satisfies the first three properties of Theorem 3.1.

Theorem 3.1. *Let $(N, (A_i)_{i \in N}, U)$ be a population game. The target projection dynamic satisfies the following properties:*

NASH STATIONARITY: *The stationary points of the target projection dynamic and the game's Nash equilibria coincide.*

BASIC SOLVABILITY: *For every initial state, a solution to the target projection dynamic exists, is unique, Lipschitz continuous in the initial state, and remains inside $\Delta(A)$ at all times.*

POSITIVE CORRELATION: *Within each population, growth rates are positively correlated with payoffs: for each $i \in N$, if $\dot{\alpha}_i \neq 0$, then $\langle \dot{\alpha}_i, U_i(\alpha) \rangle > 0$.*

INNOVATION: *If some population has not yet reached a stationary state, but it has an unused best response, then a positive mass of individuals switch to it. Formally, for each $\alpha \in \Delta(A)$ and $i \in N$, if $\dot{\alpha}_i \neq 0$, but there is an action $a_i^j \in A_i$ with $U_i^j(\alpha) = \max_{k \in \{1,\ldots,J_i\}} U_i^k(\alpha)$ and $\alpha_i^j = 0$, then $\dot{\alpha}_i^j > 0$.*

Proof. NASH STATIONARITY: Let $\alpha \in \Delta(A)$. By (1), Proposition 2.1, Proposition 3.1, and (5), the following chain of equivalences holds:

$$
\begin{aligned}
\alpha \text{ is a Nash equilibrium} \quad &\Leftrightarrow \quad \forall i \in N : \alpha_i \in \arg \max_{\beta_i \in \Delta(A_i)} \langle \beta_i, U_i(\alpha) \rangle \\
&\Leftrightarrow \quad \forall i \in N : \alpha_i \in \arg \max_{\beta_i \in \Delta(A_i)} \langle \beta_i, U_i(\alpha) \rangle - \frac{1}{2}\|\beta_i - \alpha_i\|^2 \\
&\Leftrightarrow \quad \forall i \in N : \alpha_i = P_{\Delta(A_i)}[\alpha_i + U_i(\alpha)] \\
&\Leftrightarrow \quad \forall i \in N : \dot{\alpha}_i = 0.
\end{aligned}
$$

BASIC SOLVABILITY: The target projection dynamic (5) is Lipschitz continuous: let $i \in N$. By assumption, the payoff U_i is Lipschitz continuous, say with expansion factor $C > 0$. By Proposition 2.2, the projection is Lipschitz continuous with expansion factor 1. Using the triangle inequality, it follows for each $\alpha, \beta \in \Delta(A)$ that

$$
\begin{aligned}
\|P[\alpha_i + U_i(\alpha)] - \alpha_i - P[\beta_i + U_i(\beta)] + \beta_i\| \quad &\leq \quad \|P[\alpha_i + U_i(\alpha)] - P[\beta_i + U_i(\beta)]\| + \|\alpha_i - \beta_i\| \\
&\leq \quad \|\alpha_i + U_i(\alpha) - \beta_i - U_i(\beta)\| + \|\alpha_i - \beta_i\| \\
&\leq \quad \|U_i(\alpha) - U_i(\beta)\| + 2\|\alpha_i - \beta_i\| \\
&\leq \quad (C + 2)\|\alpha_i - \beta_i\|,
\end{aligned}
$$

establishing Lipschitz continuity of the vector field in (5). Since $P_{\Delta(A_i)}$ maps onto $\Delta(A_i)$, it follows that $\sum_{j=1}^{J_i} \dot{\alpha}_i^j = 0$. Moreover, if $\alpha_i^j = 0$, then $\dot{\alpha}_i^j \geq 0$. This makes $\Delta(A)$ forward-invariant. Together, these properties imply (Hirsch and Smale, 1974, Ch. 8) that for every initial state, a solution exists, is unique, Lipschitz continuous in the initial state, and remains in $\Delta(A)$ at all times.

Positive correlation: Let $\alpha \in \Delta(A)$ and $i \in N$. Suppose $\dot{\alpha}_i = P_{\Delta(A_i)}[\alpha_i + U_i(\alpha)] - \alpha_i \neq 0$. Let $\beta_i = P_{\Delta(A_i)}[\alpha_i + U_i(\alpha)] \neq \alpha_i$. Then, using Proposition 3.1, one obtains:

$$\begin{aligned}\langle \beta_i, U_i(\alpha)\rangle &> \langle \beta_i, U_i(\alpha)\rangle - \frac{1}{2}\|\alpha_i - \beta_i\|^2 \\ &\geq \langle \alpha_i, U_i(\alpha)\rangle - \frac{1}{2}\|\alpha_i - \alpha_i\|^2 \\ &= \langle \alpha_i, U_i(\alpha)\rangle,\end{aligned}$$

So $\langle \dot{\alpha}_i, U_i(\alpha)\rangle = \langle \beta_i - \alpha_i, U_i(\alpha)\rangle > 0$.

Innovation: Assume that the premises of the innovation property hold, but that $\dot{\alpha}_i^j \leq 0$. We derive a contradiction. By Proposition 2.2 there is a $\lambda \in \mathbb{R}$ such that

$$P_{\Delta(A_i)}[\alpha_i + U_i(\alpha)] = ([\alpha_i^1 + U_i^1(\alpha) + \lambda]_+, \ldots, [\alpha_i^{J_i} + U_i^{J_i}(\alpha) + \lambda]_+).$$

By assumption, action j is unused ($\alpha_i^j = 0$) and $\dot{\alpha}_i^j \leq 0$, so

$$0 \geq \dot{\alpha}_i^j = [\alpha_i^j + U_i^j(\alpha) + \lambda]_+ - \alpha_i^j = [U_i^j(\alpha) + \lambda]_+ \geq 0,$$

i.e., $\dot{\alpha}_i^j = 0$ and $U_i^j(\alpha) + \lambda \leq 0$. But action j is a best response: $U_i^j(\alpha) = \max_{k \in \{1,\ldots,J_i\}} U_i^k(\alpha)$. Consequently, for every action $k \in \{1, \ldots, J_i\}$:

$$\dot{\alpha}_i^k = [\alpha_i^k + U_i^k(\alpha) + \lambda]_+ - \alpha_i^k \leq [\alpha_i^k + U_i^j(\alpha) + \lambda]_+ - \alpha_i^k \leq [\alpha_i^k + 0]_+ - \alpha_i^k = 0.$$

Since $\sum_{k=1}^{J_i} \dot{\alpha}_i^k = 0$, this implies that $\dot{\alpha}_i^k = 0$ for all $k \in \{1, \ldots, J_i\}$, in contradiction with the assumption that $\dot{\alpha}_i \neq 0$. $\square$

3.2 Strict domination: mind the gap

Berger and Hofbauer (2006) show that under the Brown-von Neumann-Nash (BNN) dynamic, introduced in Brown and von Neumann (1950), there are games where a strictly dominated strategy survives. Hofbauer and Sandholm (2006) generalize this example:

for each evolutionary dynamic satisfying the properties in Theorem 3.1 — actually, they restrict attention to single-population games — it is possible to construct a game with a strictly dominated strategy that survives along solutions of most initial states.

As their result applies to our target projection dynamic, it is of interest to investigate whether there are additional conditions under which such "bad" actions *are* wiped out. The next result shows that this is the case if one action strictly dominates another and the "gap" between them is sufficiently large.

Proposition 3.2. *Let $(N, (A_i)_{i \in N}, U)$ be a population game and let $i \in N$. Suppose there are actions $k, \ell \in \{1, \ldots, J_i\}$ such that $U_i^k - U_i^\ell \geq 2$, i.e., action k strictly dominates action ℓ, and the gap between the payoffs is at least two. Then the probability α_i^ℓ converges to zero in the target projection dynamic.*

Proof. We show that the differential equation for the probability α_i^ℓ of action ℓ is given by $\dot{\alpha}_i^\ell = -\alpha_i^\ell$, because then $\alpha_i^\ell(t) = \alpha_i^\ell(0)e^{-t} \to 0$ as $t \to \infty$. Let $\alpha \in \Delta(A)$. By (5), it suffices to show that the ℓ-th coordinate of the projection $P_{\Delta(A_i)}[\alpha_i + U_i(\alpha)]$ is zero. By Proposition 2.2(*ii*), there is a $\lambda \in \mathbb{R}$ such that its ℓ-th and k-th coordinate can be written as $[\alpha_i^\ell + U_i^\ell(\alpha) + \lambda]_+$ and $[\alpha_i^k + U_i^k(\alpha) + \lambda]_+$. Suppose, contrary to what we want to prove, that $[\alpha_i^\ell + U_i^\ell(\alpha) + \lambda]_+ > 0$. Then

$$\begin{aligned} [\alpha_i^k + U_i^k(\alpha) + \lambda]_+ - [\alpha_i^\ell + U_i^\ell(\alpha) + \lambda]_+ &\geq \alpha_i^k - \alpha_i^\ell + U_i^k(\alpha) - U_i^\ell(\alpha) \\ &\geq \alpha_i^k - \alpha_i^\ell + 2 \\ &\geq 1, \end{aligned} \tag{10}$$

since the difference between probabilities is bounded in absolute value by one. However, since $[\alpha_i^\ell + U_i^\ell(\alpha) + \lambda]_+ > 0$, the left-hand side of (10) is smaller than one, a contradiction. □

Also if a population has only two actions to choose from, and one of them is strictly dominated, then it is eventually eliminated:

Proposition 3.3. *If a player's action set is $A_i = \{a_i^1, a_i^2\}$, and a_i^1 strictly dominates a_i^2, the probability assigned to a_i^2 converges to zero in the target projection dynamic.*

Proof. Let $\alpha \in \Delta(A)$. By Proposition 2.2, there is a $\lambda(\alpha) \in \mathbb{R}$ such that the target projection dynamic for each of the two actions $j = 1, 2$ of population i can be rewritten as

$$\dot{\alpha}_i^j = [\alpha_i^j + U_i^j(\alpha) + \lambda(\alpha)]_+ - \alpha_i^j \tag{11}$$

Then $[\alpha_i^1 + U_i^1(\alpha) + \lambda(\alpha)]_+ > 0$. Suppose, to the contrary, that

$$\alpha_i^1 + U_i^1(\alpha) + \lambda(\alpha) \leq 0.$$

Since we project the two-dimensional vector onto the simplex, this implies

$$[\alpha_i^2 + U_i^2(\alpha) + \lambda(\alpha)]_+ = \alpha_i^2 + U_i^2(\alpha) + \lambda(\alpha) = 1.$$

Combining these two expressions gives

$$\alpha_i^2 - \alpha_i^1 \geq 1 + U_i^1(\alpha) - U_i^2(\alpha) > 1,$$

a contradiction, since the left-hand side is at most one.

By continuity of the payoffs on the compact set $\Delta(A)$ and strict domination, there is an $\varepsilon > 0$ such that $U_i^1(\alpha) - U_i^2(\alpha) > \varepsilon$ for each $\alpha \in \Delta(A)$.

Distinguish two cases. First, if $[\alpha_i^2 + U_i^2(\alpha) + \lambda(\alpha)]_+ = 0$, then $\dot{\alpha}_i^2 = -\alpha_i^2$, so the probability α_i^2 decreases at an exponential rate. Second, if $[\alpha_i^2 + U_i^2(\alpha) + \lambda(\alpha)]_+ > 0$, combine this with the facts that $[\alpha_i^1 + U_i^1(\alpha) + \lambda(\alpha)]_+ > 0$ and that these two numbers add up to one, to deduce that $\lambda(\alpha) = -\frac{1}{2}(U_i^1(\alpha) + U_i^2(\alpha))$. So $\dot{\alpha}_i^2 = \frac{1}{2}(U_i^2(\alpha) - U_i^1(\alpha)) < -\frac{1}{2}\varepsilon$, i.e., the probability α_i^2 decreases at a rate bounded away from zero. Hence, along any solution trajectory, the probability α_i^2 of the dominated action converges to zero. □

3.3 The projection dynamic and the target projection dynamic

The projection dynamic was first developed by Nagurney and Zang (1997) as part of the transportation literature, and was later introduced to game theory by Sandholm (2006), and Sandholm et al. (2006). The underlying dynamic system is defined as follows

$$\dot{\alpha}_i = P_{T(A_i)}[U_i(\alpha)], \tag{12}$$

where $T(A_i) := \{\beta \in \mathbb{R}^{J_i} : \sum_{j=1}^{J_i} \beta_i^j = 0\}$, where $T(A_i)$ denotes the tangent cone of $\Delta(A_i)$. As the following results states the projection dynamic and the target projection dynamic coincide at certain subsets of $\Delta(A_i)$.

Proposition 3.4. *Let $\alpha \in int(\Delta(A))$ be a Nash equilibrium of a population game. Then there is a neighborhood $\mathcal{O}$ of α, such that the projection dynamic and the target projection dynamic coincide for every $\beta \in \mathcal{O}$.*

Proof. It follows from Theorem 3.1 that $\dot{\alpha}_i = 0$. Since $\alpha \in \mathrm{int}(\Delta(A))$, it follows that $\alpha_i^j > 0$, for every $i \in N$, and every $j \in J_i$. Then it follows from Proposition 2.2 that $\alpha_i^j + U_i^j(\alpha) + \lambda(\alpha) > 0$, and from continuity it follows that there is a neighborhood $\mathcal{O}$ of α such that $\beta_i^j + U_i^j(\beta) + \lambda(\beta) > 0$, for every $\beta \in \mathcal{O}$. Again from Proposition 2.2 it follows that

$$\lambda(\beta) = -\frac{1}{J_i}\sum_{j=1}^{J_i} U_i^j(\beta),$$

which implies that

$$\dot{\beta}_i^j = U_i^j(\beta) - \frac{1}{J_i}\sum_{k=1}^{J_i} U_i^k(\beta),$$

for every $i \in N$, every $j \in J_i$, and every $\beta \in \mathcal{O}$. The previous formula is the projection dynamic for all interior population states, and therefore for every state in $\mathcal{O}$ (Sandholm et al., 2006), which proves the proposition. $\square$

4 The target projection dynamic in strategic games

Much of the literature on strategic adjustment deals with (mixed extensions of) finite strategic games. In the setting of population games $(N, (A_i)_{i\in N}, U)$, this simply means that each population is associated with a different player role and payoffs are defined on the set of pure strategy profiles $A = \times_{i\in N} A_i$ and then extended to mixed strategies by taking expectations. Formally, for each $i \in N$ and each $\alpha \in \Delta(A)$: $U_i(\alpha) = \sum_{a\in A}\left(\prod_{j\in N}\alpha_j(a_j)\right)U_i(a)$.

4.1 Equilibrium refinements

4.1.1 Strict Nash equilibrium

Recall that in finite strategic games a Nash equilibrium is called ***strict*** if each player chooses the unique best reply. Formally:

Definition 4.1. A state α is a ***strict Nash equilibrium*** if every strategy used in α is the unique best response to α:

$$\text{For all } i \in N, j \in \{1,\dots,J_i\}, \text{ if } \alpha_i^j > 0, \text{ then } U_i^j(\alpha) > U_i^k(\alpha) \text{ for all } k \in \{1,\dots,J_i\}.$$

Consequently, strict Nash equilibria are always Nash equilibria in pure strategies.

Proposition 4.1. *In a finite strategic game, each strict Nash equilibrium is asymptotically stable under the target projection dynamic.*

Proof. Let β be a strict Nash equilibrium. W.l.o.g., each $i \in N$ plays his first action: $\beta_i = e_1$. By definition, for each $i \in N$ and $j \in \{2,\dots,J_i\}$: $U_i^1(\beta) > U_i^j(\beta)$, so that $(\beta_i^1 + U_i^1(\beta)) - (\beta_i^j + U_i^j(\beta)) = 1 + U_i^1(\beta) - U_i^j(\beta) > 1$. By continuity, there is a neighborhood $\mathcal{O}$ of β such that for all $\alpha \in \mathcal{O}$, $i \in N$, and $j \in \{2,\dots,J_i\}$:

$$(\alpha_i^1 + U_i^1(\alpha)) - (\alpha_i^j + U_i^j(\alpha)) \geq 1.$$

For all $\alpha \in \mathcal{O}$ and $i \in N$, Remark 2.2 implies that $P_{\Delta(A_i)}(\alpha_i + U_i(\alpha)) = e_1 = \beta_i$; so $\dot{\alpha}_i = \beta_i - \alpha_i$. Hence, the function $L : \mathcal{O} \to \mathbb{R}$ with $L(\alpha) := \sum_{i \in N} \|\alpha_i - \beta_i\|^2$ is a Lyapunov function: It is nonnegative, zero only in β, and if $\alpha \in \mathcal{O} \setminus \{\beta\}$:

$$\dot{L} = 2\sum_{i \in N}\langle \alpha_i - \beta_i, \dot{\alpha}_i\rangle = 2\sum_{i \in N}\langle \alpha_i - \beta_i, \beta_i - \alpha_i\rangle = -2\sum_{i \in N}\|\alpha_i - \beta_i\|^2 < 0. \qquad \square$$

4.1.2 Evolutionarily stable strategies

In symmetric 2-player games strict Nash equilibria are a subset of the evolutionarily stable strategies, which are defined as population states robust under behavioral mutations. That is, if a small group of players from a different population (in terms of their strategy) invade the original one, their post-entry payoff will be strictly lower than the one entailed by the incumbent strategy. Thus the mutants will have incentive to switch to the original strategy.

Formally, consider a symmetric 2-player strategic form game with payoff matrix Π, and let $U_i(\alpha) = \Pi\alpha$ denote the payoff vector of the corresponding (single) population random matching game.

Definition 4.2. A strategy $\alpha \in \Delta(A)$ is an ***evolutionary stable strategy (ESS)*** if, for each $\beta \in \Delta(A)$, with $\beta \neq \alpha$, either $\alpha'\Pi\alpha > \beta'\Pi\alpha$ or $\alpha'\Pi\alpha = \beta'\Pi\alpha$ and $\alpha'\Pi\beta > \beta'\Pi\beta$. Equivalently (Hofbauer et al., 1979, p. 610), α is an ESS if there is a neighborhood $\mathcal{O}$ of α such that $\alpha'\Pi\beta > \beta'\Pi\beta$ for all $\beta \in \mathcal{O} \setminus \{\alpha\}$.

ESS is a refinement of Nash equilibrium, in the sense that the set of strategies that satisfy evolutionary stability is a subset of the Nash equilibria. That is all ESS are rest points of every dynamic that satisfies Nash stationarity. However the converse is not always true. Since the definition of evolutionary stability is conceptually based on the idea that the mutants eventually assimilate to the original population, one would expect ESS to attract the trajectories that get sufficiently close to them under dynamic processes of myopic adjustment. Taylor and Jonker (1978), and Hofbauer et al. (1979) show that every ESS is asymptotically stable under the replicator dynamic. The following proposition extends this result to 2×2 games under the target projection dynamic.

Proposition 4.2. *Every ESS is asymptotically stable in 2×2 games under the target projection dynamic.*

Proof. Let $\alpha \in \Delta(A)$ be an ESS. If $\alpha \in \text{int}(\Delta(A))$, apply Corollary 4.1. If not, assume, w.l.o.g., that $\alpha = e_1$. If $e_1'\Pi\alpha > e_2'\Pi\alpha$, it follows (from the same argument as in Proposition 4.1), that $P(\beta + \Pi\beta) = e_1$ for all β close to α and, hence, that $L(\beta) := \|\beta - \alpha\|^2$ is a Lyapunov function. Consider then the case where $u(a_1, a_1) = u(a_2, a_1)$. We apply $\dot{\alpha}_i = 0$, which holds in equilibrium, for $i = 1$ and we obtain $[\alpha_1 + U_1(\alpha) + \lambda]_+ = \alpha_1 > 0$. Given continuity of the projection, there is $\rho > 0$ such that $[\beta_1 + U_1(\beta) + \lambda]_+ > 0$, for every β close to α.

Consider now the function

$$L(\beta) = \alpha_1 - \beta_1 = 1 - \beta_1,$$

which satisfies all the Lyapunov requirements: it is continuously differentiable, positive definite, and is equal to zero if and only if $\beta = \alpha$. Then,

$$\dot{L} = -\dot{\beta}_1 = -[U_1(\beta) + \lambda]. \tag{13}$$

If $[\beta_2 + U_2(\beta) + \lambda]_+ \leq 0$, it follows that $\lambda = 1 - \beta_1 - U_1(\beta)$. Substituting into (13) entails

$$\dot{L} = \beta_1 - 1 < 0. \tag{14}$$

If on the other hand $[\beta_2 + U_2(\beta) + \lambda]_+ > 0$, it follows that $\lambda = -\big(U_1(\beta) + U_2(\beta)\big)/2$. Substituting into (13) entails

$$\begin{aligned}\dot{L} &= -\frac{1}{2}[U_1(\beta) - U_2(\beta)] = -\frac{1}{2}[\beta_1 u(a_1,a_2) + \beta_2 u(a_1,a_2) - \beta_1 u(a_2,a_1) - \beta_2 u(a_2,a_2)] \\ &= -\frac{1}{2}\ \big[\ \beta_1\big(u(a_1,a_1) - u(a_2,a_1)\big) + \beta_2\big(u(a_1,a_2) - u(a_2,a_2)\big)\ \big] < 0, \end{aligned} \tag{15}$$

since $u(a_1,a_1) = u(a_2,a_1)$ (by assumption), and $(a_1,a_2) > u(a_2,a_2)$ (due to α being ESS). Combining (14) and (15) completes the proof. □

In the next section we extend this result to larger strategic games.

4.2 Special classes of games

Sandholm et al. (2006) prove a number of stability results for potential and stable games under the projection dynamic. Stable games (Sandholm, 2003) are a family of population games that include zero-sum games, and games with an interior ESS, and are defined by

$$\langle \alpha_i - \beta_i, U_i(\alpha) - U_i(\beta) \rangle \leq 0, \tag{16}$$

for every $\alpha, \beta \in \Delta(A_i)$, and every $i \in N$. The game is called null (strictly) stable if equality (strict inequality) holds for every $\alpha, \beta \in \Delta(A_i)$, and every $i \in N$.

Proposition 4.3. *Let $\alpha \in int(\Delta(A))$ be a Nash equilibrium in a population game.*

(i) *If the game is stable then α is Lyapunov stable under the TPD.*

(ii) *If the game is strictly stable then α is asymptotically stable under the TPD.*

(iii) *If the game null stable then the TPD defines a constant motion around α.*

Proof. It follows from Proposition 3.4 that there is a neighborhood $\mathcal{O}$ of α where the projection dynamic agrees with the target projection dynamic. Then there is some $\epsilon > 0$ such that

$$\mathcal{O}_\epsilon := \{\beta \in \Delta(A) : \|\beta - \alpha\|^2 \leq \epsilon\} \subseteq \mathcal{O}.$$

Consider now the function

$$L(\beta) := \|\beta - \alpha\|^2.$$

Sandholm et al. (2006) have shown that L is (i) a Lyapunov function, (ii) a strict Lypunov function, and (iii) defines a constant motion around α under the projection dynamic. Since $\mathcal{O}_\epsilon \subseteq \mathcal{O}$, then every trajectory that starts in $\mathcal{O}_\epsilon$, will remain in it forever, under the projection dynamic, and therefore under the target projection dynamic, which completes the proof. $\square$

The following two results are a straightforward application of the previous proposition.

Corollary 4.1. *A completely mixed ESS is asymptotically stable in the target projection dynamic.*

In the previous section we showed that ESS in 2×2 strategic games are asymptotically stable under the target projection dynamic. The previous corollary extends this result to a larger games.

The following result proves that every trajectory which gets sufficiently close to a completely mixed equilibrium in zero-sum games forms a closed cyclical orbit around it. That is neither it converges to the equilibrium point, nor is it driven away from it. Instead the trajectory maintains a constant distance from it. Typical examples where this kind of behavior is exhibited are the matching pennies and the rock-paper-scissors games.

Corollary 4.2. *The target projection dynamic forms a cyclical motion around every completely mixed Nash equilibrium in a zero-sum games.*

5 Concluding remarks

This paper studies the target projection dynamic, which is a model of myopic adjustment for population games. This dynamic originally drew attention due to its geometrically appealing formulation. We put it into the standard microeconomic framework of utility maximization with control costs. We also show that it is well-behaved, since it satisfies the desirable properties introduced by Sandholm (2005): Nash stationarity, positive correlation, and existence, uniqueness, and continuity of solutions. We also show that, similarly to other well-behaved dynamics, a general result for elimination of strictly dominated strategies cannot be established. Instead we rule out survival of strictly dominated strategies in certain classes of games. We relate it to the projection dynamic (Sandholm et al., 2006), by showing that the two dynamics coincide in a subset of the strategy space.

In the second part of the paper, we study the target projection dynamic as a rule of myopic adjustment in the framework of finite strategic games. We show that strict equilibria, and ESS in 2×2 games are asymptotically stable under the target projection dynamic. Finally, we show that the stability results that hold under the projection dynamic for stable games, hold under the target projection dynamic too, for interior Nash equilibria.

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