Towards a General Theory of Good Deal Bounds *

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Abstract

We consider an incomplete market in the form of a multidimensional Markovian factor model, driven by a general marked point process (representing discrete jump events) as well as by a standard multidimensional Wiener process. Within this framework we study arbitrage free good deal pricing bounds for derivative assets along the lines of Cochrane and Saa-Requejo [4], extending the results from [4] to the point process case. As a concrete application we present numerical results for the classic Merton jump-diffusion model. As a by product of the general theory we also extend the Hansen-Jagannathan bounds [5] for the Sharpe Ratio to the point process setting.

Key words: Incomplete markets, good deal bounds, derivatives pricing.
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1 Introduction

Most realistic models of financial markets are by nature and construction highly incomplete. This holds for stochastic volatility models, models for energy and weather derivatives, as well as for stock price models driven by a nontrivial (marked) point process. Suppose now that we would like to compute an arbitrage free price process for a financial derivative within one of the model classes mentioned above. Then we are faced with the following well known facts.

- Since the underlying market is incomplete, there will not exists a unique martingale measure (or a unique stochastic discount factor). Thus there will exist infinitely many arbitrage free price processes for a given derivative.
- In incomplete settings like this, the pricing bounds provided by merely requiring absence of arbitrage are extremely wide, and such bounds are thus useless from a practical point of view.

There is thus a clear need for “reasonable” pricing bounds for derivative assets, and to this end Cochrane and Saa-Requejo introduced, in the seminal paper [4], the completely new idea of ruling out, not only those prices which are violating the no arbitrage restriction, but also those prices which in some sense would represent “deals which are too good”(henceforth referred to as “good deals”). Cochrane and Saa-Requejo formalized in [4] the idea of a good deal essentially as an asset price process with a high Sharpe ratio and posed the problem of finding the upper and lower bound for all arbitrage free price processes, given a bound on the Sharpe ratio of the derivative.

In [4] this problem was analyzed in great detail for the one-period, multi period, and also the continuous time setting. For the continuous time models, which we focus on in the present paper, the setting in [4] is that of a diffusion model driven by a multidimensional Wiener process, the technical language is that of stochastic discount factors, and the basic technique is dynamic programming. Within this framework Cochrane and Saa-Requejo derive a pricing PDE, which is then studied in detail and, in some cases, solved numerically.

For an interesting, but slightly different, view of good deal bounds, see [2] and [3].

A related approach to obtain asset price bounds, based on gains-loss-ratios, is presented in [1]. See [9] for an interesting connection of [1] to linear programming.

The main object of the present paper is to extend the analysis of [4] to allow also the possibility of jumps in the random processes describing the financial market under consideration. Thus; in the setup of the present paper all processes are allowed to be driven, not only by a multidimensional standard Wiener process, but also by a general marked point process (henceforth referred to as an “MPP”).

The structure of the paper is as follows.

- In Section 2 we present a very general probabilistic framework for the rest of the paper.
- In Section 3 we derive expressions for the risk premium, the total volatility, and the Sharpe ratio for an asset price process within the general framework of Section 2. We also provide an explicit representation of the class of equivalent martingale measures. The main result of the section is that we extend the Hansen-Jagannathan bounds from [5] to the general setup of Section 2. The HJ bounds provide an inequality for the Sharpe ratio in terms of the various market prices of risk, and this inequality is at the heart of the good deal pricing project.
- In Section 4.1, we present our basic factor market model. The model consists of a vector price process for traded assets as well as a random vector process describing non traded underlying factors. The dynamics are described in terms of a system of SDEs, driven by a vector Wiener process and an MPP.

- The pricing problem is formalized in Section 4.2, and in Section 4.3 we derive the fundamental Dynamic Programming Equation for the upper and lower good deal bounds. In Section 4.4 we discuss the special structure of this equation in some detail and also connect the good deal pricing bounds to the so called "minimal martingale measure".

- Section 5 is devoted to a fairly detailed study of some concrete point proces driven models. We start by analyzing the simple case of an asset price driven by a scalar Wiener process and a standard Poison process, and for this case we provide formulas for the optimal market prices of risk. We then extend the analysis to the case of a driving compound Poisson process, and also for this case we can provide a fairly explicit representation of the optimal market prices of risk. The classical Merton jump-diffusion model in [8] falls within this class, and we present numerical results for that particular model, where we can compare the Merton pricing formulas, as well as the pricing formula obtained by using the minimal martingale measure, with the good deal bounds.

- For completeness sake we finish the paper by studying, in Appendix A, the special case of a purely driven model, and show how the Cochrane and Saa-Requejo setup is nested within our framework.

From a more technical point of view, we note that the technique used in the present paper, as opposed to the one used in [4], is very much focused on martingale measures, rather than on stochastic discount factors. Since martingale measures and stochastic discount factors are mathematically equivalent, it is largely a matter of taste which method to use for any particular problem. However, for this class of problems the use of martingale measures is, in our opinion, to be preferred as a technical tool. Firstly, it allows us to draw upon the huge technical machinery of general martingale theory and, secondly, it streamlines the arguments considerably. In particular this can be seen in the fact that with the martingale formulation, the good deal pricing problem appears directly as a well formulated standard stochastic control problem. The relevant Bellman equation can thus be written down immediately, without any need of a separate argument.

2 General Setup

Our formal setup (see below for a more intuitive description) consists of a financial market on a fixed time interval $[0, T]$, living on a stochastic basis (filtered probability space) $(\Omega, \mathcal{F}, \mathbf{F}, P)$ where $\mathbf{F} = \{\mathcal{F}_t\}_{0 \le t \le T}$, where the measure P is interpreted as the objective (or "physical") probability measure. The basis is assumed to carry a d-dimensional standard Wiener process W as well as a marked point process $\mu(dt, dx)$ on a measurable Lusin mark space $(X, \mathcal{X})$. The predictable σ-algebra is denoted by $\mathcal{P}$, and we make the definition $\tilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{X}$. We assume that the predictable compensator $\nu(dt, dx)$ admits an intensity, i.e. that we can write $\nu(dt, dx) = \lambda_t(dx)dt$. The compensated point process $\mu(dt, dx) - \lambda_t(dx)dt$ is denoted by $\tilde{\mu}(dt, dx)$. We assume that $\nu([0, t] \times X) < \infty$ P-a.s. for all finite t, i.e. μ is a multivariate point process in the terminology of [6].

Remark 2.1 *The intuitive interpretation of the point process μ is that we are modeling "events" which are occuring at discrete points in time, and a very concrete example could be the modeling of earth quakes (or stock market crashes). As opposed to a more standard counting process setting, these discrete events are not all of the same type. Instead; every event is identified by it's "mark" $x \in X$. In the earth quake example, a natural mark would be the strength of the earthquake on the Richter scale, and in this case the mark space X would be the positive real line. The informal interpretation of the point process μ is that μ is an integer valued (random) measure such that μ has a unit point mass at the point $(t, x) \subseteq R_+ \times X$, if at time t there is an event of the type x. The interpretation of the intensity measure λ is loosely speaking that $\lambda_t(dx)$ is the expected number of events with marks in a "small set" dx, per unit time, conditional on the information in $\mathcal{F}_{t-}$. Thus the compensated point process $\tilde{\mu}(dt, dx)\mu(dt, dx) - \lambda_t(dx)dt$ is "detrended" and possesses a natural martingale property.*

3 Extended Hansen-Jagannathan Bounds

Before moving on to the main problem of pricing derivatives subject to a bound on the Sharpe ratio, we make a slight detour in order to derive an extension of the Hansen-Jagannathan bounds (see [5]) to the present point process setting. The HJ bounds, which will be needed below, provide an inequality for the Sharpe ratio of any traded asset (underlying or derivative) in terms of the "market prices of risk" of the driving random sources, and to make this idea precise we consider the arbitrage free price process S of an arbitrary asset (derivative or underlying) with P-dynamics given by

$$dS_t = S_t\alpha_t dt + S_t\sigma_t dW_t + S_{t-}\int_X \delta_t(x)\mu(dt, dx). \qquad (1)$$

Here α and σ are optional processes, whereas δ is predictable. In order to avoid negative asset prices we must also assume that $\delta_t(x) \geq -1$.

Remark 3.1 *The informal interpretation of the point process integral in (1) above is very simple: If there is an event at time t with mark x, then the stock price will have a jump with relative jump size given by $\delta_t(x)$.*

3.1 Risk Premium, Volatility, and the Sharpe Ratio

Compensating the point process μ in (1), we obtain the P-semimartingale dynamics of S as

$$dS_t = S_t\left\{\alpha_t + \int_X \delta_t(x)\lambda_t(dx)\right\}dt + S_t\sigma_t dW_t + S_{t-}\int_X \delta_t(x)\tilde{\mu}(dt, dx). \qquad (2)$$

Since the last two terms in this equation are martingale differentials, we see that the local mean rate of return under P is given by the expression

$$\alpha_t + \int_X \delta_t(x)\lambda_t(dx),$$

so, denoting the possibly stochastic short rate process by r, the **risk premium** process R is given by the formula

$$R_t = \alpha_t + \int_X \delta_t(x)\lambda_t(dx) - r_t. \tag{3}$$

We now go on to define the predictable (total) **volatility** process v, which intuitively should equal the conditional variance of the return of the stock price, i.e. it should roughly be given by the expression

$$v_t^2 dt = Var^P\left[\left.\frac{dS_t}{S_{t-}}\right|\mathcal{F}_{t-}\right]. \tag{4}$$

We need to make this notion mathematically precise and this is done by formally defining v through the relation

$$d\langle S, S\rangle_t = S_{t-}^2 v_t^2 dt, \tag{5}$$

where $\langle , \rangle$ denotes the usual predictable bracket process (see [6]). From (2) it is not hard to obtain

$$d\langle S, S\rangle_t = S_{t-}^2\left\{\|\sigma_t\|_{R^d}^2 + \int_X \delta_t^2(x)\lambda_t(dx)\right\}dt \tag{6}$$

so, by comparing (6) with (5), we see that the squared volatility process is given by

$$v_t^2 = \|\sigma_t\|_{R^d}^2 + \|\delta_t\|_{\lambda_t}^2, \tag{7}$$

where $\|\cdot\|_{\lambda_t}$ denotes the norm in the Hilbert space $L^2[X, \lambda_t(dx)]$. We can also, for future use, express the volatility v as

$$v_t = \|(\sigma_t, \delta_t)\|_{\mathcal{H}}, \tag{8}$$

where we view (σ_t, δ_t) as a vector in the Hilbert space $\mathcal{H} = R^d \times L^2[X, \lambda_t(dx)]$. We finally define the **Sharpe Ratio** process SR by

$$SR_t = \frac{R_t}{v_t}, \tag{9}$$

and our goal is to derive an inequality for the Sharpe Ratio in terms of the set of market prices of risk which turn up in connection with the class of risk neutral martingale measures Q. To this end we go on to study the class of equivalent martingale measures, but first we summarize our findings.

Proposition 3.1 *For a price process of the form (1) the following hold.*

1. *The risk premium is given by*

$$R_t = \alpha_t + \int_X \delta_t(x)\lambda_t(dx) - r_t. \tag{10}$$

2. *The (squared) total volatility is given by*

$$v_t^2 = \|\sigma_t\|^2_{R^d} + \|\delta_t\|^2_{\lambda_t}, \tag{11}$$

which also can be written

$$v_t^2 = \|\sigma_t\|^2_{R^d} + \int_X \delta_t^2(x)\lambda_t(dx). \tag{12}$$

3.2 Equivalent Martingale Measures

Given the process S above we now search for an equivalent martingale measure Q, and for any Q equivalent to P (martingale measure or not) we define the the likelihood process L by

$$\frac{dQ}{dP} = L_t, \quad \text{on } \mathcal{F}_t; \quad 0 \le t \le T. \tag{13}$$

Since L is always a P-martingale, and since every martingale within the present framework admits a stochastic integral representation (see [6]) we know that L must have dynamics of the form

$$\begin{cases} dL_t &= L_t h_t^\star dW_t + L_{t-} \int_X \varphi_t(x) \left\{\mu(dt, dx) - \lambda_t(dx)dt\right\}, \\ L_0 &= 1, \end{cases} \tag{14}$$

where the **Girsanov kernel** processes h and φ (where we view h as a column vector process, hence the transpose *) are predictable, suitably integrable (see [6] for details), and where φ must satisfy the condition

$$\varphi_t(x) \ge -1, \quad \forall t, x \quad P - a.s. \tag{15}$$

in order to ensure the positivity of the measure Q. From the Girsanov Theorem we also recall the following facts.

- We can write
$$dW = h_t dt + dW^Q, \tag{16}$$
where W^Q is a Q-Wiener process.

- The point process μ will under Q have an intensity λ^Q, given by
$$\lambda_t^Q(dx) = \left\{1 + \varphi_t(x)\right\} \lambda_t^P(dx). \tag{17}$$

The immediate problem is to find out how the kernel processes h and φ above must be chosen in order to guarantee that Q actually is a martingale measure for S. To this end we apply the Girsanov Theorem to obtain the Q-dynamics of S as

$$dS_t = S_t \left\{\alpha_t + \sigma_t h_t\right\} dt + S_t \sigma_t dW_t^Q + S_{t-} \int_X \delta_t(x) \mu(dt, dx).$$

We then compensate the point process μ under Q to obtain the Q semimartingale representation of S as

$$\begin{aligned} dS_t &= S_t \left\{\alpha_t + \sigma_t h_t + \int_X \delta_t(x) \lambda_t^Q(dx)\right\} dt + S_t \sigma_t dW_t^Q \\ &\quad + S_{t-} \int_X \delta_t(x) \left\{\mu(dt, dx) - \lambda_t^Q(dx) dt\right\}. \end{aligned} \tag{18}$$

Recalling that the measure Q is a martingale measure if and only if the local rate of return of S under Q equals the short rate r, we thus obtain the following **martingale condition**.

Proposition 3.2 *Assume that the measure Q is generated by the Girsanov kernels h, φ. Then Q is a martingale measure if and only if the following conditions are satisfied.*

$$\alpha_t + \sigma_t h_t + \int_X \delta_t(x) \lambda_t^Q(dx) = r_t, \tag{19}$$

$$\varphi(t, x) \geq -1. \tag{20}$$

Condition (19) can also be written as.

$$\alpha_t + \sigma_t h_t + \int_X \delta_t(x) \left\{1 + \varphi_t(x)\right\} \lambda_t(dx) = r_t. \tag{21}$$

A Girsanov kernel process (h, φ) for which the induced measure Q, is a martingale measure, i.e. a kernel process satisfying the martingale condition (19)-(20) will be referred to as an **admissible** Girsanov kernel.

3.3 The Extended Hansen-Jagannathan Bounds

We now go on to derive an inequality for the Sharpe ratio SR, and we start by noting that can rewrite the martingale condition (21) as

$$\alpha_t + \int_X \delta_t(x) \lambda_t(dx) - r_t = -\sigma_t h_t - \int_X \varphi_t(x) \delta_t(x) \lambda_t(dx). \tag{22}$$

From (10) we recognize the risk premium R in the left hand side of this equation so we can write R as

$$R_t = -\sigma_t h_t - \int_X \varphi_t(x) \delta_t(x) \lambda_t(dx). \tag{23}$$

From this expression we see that the Girsanov kernel process (h, φ) has a natural economic interpretation. The component $-h^i$ can be interpreted as the market price of risk for the i:th Wiener process, and $-\delta(x)$ is the market price of risk for a jump event of type x. Using (23) we may also state and prove the main result of this section.

Theorem 3.1 (Extended Hansen-Jagannathan Bounds)
For any arbitrage free price processes S and for every admissible Girsanov kernel (market price of risk) process (h, φ) the following inequality holds.

$$|SR_t| \leq \|(h_t, \varphi_t)\|_{\mathcal{H}}. \tag{24}$$

In more detail this inequality can be written as

$$|SR_t|^2 \leq \|h_t\|^2_{R^d} + \int_X \varphi_t^2(x)\lambda_t(dx). \tag{25}$$

Proof. A closer look at (23) reveals that the right hand side can be viewed as an inner product in the Hilbert space $\mathcal{H}$. Denoting this inner product by $\langle , \rangle_{\mathcal{H}}$ we can thus write

$$R_t = \langle (h_t, \varphi_t), (\sigma_t, \delta_{St}) \rangle_{\mathcal{H}} \tag{26}$$

and from the Schwartz inequality we obtain

$$|R_t| \leq \|(h_t, \varphi_t)\|_{\mathcal{H}} \cdot \|(\sigma_t, \delta_t)\|_{\mathcal{H}}. \tag{27}$$

The inequality (24) now follows immediately from (8), (9), and (27). ∎

Remark 3.2 *It is important to note that the HJ inequality not only holds for the given underlying asset prices. To be precise; suppose that we have chosen a fixed pair of Girsanov kernels (h, φ), and that we use the martingale measure induced by these to price various derivatives. Then the inequality holds for all underlying assets, for all derivatives, and for all self financing portfolios based on the underlying and the derivatives. In other words; for a given choice of (h, φ) the HJ inequality gives us a uniform upper bound of Sharpe ratios for the entire economy.*

4 A Factor Market Model

We now specialize the general setup above to that of a Markovian factor market model, and we also formalize our pricing problem.

4.1 The Model

We consider a financial market built up by the following objects, where * denotes transpose.

- An n-dimensional price process $S = (S^1, \dots, S^n)^\star$
- A k-dimensional factor process $Y = (Y^1, \dots, Y^k)^\star$.

The interpretation of this is that $S^1, \dots, S^n$ are prices of underlying traded assets without dividends, whereas the components of Y are underlying non traded factors. The precise probabilistic specification of the market model is given by the following standing assumption.

Assumption 4.1

1. *Under the objective measure P we assume that (S, Y) satisfies the following stochastic differential equations (SDEs)*

$$\begin{aligned} dS_t^i &= S_t^i \alpha_i\left(S_t, Y_t\right) dt + S_t^i \sigma_i(S_t, Y_t) dW_t \\ &\quad + S_{t-}^i \int_X \delta_i(S_{t-}, Y_{t-}, x) \mu(dt, dx), \quad i = 1, \dots, n \qquad (28) \\ dY_t^j &= a_j\left(S_t, Y_t\right) dt + b_j(S_t, Y_t) dW_t \\ &\quad + \int_X c_j(S_{t-}, Y_{t-}, x) \mu(dt, dx). \quad j = 1, \dots, k \qquad (29) \end{aligned}$$

2. *We assume that for each i and j, $\alpha_i(s, y)$ and $a_j(s, y)$ are deterministic scalar functions, $\sigma_i(s, y)$ and $b_j(s, y)$ are deterministic row vector functions, and $\delta_i(s, y, x)$ and $c_j(s, y, x)$ are deterministic scalar functions. In order to avoid negative asset prices we also assume that $\delta_i(s, y, x) \geq -1$ for all i and all (s, y, x).*

3. *All functions above are assumed to be regular enough to allow for the existence of a unique strong solution for the system of SDEs.*

4. *The point process μ has a predictable P-intensity measure λ. More precisely we assume that the P-compensator $\nu(dt, dx)$ has the form*

$$\nu(dt, dx) = \lambda(S_{t-}, Y_{t-}, dx) dt. \qquad (30)$$

   For brevity of notation we will often denote $\lambda(S_{t-}, Y_{t-}, dx)$ by $\lambda_t(dx)$. The compensated point process $\tilde{\mu}$ is defined by

$$\tilde{\mu}(dt, dx) = \mu(dt, dx) - \lambda(S_{t-}, X_{t-}, dx) dt \qquad (31)$$

5. *We assume the existence of a short rate r of the form*

$$r_t = r(S_t, Y_t).$$

6. *We assume that the model is free of arbitrage in the sense that there exists a (not necessarily unique) risk neutral martingale measure Q.*

The present setup extends the one in [4] in two ways.

- In [4] the continuous time model is purely Wiener driven. The main contribution of the present paper is that we extend the framework of [4] to also include a driving point process.
- Even in the purely Wiener driven case, our setup extends that of [4] by not making any rank assumptions for the diffusion matrices σ and b. This, as opposed to the item above, is a minor extension.

For future use we introduce some more compact notation.

Definition 4.1 *The column vector functions, α, δ, a, and c are defined by*

$$\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}, \quad \delta = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_n \end{bmatrix}, \quad a = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}. \tag{32}$$

The $n \times d$ matrix σ and the $k \times d$ matrix b are defined by

$$\sigma = \begin{bmatrix} -\sigma_1- \\ \vdots \\ -\sigma_n- \end{bmatrix}, \quad b = \begin{bmatrix} -b_1- \\ \vdots \\ -b_k- \end{bmatrix}. \tag{33}$$

4.2 The Problem

On the market specified above we consider an arbitrarily chosen contingent T-claim Z of the form

$$\mathcal{Z} = \Phi(S_T, Y_T), \tag{34}$$

and the problem is to compute a "reasonable" price process $\Pi(t; \mathcal{Z})$ for the claim $\mathcal{Z}$. Since the market in the general case is incomplete, the martingale measure Q will generically not be unique, so there will not be a uniquely determined arbitrage free price for $\mathcal{Z}$. It is also well known that in incomplete settings like this, the pricing bounds provided by merely requiring absence of arbitrage are extremely wide and thus useless from a practical point of view.

There is thus a clear need for "reasonable" pricing bounds for derivative assets, and to this end Cochrane and Saa-Requejo introduced, in the seminal paper [4], the completely new idea of ruling out, not only those prices which are violating the no arbitrage restriction, but also those prices which in some sense would represent "deals which are too good". The problem is now to define when a deal in this sense is a "good deal" and Cochrane and Saa-Requejo argued that a reasonable formalization of a (too) good deal is a deal for which the Sharpe ratio is very high.

In a first attempt, a mathematical formalization of the pricing problem would then be to find the maximum (minimum) arbitrage free price process for the derivative, subject to an upper bound on the Sharpe ratio. However; this way of formalizing the problem turns out have two major drawbacks.

1. The optimization problem turns out to be mathematically intractable.
2. A much more serious problem is the following: Suppose that we have found upper and lower pricing bounds on a derivative, subject to a bound on the Sharpe ratio of the derivative. Then it may in principle still be possible to form a self financing portfolio, based on the underlying assets and the newly introduced derivative, such that the portfolio has a very high Sharpe ratio.

What we need is thus a formalization of the pricing problem which gives us a mathematically tractable problem, and which at the same time allows us to have complete control over the Sharpe ratios of *all portfolios* based on the underlying assets and the derivative.

This is precisely where the Hansen-Jagannathan comes in useful (see Remark 3.2), and thus Cochrane and Saa-Requejo suggested that instead of putting a bound on the Sharpe ratio of the derivative under study, we put a bound on the right hand side of the Hansen-Jagannathan inequality (i.e. the norm of the market price of risk vector). In the final formulation, the pricing problem is thus that of finding the maximum (minimum) arbitrage free price process for a given derivative, subject to a bound on the norm of the market price of risk vector. The procedure is formalized in the following definition, where for brevity of notation we write σ_i as shorthand for $\sigma_i(S_t, Y_t)$ and similarly for other terms.

Definition 4.2 *Given a bound A for the market prices of risk, the upper good deal price bound process is defined as the optimal value process for the following optimal control problem.*

$$\max_{h,\varphi} \quad E^Q\left[e^{-\int_t^T r_u du} \Phi\left(S_T, Y_T\right) \middle| \mathcal{F}_t \right] \tag{35}$$

with Q dynamics

$$\begin{aligned} dS_t^i &= S_t^i \left\{ r_t - \int_X \delta_i(x) \left\{1 + \varphi_t(x)\right\} \lambda_t(dx) \right\} dt + S_t^i \sigma_i dW_t^Q \\ &\quad + S_{t-}^i \int_X \delta_i(x) \mu(dt, dx), \quad i = 1, \dots, n \end{aligned} \tag{36}$$

$$\begin{aligned} dY_t^j &= \left\{a_j + b_j h_t\right\} dt + b_j dW_t^Q \\ &\quad + \int_X c_j(x) \mu(dt, dx). \quad j = 1, \dots, k \end{aligned} \tag{37}$$

where the Q-compensator of μ is given by

$$\nu^Q(dt, dx) = \left\{1 + \varphi_t(x)\right\} \lambda_t(dx). \tag{38}$$

The predictable processes h and φ are subject to the constraints

$$\alpha_i + \sigma_i h_t + \int_X \delta_i(x)\left\{1 + \varphi_t(x)\right\} \lambda_t(dx) = r_t, \quad i = 1, \ldots, n \tag{39}$$

$$\|h_t\|_{R^d} + \int_X \varphi_t^2(x)\lambda_t(dx) \leq B^2, \tag{40}$$

$$\varphi_t(x) \geq -1, \quad \forall t, x. \tag{41}$$

Some comments are perhaps in order.

- The expected value in (35) is the standard risk neutral valuation formula for contingent claims.
- In (39) we have the conditions on h and φ, guaranteeing that the induced measure Q is indeed a martingale measure for $S^1, \ldots, S^n$. The calculations are identical to those in Section 3.2, and (39) is in fact identical to (19).
- The induced Q dynamics of $S^1, \ldots, S^n$ are given in (36), and derived exactly along the lines of Section 3.2.
- The induced Q dynamics of $Y^1, \ldots, Y^k$ are given in (37).
- The constraint (40) is the constraint to rule out "good deals".
- The constraint (41) is needed to ensure that Q is a positive measure.
- Formula (38) specifies the Q distribution of μ.
- In order to obtain the lower pricing bound, we solve the corresponding minimum problem.

4.3 The Pricing Equation

In order to allow us to treat the optimal control problem above with dynamic programming methods we have to make an extra assumption, which will ensure that the Markovian structure is preserved also under the martingale measure Q.

Assumption 4.2 *We assume henceforth that the Girsanov kernel processes h and φ are of the restricted form*

$$h_t = h(t, S_t, Y_t), \tag{42}$$

$$\varphi_t(x) = \varphi(t, S_{t-}, Y_{t-}, x). \tag{43}$$

Here, with a slight abuse of notation, the right hand side occurences of h and φ denote deterministic functions of the form $h : R_+ \times R^n \times R^k \to R^n$ and $\varphi : R_+ \times R^n \times R^k \times X \to R$ respectively.

We now go on to present the basic pricing equation for the upper and lower good deal bounds, and in the present setting this is quite straightforward. Under Assumption 4.2, the optimal expected value in (35) can in fact be written as $V(t, S_t, Y_t)$, where the deterministic mapping $V : R_+ \times R^n \times R^k \rightarrow R$ is known as the **optimal value function**. Since we are in a standard setting for dynamic programming (DynP), we know from general DynP-theory that the optimal value function will satisfy the following Bellman-Hamilton-Jacobi equation on the time interval $[0, T]$.

$$\frac{\partial V}{\partial t} + \sup_{h,\varphi} \mathbf{A}^{h,\varphi} V - rV = 0, \tag{44}$$

$$V(T, s, y) = \Phi(s, y), \tag{45}$$

where the sup is subject to constraints of the form (39)-(41), and where $\mathbf{A}^{h,\varphi}$ denotes the infinitesimal operator for the process (S, Y), under the measure Q defined by h and φ.

We recall that from an operational point of view, the infinitesimal operator $\mathbf{A}^{h,\varphi}$ is nothing else than the the integro-differential operator which turns up in the dt term in the stochastic differential $dV(t, S_t, Y_t)$ (when the point process increment has been compensated). A standard application of the Itô formula for semimartingales will in fact give us the following result.

Proposition 4.1 *The infinitesimal operator* $\mathbf{A}^{h,\varphi}$ *is given by*

$$\begin{aligned} &\mathbf{A}^{h,\varphi} V(t, s, y) = \qquad (46) \\ = \;& \sum_{i=1}^{n} \frac{\partial V}{\partial s_i}(t, s, y) s_i \left\{ r - \int_X \delta_i(s, y, x) \left\{1 + \varphi(t, s, y, x)\right\} \lambda_t(s, y, dx) \right\} \\ + \;& \sum_{j=1}^{k} \frac{\partial V}{\partial y_j}(t, s, y) \left\{ a_j(s, y) + b_j(s, y) h(t, s, y) \right\} \\ + \;& \int_X \Delta V(t, s, y, x) \left\{1 + \varphi(t, s, y, x)\right\} \lambda_t(s, y, dx) \\ + \;& \frac{1}{2} \sum_{i,l=1}^{n} \frac{\partial^2 V}{\partial s_i \partial s_l}(t, s, y) s_i s_l \sigma_i^\star(s, y) \sigma_l(s, y) + \frac{1}{2} \sum_{j,l=1}^{k} \frac{\partial^2 V}{\partial y_j \partial y_l}(t, s, y) b_j^\star(s, y) b_l(s, y) \\ + \;& \sum_{i,j=1}^{k} \frac{\partial^2 V}{\partial s_i \partial y_j}(t, s, y) s_i \sigma_i^\star(s, y) b_j(s, y). \qquad (47) \end{aligned}$$

Here ΔV *is defined by*

$$\Delta V(t, s, y, x) = V\left(t, s(1 + \delta(s, y, x)), y + c(s, y, x)\right) - V(t, s, y), \tag{48}$$

where addition and multiplication in $s(1 + \delta(s, y, x))$ *and* $y + c(s, y, x)$ *are interpreted component wise.*

Proof. An easy application of the Itô formula. ∎

Collecting the facts above, we can finally present the basic equation for the upper good deal bound.

Theorem 4.1 *The upper good deal bound function is the solution V to the following boundary value problem.*

$$
\begin{aligned}
\frac{\partial V}{\partial t}(t,s,y) + \sup_{h,\varphi}\left\{\mathbf{A}^{h,\varphi}V(t,s,y)\right\} - r(s,y)V(t,s,y) &= 0, &(49)\\
V(T,s,y) &= \Phi(s,y). &(50)
\end{aligned}
$$

Here, $\mathbf{A}^{h,\varphi}V$ is given by (47), and the supremum in (49) should be taken over all functions $h(t,s,y)$ and $\varphi(t,s,y,x)$ satisfying, for all (t,s,y), the constraints

$$
\begin{aligned}
\alpha_i + \sigma_i h + \int_X \delta_i(x)\left\{1+\varphi(x)\right\}\lambda_t(dx) &= r, \quad i=1,\dots,n &(51)\\
\|h\|_{R^d} + \int_X \varphi^2(x)\lambda_t(dx) &\leq B^2, &(52)\\
\varphi(x) &\geq -1. &(53)
\end{aligned}
$$

The lower bound price functions satisfies the same equation with the supremum operator replaced by $\inf_{h,\varphi}$.

4.4 On the Structure of the Pricing Equation

The pricing equation (49)-(50) is a partial integro-differential equation (PIDE), and in the general case there is of course no hope at all of finding an analytical solution. There are however some very particular features of the equation which we want to stress.

As in all applications of stochastic dynamic programming, we note that the stochastic intertemporal optimal control problem (35)-(41) is reduced to the following two purely deterministic problems:

1. The static optimization problem of finding, for each fixed (t,s,y), the optimal h and φ in the constrained maximization problem

$$\sup_{h,\varphi}\left\{\mathbf{A}^{h,\varphi}V(t,s,y)\right\} \tag{54}$$

   appearing in (49).

2. Having solved the static problem above, and denoting the optimal h,φ by $\hat{h},\hat{\varphi}$, we have to solve the PIDE

$$
\begin{aligned}
\frac{\partial V}{\partial t} + \mathbf{A}^{\hat{h},\hat{\varphi}}V - rV &= 0, &(55)\\
V(T,s,y) &= \Phi(s,y), &(56)
\end{aligned}
$$

Obviously; if we ever want to be able to solve the PIDE in step 2 above, then we first have to solve the static optimization problem in step 1, so it is of great importance to understand the structure of the static problem. We then note that this problem is in fact an infinite dimensional one. More precisely; the problem (54) has to be solved for every fixed choice of (t, s, y), and the control variables are h and φ, but whereas the diffusion kernel $h(t, s, y)$ (for fixed t, s and y) is merely a d-dimensional vector, the point process kernel $\varphi(t, s, y, \cdot)$ has to be determined as a function of x and hence $\varphi(t, s, y, \cdot)$ is an infinite dimensional control variable. We thus see that the static optimization problem is in fact not a standard finite dimensional mathematical programming problem, but a full fledged variational problem.

The infinite dimensionality of the static optimization problem is intimately connected to the cardinality of the mark space X (or rather to the cardinality of the support of the measure $\lambda_t(dx)$). If the mark space has an infinite number of elements then the static problem is infinite dimensional. If, on the other hand, X has a finite number of elements then the static problem is a finite dimensional problem. From a more modeling point of view this basically means that if we want to model a situation with an infinite number of possible jump sizes, then the static problem is a variational problem.

Even if the static problem is an infinite dimensional one, it has a very particular structure. Looking closer at the expression (47) for the infinitesimal operator $\mathbf{A}^{h,\varphi}$ we see that in fact only three terms involve the control variables h and φ and that in fact the control variables enter linearly. With notation as in Definition 4.1 we formalize this observation in a lemma.

Lemma 4.1 *The static optimization problem in (49) and (54) can be written as*

$$\max_{h,\varphi} \quad \langle \Delta V, \varphi \rangle_{\lambda_t} - V_s D(s) \langle \delta, \varphi \mathbf{1} \rangle_{\lambda_t} + V_y b h \tag{57}$$

subject to the constraints

$$\alpha + \sigma h + \langle \delta, \mathbf{1} \rangle_{\lambda_t} + \langle \delta, \varphi \mathbf{1} \rangle_{\lambda_t} \quad = \quad r\mathbf{1}, \tag{58}$$

$$\|h\|^2_{R^d} + \|\varphi\|^2_{\lambda_t} \quad \leq \quad B^2, \tag{59}$$

$$\varphi \quad \geq \quad -1. \tag{60}$$

Here V_s and V_y denote the gradients of V w.r.t. the vector variables s and y respectively, $D(s)$ denotes the diagonal matrix with the components of s on the diagonal, $\mathbf{1}$ denotes the column vector with 1 in all components, and the inner products $\langle \delta, \varphi \mathbf{1} \rangle_{\lambda_t}$ and $\langle \delta, \mathbf{1} \rangle_{\lambda_t}$ are interpreted component wise.

Writing the static problem on this form we see very clearly that we have a linear objective function, an infinite dimensional linear equality constraint, a scalar quadratic inequality constraint, and an infinite dimensional linear inequality constraint. Since the set of admissible points is convex, the linearity of the objective function will thus imply that the optimal point is an extremal point of the admissible set. It is also clear that at least one of the inequality constraints has to be binding.

4.5 Positivity and the Minimal Martingale Measure

The static problem (57) is, apart from the positivity constraint (60), a fairly standard linear quadratic problem in the space $L^2[X, \lambda(dx)]$. The really problematic part is the generically infinite dimensional positivity constraint (60), and the reason why this is problematic is that the positive cone in L^2 does not contain any interior points, which effectively prohibits us form using standard infinite dimensional Kuhn-Tucker methodology. We now go on to discuss the positivity constraint in some more detail.

As a first approach to solving Problem 57 one could of course hope that the positivity constraint is not binding in the optimal solution. It would thus be natural to solve a relaxed version of Problem 57, where the positivity constraint is not present and, having found the optimal solution, we could then check whether the positivity constraint is binding or not. If the positivity constraint turns out not to be binding at the relaxed optimal point, then all is well and we have found our optimal solution. If the constraint is violated at the optimal point of the relaxed problem, then the induced measure will in fact not be a positive measure. Using this measure for pricing will still give us pricing bounds, but these will be wider than the optimal ones. We now formalize these ideas, and we will also relate them to the concept of the "minimal martingale measure" from the theory of local risk minimization.

Definition 4.3

- *Denote the optimal upper and lower bound Girsanov kernels from Theorem 4.1 by (h^s, φ^s) and (h^i, φ^i), denote the corresponding* **optimal martingale measures** *by Q^s, Q^i, and define the pricing functions V^s and V^i correspondingly. Here "s" is standing for "sup" and "i" is standing for "inf".*

- *Denote by $(\bar{h}^s, \bar{\varphi}^s)$ the optimal kernels for the relaxed static problem*

$$\sup_{h,\varphi} \quad \mathbf{A}^{h,\varphi} V(t, s, y) \tag{61}$$

subject to the constraints

$$\alpha_i + \sigma_i h + \int_X \delta_i(x) \{1 + \varphi(x)\} \lambda_t(dx) \quad = \quad r, \quad i = 1, \ldots, n \tag{62}$$

$$\|h\|_{R^d} + \int_X \varphi^2(x)\lambda_t(dx) \quad \leq \quad B^2, \tag{63}$$

and denote by $(\bar{h}^i, \bar{\varphi}^i)$ the optimal solutions to the corresponding minimization problem. Denote the solution to the PIDE

$$\frac{\partial V}{\partial t} + \mathbf{A}^{\bar{h}^s, \bar{\varphi}^s} V - rV \quad = \quad 0, \tag{64}$$

$$V(T, s, y) \quad = \quad \Phi(s, y), \tag{65}$$

by $\bar{V}^s$ and define $\bar{V}^i$ in the same way. The **relaxed martingale measures** *induced by $(\bar{h}^s, \bar{\varphi}^s)$ and $(\bar{h}^i, \bar{\varphi}^i)$ are denoted by $\bar{Q}^s$, and $\bar{Q}^i$ respectively.*

- *Denote by (h^m, φ^m) the Girsanov kernels obtained by solving the problem*

$$\min_{h,\varphi} \quad \|h\|^2_{R^d} + \|\varphi\|^2_{\lambda_t} \tag{66}$$

  subject to

$$\alpha_i + \sigma_i h + \int_X \delta_i(x) \{1 + \varphi(x)\} \lambda_t(dx) = r, \quad i = 1, \dots, n \tag{67}$$

  Denote the solution to the PIDE

$$\frac{\partial V}{\partial t} + \mathbf{A}^{h^m, \varphi^m} V - rV = 0, \tag{68}$$

$$V(T, s, y) = \Phi(s, y), \tag{69}$$

  by V^m. The measure Q^m, henceforth referred to as the **minimal martingale measure** *("MMM" for short) is defined as the (possibly signed) measure induced by (h^m, φ^m).*

Of the measures defined above, the first four will depend upon the choice of derivative to be priced, whereas the minimal martingale measure Q^m is independent of the choice of derivative. We see that the MMM kernels (h^m, φ^m) are obtained by minimizing the right hand side of the HJ inequality, subject to the "martingale condition" (67), but without the positivity constraint, so the MMM is the P-equivalent measure with pointwise minimal L^2 norm satisfying the martingale constraint. It may thus happen that the MMM is a signed measure. The minimal martingale measure was first defined in connection with local risk minimization (see [10]) where it plays a fundamental role. The original definition of the MMM in [10] is not the one given above, but it is fairly easy to see that in the present context the two definitions coincide. We now have the following easy result.

Proposition 4.2

- *We always have the relations*

$$\bar{V}^i \leq V^i \leq V^s \leq \bar{V}^s, \tag{70}$$

  and

$$\bar{V}^i \leq V^m \leq \bar{V}^s, \tag{71}$$

- *If the positivity constraint is satisfied by $\bar{h}^i$, $\bar{h}^s$ and h^m, then $\bar{Q}^i$, $\bar{Q}^s$ and Q^m are probability measures (and not just signed measures) and we have*

$$\bar{V}^i \leq V^i \leq V^m \leq V^s \leq \bar{V}^s. \tag{72}$$

Proof. Obvious from the definitions. ∎

The moral of this can be summarized as follows.

- The minimal martingale measure provides us with a canonical benchmark for pricing any derivative. Furthermore, since the MMM is the solution to a standard minimum norm problem in L^2, it can easily be computed.
- The relaxed martingale measures $\bar{Q}^i$ and $\bar{Q}^s$ are much easier to compute than the optimal measures Q^i and Q^s.
- The pricing bounds provided by the relaxed measures $\bar{Q}^i$ and $\bar{Q}^s$ are generically not optimal but, for reasonable values of the Sharpe ratio constraint B, they turn out to be much tighter than the no arbitrage bounds.
- The bounds obtained by the (harder to compute) optimal measures Q^i and Q^s are in their turn considerably tighter than those obtained from $\bar{Q}^i$ and $\bar{Q}^s$.
- Both the minimal martingale measure Q^m and the relaxed measures $\bar{Q}^i$ and $\bar{Q}^s$ can be computed explicitly in terms of input data. However; the general formulas are rather messy so we have chosen not to include them. See Section 5.4 for a concrete worked out example.

5 Point Process Examples

In this section we study a number of illustrative concrete examples, and since the main focus of the present paper is on models including jumps, we restrict ourselves to these. See Appendix A for the purely Wiener driven case.

As opposed to a purely Wiener driven model, the introduction of a driving point process (together with a Wiener process) will produce a nontrivial incomplete market model even without including the factor model Y. For this reason, but of course also for reasons of tractability, we will therefore confine ourselves to study pure jump-diffusion stock price models without any external factors. More precisely; all models studied in this section will be assumed to have the following structure.

Assumption 5.1 *We consider a financial market and a scalar price process S satisfying the SDE*

$$dS_t = S_t \alpha dt + S_t \sigma dW_t + S_{t-} \int_X \delta(x) \mu(dt, dx). \tag{73}$$

For this model we furthermore assume that

1. *The Wiener process W is one-dimensional.*
2. *The drift α and diffusion volatility σ are deterministic constants.*

3. *The jump function δ is a time invariant deterministic function of x only, i.e. δ is a mapping $\delta : X \rightarrow R$.*

4. *The point process μ has a P-compensator of the form*

$$\nu^P(dt, dx) = \lambda(dx)dt$$

where λ is a time invariant deterministic finite nonnegative measure on $(X, \mathcal{X})$.

5. *The short rate r is constant.*

Under this assumption the model parameters α, σ, δ, and λ are thus deterministic object which do not depend on the stock price S. In particular the assumption about λ implies that the point process μ has the following properties under P.

- The jump events (disregarding the mark) will occur according to a standard Poisson process with the constant intensity $\lambda(X)$.
- If X_n denotes the mark of event number n then the sequence $X_1, X_2, \ldots$ is i.i.d. with the common probability distribution

$$\frac{1}{\lambda(X)}\lambda(dx). \tag{74}$$

  The sequence above is also independent of the inter arrival times of the events.

In order to get a feeling for the techniques used, we start with a very simple example and then go on to consider more complicated cases.

5.1 The Poisson-Wiener Model

The simplest special case in the jump-diffusion setting above is when we define the point process μ as a standard Poisson process with constant intensity. In terms of the notation above this means that the mark space X contains a single point denoted by x_0. Hence $X = \{x_0\}$, the measure $\lambda(dx)$ is just a point mass $\lambda(x_0)$ at x_0, and the jump function δ is just a real number $\delta(x_0)$. For brevity we will denote $\lambda(x_0)$ by λ and $\delta(x_0)$ by δ. We thus have the following P dynamics of S.

$$dS_t = S_t\alpha dt + S_t\sigma dW_t + S_{t-}\delta dN_t \tag{75}$$

where N is Poisson with constant intensity λ.

In this case the kernel function $h(t, s)$ is scalar, and the kernel $\varphi(t, s)$ does not depend upon x. The upper good deal bound function $V(t, s)$ is the solution to the following boundary value problem.

$$\begin{aligned} \frac{\partial V}{\partial t}(t,s) + \sup_{h,\varphi}\left\{\mathbf{A}^{h,\varphi}V(t,s)\right\} - rV(t,s) &= 0, & (76) \\ V(T,s) &= \Phi(s), & (77) \end{aligned}$$

were we for the moment suppress the constraints, and where

$$\mathbf{A}^{h,\varphi}V(t,s) = \frac{\partial V}{\partial s}s\left\{r-\delta\lambda(1+\varphi)\right\}+\frac{1}{2}s^2\sigma^2\frac{\partial^2 V}{\partial s^2} + \left\{V(t,s(1+\delta))-V(t,s)\right\}\lambda(1+\varphi). \tag{78}$$

The static optimization problem in Lemma 4.1 thus becomes

Problem 5.1

$$\max_{h,\varphi} \quad \lambda\left\{V(t,s(1+\delta))-V(t,s)-V_s(t,s)s\delta\right\}\varphi \tag{79}$$

subject to the constraints

$$\alpha+\sigma h+\delta\lambda\left\{1+\varphi\right\} = r, \tag{80}$$
$$h^2+\varphi^2\lambda \leq B^2, \tag{81}$$
$$\varphi \geq -1. \tag{82}$$

To study the static problem in more detail we need some notation.

Definition 5.1 *Define $(h_{\max},\varphi_{\max})$ as the optimal solution to the programming problem*

$$\max_{h,\varphi}\varphi \tag{83}$$

subject to the constraints (80)-(82) and $(h_{\min},\varphi_{\min})$ as the optimal solution to the problem

$$\min_{h,\varphi}\varphi \tag{84}$$

subject to the same constraints.

We will need $h_{\max}$, $\varphi_{\max}$, $h_{\min}$, and $\varphi_{\min}$ below, so we should describe these constants in terms of the given model parameters. This is a simple exercise in constrained optimization theory, but a bit messy, and the result is as follows.

Lemma 5.1 *Denote the excess return $\alpha+\delta\lambda-r$ by R. Then the following hold.*

- *The constants $h_{\max}$ and $\varphi_{\max}$ are given by*

$$h_{\max} = -\frac{\sigma R}{(\sigma^2+\delta^2\lambda)\lambda}-\frac{\delta\sqrt{B^2\left(\sigma^2+\delta^2\lambda\right)-R^2)}}{\left(\sigma^2+\delta^2\lambda\right)\sqrt{\lambda}} \tag{85}$$

$$\varphi_{\max} = -\frac{\delta R}{\sigma^2+\delta^2\lambda}+\frac{\sigma\sqrt{B^2\left(\sigma^2+\delta^2\lambda\right)-R^2)}}{\left(\sigma^2+\delta^2\lambda\right)\sqrt{\lambda}} \tag{86}$$

- *The constants $h_{\min}$ and $\varphi_{\min}$ are given by the following expressions.*

1. *If*

$$-\frac{\delta R}{\sigma^2+\delta^2\lambda}-\frac{\sigma\sqrt{B^2\left(\sigma^2+\delta^2\lambda\right)-R^2)}}{\left(\sigma^2+\delta^2\lambda\right)\sqrt{\lambda}}>-1, \tag{87}$$

then

$$h_{\min} = -\frac{\sigma R}{(\sigma^2+\delta^2\lambda)\lambda}+\frac{\delta\sqrt{B^2\left(\sigma^2+\delta^2\lambda\right)-R^2)}}{\left(\sigma^2+\delta^2\lambda\right)\sqrt{\lambda}} \tag{88}$$

$$\varphi_{\min} = -\frac{\delta R}{\sigma^2+\delta^2\lambda}-\frac{\sigma\sqrt{B^2\left(\sigma^2+\delta^2\lambda\right)-R^2)}}{\left(\sigma^2+\delta^2\lambda\right)\sqrt{\lambda}}. \tag{89}$$

2. *If*

$$-\frac{\delta R}{\sigma^2+\delta^2\lambda}-\frac{\sigma\sqrt{B^2\left(\sigma^2+\delta^2\lambda\right)-R^2)}}{\left(\sigma^2+\delta^2\lambda\right)\sqrt{\lambda}}\leq -1, \tag{90}$$

then

$$h_{\min} = \frac{r-\alpha}{\sigma}, \tag{91}$$

$$\varphi_{\min} = -1. \tag{92}$$

Proof. A direct application of Kuhn-Tucker. ∎

We can now present a preliminary description of the optimal kernels.

Proposition 5.1 *The optimal kernels $(\hat{h},\hat{\varphi})$ for the static problem (79)-(82) have the following structure.*

1. *For all (t,s) such that*

$$V(t,s(1+\delta))-V(t,s)-V_s(t,s)s\delta\geq 0, \tag{93}$$

the optimal kernels $(\hat{h},\hat{\varphi})$ are given by

$$\hat{h}(t,s)=h_{\max},\quad \hat{\varphi}(t,s)=h_{\max}. \tag{94}$$

2. *For all (t,s) such that*

$$V(t,s(1+\delta))-V(t,s)-V_s(t,s)s\delta< 0, \tag{95}$$

the optimal kernels $(\hat{h},\hat{\varphi})$ are given by

$$\hat{h}(t,s)=h_{\min},\quad \hat{\varphi}(t,s)=h_{\min}. \tag{96}$$

Proof. Obvious from the arguments above. ∎

We thus see that the optimal kernels have a so called bang-bang structure, i.e. they switch between the extremal choices $(h_{\max}, \varphi_{\max})$ and $(h_{\min}, \varphi_{\min})$. For an arbitrarily chosen problem, switches will indeed occur, and the number of switches will of course depend upon the optimal value function V through the conditions (93) and (95), but there is an interesting special case when there are no switches and the optimal kernels thus are constant. Before proving the main result in this direction, we need some preliminary lemmas.

Lemma 5.2

1. *If the optimal value function $V(t,s)$ is convex in the s-variable for all fixed values of t, then*

$$\hat{h}(t,s) = h_{\max}, \quad \hat{\varphi}(t,s) = \varphi_{\max}, \quad \forall t, s. \tag{97}$$

2. *If the optimal value function $V(t,s)$ is concave in the s-variable for all fixed values of t, then*

$$\hat{h}(t,s) = h_{\min}, \quad \hat{\varphi}(t,s) = \varphi_{\min}, \quad \forall t, s. \tag{98}$$

Proof. If V is convex in s then (as a function of s) the tangent of V lies below the graph at each point, which implies condition (93). The concave case is similar. ∎

Lemma 5.3 *Consider the Poisson-Wiener model in (75), a fixed martingale measure Q generated by some choice of kernel functions (h, φ), and a contract function $\Phi(s)$. Assume the following:*

1. *The contract function Φ is convex (concave).*
2. *The kernel functions h and φ are deterministic functions of time, i.e. they are of the form*

$$h(t,s) = h(t), \quad \varphi(t,s) = \varphi(t). \tag{99}$$

Then the arbitrage free pricing function $F(t,s)$ defined by

$$F(t,s) = e^{-r(T-t)} E^{Q}_{ts} \left[\Phi(S_T) \right] \tag{100}$$

is convex (concave) in the s variable.

Proof. Using the Itô formula it is easy to see that, given $S_t = s$, the SDE (75) has the solution

$$S_T = s e^{(\alpha - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t) + (N_T - N_t)\ln(1+\delta)}, \tag{101}$$

which we will write as

$$S_T = s \cdot Z, \tag{102}$$

with the random variable Z defined as the exponential above. We thus have

$$F(t,s) = e^{-r(T-t)} E^Q_{ts}\left[\phi(s \cdot Z)\right], \tag{103}$$

and from the assumptions on h and φ it follows that the Q-distribution of Z does not depend upon the value of s. The assumed convexity of Φ now immediately implies the convexity of F. ∎

We can now state and prove the main theoretical result concerning the Poisson Wiener model.

Proposition 5.2 *Assume that the contract function Φ is convex. Then the following hold:*

1. *The optimal upper bound value function V is convex.*
2. *The optimal kernels $\hat{h}$ and $\hat{\varphi}$ are constant and given by*

$$\hat{h} = h_{\max}, \quad \hat{\varphi} = \varphi_{\max}. \tag{104}$$

3. *V satisfies the PIDE*

$$\begin{aligned} \frac{\partial V}{\partial t}(t,s) + \mathbf{A}^{\hat{h},\hat{\varphi}} V(t,s) - rV(t,s) &= 0, & (105)\\ V(T,s) &= \Phi(s), & (106) \end{aligned}$$

where $\hat{h}, \hat{\varphi}$ are defined by (104) and where

$$\begin{aligned} \mathbf{A}^{\hat{h},\hat{\varphi}} V &= \frac{\partial V}{\partial s} s\left\{r - \delta\lambda(1+\hat{\varphi})\right\} + \frac{1}{2} s^2 \sigma^2 \frac{\partial^2 V}{\partial s^2} \\ &\quad + \left\{V(t, s(1+\delta)) - V(t,s)\right\} \lambda(1+\hat{\varphi}). \end{aligned} \tag{107}$$

If the contract function Φ instead is concave, then V is concave, and items 2-3 above still hold, with the only change that $h_{\max}$ and $\varphi_{\max}$ are replaced by $h_{\min}$ and $\varphi_{\min}$.

Proof. Define the function F as the solution of

$$\begin{aligned} \frac{\partial F}{\partial t}(t,s) + \mathbf{A}^{\hat{h},\hat{\varphi}} F(t,s) - rF(t,s) &= 0, & (108)\\ F(T,s) &= \Phi(s), & (109) \end{aligned}$$

with $\mathbf{A}^{\hat{h},\hat{\varphi}}$ defined as above. We now want to show that $F = V$ i.e. that F is in fact equal to the optimal value function of the control problem for the upper bound. To do this we first apply a Feynman-Kac Representation Theorem to deduce that we can write F as

$$F(t,s) = e^{-r(T-t)} E_{ts}^{\hat{Q}} \left[\Phi(S_T) \right] \tag{110}$$

where $\hat{Q}$ is generated by $\hat{h}$ and $\hat{\varphi}$. From Lemma 5.3 we then deduce that F is convex in the s-variable, and this implies, as in the Proof of Lemma 5.2, F also satisfies the PIDE

$$\frac{\partial F}{\partial t}(t,s) + \sup_{h,\varphi} \left\{ \mathbf{A}^{h,\varphi} F(t,s) \right\} - rF(t,s) = 0, \tag{111}$$

$$F(T,s) = \Phi(s), \tag{112}$$

with obvious notation and standard constraints on h and φ. We have thus shown that F, defined by (108)-(109), satisfies the Hamilton-Jacobi-Bellman equation for the optimal control problem for the upper good deal bound, and we can then apply a standard verification theorem to deduce that $F = V$. ∎

The moral of this result is that for convex contract functions, like European puts and calls, we now have derived a well behaved standard pricing equation without any supremum operator. For non-convex contract functions, like that of a digital option, the situation is much more complicated and we must solve the full Hamilton-Jacobi-Bellman equation numerically.

5.2 The Compound Poisson-Wiener Model

We now turn to the general Compound Poisson-Wiener Model specified by Assumption A.1. For this model, the static problem of Lemma 4.1 has the following form

Problem 5.2

$$\max_{h,\varphi} \quad \int_X \Delta V(t,s,x)\varphi(t,s,x)\lambda(dx) - sV_s(t,s)\int_X \delta(x)\varphi(t,s,x)\lambda(dx), \tag{113}$$

subject to

$$\alpha + \sigma h + \int_X \delta(x)\lambda(dx) + \int_X \delta(x)\varphi(x)\lambda(dx) = r, \tag{114}$$

$$h^2 + \int_X \varphi^2(x)\lambda(dx) \leq B^2, \tag{115}$$

$$\varphi(x) \geq -1, \tag{116}$$

where, as before,

$$\Delta V(t,s,x) = V(t, s(1+\delta(x))) - V(t,s). \tag{117}$$

Using, as above, the notation R for the risk premium

$$R = \alpha + \int_X \delta(x)\lambda(dx) - r, \tag{118}$$

we can express the problem in functional analytical terms as follows.

Problem 5.3

$$\max_{h,\varphi} \quad \langle H, \varphi \rangle_\lambda, \tag{119}$$

subject to

$$\begin{aligned} \sigma h + \langle \delta, \varphi \rangle_\lambda + R &= 0, & (120)\\ h^2 + \|\varphi\|_\lambda &\leq B^2, & (121)\\ \varphi &\geq -1, & (122) \end{aligned}$$

where

$$H(t, s, x) = \Delta V(t, s, x) - V_s(t, s)s\delta(x). \tag{123}$$

We now go on to study this problem in some detail.

5.3 The Minimal Martingale Measure

We start by computing the minimal martingale measure Q^m for this model. The measure Q^m is generated by the optimal kernels (h^m, φ^m) for the following programming problem.

Problem 5.4

$$\min_{h,\varphi} \quad h^2 + \|\varphi\|_\lambda^2, \tag{124}$$

subject to

$$\sigma h + \langle \delta, \varphi \rangle_\lambda + R = 0, \tag{125}$$

This is a standard minimum norm problem in L^2, so from general theory we know that the optimal (h, φ) has to be of the form

$$(h, \varphi) = c \cdot (\sigma, \delta),$$

for some scalar c. Plugging this into the constraint (125) gives us the following result.

Proposition 5.3 *The minimal martingale measure Q^m is generated by the kernels (h^m, φ^m) given by*

$$\begin{aligned} h^m &= -\frac{R \cdot \sigma}{h^2 + \|\varphi\|_\lambda^2}, & (126)\\ \varphi^m &= -\frac{R \cdot \delta}{h^2 + \|\varphi\|_\lambda^2}. & (127) \end{aligned}$$

5.4 The Positivity Constraint

As we noted in Section 4.5, the Problem 5.3 above is, apart from the constraint (122), a fairly standard linear quadratic problem in the space $L^2[X, \lambda(dx)]$. Following the arguments of Section 5.4 we are thus led to study Problem 5.3 without the constraint (122) in order to compute the "relaxed measures" $\bar{Q}^s$ and $\bar{Q}^i$. The problem for determining $\bar{Q}^s$ is thus as follows.

Problem 5.5

$$\max_{h,\varphi} \quad \langle H, \varphi \rangle_\lambda, \tag{128}$$

subject to

$$\begin{aligned} \sigma h + \langle \delta, \varphi \rangle_\lambda + R &= 0, & (129) \\ h^2 + \|\varphi\|_\lambda^2 &\leq B^2. & (130) \end{aligned}$$

This is now a standard optimization problem in $L^2[X, \lambda(dx)]$ and we can solve it by finding the extremal points of the associated Lagrangian. See [7] for details and all unexplained terminology from functional analysis below. First, however, we can simplify the problem by using (120) to eliminate h. Since the remaining constraint has to be binding we then have the following problem.

Problem 5.6

$$\max_{\varphi} \quad \langle H, \varphi \rangle, \tag{131}$$

subject to

$$2R\langle \delta, \varphi \rangle + \langle \delta, \varphi \rangle^2 + \sigma^2 \|\varphi\|^2 + R^2 - \sigma^2 B^2 = 0, \tag{132}$$

where we have suppressed λ in $\|\cdot\|_\lambda$ and $\langle \cdot, \cdot \rangle_\lambda$.

This is a standard programming problem in L^2, and the Lagrangian function L is given by

$$L(\varphi, \gamma) = \langle H, \varphi \rangle + \gamma \left\{ 2R\langle \delta, \varphi \rangle + \langle \delta, \varphi \rangle^2 + \sigma^2 \|\varphi\|^2 + R^2 - \sigma^2 B^2 \right\}. \tag{133}$$

From the Kuhn-Tucker Theorem (see [7]) we know that the optimal solution $\hat{\varphi}$ is an extremal point of L, i.e. a point where the Frechet derivative vanishes. Denoting the Frechet derivative of L w.r.t. φ by L_φ, we easily obtain

$$L_\varphi(\varphi, \gamma) = H + 2\gamma \left\{ R\delta + \langle \delta, \varphi \rangle \delta + \sigma^2 \varphi \right\}. \tag{134}$$

Thus the first order conditions for Problem (5.6) are

$$H + 2\gamma \left\{ R\delta + \langle \delta, \varphi \rangle \delta + \sigma^2 \varphi \right\} = 0. \tag{135}$$

Taking the inner product with δ in (135) gives us the relation

$$\langle H, \delta \rangle + 2\gamma \left\{ R\|\delta\|^2 + \langle \delta, \varphi \rangle \|\delta\|^2 + \sigma^2 \langle \delta, \varphi \rangle \right\} = 0, \tag{136}$$

and solving for $\langle \delta, \varphi \rangle$ we obtain

$$\langle \delta, \varphi \rangle = -\frac{\langle H, \delta \rangle + 2\gamma R \|\delta\|^2}{2\gamma \left\{ \|\delta\|^2 + \sigma^2 \right\}}. \tag{137}$$

Plugging this expression into (135) gives us the optimal φ as

$$\varphi = \frac{\langle H, \delta \rangle \delta}{2\gamma\sigma^2 \left\{ \|\delta\|^2 + \sigma^2 \right\}} - \frac{R\delta}{\|\delta\|^2 + \sigma^2} - \frac{H}{2\gamma\sigma^2}. \tag{138}$$

In order to determine the Lagrange multiplier γ we plug (137)-(138) into the constraint (132). After tedious calculations, we obtain the following quadratic equation for γ

$$\gamma^2 = \frac{\sigma^2 \langle H, \delta \rangle^2 + \langle H, \delta \rangle \|\delta\|^2 + \|H\|^2 K^2 - 2\langle H, \delta \rangle^2 K^2}{4\sigma^4 K \left\{ B^2 K - R^2 \right\}}, \tag{139}$$

where

$$K = \|\delta\|^2 + \sigma^2. \tag{140}$$

The Lagrange multiplier is the positive root of this equation. We thus have the following pricing result.

Theorem 5.1 *The upper and lower relaxed good deal bound pricing functions, $\bar{V}^s(t,s)$ and $\bar{V}^i(t,s)$ are given as the solutions of the PIDE*

$$\frac{\partial V}{\partial t} + \mathbf{A}^\varphi V - rV = 0, \tag{141}$$

$$V(T,s) = \Phi(s,y), \tag{142}$$

where $\mathbf{A}^\varphi$ is given by

$$\begin{aligned} \mathbf{A}^\varphi V(t,s) &= \frac{\partial V}{\partial s}(t,s)s \left\{ r - \int_X \delta(s,x) \left\{ 1 + \varphi(t,s,x) \right\} \lambda(dx) \right\} \\ &+ \int_X \Delta V(t,s,y,x) \left\{ 1 + \varphi(t,s,x) \right\} \lambda(dx) + \frac{1}{2} s^2 \sigma^2 \frac{\partial^2 V}{\partial s^2}(t,s), \end{aligned}$$

with $\varphi = \bar{\varphi}^s$ and $\varphi = \bar{\varphi}^i$ respectively. The optimal relaxed kernels $\bar{\varphi}^s$ and $\bar{\varphi}^i$ are given by (138), where γ is the positive, resepctively negative, root of (139).

The question now arises if the positivity constraint really is binding or not at the optimal point. We have no general theoretical results concerning this question, but our numerical experience (see Section 5.5 below) indicates strongly that the constraint is indeed binding in the generic case. The implication of this negative fact is that, in the generic case, the static problem has to be solved numerically. In the next section we study a concrete numerical example where we discuss the numerical solution of the static problem in some more detail.

5.5 A Numerical Example

In the graphs below we provide the numerical results for a special case of the Wiener compound Poisson model described above. The model under consideration is the Merton jump diffusion stock price model of [8], where relative jump size has a lognormal distribution. In terms of the notation above this means that $X = [-1, \infty)$, $\delta(x) = x$ and

$$\lambda(dx) = \lambda_0 f(x)dx \tag{143}$$

where λ_0 is the intensity of the underlying Poisson process, and f is the density of the lognormal distribution. In the first graph we see the upper and lower relaxed bounds. In between these we find the optimal bounds, and in the middle we have the price generated by the minimal martingale measure. In the second figure we show how the MMM price curve relates to the price curve generated by the merton optinon pricing formula (where, by assumption, the market price of jump risk equals zero).

The minimal martingale measure price and the relaxed pricing bounds have been obtained by plugging the relevant kernels from the previous sections into the pricing PIDE, and then solving the PIDE numerically. In order to obtain the optimal pricing bounds (including the positivity constraint) we have solved the static problem numerically (for each point in the discretized state space), using an interior point algorithm, kindly provided to us by Mathias Stolpe. The kernels thus obtaind have then been fed into the PIDE which has been soled numerically.

We have used the following parameter values. Maximum grid size $M = 120$, grid stock price step $\delta S = 1$, grid time step $\delta t = 0,0003125$, time to maturity $TT = 0,25$, number of steps $T = 800$, interest rate $r = 0,05$, strike price $K = 100$, volatility $\sigma = 0,15$, Poisson intensity $\lambda_0 = 0,1$, $\alpha = -0,1$, $B = 1$, the parameters for the normal distribution generating the lognormal jump distribution were: mean $0,89$, standard deviation $0,45$

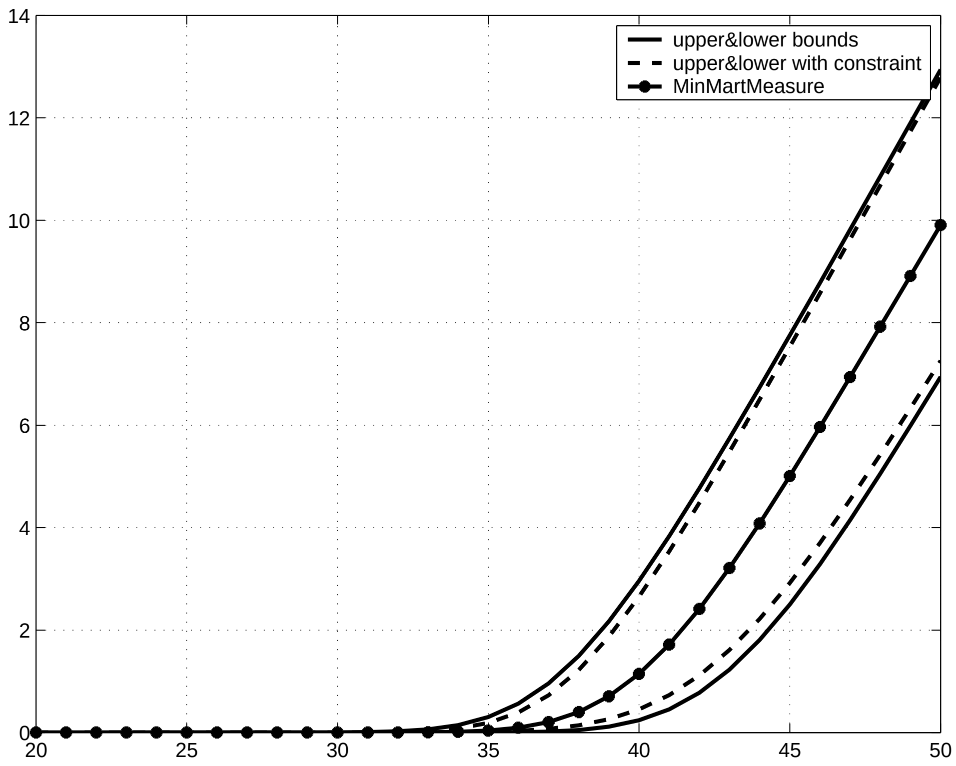

Figure 1: Good deal pricing bounds

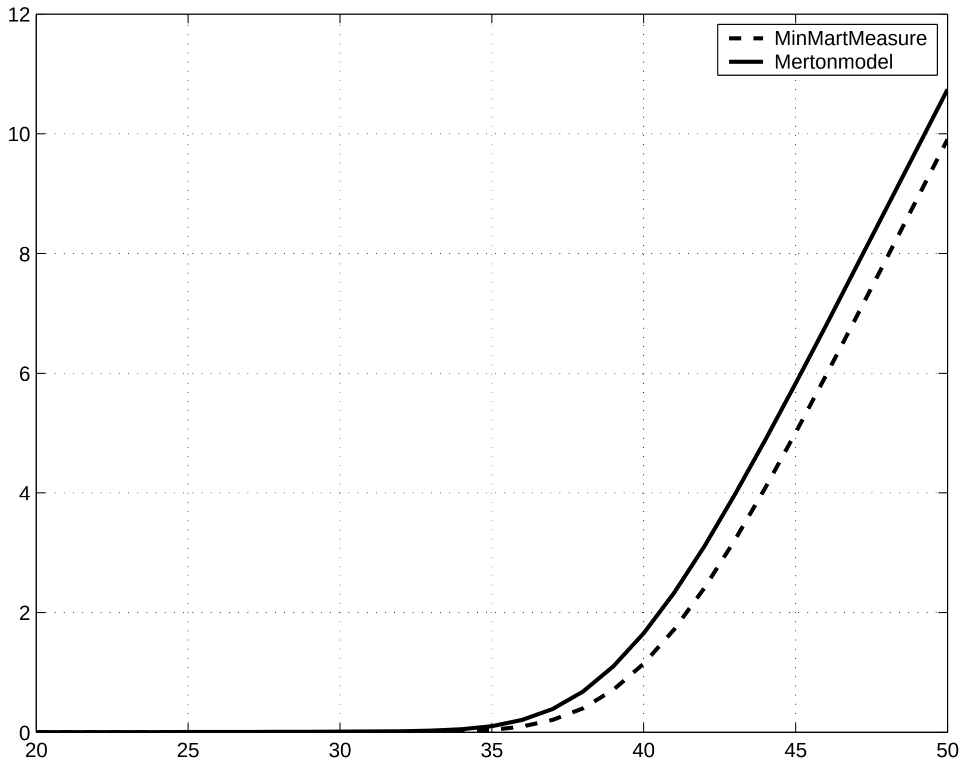

Figure 2: The minimal martingale measure and the Merton model

A Purely Wiener-Driven Models

Although the main object of the present paper is to study good deal bounds in the presence of a marked point process we will, in this appendix and for completeness sake, also study pricing in our model without a driving point process, i.e we will study the special case of a purely Wiener-driven model. This is an extension, although a very modest one, of the model originally considered in Cochrane and Saa-Requejo (2000), the main difference being that we do not need a certain rank condition assumed by Cochrane and Saa-Requejo. We derive the general HJB equation for the upper (lower) price bounds and given the appropriate rank condition we derive the pricing PDE presented by Cochrane and Saa-Requejo.

A.1 The General Case

We recall our model for the purely Wiener driven situation.

Assumption A.1

1. *The price and factor dynamics under objective probability measure P are given by*

$$\begin{aligned} dS_t^i &= S_t^i \alpha_i \left(S_t, Y_t\right) dt + S_t^i \sigma_i(S_t, Y_t) dW_t, \quad i = 1, \dots, n \\ dY_t^j &= a_j \left(S_t, Y_t\right) dt + b_j(S_t, Y_t) dW_t, \quad j = 1, \dots, k \end{aligned}$$

2. *We assume that for each i and j, $\alpha_i(s, y)$ and $a_j(s, y)$ are deterministic scalar functions, $\sigma_i(s, y)$ and $b_j(s, y)$ are deterministic row vector functions.*

3. *All functions above are assumed to be regular enough to allow for the existence of a unique strong solution for the system of SDEs.*

4. *We assume the existence of a short rate r of the form*

$$r_t = r(S_t, Y_t).$$

5. *We assume that the model is free of arbitrage in the sense that there exists a (not necessarily unique) risk neutral martingale measure Q.*

From Theorem 4.1 we see that the upper good deal bound function $V(t, s, y)$ satisfies the following boundary value problem

$$\frac{\partial V}{\partial t}(t, s, y) + \sup_h \left\{ \mathbf{A}^h V(t, s, y) \right\} - rV(t, s, y) = 0, \tag{144}$$

$$V(T, s, y) = \Phi(s, y), \tag{145}$$

where we for the moment suppress all the constraints, and where the infinitesimal operator $A^{h,\varphi}$ is given by

$$\begin{aligned} \mathbf{A}^h V(t, s, y) &= \sum_{i=1}^{n} \frac{\partial V}{\partial s_i}(t, s, y) s_i r \\ &+ \sum_{j=1}^{k} \frac{\partial V}{\partial y_j}(t, s, y) \left\{ a_j(s, y) + b_j(s, y) h(t, s, y) \right\} \\ &+ \frac{1}{2} \sum_{i,l=1}^{n} \frac{\partial^2 V}{\partial s_i \partial s_l}(t, s, y) s_i s_l \sigma_i^\star(s, y) \sigma_l(s, y) \\ &+ \frac{1}{2} \sum_{j,l=1}^{k} \frac{\partial^2 V}{\partial y_j \partial y_l}(t, s, y) b_j^\star(s, y) b_l(s, y) \\ &+ \sum_{i,j=1}^{k} \frac{\partial^2 V}{\partial s_i \partial y_j}(t, s, y) s_i \sigma_i^\star(s, y) b_j(s, y). \end{aligned} \tag{146}$$

Thus, we have the following static optimization problem

Problem A.1

$$\max_h \sum_{j=1}^{k} \frac{\partial V}{\partial y_j}(t,s,y)\left\{b_j(s,y)h(t,s,y)\right\} \tag{147}$$

subject to the constraints

$$\alpha_i + \sigma_i h \quad = \quad r, \quad i = 1, \ldots, n \tag{148}$$

$$\|h\|^2_{R^d} \quad \leq \quad A^2. \tag{149}$$

This is a very simple finite dimensional optimization problem and, using standard Kuhn-Tucker techniques, we have the following result.

Proposition A.1 *Denote the excess return $\alpha - r$ by R.*

- *The upper (lower) good-deal bound function $V(t,s,y)$ satisfies the following boundary value problem*

$$\frac{\partial V}{\partial t} + r\sum_{i=1}^{n} \frac{\partial V}{\partial s_i} s_i + \sum_{j=1}^{k} \frac{\partial V}{\partial y_j}\left\{a_j + b_j \hat{h}\right\} + \frac{1}{2}\sum_{j,l=1}^{k} \frac{\partial^2 V}{\partial s_i \partial s_l} s_i s_l \sigma_i^\star \sigma_l$$

$$+\frac{1}{2}\sum_{j,l=1}^{k} \frac{\partial^2 V}{\partial y_j \partial y_l} b_j^\star b_l + \sum_{i,j=1}^{k} \frac{\partial^2 V}{\partial s_i \partial y_j} s_i \sigma_i^\star b_j - rV = 0 \tag{150}$$

$$V(T,s,y) = \Phi(s,y). \tag{151}$$

- *For the upper bound, the kernel $\hat{h} = \hat{h}_{\max}$ is given by*

$$\hat{h}_{\max} = b'V_y - \sigma'(\sigma\sigma')^{-1}\sigma b'V_y + \frac{\sigma'(\sigma\sigma')^{-1}R\sqrt{V_y'b\left\{I - \sigma'(\sigma\sigma')^{-1}\sigma\right\}b'V_y}}{\sqrt{A^2 - R'(\sigma\sigma')^{-1}R}} \tag{152}$$

- *For the lower bound, the kernel $\hat{h} = \hat{h}_{\min}$ is given by*

$$\hat{h}_{\min} = b'V_y - \sigma'(\sigma\sigma')^{-1}\sigma b'V_y - \frac{\sigma'(\sigma\sigma')^{-1}R\sqrt{V_y'b\left\{I - \sigma'(\sigma\sigma')^{-1}\sigma\right\}b'V_y}}{\sqrt{A^2 - R'(\sigma\sigma')^{-1}R}} \tag{153}$$

Here we have used the notation $V_y = \left(\frac{\partial V}{\partial y_1}, \ldots, \frac{\partial V}{\partial y_k}\right)^\star$

A.2 The Cochrane and Saa-Requejo Model

The pricing PDE of Cochrane and Saa-Requejo (2000) can now be obtained as a particular case of our slightly more general model above. In terms of our notation, the Cochrane and Saa-Requejo model is specified by the following set of assumptions.

Assumption A.2

- *The price dynamics are assumed to be of the form*

$$dS_t^i = S_t^i \alpha_i(S_t, Y_t, t)dt + S_t^i \sigma_i(S_t, Y_t, t)dZ_t, \quad i = 1, \ldots, n, \tag{154}$$

were Z is an n-dimensional standard Wiener process.

- *The factor dynamics are assumed to be of the form*

$$dY_t^j = a_j(S_t, Y_t, t)dt + b_j^z(S_t, Y_t, t)dZ + b_j^w(S_t, Y_t, t)dW_t, \quad j = 1, \ldots, k, \tag{155}$$

where W is a k-dimensional standard Wiener process orthogonal to Z.

- *The $n \times n$ volatility matrix σ is assumed to be invertible.*

Given this assumption we have a d dimensional driving Wiener process (Z, W), where $d = n + k$, so the kernel process h is now $(n+k)$-dimensional and can be decomposed as $h = (h_z, h_w)$. The martingale condition (148) will take the form

$$\alpha_i + \sigma_i h^z = r, \quad i = 1, \ldots, n \tag{156}$$

and the invertibility assumption above allows us to solve for h^z to obtain

$$h_z = \sigma^{-1}(r - \alpha). \tag{157}$$

Thus, the static problem simplifies as follows.

Problem A.2

$$\max_h \sum_{j=1}^k \frac{\partial V}{\partial y_j}(t, s, y) \{b_j(s, y) h_w(t, s, y)\} \tag{158}$$

subject to the constraints

$$h_w' h_w \leq A^2 - R'R.$$

This is a trivial linear-quadratic optimization problem and the optimal h_w can be easily found as

$$h_w = \frac{\sqrt{A^2 - R'R}}{\sqrt{b'V_y V_y' b}} \cdot b'V_y.$$

We have thus determined the entire optimal vector $h = (h_z, h_w)$, and substituting this into the pricing equation of Proposition A.1 one obtains the pricing PDE of Cochrane and Saa-Requejo (2000).

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