An Application of the Analogy between Vector ARCH and Vector Random Coefficient Autoregressive Models

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Abstract

In this paper we derive conditions for the conditional covariance matrix to be positive definite in a general vector ARCH model. The conditions can be easily extended to the diagonal vector GARCH model. For the general vector GARCH model, analytical expressions for the conditions in terms of the parameters become complicated, but their validity can in principle be checked numerically once the values of the parameters are given.

Keywords: conditional covariance matrix, multivariate GARCH, multivariate volatility model, random coefficient model, volatility forecasting

JEL Classification Code: C32

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1. Introduction

During the last fifteen years or so, the focus in modelling volatility has partly shifted from univariate to multivariate relationships. A number of vector models has been introduced for the purpose. Among others, generalizations of the univariate GARCH model to the multivariate (vector) case have appeared in the literature. For brief surveys, see Bollerslev, Engle & Nelson (1994) and Gouriéroux (1996), Chapter 6.

The properties of the vector ARCH and GARCH models are not well-known. In order to shed light on certain aspects of them we shall consider vector random coefficient autoregressive models. It is well-known that univariate vector ARCH and GARCH models may be viewed as autoregressive models with random coefficients such that each random coefficient has mean zero, see Bera, Higgins & Lee (1992). In this work we make use of the corresponding analogy in the multivariate case; see Wong & Li (1997). This is done for the following reasons. First, we want to find conditions for the conditional variance of a general vector ARCH model to be positive definite almost surely for all t. Second, vector random coefficient models are useful in characterizing parameter restrictions that certain vector ARCH models impose on the general vector ARCH model. The ARCH version of the BEKK-GARCH model of Engle & Kroner (1995) constitutes an example. Finally, viewing a general ARCH models that way helps one to interpret parameters in vector ARCH models and to define new, potentially useful parsimonious vector ARCH models.

The plan of the paper is as follows. In Section 2 we introduce the Vector Random Coefficient Autoregressive model and describe its connection with the general vector ARCH model. This connection is used in Section 3 to derive a condition for the conditional covariance matrix of the general vector ARCH model of Bollerslev, Engle & Wooldridge (1988) to be positive definite. In Section 4 the BEKK-ARCH model is interpreted as the vector random autoregressive model. Sections 5 and 6 contain a discussion of generalizing the condition to the vector GARCH model. In Section 7 the usefulness of the theory is demonstrated by a small empirical example. Finally, the conclusions can be found in Section 8.

2. Vector random coefficient autoregressive and vector ARCH models

Consider the following vector autoregressive model

$$\mathbf{y}_t = \boldsymbol{\mu} + \sum_{j=1}^{q} \mathbf{B}_j \mathbf{y}_{t-j} + \boldsymbol{\varepsilon}_t$$

where $\{\boldsymbol{\varepsilon}_t\}$ is a martingale difference sequence of $m \times 1$ random vectors with respect to the increasing set of sigma fields $\mathcal{F}_{t-1} = \sigma(\boldsymbol{\varepsilon}_{t-1}, \boldsymbol{\varepsilon}_{t-2}, ...)$.Assume, furthermore, that the error process

$$\boldsymbol{\varepsilon}_t = \sum_{j=1}^{q} \boldsymbol{\Phi}_{jt} \boldsymbol{\varepsilon}_{t-j} + \boldsymbol{\eta}_t \tag{1}$$

where $\{\boldsymbol{\Phi}_{jt}\} = \{[\phi_{jikt}]\}$ is a sequence of $m \times m$ independent random matrices such that $\mathsf{E}\boldsymbol{\Phi}_{jt} = \mathbf{0}, j = 1, ..., q$. Setting $\boldsymbol{\phi}_{jt} =$vec$(\boldsymbol{\Phi}'_{jt})$ it is assumed that $\{\boldsymbol{\phi}_{jt}\}$ is a sequence of independent identically distributed random vectors such that $\mathsf{E}\boldsymbol{\phi}_{jt}\boldsymbol{\phi}'_{jt} = \boldsymbol{\Sigma}_j = [\sigma_{j,ik,ln}]\,,\ j = 1, ..., q$. In particular, we denote $\sigma_{j,ik,ik} = \sigma^2_{j,ik}$. It is assumed that these covariance matrices are positive definite. Random vectors $\boldsymbol{\phi}_{it}$ and $\boldsymbol{\phi}_{jt}$, $i \neq j$, are assumed independent for every t. The ith equation of (1) is

$$\varepsilon_{it} = \sum_{j=1}^{q} \sum_{k=1}^{m} \phi_{jikt} \varepsilon_{k,t-j} + \eta_{it}, i = 1, ..., m. \tag{2}$$

Model (1) is a special case of a more general random coefficient model in Wong & Li (1997). It is a multivariate counterpart of the model considered in Bera et al. (1992). In the model of Wong and Li, among other things, $\boldsymbol{\phi}_{it}$ and $\boldsymbol{\phi}_{jt}$, $i \neq j$, are not independent. The conditional heteroskedasticity counterparts of that model are more general than vector ARCH ones considered here.

The conditional variances of the elements of $\boldsymbol{\varepsilon}_t$ with respect to $\mathcal{F}_{t-1}$ can now be defined as follows. Let $\mathbf{H}_t = [h_{ijt}]$ be the $m \times m$ conditional covariance matrix of $\boldsymbol{\varepsilon}_t$. Then, from (1) it follows that

$$\begin{aligned} \mathbf{H}_t &= \mathsf{E}\boldsymbol{\eta}_t\boldsymbol{\eta}'_t + \sum_{j=1}^{q} \mathsf{E}\{\boldsymbol{\Phi}_{jt}\boldsymbol{\varepsilon}_{t-j}\boldsymbol{\varepsilon}'_{t-j}\boldsymbol{\Phi}'_{jt}|\mathcal{F}_{t-1}\} \\ &= \mathsf{E}\boldsymbol{\eta}_t\boldsymbol{\eta}'_t + \sum_{j=1}^{q} (\mathbf{I}_m \otimes \boldsymbol{\varepsilon}'_{t-j})\boldsymbol{\Sigma}_j(\mathbf{I}_m \otimes \boldsymbol{\varepsilon}_{t-j}) \end{aligned} \tag{3}$$

where $\mathsf{E}\boldsymbol{\eta}_t\boldsymbol{\eta}_t' = [\omega_{ij}]$. Assumption $\mathsf{E}\boldsymbol{\phi}_{it}\boldsymbol{\phi}_{jt}' = \mathbf{0}, i \neq j$, implies $\mathsf{E}\{\boldsymbol{\Phi}_{it}\boldsymbol{\varepsilon}_{t-i}\boldsymbol{\varepsilon}_{t-j}'\boldsymbol{\Phi}_{jt}'|\mathcal{F}_{t-1}\} = \mathbf{0}$, $i \neq j$, in (3). For example, when $m = 2$ in (1), the elements of $\mathbf{H}_t$ have the following form:

$$
\begin{aligned}
h_{1t} &= \mathsf{E}(\varepsilon_{1t}^2|\mathcal{F}_{t-1}) = \omega_{11} + \sum_{j=1}^{q}(\sigma_{j,11}^2\varepsilon_{1,t-j}^2 + \sigma_{j,12}^2\varepsilon_{2,t-j}^2 + 2\sigma_{j,11,12}\varepsilon_{1,t-j}\varepsilon_{2,t-j}) \\
h_{12t} &= \mathsf{E}(\varepsilon_{1t}\varepsilon_{2t}|\mathcal{F}_{t-1}) = \omega_{12} + \sum_{j=1}^{q}\{\sigma_{j,11,21}\varepsilon_{1,t-j}^2 + \sigma_{j,12,22}\varepsilon_{2,t-j}^2 \\
&\quad +(\sigma_{j,11,22} + \sigma_{j,12,21})\varepsilon_{1,t-j}\varepsilon_{2,t-j}\}. \\
h_{2t} &= \mathsf{E}(\varepsilon_{1t}^2|\mathcal{F}_{t-1}) = \omega_{22} + \sum_{j=1}^{q}(\sigma_{j,21}^2\varepsilon_{1,t-j}^2 + \sigma_{j,22}^2\varepsilon_{2,t-j}^2 + 2\sigma_{j,21,22}\varepsilon_{1,t-j}\varepsilon_{2,t-j})
\end{aligned} \tag{4}
$$

where ω_{ij} =cov$(\eta_{it}, \eta_{jt}), i, j = 1, 2$. Assuming the elements of $\boldsymbol{\phi}_{jt}$ independent for all j leads to $h_{12t} = \omega_{12}$, and the corresponding model could be called the "constant conditional covariance" model. Assuming, furthermore, that $\omega_{12} = 0$ gives the result that ε_{1t} and ε_{2t} are conditionally independent given $\mathcal{F}_{t-1}$.

Writing (3) in vec form one obtains

$$
\text{vec}(\mathbf{H}_t) = \text{vec}(\mathsf{E}\boldsymbol{\eta}_t\boldsymbol{\eta}_t') + \sum_{j=1}^{q}\mathsf{E}(\boldsymbol{\Phi}_{jt}\otimes\boldsymbol{\Phi}_{jt})\text{vec}(\boldsymbol{\varepsilon}_{t-j}\boldsymbol{\varepsilon}_{t-j}'). \tag{5}
$$

On the other hand, Bollerslev et al. (1988) defined a multivariate GARCH model in which all conditional variances and covariances have their own equations. The ARCH version of their model, which we shall call BEW-ARCH for brevity, can be written in the vector form as follows:

$$
\text{vech}(\mathbf{H}_t) = \boldsymbol{\omega} + \sum_{j=1}^{q}\mathbf{A}_j\text{vech}(\boldsymbol{\varepsilon}_{t-j}\boldsymbol{\varepsilon}_{t-j}'). \tag{6}
$$

where $\mathbf{A}_j = [a_{jik}]$, $j = 1, ..., q$, and $\boldsymbol{\omega} = (\omega_1, ..., \omega_{m(m+1)/2})'$. As an example, let

$m = 2$. Then the equations in (6) are

$$\begin{aligned} h_{1t} &= \omega_1 + \sum_{j=1}^{q}(a_{j11}\varepsilon_{1,t-j}^2 + a_{j12}\varepsilon_{1,t-j}\varepsilon_{2,t-j} + a_{j13}\varepsilon_{2,t-j}^2) \\ h_{12t} &= \omega_2 + \sum_{j=1}^{q}(a_{j21}\varepsilon_{1,t-j}^2 + a_{j22}\varepsilon_{1,t-j}\varepsilon_{2,t-j} + a_{j23}\varepsilon_{2,t-j}^2) \\ h_{2t} &= \omega_3 + \sum_{j=1}^{q}(a_{j31}\varepsilon_{1,t-j}^2 + a_{j32}\varepsilon_{1,t-j}\varepsilon_{2,t-j} + a_{j33}\varepsilon_{2,t-j}^2). \end{aligned} \quad (7)$$

A comparison of (5) and (6) suggests that their is an analogy between the VRCAR model and the BEW-ARCH model: the latter is a special case of the former, obtained by imposing parameter restrictions on the covariances of the random coefficients in the VRCAR model. A comparison of (4) and (7) constitutes an illuminating example.

3. Condition for the conditional covariance matrix of the BEW-ARCH model to be positive definite

3.1. General case

In practice, it has turned out to be difficult to find conditions for the conditional covariance matrix $\mathbf{H}_t$ of the BEW-GARCH model to be positive definite almost surely. The special case in which the BEW-GARCH model is diagonal is easier to handle and has been discussed in Bollerslev et al. (1994). The usefulness of the analogy between the VCRAR model and the BEW-ARCH one lies in the fact that it can be used for finding such conditions for the latter model. Corresponding conditions for the BEW-GARCH model can then in principle be obtained through an infinite-order BEW-ARCH model. This will be discussed in Section 5.

In what follows we shall call "positive definiteness" of a square matrix "positivity" for short and shall also use the term "positivity conditions" for conditions of a square matrix to be positive definite. We begin by formulating the following proposition:

Proposition 1. *Consider model* (1) *with conditional covariance matrix* (3) *. The conditional covariance matrix is positive definite if* $\mathsf{E}\boldsymbol{\eta}_t\boldsymbol{\eta}_t'$ *is a positive definite matrix and the covariance matrices* $\boldsymbol{\Sigma}_j$, $j = 1, ..., q$, *are at least positive semidefinite.*

Alternatively, (1) *is positive definite if* $\mathsf{E}\boldsymbol{\eta}_t\boldsymbol{\eta}_t'$ *is positive semidefinite and at least one of matrices* $\boldsymbol{\Sigma}_j$, $j = 1, ..., q$, *is positive definite, whereas the remaining ones are at least positive semidefinite.*

In order to apply the proposition, it is necessary to find the connection between the covariance matrix $\boldsymbol{\Sigma}_j$ and the coefficient matrix $\mathbf{A}_j$ in (6) . For this purpose we express (6) in vec form. This is done by defining the $m^2 \times m(m+1)/2$ duplication matrix $\mathbf{G}_m$ of rank $m(m+1)/2$ such that $\mathrm{vec}(\mathbf{C}) = \mathbf{G}_m\mathrm{vech}(\mathbf{C})$. Furthermore, let $\mathbf{L}_m$ be an $m(m+1)/2 \times m^2$ left-inverse of $\mathbf{G}_m$, that is, $\mathbf{L}_m\mathbf{G}_m = \mathbf{I}_{m(m+1)/2}$. We choose $\mathbf{L}_m = (\mathbf{G}_m'\mathbf{G}_m)^{-1}\mathbf{G}_m'$ (see, for example, Harville (1997), pp. 352-354). Then (6) can be written in vec form as

$$\begin{aligned}\mathrm{vec}(\mathbf{H}_t) &= \mathbf{G}_m\boldsymbol{\omega} + \sum_{j=1}^{q}\mathbf{G}_m\mathbf{A}_j\mathbf{L}_m\mathbf{G}_m\mathrm{vech}(\boldsymbol{\varepsilon}_{t-j}\boldsymbol{\varepsilon}_{t-j}') \\ &= \mathbf{G}_m\boldsymbol{\omega} + \sum_{j=1}^{q}\mathbf{G}_m\mathbf{A}_j(\mathbf{G}_m'\mathbf{G}_m)^{-1}\mathbf{G}_m'\mathrm{vec}(\boldsymbol{\varepsilon}_{t-j}\boldsymbol{\varepsilon}_{t-j}'). \qquad (8)\end{aligned}$$

By comparing (5) and (8) it follows that

$$\mathsf{E}(\boldsymbol{\Phi}_{jt} \otimes \boldsymbol{\Phi}_{jt}) = \mathbf{G}_m\mathbf{A}_j(\mathbf{G}_m'\mathbf{G}_m)^{-1}\mathbf{G}_m'. \qquad (9)$$

We have now established a link between the variances and covariances in $\mathsf{E}(\boldsymbol{\Phi}_{jt} \otimes \boldsymbol{\Phi}_{jt})$ and the elements of $\mathbf{A}_j$. It is seen from (9) that the BEW-GARCH model imposes parameter restrictions on $\mathsf{E}(\boldsymbol{\Phi}_{jt} \otimes \boldsymbol{\Phi}_{jt})$. The difference in the number of parameters in $\mathsf{E}(\boldsymbol{\Phi}_{jt} \otimes \boldsymbol{\Phi}_{jt})$ and $\mathbf{A}_j$ equals $\{m(m-1)\}^2/4$.

Next, we observe that $\boldsymbol{\Sigma}_j$ and $\mathsf{E}(\boldsymbol{\Phi}_{jt} \otimes \boldsymbol{\Phi}_{jt})$ contain exactly the same elements. Omitting subscript j, write the $m^2 \times m^2$ matrix

$$\mathsf{E}(\boldsymbol{\Phi}_t \otimes \boldsymbol{\Phi}_t) = \begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} & ... & \mathbf{F}_{mm} \\ \mathbf{F}_{21} & \mathbf{F}_{22} & ... & \mathbf{F}_{2m} \\ ... & & & \\ \mathbf{F}_{m1} & \mathbf{F}_{m2} & ... & \mathbf{F}_{mm} \end{bmatrix}$$

where all submatrices are $m \times m$. Let $\mathcal{T}$ be the operator that permutes rows of $\mathbf{F}_{ii}$ with rows of $\mathbf{F}_{ki}$, $i \neq k$, in such a way that the kth row of $\mathbf{F}_{ii}$ is permuted with the ith row of $\mathbf{F}_{ki}$, $k = 1, ..., i-1, i+1, ..., m$. Using this notation, we obtain

$$\boldsymbol{\Sigma}_j = \mathcal{T}(\mathbf{G}_m\mathbf{A}_j(\mathbf{G}_m'\mathbf{G}_m)^{-1}\mathbf{G}_m').$$

As an example, for $m = 2$, we have

$$\boldsymbol{\Sigma}_j = \begin{bmatrix} \alpha_{j11} & \alpha_{j12}/2 & \alpha_{j21} & \alpha_{j22}/2 \\ \alpha_{j12}/2 & \alpha_{j13} & \alpha_{j22}/2 & \alpha_{j23} \\ \alpha_{j21} & \alpha_{j22}/2 & \alpha_{j31} & \alpha_{j32}/2 \\ \alpha_{j22}/2 & \alpha_{j23} & \alpha_{j32}/2 & \alpha_{j33} \end{bmatrix}, j = 1, ..., q. \tag{10}$$

It is seen from (10) that the restrictions imposed on the non-diagonal elements of $\boldsymbol{\Sigma}_j$ for (4) to be a genuine VGARCH model of order two equals $\sigma_{j,11,22} = \sigma_{j,12,21}$. The requirement that $\boldsymbol{\Sigma}_j \geq 0$ further restricts the values of the parameters. Explicit expressions for the positivity conditions as functions of the elements of $\mathbf{A}_j$ are obtained by applying the result that the principal minors of a positive semidefinite matrix have to be at least positive semidefinite, but they are hardly particularly intuitive. Nevertheless, (10) can be used for checking numerically whether or not an estimated BEW-ARCH model satisfies these conditions.

Furthermore, as already discussed, the intercepts in also have to satisfy certain conditions such that $\mathsf{E}\boldsymbol{\eta}_t\boldsymbol{\eta}_t' \geq 0$. For $m = 2$, this condition is $\omega_1\omega_3 \geq \omega_2^2$, $\omega_1, \omega_3 > 0$, in (7).

3.2. Diagonal model

Bollerslev et al. (1988) also suggested a parsimonious version of their model by assuming that $\mathbf{A}_j, j = 1, ..., q$, be diagonal matrices. The counterpart of this assumption in the VRCAR framework is that $\boldsymbol{\Phi}_{jt}$ =diag$(\phi_{j,11,t}, ..., \phi_{j,mm,t})', j = 1, ..., q$, in (1). Then, dropping the subscript j, it follows from (1) that

$$\mathsf{E}(\boldsymbol{\Phi}_t \otimes \boldsymbol{\Phi}_t) = \text{diag}(\sigma_{11}^2, \sigma_{11,22}, ..., \sigma_{11,mm}, \sigma_{22,11}, \sigma_{22}^2, ..., \sigma_{22,mm}, ..., \sigma_{mm}^2). \tag{11}$$

There now exists a one-to-one correspondence between the diagonal elements of (11) and the ones of $\mathbf{A}_j$: $\sigma_{11}^2 = \alpha_{11}, \sigma_{11,22} = \alpha_{22}, ..., \sigma_{mm}^2 = \alpha_{m(m+1)/2,m(m+1)/2}$. Using the permutation operator $\mathcal{T}$ allows one to find the relevant positivity conditions. As an example, let $m = 2$. Then, $\mathsf{E}(\boldsymbol{\Phi}_t \otimes \boldsymbol{\Phi}_t)$ =diag$(\sigma_{11}^2, \sigma_{1122}, \sigma_{2211}, \sigma_{22}^2)$, so that

$$\mathcal{T}(\mathsf{E}(\boldsymbol{\Phi}_t \otimes \boldsymbol{\Phi}_t)) = \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11}^2 & 0 & 0 & \sigma_{11,22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sigma_{11,22} & 0 & 0 & \sigma_{22}^2 \end{bmatrix} = \begin{bmatrix} \alpha_{j11} & 0 & 0 & \alpha_{j22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha_{j22} & 0 & 0 & \alpha_{j33} \end{bmatrix}. \tag{12}$$

Removing the zero rows and columns corresponding to the nondiagonal random variables ϕ_{12t} and ϕ_{21t} that are identically zero leads to

$$\overline{\boldsymbol{\Sigma}}_j = \mathsf{E}\boldsymbol{\phi}_{jt}\boldsymbol{\phi}_{jt}' = \begin{bmatrix} \alpha_{j11} & \alpha_{j22} \\ \alpha_{j22} & \alpha_{j33} \end{bmatrix}, j = 1, ..., q. \tag{13}$$

Matrices (12) and (13) have the same nonzero eigenvalues. It follows from (13) that the positivity condition for the conditional variances requires $a_{j11}a_{j33} \geq a_{j22}^2$; $a_{j11}, a_{j33} > 0, j = 1, ...q$. The condition involving the intercepts is not affected by the diagonality assumption.

A triangular model forms an intermediate case between the "full" and the diagonal model. In an upper triangular vector ARCH model, there is no volatility feedback from $\varepsilon_{i,t-j}$ to ε_{kt} for $k > i$. As an example, let $m = 2$, so that $\phi_{j21t} \equiv 0$, $j = 1, ..., q$. Then the relevant covariance matrices

$$\ddot{\boldsymbol{\Sigma}}_j = \begin{bmatrix} \alpha_{j11} & \alpha_{j12}/2 & \alpha_{j22} \\ \alpha_{j12}/2 & \alpha_{j13} & \alpha_{j23} \\ \alpha_{j22} & \alpha_{j23} & \alpha_{j33} \end{bmatrix}, j = 1, ..., q.$$

4. BEKK-ARCH model as a random coefficient autoregressive model

As the BEW-ARCH model contains a large number of parameters, more parsimonious specifications have been considered in the literature. The BEKK-GARCH model is probably the best known example of such a specification. The corresponding BEKK-ARCH(q) model has the conditional variance

$$\mathbf{H}_t = (\mathbf{C}_0^*)'\mathbf{C}_0^* + \sum_{k=1}^{K}\sum_{j=1}^{q}(\mathbf{A}_{kj}^*)'\boldsymbol{\varepsilon}_{t-j}\boldsymbol{\varepsilon}_{t-j}'\mathbf{A}_{kj}^* \tag{14}$$

or

$$\mathrm{vec}(\mathbf{H}_t) = \mathrm{vec}(\mathbf{C}_0^*)'\mathbf{C}_0^* + \sum_{k=1}^{K}\sum_{j=1}^{q}(\mathbf{A}_{kj}^* \otimes \mathbf{A}_{kj}^*)'\mathrm{vec}(\boldsymbol{\varepsilon}_{t-j}\boldsymbol{\varepsilon}_{t-j}') \tag{15}$$

where $\mathbf{C}_0^*$ is a nonsingular matrix. As an example consider the case $m = 2$ and $K = 1$ and write $\mathbf{C}_0^* = [\mathbf{c}_{01}\ \mathbf{c}_{02}]$. Then the conditional variances and covariances in (14) are

$$
\begin{aligned}
h_{1t} &= \mathbf{c}_{01}'\mathbf{c}_{01} + \sum_{j=1}^{q}(a_{j11}\varepsilon_{1,t-j} + a_{j21}\varepsilon_{2,t-j})^2 \\
h_{12t} &= \mathbf{c}_{01}'\mathbf{c}_{02} + \sum_{j=1}^{q}(a_{j11}\varepsilon_{1,t-j} + a_{j21}\varepsilon_{2,t-j})(a_{j12}\varepsilon_{1,t-j} + a_{j22}\varepsilon_{2,t-j}) \\
h_{2t} &= \mathbf{c}_{02}'\mathbf{c}_{02} + \sum_{j=1}^{q}(a_{j12}\varepsilon_{1,t-j} + a_{j22}\varepsilon_{2,t-j})^2. \qquad (16)
\end{aligned}
$$

The analogy between the VRCAR model and the BEKK-ARCH model when $K > 1$ can now be worked out as follows. First note that (15) is already in vec form so that $\mathbf{G}_m = \mathbf{I}$. Furthermore, $\mathcal{T}(\mathbf{A}_{kj}^* \otimes \mathbf{A}_{kj}^*) = \mathbf{a}^*\mathbf{a}^{*\prime}$ where $\mathbf{a}^* = (a_{11}^*, ...a_{1m}^*, a_{21}^*, ..., a_{2m}^*, ..., a_{mm}^*)'$. The covariance matrix $\mathbf{\Sigma}_j = \mathbf{a}^*\mathbf{a}^{*\prime}$ is thus of rank one, so that the random coefficients are linearly dependent. In fact, we have $\mathbf{\Phi}_{jt} = \phi_{jt}\widetilde{\mathbf{A}}_j', j = 1, ..., q$, where $\{\phi_{jt}\}$ is a sequence of independent random variables with mean zero and unit variance.

Variable ε_t is thus driven by fixed linear combinations of $\varepsilon_{1,t-j}$, ..., $\varepsilon_{m,t-j}$, $j = 1, ..., q$, that move in unison in the sense that both the sign and the size of their impact on ε_t are determined by a single random variable, and a disturbance vector $\boldsymbol{\varepsilon}_t$. We consequently have the following VRCAR model for the BEKK-ARCH(q) model:

$$
\boldsymbol{\varepsilon}_t = \sum_{j=1}^{q} \phi_{jt}\widetilde{\mathbf{A}}_{1j}'\boldsymbol{\varepsilon}_{t-j} + \boldsymbol{\eta}_t. \qquad (17)
$$

The assumptions satisfied by ϕ_{jt} are the same as the ones for $\mathbf{\Phi}_{jt}$ in (1) , completed by the extra (identifying) restriction on the variances of ϕ_{jt}. We may assume $\text{var}(\phi_{jt}) = 1, j = 1, ..., q$.

When $K > 1$, (17) becomes

$$
\boldsymbol{\varepsilon}_t = \sum_{k=1}^{K}\sum_{j=1}^{q} \phi_{jkt}\widetilde{\mathbf{A}}_{kj}'\boldsymbol{\varepsilon}_{t-j} + \boldsymbol{\eta}_t. \qquad (18)
$$

It is seen from (18) that there now exist K fixed linear combinations of $\varepsilon_{1,t-j}$, ..., $\varepsilon_{m,t-j}$ for each lag, and their weights are drawn randomly from the distributions of $\phi_{jkt}, k = 1, ..., K$. The combinations $\phi_{jkt}\widetilde{\mathbf{A}}_{kj}', k = 1, ..., K$, are exchangeable, and restrictions on the elements of $\widetilde{\mathbf{A}}_{kj}'$ are required for identification. They are discussed in Engle & Kroner (1995). In general, setting $K > 1$ increases the flexibility of the BEKK-ARCH model and brings it closer to the BEW-ARCH model of Bollerslev et al. (1988). For more discussion of this in the context of the BEW-GARCH model, see Engle & Kroner (1995).

5. Infinite-order vector RCAR and vector ARCH models

Model (1) becomes an infinite order vector ARCH model as the number of lags $q \to \infty$. These models nest vector GARCH models, but as in the univariate case, finding the appropriate parameter restrictions may be very difficult. Kokoszka & Leipus (2000) derived a stationary solution for a univariate infinite-order ARCH model under the condition that $\mathsf{E}z_t^4 < \infty$. The corresponding result for the multivariate model does not seem to be available. It is seen from (1) , however, that existence conditions for the unconditional variance of ε_t depend on the covariance structure of $\boldsymbol{\phi}_{jt}$ $=\{\text{vech}(\boldsymbol{\Phi}_{jt})\}$. Let $\mathsf{E}\boldsymbol{\phi}_{jt}\boldsymbol{\phi}'_{jt} = \boldsymbol{\Sigma}_j = [\sigma_{j,il,kn}]$. A necessary condition for weak stationarity is that the sequences $\{\sigma_{j,ii,ii}\}_{j=1}^{\infty}$ are summable for $i = 1, ..., m$. This means that the variances of the variables in the sequence $\{\boldsymbol{\phi}_{jt}\}_{j=1}^{M}$ have to converge to zero sufficiently fast as $M \to \infty$. For example, for $m = 2$ this implies $\lim_{M\to\infty} \sum_{j=1}^{M} \alpha_{jik} < \infty$, $i, k = 1, 3$.

6. The generalized ARCH model

The conditions we have derived can in principle be generalized to the GARCH model by applying an infinite-order ARCH model. However, we shall only discuss the first-order case in detail because the first-order GARCH model is the one normally used in applications. Furthermore, the algebra needed for treating higher-order cases would be quite tedious. Nelson & Cao (1992) solved the univariate higher-order case. Consider the infinite-order ARCH model

$$\text{vech}(\mathbf{H}_t) = \boldsymbol{\omega} + \mathbf{C}(L)\text{vech}(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}'_t) \tag{19}$$

where $\boldsymbol{\Gamma}(L) = \sum_{j=1}^{\infty} \boldsymbol{\Gamma}_j L^j$ and L is the lag operator. Assume that

$$\boldsymbol{\Gamma}(L) = \sum_{j=1}^{\infty} \boldsymbol{\Gamma}_j L^j = \mathbf{B}(L)^{-1}\mathbf{A}(L) \tag{20}$$

where $\mathbf{A}(L) = \sum_{j=1}^{q} \mathbf{A}_j L^j$, and $\mathbf{B}(L) = \mathbf{I} - \sum_{j=1}^{p} \mathbf{B}_j L^j$ such that $|\mathbf{I} - \sum_{j=1}^{p} \mathbf{B}_j z^j| \neq 0$ for $|z| \leq 1$. Then (19) and (20) define a general GARCH(p, q) model. In principle, the positivity conditions of the conditional variances for this model can be derived by converting the model into an infinite-order ARCH model and using the theory presented in Section 2.

In the GARCH(1,1) case, a necessary condition for the existence of the conditional variance is that the eigenvalues of $\mathbf{B}_1$ have modulus less than one and, furthermore, that the coefficients of the variances $h_{ii,t-1}$ are positive. Then the infinite ARCH representation for this model exists and has the form

$$\text{vech}(\mathbf{H}_t) = (\mathbf{I} - \mathbf{B}_1)^{-1}\boldsymbol{\omega} + \sum_{j=1}^{\infty} \mathbf{B}_1^{j-1}\mathbf{A}_1 \text{vech}(\boldsymbol{\varepsilon}_{t-j}\boldsymbol{\varepsilon}_{t-j}'). \tag{21}$$

The condition for positive definiteness of the conditional covariance matrix involves $\mathbf{A}_1$ and the matrix products $\mathbf{B}_1^j \mathbf{A}_1, j = 1, 2, ...,$ such that

$$\boldsymbol{\Sigma}_j = \mathcal{T}(\mathbf{G}_m \mathbf{B}_1^{j-1} \mathbf{A}_j (\mathbf{G}_m' \mathbf{G}_m)^{-1} \mathbf{G}_m'), j = 1, 2, ... \,. \tag{22}$$

These conditions become quite complicated already in the simplest bivariate case but for an estimated model their validity can be verified numerically using (22), at least up to some finite j. The diagonal GARCH(1,1) model, however, is an exception to this rule. As an example, let $m = 2$. Then

$$\overline{\boldsymbol{\Sigma}}_j = \begin{bmatrix} \alpha_{11}\beta_{11}^{j-1} & \alpha_{22}\beta_{22}^{j-1} \\ \alpha_{22}\beta_{22}^{j-1} & \alpha_{33}\beta_{33}^{j-1} \end{bmatrix} = \overline{\mathbf{B}}^{(j-1)} \odot \overline{\mathbf{A}}, j \geq 1 \tag{23}$$

where $\overline{\mathbf{A}} = \begin{bmatrix} \alpha_{11} & \alpha_{22} \\ \alpha_{22} & \alpha_{33} \end{bmatrix}$, $\overline{\mathbf{B}}^{(j-1)} = \begin{bmatrix} \beta_{11}^{j-1} & \beta_{22}^{j-1} \\ \beta_{22}^{j-1} & \beta_{33}^{j-1} \end{bmatrix}$ and $\odot$ denotes the Hadamard product (element-by-element multiplication). According to the Schur product theorem, see, for example, Horn & Johnson (1985), p. 458, if two matrices $\mathbf{A}$ and $\mathbf{B}$ are positive semidefinite then $\mathbf{A} \odot \mathbf{B}$ is positive semidefinite. Setting $j = 1$ in (23) yields condition $\alpha_{11}\alpha_{33} - \alpha_{22}^2 \geq 0$, whereas applying the Schur product theorem to $j = 2$ while assuming $\alpha_{11}\alpha_{33} - \alpha_{22}^2 \geq 0$ leads to $\beta_{11}\beta_{33} - \beta_{22}^2 \geq 0$.As the existence of the infinite-order ARCH representation requires $0 < \beta_{ii} < 1, i = 1, 3,$ it follows that $\beta_{11}^j\beta_{33}^j - \beta_{22}^{2j} > 0, j \geq 1$. Thus the necessary and sufficient conditions are $\alpha_{11}\alpha_{33} \geq \alpha_{22}^2$ and $\beta_{11}\beta_{33} > \beta_{22}^2$. This argument generalizes to the VGARCH model with m variables, $m > 2$.

Likewise, as $(\mathbf{I}-\mathbf{B}_1)^{-1}\boldsymbol{\omega} = \sum_{j=1}^{\infty}\mathbf{B}^{j-1}\boldsymbol{\omega}$ it follows from (21) that

$$\mathsf{E}\boldsymbol{\eta}_t\boldsymbol{\eta}_t' = \sum_{j=1}^{\infty}\mathbf{B}^{(j-1)} \odot \boldsymbol{\Omega} \tag{24}$$

where $\boldsymbol{\Omega} = \begin{bmatrix} \omega_{11} & \omega_{22} \\ \omega_{22} & \omega_{33} \end{bmatrix}$. Thus, the additional condition for positivity, following from (24), is that $\omega_{11}\omega_{33} \geq \omega_{22}^2$. Even this result generalizes to the case $m > 2$.

The higher-order diagonal models are also relatively simple to handle because the inverse $\mathbf{B}(L)^{-1}$ is a diagonal matrix polynomial. Bollerslev et al. (1994) derived, also using Hadamard products but in a different way, conditions for the conditional covariance matrix in the diagonal VGARCH model to be positive definite. Their considerations did not extend to the nondiagonal model.

7. Example

As an example, consider the application in Bollerslev et al. (1988). The authors fit a three-variable diagonal GARCH(1,1) model to a trivariate CAPM. We focus on the conditional variance process. Using the parameter estimates for the intercept vector one obtains

$$\widehat{\mathsf{E}\boldsymbol{\eta}_t\boldsymbol{\eta}_t'} = \begin{bmatrix} .026 & .438 & .013 \\ & 23.8 & 3.82 \\ & & 3.92 \end{bmatrix}$$

As $\det\widehat{\mathsf{E}\boldsymbol{\eta}_t\boldsymbol{\eta}_t'} = 1.33$, the intercept condition is satisfied. On the other hand,

$$\widehat{\boldsymbol{\Sigma}}_1 = \begin{bmatrix} .445 & .233 & .197 \\ & .188 & .165 \\ & & .078 \end{bmatrix}$$

and $\det(\widehat{\boldsymbol{\Sigma}}_1) = -0.002$ (or, the smallest eigenvalue of $\widehat{\boldsymbol{\Sigma}}_1$ equals -0.043). The estimated model thus has a positive probability of generating indefinite conditional covariance matrices. This does not necessarily mean abandoning the assumption that a trivariate diagonal GARCH model has generated the observations, because due to short series the parameter estimates are rather uncertain. However, if the estimated model is used for forecasting volatility, it may generate invalid forecasts for conditional variances and covariances.

8. Conclusions

While several multivariate GARCH models have been designed to have a positive definite conditional covariance matrix, the general multivariate GARCH model is not one of them. The necessary and sufficient conditions for the conditional covariance matrix of this model to be positive definite almost surely, derived in this paper, fill a void in the literature. The results are useful in checking whether or not an estimated multivariate ARCH or GARCH model possesses this property. This information is important when the model is used for forecasting volatility.

References

Bera, A. K., Higgins, M. L. & Lee, S. (1992). Interaction between autocorrelation and conditional heteroscedasticity: A random-coefficient approach, *Journal of Business and Economic Statistics* **10**: 133–142.

Bollerslev, T., Engle, R. F. & Nelson, D. B. (1994). ARCH models, *in* R. F. Engle & D. L. McFadden (eds), *Handbook of Econometrics*, Vol. 4, North-Holland, Amsterdam, pp. 2959–3038.

Bollerslev, T., Engle, R. F. & Wooldridge, J. (1988). A capital asset-pricing model with time-varying covariances, *Journal of Political Economy* **96**: 116–131.

Engle, R. F. & Kroner, K. F. (1995). Multivariate simultaneous generalized ARCH, *Econometric Theory* **11**: 122–150.

Gouriéroux, C. (1996). *ARCH Models and Financial Applications*, Springer, Berlin.

Harville, D. A. (1997). *Matrix Algebra from a Statistician's Perspective*, Springer, New York.

Horn, R. A. & Johnson, C. A. (1985). *Matrix Analysis*, Cambridge University Press, Cambridge.

Kokoszka, P. & Leipus, R. (2000). Change-point estimation in ARCH models, *Bernoulli* **6**: 513–539.

Nelson, D. B. & Cao, C. Q. (1992). Inequality constraints in the univariate GARCH model, *Journal of Business and Economic Statistics* **10**: 229–235.

Wong, H. & Li, W. K. (1997). On a multivariate conditional heteroscedastic model, *Biometrika* **84**: 111–123.