Bond Market Structure in the Presence of Marked Point Processes

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Abstract

We investigate the term structure of zero coupon bonds when interest rates are driven by a general marked point process as well as by a Wiener process. Developing a theory which allows for measure-valued trading portfolios we study existence and uniqueness of a martingale measure. We also study completeness and its relation to the uniqueness of a martingale measure. For the case of a finite jump spectrum we give a fairly general completeness result and for a Wiener–Poisson model we prove the existence of a time- independent set of basic bonds. We also give sufficient conditions for the existence of an affine term structure.

Key words: bond market, term structure of interest rates, jump-diffusion model, measure-valued portfolio, arbitrage, market completeness, martingale operator, hedging operator, affine term structure.

1 Introduction

One of the most challenging mathematical problems arising in the theory of financial markets concerns market completeness, i.e. the possibility of duplicating a contingent claim by a self-financing portfolio. Informally, such a possibility arises whenever there are as many risky assets available for hedging as there are independent sources of randomness in the market.

In bond markets as well as in stock markets it seems reasonable to take into account the possible occurrence of jumps, considering not only the simple Poisson jump models, but also marked point process models allowing a continuous jump spectrum. However, introducing a continuous jump spectrum also introduces a possibly infinite number of independent sources of randomness and, as a consequence, completeness may be lost.

In traditional stock market models there are usually only a finite number of basic assets available for hedging, and in order to have completeness one usually assumes that their prices are driven by a finite number (equaling the number of basic assets) of Wiener processes. More realistic jump-diffusion models seem to encounter some skepticism precisely due to the completeness problems mentioned above.

There is, however, a fundamental difference between stock and bond markets: while in stock markets portfolios are naturally limited to a finite number of basic assets, in bond markets there is at least the theoretical possibility of having portfolios with an infinite number of assets, namely bonds with a continuum of possible maturities. Since all modern continuous time models of bond markets assume the existence of bonds with a continuum of maturities, it seems reasonable to require that a coherent theory of bond markets should allow for portfolios consisting of uncountably many bonds. We also see from the discussion above that, in models with a continuous jump spectrum, such portfolios are indeed necessary if completeness is not to be lost.

It is worth noticing that also in stock market models one may consider a continuum of derivative securities, such as e.g. options parameterized by maturities and/or strikes.

The purpose of our paper is to present an approach which, on one hand, allows bond prices to be driven also by marked point processes while, on the other hand, admitting portfolios with an infinite number of securities. As such, this approach appears to be new and leads to the two mathematical problems of:

- an appropriate modeling of the evolution of bond prices and their forward rates;
- a correct definition of infinite-dimensional portfolios of bonds and the corresponding value processes by viewing trading strategies as measure-valued processes.

A further point of interest in this context is that, in stock markets and under general assumptions, completeness of the market is equivalent to uniqueness of the martingale measure. The question now arises whether this fact remains true also in bond markets when marked point processes with continuous mark spaces, i.e. an infinite number of sources of randomness, are allowed? One of the main results of this paper is that, at this level of generality, uniqueness of the martingale measure implies only that the set of hedgeable claims is dense in the set of all contingent claims. This phenomenon is not entirely unexpected and has been observed by different authors (see, e.g., definition of quasicompleteness in [24]); its nature is transparent on the basis of elementary functional analysis which we rely upon in Section 4.

The main results of the paper are as follows.

- We give conditions for the existence of a martingale measure in terms of conditions on the coefficients for the bond- and forward rate dynamics. In particular we extend the Heath–Jarrow–Morton "drift condition" to point process models.
- We show that the martingale measure is unique if and only if certain integral operators of the first kind (the "martingale operators") are injective.

- We show that a contingent claim can be replicated by a self-financing portfolio if and only if certain integral equations of the first kind (the "hedging equations") have solutions. Furthermore, the integral operators appearing in these equations (the "hedging operators") turn out to be adjoint of the martingale operators.

- We show that uniqueness of the martingale measure is equivalent to the denseness of the image space of the hedging operators. In particular, it turns out that in the case with a continuous jump spectrum, uniqueness of the martingale measure does not imply completeness of the bond market. Instead, uniqueness of the martingale measure is shown to be equivalent to approximate completeness of the market.

- Under additional conditions on the forward rate dynamics we can give a rather explicit characterization of the set of hedgeable claims in terms of certain Laplace transforms.

- In particular, we study the model with a finite mark space (for the jumps) showing that in this case one may hedge an arbitrary claim by a portfolio consisting of a finite number of bonds, having essentially arbitrary but different maturities. This considerably extends and clarifies a previous result by Shirakawa [28].

- We give sufficient conditions for the existence of a so-called affine term structure (ATS) for the bond prices.

The paper has the following structure. In Section 2 we lay the foundations and we present a "toolbox" of propositions which explain the interrelations between the dynamics of the forward rates, the bond prices and the short rate of interest.

In Section 3 we define our measure-valued portfolios with their value processes and investigate the existence and uniqueness of a martingale measure. We also give the martingale dynamics of the various objects, leading among other things to a HJM-type "drift condition".

In a stock market, the current state of a portfolio is a vector of quantities of securities held at time t which can be identified with a linear functional; it gives the portfolio value being applied to the current asset price vector. In a bond market, the latter is substituted by a price curve which one can consider as a vector in a space of continuous functions. By analogy, it is natural to identify a current state of a portfolio with a linear functional, i.e. with an element of the dual space, a signed finite measure. So, our approach is based on a kind of stochastic integral with respect to the price curve process though we avoid a more technical discussion of this aspect here (see [4]).

In Section 4 we study uniqueness of the martingale measure and its relation to the completeness of the bond market. Section 5 is devoted to a more detailed study of two cases when we can characterize the set of hedgeable claims. In 5.1 we consider a class of models with infinite mark space which leads us to Laplace transform theory and in 5.2 we explore the case of a finite mark space. We end by discussing the existence of affine term structures in Section 6.

For the case of Wiener-driven interest rates there is an enormous number of papers. For general information about arbitrage free markets we refer to the book [13] by Duffie. Basic papers in the area are Harrison–Kreps [17], Harrison–Pliska [18]. For interest rate theory we recommend Artzner–Delbaen [1] and some other important references can be found in the bibliography; the recent book by Dana and Jeanblanc-Picqué [10] contains a comprehensive account of main models.

Very little seems to have been written about interest rate models driven by point processes. Shirakawa [28], Björk [3], and Jarrow–Madan [23] all consider an interest rate model of the type to be discussed below for the case when the mark space is finite, i.e. when the model is driven by a finite number of counting processes. (Jarrow–Madan also consider the interplay between the stock- and the bond market). In the present paper we focus primarily on the case of an infinite mark space, but the interest rate models above are included as special cases of our model, and our results for the finite case amount to a considerable extension of those in[28].

In an interesting preprint, Jarrow–Madan [24] consider a fairly general model of asset prices driven by semimartingales. Their mathematical framework is that of topological vector spaces and, using a concept of quasicompleteness, they obtain denseness results which are related to ours.

Babbs and Webber [2] study a model where the short rate is driven by a finite number of counting processes. The counting process intensities are driven by the short rate itself and by an underlying diffusion-type process. Lindberg–Orszag–Perraudin [25] consider a model where the short rate is a Cox process with a squared Ornstein–Uhlenbeck process as intensity process. Using Karhunen–Loève expansions they obtain quasi-analytic formulas for bond prices.

Structurally the present paper is based on Björk [3] where only the finite case is treated. The working paper Björk–Kabanov–Runggaldier

[5] contains some additional topics not treated here. In particular some pricing formulas are given, and the change of numéraire technique developed by Geman et. al. in [16] is applied to the bond market. In a forthcoming paper [4] we develop the theory further by studying models driven by rather general Lévy processes, and this also entails a study of stochastic integration with respect to C-valued processes. In the present exposition we want to focus on financial aspects, so we try to avoid, as far as possible, details and generalizations (even straightforward ones) if they lead to mathematical sophistications. For the present paper the main reference concerning point processes and Girsanov transformations are Brémaud [7] and Elliott [15]. For the more complicated paper [4], the excellent (but much more advanced) exposition by Jacod and Shiryaev [22] is the imperative reference.

Throughout the paper we use the Heath–Jarrow–Morton parameterization, i.e. forward rates and bond prices are parameterized by time **of** maturity T. In certain applications it is more convenient to parameterize forward rates by instead using the time **to** maturity, as is done in Brace-Musiela [6]. This can easily be accomplished, since there exists a simple set of translation formulae between the two ways of parametrization.

2 Relations between $df(t,T)$, $dp(t,T)$, and dr_t

We consider a financial market model "living" on a stochastic basis (filtered probability space) $(\Omega, \mathcal{F}, \mathbf{F}, P)$ where $\mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}$. The basis is assumed to carry a Wiener process W as well as a marked point process $\mu(dt, dx)$ on a measurable Lusin mark space $(E, \mathcal{E})$ with compensator $\nu(dt, dx)$. We assume that $\nu([0,t] \times E) < \infty$ P-a.s. for all finite t, i.e. μ is a multivariate point process in the terminology of [22].

The main assets to be considered on the market are zero coupon bonds with different maturities. We denote the price at time t of a bond maturing at time T (a "T-bond") by $p(t,T)$.

Assumption 2.1 *We assume that*

1. *There exists a (frictionless) market for T-bonds for every $T > 0$.*

2. *For every fixed T, the process $\{p(t,T);\ 0 \leq t \leq T\}$ is an optional stochastic process with $p(t,t) = 1$ for all t.*

3. *For every fixed t, $p(t,T)$ is P-a.s. continuously differentiable in the T-variable. This partial derivative is often denoted by*

$$p_T(t,T) = \frac{\partial p(t,T)}{\partial T}.$$

We now define the various interest rates.

Definition 2.2 *The instantaneous* **forward rate** *at T, contracted at t, is given by*

$$f(t,T) = -\frac{\partial \log p(t,T)}{\partial T}.$$

The **short rate** *is defined by*

$$r_t = f(t,t).$$

The **money account** *process is defined by*

$$B_t = \exp\left\{\int_0^t r_s ds\right\},$$

i.e.

$$dB_t = r_t B_t dt, \quad B_0 = 1.$$

For the rest of the paper we shall, either by implication or by assumption, consider dynamics of the following type.

Short rate dynamics

$$dr(t) = a_t dt + b_t dW_t + \int_E q(t,x)\mu(dt,dx), \tag{1}$$

Bond price dynamics

$$\begin{aligned} dp(t,T) &= p(t,T)m(t,T)dt + p(t,T)v(t,T)dW_t \\ &+ p(t-,T)\int_E n(t,x,T)\mu(dt,dx), \end{aligned} \tag{2}$$

Forward rate dynamics

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t + \int_E \delta(t,x,T)\mu(dt,dx). \tag{3}$$

In the above formulas the coefficients are assumed to meet standard conditions required to guarantee that the various processes are well defined.

We shall now study the formal relations which must hold between bond prices and interest rates. These relations hold regardless of the measure under consideration, and in particular we do *not* assume that markets are free of arbitrage. We shall, however, need a number of technical assumptions which we collect below in an "operational" manner.

Assumption 2.3

1. *For each fixed ω, t and, (in appropriate cases) x, all the objects $m(t,T)$, $v(t,T)$, $n(t,x,T)$, $\alpha(t,T)$, $\sigma(t,T)$, and $\delta(t,x,T)$ are assumed to be continuously differentiable in the T -variable. This partial T-derivative sometimes is denoted by $m_T(t,T)$ etc.*

2. *All processes are assumed to be regular enough to allow us to differentiate under the integral sign as well as to interchange the order of integration.*

3. *For any t the price curves $p(\omega, t, .)$ are bounded functions for almost all ω.*

This assumption is rather *ad hoc* and one would, of course, like to give conditions which *imply* the desired properties above. This can be done but at a fairly high price as to technical complexity. As for the point process integrals, these are made trajectorywise, so the standard Fubini theorem can be applied. For the stochastic Fubini theorem for the interchange of integration with respect to dW and dt see Protter [26] and also Heath–Jarrow–Morton [19] for a financial application.

Proposition 2.4

1. *If $p(t,T)$ satisfies (2), then for the forward rate dynamics we have*

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t + \int_E \delta(t,x,T)\mu(dt,dx),$$

where α, σ and δ are given by

$$\left\{ \begin{array}{rcl} \alpha(t,T) & = & v_T(t,T) \cdot v(t,T) - m_T(t,T), \\ \sigma(t,T) & = & -v_T(t,T), \\ \delta(t,x,T) & = & -n_T(t,x,T) \cdot [1 + n(t,x,T)]^{-1}. \end{array} \right. \tag{4}$$

2. If $f(t,T)$ satisfies (3) then the short rate satisfies

$$dr_t = a_t dt + b_t dW_t + \int_E q(t,x)\mu(dt,dx),$$

where

$$\begin{cases} a_t &= f_T(t,t) + \alpha(t,t), \\ b_t &= \sigma(t,t), \\ q(t,x) &= \delta(t,x,t). \end{cases} \tag{5}$$

3. If $f(t,T)$ satisfies (3) then $p(t,T)$ satisfies

$$\begin{aligned} dp(t,T) &= p(t,T)\left\{r_t + A(t,T) + \frac{1}{2}S^2(t,T)dt\right\} + p(t,T)S(t,T)dW_t \\ &+ p(t-,T)\int_E \left\{e^{D(t,x,T)} - 1\right\}\mu(dt,dx), \end{aligned}$$

where

$$\begin{cases} A(t,T) &= -\int_t^T \alpha(t,s)ds, \\ S(t,T) &= -\int_t^T \sigma(t,s)ds, \\ D(t,x,T) &= -\int_t^T \delta(t,x,s)ds. \end{cases} \tag{6}$$

Proof. The first part of the Proposition follows immediately if we apply the Itô formula to the process $\log p(t,T)$, write this in integrated form and differentiate with respect to T.

For the second part we integrate the forward rate dynamics to get

$$\begin{aligned} r_t &= f(0,t) + \int_0^t \alpha(s,t)ds + \int_0^t \sigma(s,t)dW_s \\ &+ \int_0^t \int_E \delta(s,x,t)\mu(ds,dx). \end{aligned} \tag{7}$$

Now we can write

$$\begin{aligned} \alpha(s,t) &= \alpha(s,s) + \int_s^t \alpha_T(s,u)du, \\ \sigma(s,t) &= \sigma(s,s) + \int_s^t \sigma_T(s,u)du, \\ \delta(s,x,t) &= \delta(s,x,s) + \int_s^t \delta_T(s,x,u)du, \end{aligned}$$

and, inserting this into (7) we have

$$\begin{aligned} r_t &= f(0,t) + \int_0^t \alpha(s,s)ds + \int_0^t \int_s^t \alpha_T(s,u)duds \\ &+ \int_0^t \sigma(s,s)dW_s + \int_0^t \int_s^t \sigma_T(s,u)dudW_s \\ &+ \int_0^t \int_E \delta(s,x,s)\mu(ds,dx) + \int_0^t \int_E \int_s^t \delta_T(s,x,u)du\mu(ds,dx). \end{aligned}$$

Changing the order of integration and identifying terms gives us the result.

For the third part we adapt a technique from Heath–Jarrow–Morton [19]. Using the definition of the forward rates we may write

$$p(t,T) = \exp\{Z(t,T)\} \tag{8}$$

where Z is given by

$$Z(t,T) = -\int_t^T f(t,s)ds. \tag{9}$$

Writing (3) in integrated form, we obtain

$$f(t,s) = f(0,s) + \int_0^t \alpha(u,s)du + \int_0^t \sigma(u,s)dW_u + \int_0^t \int_E \delta(u,x,s)\mu(du,dx).$$

Inserting this expression into (9), splitting the integrals and changing the order of integration gives us

$$\begin{aligned}
&Z(t,T) = -\int_t^T f(0,s)ds - \int_0^t \int_t^T \alpha(u,s)dsdu - \int_0^t \int_t^T \sigma(u,s)dsdW_u \\
&- \int_0^t \int_t^T \int_E \delta(u,x,s)ds\mu(du,dx) \\
&= -\int_0^T f(0,s)ds - \int_0^t \int_u^T \alpha(u,s)dsdu - \int_0^t \int_u^T \sigma(u,s)dsdW_u \\
&- \int_0^t \int_u^T \int_E \delta(u,x,s)ds\mu(du,dx) \\
&+ \int_0^t f(0,s)ds + \int_0^t \int_u^t \alpha(u,s)dsdu + \int_0^t \int_u^t \sigma(u,s)dsdW_u \\
&+ \int_0^t \int_u^t \int_E \delta(u,x,s)ds\mu(du,dx) \\
&= Z(0,T) - \int_0^t \int_u^T \alpha(u,s)dsdu - \int_0^t \int_u^T \sigma(u,s)dsdW_u \\
&- \int_0^t \int_u^T \int_E \delta(u,x,s)ds\mu(du,dx) \\
&+ \int_0^t f(0,s)ds + \int_0^t \int_0^s \alpha(u,s)duds + \int_0^t \int_0^s \sigma(u,s)dW_u ds \\
&+ \int_0^t \int_0^s \int_E \delta(u,x,s)\mu(du,dx)ds.
\end{aligned}$$

Now we can use the fact that $r_s = f(s,s)$ and, integrating the forward rate dynamics (3) over the interval $[0,s]$, we see that the last two lines above equal $\int_0^t r_s ds$ so we finally obtain

$$\begin{aligned} Z(t,T) &= Z(0,T) + \int_0^t r_s ds - \int_0^t \int_u^T \alpha(u,s) ds du - \int_0^t \int_u^T \sigma(u,s) ds dW_u \\ &- \int_0^t \int_u^T \int_E \delta(u,x,s) ds \mu(du,dx). \end{aligned}$$

Thus, with A, S and D as in the statement of the proposition, the stochastic differential of Z is given by

$$dZ(t,T) = \{r_t + A(t,T)\} dt + S(t,T) dW_t + \int_E D(t,x,T) \mu(dt,dx),$$

and an application of the Itô formula to the process $p(t,T) = \exp\{Z(t,T)\}$ completes the proof. ∎

Remark 2.5 To fit reality, a "good" model of bond price dynamics or interest rates must satisfy other important conditions. A bond price process "should" e.g. take values in the interval $[0,1]$ and forward rates "ought" to be positive (see [27]). We do not restrict ourselves to the class of "realistic models" (obviously the most important ones) since we also want to treat generalizations of "bad" models (like the various Gaussian models for the short rate) which are useful because their simplicity leads to instructive explicit formulas.

3 Absence of arbitrage

3.1 Generalities

The purpose of this section is to give the appropriate definitions of self-financing measure-valued portfolios, contingent claims, arbitrage possibilities and martingale measures. We then proceed to show that the existence of a martingale measure implies absence of arbitrage, and we end the section by investigating existence and uniqueness of martingale measures.

We make the following standing assumption for the rest of the section.

Assumption 3.1 *We assume that*

(i) *There exists an asset (usually referred to as locally risk-free) with the price process*

$$B_t = \exp\left\{\int_0^t r_s ds\right\}.$$

(ii) *The filtration* $\mathbf{F} = (\mathcal{F}_t)$ *is the natural filtration generated by* W *and* μ, *i.e.*

$$\mathcal{F}_t = \sigma\{W_s,\ \mu([0,s] \times A),\ B;\ 0 \le s \le t,\ A \in \mathcal{E},\ B \in \mathcal{N}\}$$

where $\mathcal{N}$ *is the collections of* P*-null sets from* $\mathcal{F}$.

(iii) *The point process* μ *has an intensity* λ, *i.e. the* P*-compensator* ν *has the form*

$$\nu(dt, dx) = \lambda(t, dx)dt$$

where $\lambda(t, A)$ *is a predictable process for all* $A \in \mathcal{E}$.

(iv) *The stochastic basis has the predictable representation property: any local martingale* M *is of the form*

$$M_t = M_0 + \int_0^t f_s dW_s + \int_0^t \int_E \psi(s,x)(\mu(ds,dx) - \nu(ds,dx))$$

where f *is a process measurable with respect to the predictable* σ*-algebra* $\mathcal{P}$ *and* ψ *is a* $\tilde{\mathcal{P}}$*-measurable function* ($\tilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{E}$) *such that for all finite* t

$$\int_0^t |f_s|^2 ds < \infty, \qquad \int_0^t \int_E |\psi(s,x)|\nu(ds,dx) < \infty.$$

We need (ii) and (iv) above in order to have control over the class of absolute continuous measure transformations of the basic ("objective") probability measure P. These assumptions are made largely for convenience, but if we omit them, some of the equivalences proved below will be weakened to one-side implications. See [4] for further information. The assumption (iii) is not really needed at all from a logical point of view, but it makes some of the formulas below much easier to read.

3.2 Self-financing portfolios

Definition 3.2 *A* **portfolio** *in the bond market is a pair* $\{g_t, h_t(dT)\}$, *where*

1. *The component* g *is a predictable process.*

2. *For each* ω, t, *the set function* $h_t(\omega, \cdot)$ *is a signed finite Borel measure on* $[t, \infty)$.

3. *For each Borel set* A *the process* $h_t(A)$ *is predictable.*

The intuitive interpretation of the above definition is that g_t is the number of units of the risk-free asset held in the portfolio at time t. The object $h_t(dT)$ is interpreted as the "number" of bonds, with maturities in the interval $[T, T+dT]$, held at time t.

We will now give the definition of an admissible portfolio.

Definition 3.3

1. *The* **discounted bond prices** $Z(t,T)$ *are defined by*

$$Z(t,T) = \frac{p(t,T)}{B(t)}.$$

2. *A portfolio* $\{g,h\}$ *is said to be* **feasible** *if the following conditions hold for every* t*:*

$$\int_0^t |g_s| ds < \infty, \tag{10}$$

$$\int_0^t \int_s^\infty |m(s,T)||h_s(dT)| ds < \infty, \tag{11}$$

$$\int_0^t \int_E \int_s^\infty |n(s,x,T)||h_s(dT)| \nu(ds,dx) < \infty, \tag{12}$$

$$\int_0^t \left\{ \int_s^\infty |v(s,T)||h_s(dT)| \right\}^2 ds < \infty. \tag{13}$$

3. *The* **value process** *corresponding to a feasible portfolio* $\{g,h\}$ *is defined by*

$$V_t = g_t B_t + \int_t^\infty p(t,T) h_t(dT). \tag{14}$$

4. *The* **discounted value process** *is*

$$V_t^Z = B_t^{-1} V_t. \tag{15}$$

5. *A feasible portfolio is said to be* **admissible** *if there is a number* $a \geq 0$ *such that* $V_t^Z \geq -a \ P-a.s.$ *for all* t.

6. *A feasible portfolio is said to be* **self-financing** *if the corresponding value process satisfies*

$$\begin{aligned} V_t &= V_0 + \int_0^t g_s dB_s + \int_0^t \int_s^\infty m(s,T) p(s,T) h_s(dT) ds \\ &+ \int_0^t \int_s^\infty v(s,T) p(s,T) h_s(dT) dW_s \\ &+ \int_0^t \int_s^\infty \int_E n(s,x,T) p(s-,T) h_s(dT) \mu(ds,dx). \end{aligned} \tag{16}$$

There are obvious modifications of these definitions like "admissible on the interval $[0, T_0]$".

The relation (16) is a way of making mathematical sense out of the expression

$$dV_t = g_t dB_t + \int_t^\infty h_t(dT) dp(t,T) \tag{17}$$

which is the formal generalization of the standard self-financing condition. We shall sometimes use equation (17) as a shorthand notation for the equation (16). It seems natural that the adequate stochastic calculus for the theory of bond market has to include an integration of measure-valued processes with respect to jump-diffusion processes with values in some Banach space of continuous functions. Some versions of such a calculus are given in our paper [4].

We shall as usual be working much with discounted prices, and the following lemma shows that the self-financing condition is the same for the discounted bond prices $Z(t,T)$ as for the undiscounted ones.

Lemma 3.4 *For an admissible portfolio the following conditions are equivalent.*

(i) $dV_t = g_t dB_t + \int_t^\infty h_t(dT) dp(t,T)$,

(ii) $dV_t^Z = \int_t^\infty h_t(dT) dZ(t,T)$.

Proof. The Itô formula. ∎

Notice that for a self-financing portfolio the g-component is automatically defined by the initial endowment V_0 and the h-component; the pair (V_0, h) is sometimes called the investment strategy of a self-financing portfolio.

For technical purposes it is sometimes convenient to extend the definition of the bond price process $p(t,T)$ (as well as other processes) from the interval $[0,T]$ to the whole half-line. It is then natural to put $Z(t,T) = 1$, $A(t,T) = 0$ etc. for $t \geq T$, i.e. one can think that after the time of maturity the money is transferred to the bank account.

Remark 3.5 From the point of view of economics, discounting means that the locally risk-free asset is chosen as the "numéraire", i.e. the prices of all other assets are evaluated in the units of this selected one. Some mathematical properties may however change under a change of the numéraire, see [11].

We now go on to define contingent claims and arbitrage portfolios, modifying somewhat the standard concepts.

Definition 3.6

1. *A* **contingent** T**-claim** *is a random variable* $X \in L^0_+(\mathcal{F}_T, P)$ *(i.e. an arbitrary non-negative* $\mathcal{F}_T$*-measurable random variable). We shall use the notation* $L^0_{++}(\mathcal{F}_T, P)$ *for the set of elements* X *of* $L^0_+(\mathcal{F}_T, P)$ *with* $P(X > 0) > 0$*.*

2. *An* **arbitrage portfolio** *is an admissible self-financing portfolio* $\{g, h\}$ *such that the corresponding value process has the properties*

   (a) $V_0 = 0$,

   (b) $V_T \in L^0_{++}(\mathcal{F}_T, P)$.

   If no arbitrage portfolios exist for any $T \in \mathbf{R}_+$ *we say that the model is "free of arbitrage" or "arbitrage-free" (AF).*

We now want to tie absence of arbitrage to the existence of a martingale measures. Since we do not fix a (finite deterministic) time horizon, it turns out to be convenient to consider a martingale *density* process as a basic object (rather than a martingale **measure**).

Definition 3.7 *Take the measure* P *as given. We say that a positive martingale* $L = (L_t)_{t \geq 0}$ *with* $E^P[L_t] = 1$ *is a* **martingale density** *if for every* $T > 0$ *the process* $\{Z(t,T)L_t;\ 0 \leq t \leq T\}$ *is a* P*-local martingale. If, moreover,* $L_t > 0$ *for all* $t \in \mathbf{R}_+$ *we say that* L *is a* **strict martingale density**.

Definition 3.8 *We say that a probability* Q *on* $(\Omega, \mathcal{F})$ *is a* **martingale measure** *if* $Q_t \sim P_t$ *(where* $Q_t = Q|\mathcal{F}_t$*,* $P_t = P|\mathcal{F}_t$*) and the process* $\{Z(t,T);\ 0 \leq t \leq T\}$ *is a* Q*-local martingale for every* $T > 0$ *.*

In other words, Q is a martingale measure if it is locally equivalent to P and the density process dQ_t/dP_t is a strict martingale density.

Proposition 3.9 *Suppose that there exists a strict martingale density* L*. Then the model is arbitrage-free.*

Proof. Fix any admissible self-financing portfolio $\{g, h\}$ and assume that for some finite T the corresponding value process is such that $V_T \in L^0_{++}(\mathcal{F}_T, P)$. By admissibility, $V^Z \geq -a$ for some $a > 0$. The process $(V^Z + a)L$ is a positive local martingale hence a supermartingale. As L is a martingale, $V^Z L$ is a supermartingale. Thus, $E^P\left[V^Z_0 L_0\right] \geq E^P\left[V^Z_T L_T\right] > 0$, which is impossible because we assume that $V^Z_0 = 0$. ∎

Remark 3.10 Notice that for the model restricted to some finite time horizon T, a strict martingale density defines an **equivalent martingale measure** $Q^T = L_T P$, i.e. a probability which is equivalent to P on $\mathcal{F}_T$ (in symbols: $Q^T_T \sim P_T$) such that all discounted bond prices are martingales on $[0, T]$. If $E^P[L_\infty] = 1$, there exists an equivalent martingale measure also for the infinite horizon and the above proposition can be easily extended to this case in an obvious way. In general, when L is not uniformly integrable, a measure Q on $\mathcal{F}$ such that $L_t = dQ_t/dP_t$, may not exist. The following simple example when a martingale density does not define Q explains the situation.

Let the stochastic basis be the coordinate space of counting functions $N = (N_t)$ equipped with the measure of the unit rate Poisson process. Let us modify this space by excluding only one point: the function which is identically zero. It is clear that the process $L_t = I_{\{N_t=0\}}e^t$ is a martingale density defining Q^T for every finite T (under Q^T the coordinate process has the intensity zero on $I_{[T,\infty]}$) but the measure Q such that $Q|\mathcal{F}_T = Q^T|\mathcal{F}_T$ for all T does not exist.

This example reveals that the origin of such an undesirable property lies in a certain pathology of the stochastic basis while Proposition 3.9 shows that one can work with a strict martingale density without any reference to the martingale measure. Facing the choice between an insignificant supplementary requirement and a perspective to be far away from the traditional language we prefer the first option. So we impose

Assumption 3.11 *For any positive martingale $L = (L_t)$ with $E^P[L_t] = 1$ there exists a probability measure Q on $\mathcal{F}$ such that $L_t = dQ_t/dP_t$.*

Remark 3.12 In numerous papers devoted to the term structure of interest rates one can observe a rather confusing terminology : the model is said to be arbitrage-free if there exists a martingale measure. The origin of this striking difference with the theory of stock markets (where arbitrage means the possibility to get a profit which in some sense is riskless) is clear, because in continuous-time bond market models there are uncountably many basic securities and the key question is : what are portfolios of bonds ? The discussion of the latter problem is avoided since the straightforward use of finite-dimensional stochastic integrals does not allow to define a general portfolio in a correct way (see the apparent difficulties with the basic bonds in [28]). Interesting mathematical problems concerning relations between different definitions of arbitrage are almost untouched in the theory of bond markets; this subject is beyond the scope of the present paper as well.

3.3 Existence of martingale measures

Suppose that the bond prices and forward rates have P-dynamics given by the equations (2) and (3). We now ask how various coefficients in these equations must be related in order to ensure the existence of a martingale measure (or, in view of the Assumption 3.11, of a strict martingale density). The main technical tool is, as usual, a suitable version of the Girsanov theorem, which we now recall. The first (direct) part (I) below holds true regardless of how large the filtration is chosen to be, but the converse part (II) depends heavily on the fact that we have assumed the predictable representation property.

Theorem 3.13 (Girsanov)
I. *Let Γ be a predictable process and $\Phi = \Phi(\omega, t, x)$ be a strictly positive $\tilde{\mathcal{P}}$-measurable function such that for finite t*

$$\int_0^t |\Gamma_s|^2 ds < \infty, \qquad \int_0^t \int_E |\Phi(s,x)| \lambda(s, dx) ds < \infty.$$

Define the process L by

$$\begin{aligned} \log L_t &= \int_0^t \Gamma_s dW_s - \frac{1}{2} \int_0^t |\Gamma_s|^2 dW_s \\ &+ \int_0^t \int_E \log \Phi(s,x) \mu(ds, dx) + \int_0^t \int_E (1 - \Phi(s,x)) \nu(ds, dx), \end{aligned} \tag{18}$$

or, equivalently, by

$$dL_t = L_t \Gamma_t dW_t + L_{t-} \int_E (\Phi(t,x) - 1) \left\{ \mu(dt, dx) - \nu(dt, dx) \right\}, \quad L_0 = 1, \tag{19}$$

and suppose that for all finite t

$$E^P \left[L_t \right] = 1. \tag{20}$$

Then there exists a probability measure Q on $\mathcal{F}$ locally equivalent to P with

$$dQ_t = L_t dP_t \tag{21}$$

such that:

(i) *We have*

$$dW_t = \Gamma_t dt + d\tilde{W}_t, \tag{22}$$

where $\tilde{W}$ is a Q-Wiener process.

(ii) *The point process μ has a Q-intensity, given by*

$$\lambda_Q(t, dx) = \Phi(t, x)\lambda(t, dx). \tag{23}$$

II. *Every probability measure Q locally equivalent to P has the structure above.*

We now come to the main results concerning the existence of a martingale measure. They generalize the corresponding results of Heath–Jarrow–Morton and can be easily extended to the case of a multidimensional Wiener process. The identities between processes are understood $dPdt$-a.e.

Theorem 3.14

I. *Let the bond price dynamics be given by (2). Assume that $n(t, x, T)$ for any fixed T is bounded by a constant (depending on T). Then there exists a martingale measure Q if and only if the following conditions hold:*

(i) *There exists a predictable process Γ and a $\tilde{\mathcal{P}}$-measurable function $\Phi(t, x)$ with $\Phi > 0$ satisfying the integrability conditions of Theorem 3.13 and such that $E^P[L_t] = 1$ for all finite t, where L is defined by (19).*

(ii) *For all $T > 0$ on $[0, T]$ we have*

$$m(t, T) + \Gamma_t v(t, T) + \int_E \Phi(t, x) n(t, x, T)\lambda(t, dx) = r_t. \tag{24}$$

II. *Let the forward rate dynamics be given by (3). Assume that $e^{D(t,x,T)}$ for any fixed T is bounded by a constant (depending on T). Then there exists a martingale measure if and only if the following conditions hold:*

(iii) *There exist a predictable process Γ and a $\tilde{\mathcal{P}}$-measurable function $\Phi(t,x)$ with $\Phi > 0$ satisfying the integrability conditions of Theorem 3.13 and such that $E^P[L_t] = 1$ for all finite t where L is defined by (19).*

(iv) *For all $T > 0$, on $[0,T]$ we have*

$$A(t,T) + \frac{1}{2}S^2(t,T) + \Gamma_t S(t,T) + \int_E \Phi(t,x)\Lambda(t,dx) = 0, \qquad (25)$$

where

$$\Lambda(t,dx) = \left\{e^{D(t,x,T)} - 1\right\}\lambda(t,dx)$$

and A, S and D are defined as in (6).

Proof.
I. First of all it is easy to see (using the Itô formula) that a measure Q is a martingale measure if and only if the bond dynamics under Q are of the form

$$dp(t,T) = r_t p(t,T)dt + dM_t^Q, \qquad (26)$$

where M^Q is a Q-local martingale. Using the Girsanov theorem we see that under any equivalent measure Q, the bond dynamics have the following form, where we have compensated μ under Q.

$$\begin{aligned} dp(t,T) &= p(t,T)m(t,T)dt + p(t,T)v(t,T)(\Gamma_t dt + d\tilde{W}_t) \\ &+ p(t-,T)\int_E n(t,x,T)\Phi(t,x)\lambda(t,dx)dt \\ &+ p(t-,T)\int_E n(t,x,T)\left\{\mu(dt,dx) - \Phi(t,x)\lambda(t,dx)dt\right\}. \end{aligned}$$

Thus we have

$$\begin{aligned} dp(t,T) &= p(t,T)\left[m(t,T) + v(t,T)\Gamma_t \int_E n(t,x,T)\Phi(t,x)\lambda(t,dx)\right]dt + \\ &+ dM_t^Q. \end{aligned}$$

Comparing this with the equation (26) gives the result.

II. If the forward rate dynamics are given by (3) then the corresponding bond price dynamics are given by Proposition 2.4. We can then apply part 1 of the present theorem. ∎

We now turn to the issue of so called "martingale modelling", and remark that one of the main morals of the martingale approach to arbitrage-free pricing of derivative securities can be formulated as follows.

- The dynamics of prices and interest rates under the objective probability measure P are, to a high degree, irrelevant. The important objects to study are the dynamics of prices and interest rates *under the martingale measure* Q.

When building a model it is thus natural, and in most cases extremely time saving, to specify all objects directly under a martingale measure Q. This will of course impose restrictions on the various parameters in, e.g., the forward rate equations, and the main results are as follows.

Proposition 3.15 *Assume that we specify the forward rate dynamics under a martingale measure* Q *by*

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)d\tilde{W}_t + \int_E \delta(t,x,T)\mu(dt,dx). \tag{27}$$

Then the following relation holds

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds - \int_E \delta(t,x,T)e^{D(t,x,T)}\lambda_Q(t,dx). \tag{28}$$

Furthermore, the bond price dynamics under Q *are given by*

$$\begin{aligned} dp(t,T) &= p(t,T)r_t dt + p(t,T)S(t,T)d\tilde{W}_t \\ &+ p(t-,T)\int_E \left\{e^{D(t,x,T)} - 1\right\}\tilde{\mu}(dt,dx), \end{aligned} \tag{29}$$

where $\tilde{\mu}$ *is the* Q*-compensated point process*

$$\tilde{\mu}(dt,dx) = \mu(dt,dx) - \lambda_Q(t,dx)dt.$$

Here λ_Q *is the* Q*-intensity of* μ *whereas* D *and* S *are defined by (6).*

Proof. Since we are working under Q we may use Theorem 3.14 with $\Gamma = 0$ and $\Phi = 1$ to obtain

$$A(t,T) + \frac{1}{2}S^2(t,T) + \int_E \left\{e^{D(t,x,T)} - 1\right\}\lambda_Q(t,dx) = 0,$$

and differentiating this equation with respect to T gives us the equation (28). The result on bond prices now follows immediately from the result above and from Proposition 2.4. ∎

The single most important formula in this section is the relation (28) which is the point process extension of the Heath–Jarrow–Morton "drift condition". We see that if we want to model the forward rates directly under the martingale measure Q, then the drift α is uniquely determined by the diffusion volatility σ, the jump volatility δ and *by the Q-intensity* λ_Q. This has important implications when it comes to parameter estimation, since we are modelling under Q while our concrete observations, of course, are made under an objective measure P. As far as volatilities are concerned they do not change under an equivalent measure transformation, so "in principle" we can determine σ and δ from actual observations of the forward rate trajectories. The intensity measure however presents a totally different problem. Suppose for simplicity that μ is a standard Poisson process (under Q) with Q-intensity λ_Q. If we could observe the forward rates under Q then we would, of course, have access to a vast machinery of statistical estimation theory for the determination of a point estimate of λ_Q, but the problem here is that we are *not* making observations under Q, but under P. Thus the estimation of the Q-intensity λ_Q is *not* a statistical estimation problem to be solved with standard statistical techniques. This fact may be regarded as a piece of bad news or as an interesting problem. We opt for the latter interpretation, and one obvious way out is to estimate λ_Q by using market data for bond prices (which contain implicit information concerning λ_Q).

4 Uniqueness of Q and market completeness

4.1 Uniqueness of the martingale measure

Throughout this section we shall work with a model specified by the forward rate dynamics under

Assumption 4.1 *The coefficient $D(t, x, T)$ is uniformly bounded.*

The main issue to be dealt with below is the relation between uniqueness of the martingale measure and completeness of the bond market. Using Theorem 3.14 we immediately have the following result.

Proposition 4.2 *Let the forward rate dynamics be given by (3) and assume that the assumptions (iii) and (iv) of Theorem 3.14 (equivalent to existence of a martingale measure Q) are satisfied. Then the martingale measure Q is unique if and only if $dPdt$-a.e.*

$$\operatorname{Ker} \mathcal{K}_t(\omega) = 0 \tag{30}$$

where the linear operator

$$\mathcal{K}_t(\omega):\ R \times L^2(E, \mathcal{E}, \lambda(\omega, t, dx)) \to C[0, \infty[\tag{31}$$

is defined by

$$\mathcal{K}_t(\omega):\ (\Gamma, \Phi) \mapsto S(\omega, t, .)\Gamma + \int_E \Phi(x)\Lambda(\omega, t, dx, T) \tag{32}$$

with

$$\Lambda(\omega, t, dx, T) = \left\{e^{D(\omega,t,x,T)} - 1\right\} \lambda(\omega, t, dx).$$

The important thing to note here is that the operators $\mathcal{K}_t(\omega)$ are integral operators of the first kind. We shall refer to $\mathcal{K}$ as "the martingale operators".

Corollary 4.3 *Suppose that the forward rate dynamics is given by (3), that the model coefficients $\alpha(t,T)$, $\sigma(t,T)$, $\delta(t,x,T)$, and $\lambda(t,dx)$ are deterministic and that the martingale measure Q is unique. Then the Girsanov transformation parameters Γ and Φ are deterministic functions, i.e. under Q the process $\tilde{W}$ is a Wiener process with constant drift, and μ is a Poisson measure.*

Proof. It is sufficient to notice that the operators $\mathcal{K}_t$ do not depend of ω and hence (outside the exclusive $dPdt$-null sets) values of the Girsanov transformation parameters corresponding to a fixed t but different ω must satisfy the *same* equation (25), which has a unique solutions because of (30). ∎

Corollary 4.4 *If we add to the hypotheses of Corollary 4.3 the assumption that $\alpha(t,T) = \alpha(T-t)$, $\sigma(t,T) = \sigma(T-t)$, $\delta(t,x,T) = \delta(T-t,x)$, and $\lambda(t,dx) = \lambda(dx)$ then Γ and Φ do not depend on t, i.e. under Q the process $\tilde{W}$ is a Wiener process with a constant drift and μ is a Poisson measure invariant under time translations.*

Of course, the above assertions are almost trivial but they can be considered as an overture to a more systematic use of classical functional analysis, which appears to be an adequate tool in the considered setting. Clearly, instead of considering families of operators as we do, one can chose slightly different definitions and e.g. consider a single operator acting from one space of random processes to another.

In spite of the simplicity of the definition (31)-(32), it has a drawback because it uses $C[0,\infty[$ with its associated complicated dual. In order to be able to work with a more manageable dual space it is therefore natural to modify the definition of the martingale operators and impose the following constraint on the model:

Assumption 4.5 *For any t, x*

$$\lim_{T\to\infty} Z(t-,T)S(t,T) = 0, \qquad \lim_{T\to\infty} Z(t-,T)\left\{e^{D(t,x,T)} - 1\right\} = 0$$

where D and S are given by (6), and Z is the discounted price process.

Let $C_0[0,\infty[$ be the space of continuous functions on $[0,\infty[$ converging to zero at infinity. Notice that here we have the well known duality $C_0^\star[0,\infty[= \mathcal{M}[0,\infty[$, where $\mathcal{M}[0,\infty[$ is the space of measures on $[0,\infty[$.

The formula

$$\mathcal{K}_t^Z(\omega):\ (\Gamma,\Phi)\mapsto Z(\omega,t-,.)S(\omega,t,.)\Gamma + Z(\omega,t-,.)\int_E \Phi(x)\Lambda(\omega,t,dx,.)$$

defines a linear operator

$$\mathcal{K}_t^Z(\omega):\ R\times L^2(E,\mathcal{E},\lambda(\omega,t,dx)) \to C_0[0,\infty[.$$

In other words, $\mathcal{K}_t^Z(\omega)$ is the product of the operator $\mathcal{K}_t(\omega)$ and the operator of multiplication by the function $Z(\omega,t-,.)$ and one may write $\mathcal{K}_t^Z = \mathcal{Z}_t\mathcal{K}_t$. Clearly, the above results hold also with $\mathcal{K}$ substituted by $\mathcal{K}^Z$ but the modified definition leads to some nice duality arising in the context of market completeness.

As an alternative, to avoid problems with the dual space, one can suppose that there is a finite time horizon T_f and all traded bonds have maturities $T \le T_f$. In this case, in section 5.1 below, we have to restrict ourselves to measures $G_t(dT)$ with support in $[T,T_f]$.

4.2 Completeness

Let the forward rate dynamics under a martingale measure Q be given by

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)d\tilde{W}_t + \int_E \delta(t,x,T)\mu(dt,dx), \tag{33}$$

where $\tilde{W}$ is a Q-Wiener process and μ has the Q-intensity λ_Q. Our aim is now to investigate the possibility of hedging contingent claims.

Definition 4.6 *Consider a contingent claim $X \in L^\infty(\mathcal{F}_{T_0})$ expressed in terms of the numéraire. We say that it can be* **replicated** *or that we can* **hedge** *against X if there exists a self-financing portfolio with the bounded, discounted value process V^Z such that*

$$V^Z_{T_0} = X. \tag{34}$$

If every such $X \in L^\infty(\mathcal{F}_{T_0})$ (for every T_0) can be replicated, the model is said to be **complete**.

If for every such $X \in L^\infty(\mathcal{F}_{T_0})$ there exists a sequence of uniformly bounded hedgeable claims converging to X in probability, we say that the model is **approximately complete**.

It is important to notice that the spaces L^∞ and L^0 are invariant under an equivalent change of probability measures (recall also that convergence in probability can be expressed in terms of convergence a.s. along a subsequence).

Suppose now that we want to find a self-financing portfolio $\{g, h\}$ which replicates $X \in L^\infty_+(\mathcal{F}_{T_0})$. Using Lemma 3.4 we see that the problem is reduced to finding a portfolio strategy with an initial endowment V^Z_0 and a bond investment process h such that

$$dV^Z_t = \int_t^\infty h_t(dT) dZ(t, T), \tag{35}$$

$$V^Z_{T_0} = X, \tag{36}$$

Proposition 3.15 gives us the Q-dynamics of the bond prices and a simple calculation shows that for Z we have the dynamics

$$dZ(t,T) = Z(t,T)S(t,T)d\tilde{W}_t + Z(t-,T)\int_E \left\{e^{D(t,x,T)} - 1\right\}\tilde{\mu}(dt,dx), \tag{37}$$

where S and D are defined as usual. We are thus looking for a pair $\left\{V^Z_0, h\right\}$ such that

$$V^Z_{T_0} = X, \tag{38}$$

$$dV^Z_t = \int_t^\infty h(t,dT)Z(t,T)S(t,T)d\tilde{W}_t + \int_E\int_t^\infty h(t,dT)Z(t-,T)\left\{e^{D(t,x,T)} - 1\right\}\tilde{\mu}(dt,dx), \tag{39}$$

with the integrability conditions

$$\int_0^{T_0} \left\{\int_s^\infty |h(s,dT)| \cdot |Z(s,T)S(s,T)|\right\}^2 ds < \infty, \tag{40}$$

$$\int_0^{T_0} \int_E \int_s^\infty |h(s,dT)| \cdot |Z(s,T) \left\{ e^{D(t,x,T)} - 1 \right\} | \nu(ds,dx) < \infty. \qquad (41)$$

Now, since $X \in L_+^\infty(\mathcal{F}_{T_0})$ the process

$$M_t = E^Q \left[X \,|\mathcal{F}_t \right] \qquad (42)$$

is a Q-martingale. By the assumptions it has an integral representation, that is there are γ and φ such that

$$dM_t = \gamma_t d\tilde{W}_t + \int_E \varphi(t,x) \tilde{\mu}(dt,dx), \qquad (43)$$

with

$$E^Q \left[\int_0^{T_0} \gamma_t^2 dt \right] < \infty$$

and

$$E^Q \left[\int_0^{T_0} \int_E \varphi^2(t,x) d\nu(dt,dx) \right] < \infty.$$

Now we may formulate our first proposition concerning hedging.

Proposition 4.7 *We can replicate a claim $X \in L_+^\infty(\mathcal{F}_{T_0})$ if and only if there exists a predictable measure-valued process $h(t,dT)$ which satisfies the integrability conditions (40) and (41) and solves on $[0,T_0]$ (dPdt-a.e.) the equations*

$$\mathcal{K}_t^{Z\star} h = \left[\begin{array}{c} \gamma_t \\ \varphi(t,.) \end{array} \right] \qquad (44)$$

where γ and φ are defined as above and where the "hedging operators" $\mathcal{K}_t^{Z\star}$ (acting on measures) are defined by

$$\mathcal{K}_t^{Z\star}(\omega) : \, m \mapsto \left[\begin{array}{c} \int_t^\infty Z(\omega,t-,T) S(\omega,t,T) m(dT) \\ \\ \int_t^\infty Z(\omega,t-,T) \left\{ e^{D(\omega,t,.,T)} - 1 \right\} m(dT) \end{array} \right]. \qquad (45)$$

Proof. Sufficiency. Assume that $h(t,dT)$ is solution of (44). Then we have

$$\begin{aligned} dM_t &= \int_t^\infty h(t,dT) Z(t,T) S(t,T) d\tilde{W}_t \\ &+ \int_E \int_t^\infty h(t,dT) Z(t-,T) \left\{ e^{D(t,x,T)} - 1 \right\} \tilde{\mu}(dt,dx). \end{aligned} \qquad (46)$$

Now we define g by

$$g_t = M_t - \int_t^\infty h(t,dT) Z(t,T). \qquad (47)$$

We see from (47) that the value process corresponding to the portfolio $\{g, h\}$ is given by $V_t^Z = M_t$. Furthermore, it follows from (46) that the portfolio is self-financing. Finally we see from the definition of M that $V_{T_0}^Z = M_{T_0} = X$. Thus we have found a hedge against X and sufficiency is proved.
Necessity. The discounted value process V^Z of a hedging portfolio $\{g, h\}$ is a bounded martingale with $V_{T_0}^Z = X$. Thus V^Z is indistinguishable from M given by (42), and the uniqueness considerations yield (44). ∎

Note that the hedging operators

$$\mathcal{K}_t^{Z\star} : \mathcal{M}[0,\infty[\to R \times L^2(E, \mathcal{E}, \lambda_Q(t, dx)) \tag{48}$$

are indeed the adjoint of the martingale operators $\mathcal{K}_t^Z$.

To sum up we have the following conclusions.

Proposition 4.8

1. *The martingale measure is unique if and only if the mappings $\mathcal{K}^Z$ are injective (a.e.).*

2. *The market is complete if and only if the mappings $\mathcal{K}^{Z*}$ are surjective (a.e.).*

The proof of a natural extension of the second assertion which we give below involves a measurable selection technique.

Proposition 4.9 *The following conditions are equivalent.*

(i) *The market is approximately complete.*

(ii) $\mathrm{cl}\,(\mathrm{Im}\,\mathcal{K}_t^{Z*}(\omega)) = R \times L^2(E, \mathcal{E}, \lambda_Q(\omega, t, dx))$ *(a.e.).*

Proof. (**i**) $\Leftarrow$ (**ii**) Let X be a bounded discounted contingent T_0-claim to be approximated. Using truncation arguments we can suppose, without loss of generality, that X is such that γ and φ in the representation (43) are bounded. For $\varepsilon > 0$ put

$$F^\varepsilon(t, m) = |\mathcal{K}_t^{Z\star,1}(m) - \gamma_t|^2 + \|\mathcal{K}_t^{Z\star,2}(m) - \varphi(t, .)\|^2_{L^2(\lambda_Q(t,dx))}$$

where we use superscripts to denote the first and the second "coordinates" in (45). Let us consider on $\mathcal{M}[0,\infty[$ the σ-algebra $\mathcal{W}$ generated by the weak topology (more precisely, by $\sigma(\mathcal{M}[0,\infty[, C_0[0,\infty[))$. Recall that balls in $\mathcal{M}[0,\infty[$ are metrizable compacts, hence $(\mathcal{M}[0,\infty[, \mathcal{W})$ is a Lusin space as a countable union of Polish spaces. The function F^ε, being $\mathcal{P}$-measurable in (ω, t) and continuous in m, is jointly measurable. Therefore, the set-valued mapping

$$(\omega, t) \mapsto \{m \in \mathcal{M}[0,\infty[: \; F^\varepsilon(\omega, t, m) \leq \varepsilon\}$$

has a $\mathcal{P} \otimes \mathcal{W}$-measurable graph and, hence, admits a $\mathcal{P}$-measurable a.e.-selector $m^\varepsilon(t, dT)$ (see, e.g. [12]), which "almost" solves the problem: the terminal values of the processes, defined by $I_{[0,t]} m^\varepsilon(t, dT)$ and the initial endowment $E^Q[X]$, converge to X in $L^2(\mathcal{F}_{T_0}, Q)$, hence in probability, as $\varepsilon \to 0$. One can notice, however, that the construction is not accomplished since the strategy generates a value process which is not bounded (and even admissible). A standard truncation and localization arguments finally lead to the desired goal.

(i) $\Rightarrow$ **(ii)** Assume that the market is approximately complete. Then there exists a countable set $H = \{X^n\}$ of bounded hedgeable random variables, dense in the Hilbert space $L^2(\mathcal{F}_{T_0}, Q)$ and closed under linear combinations with rational coefficients; let $g^n = (\gamma^n, \varphi^n)$ be the pair of functions in the integral representation of X^n given by (43). Without loss of generality, we may assume that for all (ω, t) one has $\|g^n\|_{\omega,t} < \infty$ where $\|.\|_{\omega,t}$ and $(.,.)_{\omega,t}$ are, respectively, the norm and the scalar product in $R \times L^2(E, \mathcal{E}, \lambda_Q(\omega, t, dx))$. Let us denote by $H_{\omega,t}$ the closure in this norm of the set $\{g^n(\omega, t)\}$, which is, evidently, a linear subspace, and by $H^\perp_{\omega,t}$ its orthogonal complement.

It is easy to show that there exists a pair of functions $g = (\gamma, \varphi)$ such that γ is $\mathcal{P}$-measurable, φ is $\tilde{\mathcal{P}}$-measurable, and $\|g\|_{\omega,t} = 1$ if $H^\perp_{\omega,t} \neq 0$. Indeed, let $\{I(j)\}$ be a sequence of indicator functions generating $\mathcal{E}$ and

$$k(\omega, t) = \inf \left\{ j : \; \inf_n \|I(j, .) - \varphi^n(\omega, t, .)\|_{L^2(\lambda_Q(\omega,t,dx))} > 0 \right\}.$$

Put $\tilde{\gamma}_t(\omega) = I_{\{\sup_n |\gamma^n_t(\omega)| > 0\}}$, $\tilde{\varphi}(\omega, t, x) = I(k(\omega, t), x)$ if $k(\omega, t) < \infty$ and $\tilde{\varphi}(\omega, t) = 0$ otherwise. The pair of functions $\tilde{g} = (\tilde{\gamma}, \tilde{\varphi})$ meets the necessary measurability requirements. Furthermore, there is $\tilde{g}^\pi = (\tilde{\gamma}^\pi, \tilde{\varphi}^\pi)$ which is measurable in the same way and such that all the sections $\tilde{g}^\pi(\omega, t)$ are representatives of the projections of $\tilde{g}(\omega, t)$ onto $H_{\omega,t}$ (one can orthogonalize $\{g^n(\omega, t)\}$ preserving measurability and notice that in this case the Fourier coefficients are obviously predictable). Normalizing the difference $\tilde{g} - \tilde{g}^\pi$ we get g with the required properties.

The pair $g = (\gamma, \varphi)$ defines by (43) with $M_0 = 0$, a random variable $M_T = X \in L^2(\mathcal{F}_{T_0}, Q)$, orthogonal, by construction, to all X^n. If **(ii)** does not hold then X is nontrivial; this leads to an apparent contradiction. ∎

By experience from the theory of financial markets with finitely many assets one could expect that the market is complete if and only if the martingale measure is unique, but in our infinite dimensional setting this is no longer true. Due to the duality relation $(\operatorname{Ker}\mathcal{K})^{\perp} = \operatorname{cl}(\operatorname{Im}\mathcal{K}^{\star})$ we obtain instead from the above assertions

Theorem 4.10 *The following statements are equivalent:*

(i) *The martingale measure is unique.*

(ii) *The market is approximately complete.*

For a model with a finite mark space E, where the hedging problem is reduced to a finite dimensional system of equations (for each (ω, t)), the duality relation is simpler: $(\operatorname{Ker}\mathcal{K})^{\perp} = \operatorname{Im}\mathcal{K}^{\star}$, so in this case we have

Corollary 4.11 *Suppose that the mark space E is finite. Then the bond market is complete if and only if the martingale measure is unique.*

The same conclusion holds if for almost all (ω, t) the measures $\lambda_t(\omega, dx)$ are concentrated in a finite number of points. In general, for an infinite E the "principle" that uniqueness of Q is equivalent to completeness of the market fails: the set of hedgeable claims may be a strict subset in the set of all claims $L^{\infty}(\mathcal{F}_{T_0})$. Clearly, it is the case when D is continuous in x and bounded (hence the image contains only continuous functions); typically, $\mathcal{K}_t^{Z*}$ is a compact operator and, hence, has no bounded inverse.

Thus the case with an infinite mark space introduces some completely new features into the theory, and we also encounter some new problems when it comes to the numerical computation of hedging portfolios. The formal result is as follows.

Corollary 4.12 *Suppose that the mark space E is infinite. Then the hedging equation (44) is ill-posed in the sense of Hadamard, i.e. the inverse of $\mathcal{K}_t^{Z\star}$ restricted to $\operatorname{Im}\mathcal{K}^{Z\star}$ is not bounded.*

Proof. This follows immediately from the fact that $\mathcal{K}^{Z\star}$ is compact. ∎

The main content of this result is that the hedging equation is numerically ill-conditioned, in the sense that a small disturbance of the right-hand side (e.g. due to a small round-off error) gives rise to large fluctuations in the solution. Thus, a naive approximation scheme for the calculation of a concrete hedge may very well lead to great numerical problems. Fortunately, there exists a large literature on stable solutions of ill-posed problems but we will not pursue this topic here.

5 Characterization of hedgeable claims

5.1 Laplace transforms

In this section we suppose that Q is unique and that E is infinite. Assume for simplicity that we have no driving Wiener process. One can also think that the model coefficients in (50) below are deterministic.

From the general theory developed in the previous section it follows that the hedging equation, symbolically written as

$$\mathcal{K}^* G = \varphi \tag{49}$$

with the measure $G_t(dT) = Z(t-, T)h_t(dT)$, can only be solved for a right-hand side φ in a dense subset of the image space. The purpose of this section is to present a class of models, for which we can give an explicit characterization of the class of hedgeable claims.

Assumption 5.1 *The forward rate dynamics under Q is given by*

$$df(t,T) = \alpha(t,T)dt + \int_E \delta(t,x,T)\,\mu(dt,dx) \tag{50}$$

with δ of the form

$$\delta(t,x,T) = -c_t(x) \cdot \delta_0(t,T) \tag{51}$$

where the functions c and δ_0 are such that

(i) *For each t the mapping $c_t : E \to R$ is injective.*

(ii) *For each t the set $c_t(E)$ is an interval $[l_t, \infty[$ or $]l_t, \infty[$ (i.e. the left endpoint l_t may or may not belong to $c_t(E)$).*

(iii) $\delta_0 > 0$.

The important restriction introduced by this assumption is the volatility structure given by (51) (see, however, Remark 6.8 of Section 6). Condition (i) simply means that different points in E really give rise to different behavior of the forward rates. Assumption (ii) guarantees that we have an infinite mark space and that $c_t(E)$ has a limit point at infinity; assumption (iii) does not seem to be severe.

Suppose now that we want to hedge against a particular bounded discounted T_0-claim X. The martingale representation result, see (43), will then provide us with a function $\varphi(t,x)$, and the hedging problem reduces, modulo a measurable selection, to the problem of finding a measure-valued process G such that for almost all $t \in [0, T_0]$

$$\int_t^\infty \left\{ e^{D(t,x,T)} - 1 \right\} G(t, dT) = \varphi(t, x) \quad \lambda_Q(t, dx)\text{-a.e.} \tag{52}$$

Using Assumption 5.1 we obtain

$$\int_t^\infty \left\{ e^{-c_t(x)\Delta(t,T)} - 1 \right\} G(t, dT) = \varphi(t, x) \quad \lambda_Q(t, dx)\text{-a.e.} \tag{53}$$

where Δ is given by

$$\Delta(t, T) = \int_t^T \delta_0(t, s)\, ds. \tag{54}$$

Since c_t is assumed to be injective we can write (53) as

$$\int_t^\infty \left\{ e^{-y\Delta(t,T)} - 1 \right\} G(t, dT) = \varphi^c(t, y). \tag{55}$$

where $\varphi^c(t, y) = \varphi(t, c_t^{-1}(y))$, $y \in c_t(E)$, and the equality is understood modulo $\lambda_Q^c(t, dy)$, the image of the measure $\lambda_Q(t, dx)$ under the mapping c_t.

Since the right-hand side of (55) is a class of equivalence, the rigorous formulation of the following necessary condition concerns, in fact, the properties of a representative of this class.

Lemma 5.2 *Necessary conditions for the existence of a solution to the hedging equation (52) are that*

(i) *For each t the function $\varphi^c(t, y)$ can be extended to an analytic function for all complex z such that* $\operatorname{Re} z > l_t$.

(ii) *For each t the limit $\lim_{y\to\infty} \varphi^c(t, y)$ exists.*

Proof Analyticity follows from the fact that, because $\Delta > 0$, the left-hand side of (55) is analytic. Existence of the limit then follows directly from (55). ∎

We thus see that it is only in rather special cases that we can solve the hedging equation, and this fact is in complete accordance with the denseness result of the preceding section.

We continue our investigation, assuming that we can actually solve the hedging equation. Then we may write (55) as

$$\int_t^\infty e^{-y\Delta(t,T)} G(t,dT) = \varphi^c(t,y) - \varphi^c(t,\infty) \tag{56}$$

and (remember that t is fixed in each equation) change the integration variable from T to u by the substitution $u = \Delta(t,T)$. Thus the measure $G(t,dT)$ will be pushed to a measure $G^\Delta(t,du)$, defined by

$$G^\Delta(t,du) = G(t,\Delta^{-1}(t,dT)). \tag{57}$$

We now have

$$\int_0^\infty e^{-yu} G^\Delta(t,du) = \varphi^c(t,y) - \varphi^c(t,\infty), \quad \forall y \in c_t(E). \tag{58}$$

This equation is a Laplace transformation for each t, and we have the following characterization of the set of φ's for which we can solve the hedging equation.

Theorem 5.3 *Consider a fixed claim X and its corresponding φ. Then we can hedge exactly against X if and only if the following conditions hold for each (ω,t):*

(i) *The function $\varphi^c(t,y)$ can be extended to an entire analytic function for all complex z such that $\operatorname{Re} z > l_t$.*

(ii) *For each t the limit $\lim_{y\to\infty} \varphi^c(t,y)$ exists.*

(iii) *The function $\varphi^c(t,y) - \varphi^c(t,\infty)$ is the Laplace transform of a signed (finite) Borel measure on $[0,\infty)$.*

For a particular claim we may also want to know if the replicating portfolio contains only bonds with maturities in a prespecified set. If X corresponds to an expiration time T_0 one can, e.g., ask whether it can be hedged with a portfolio consisting entirely of bonds with maturities greater than T_0. Properties of this kind can, in fact, be read off immediately from the structure of the predictable representation, i.e. of φ.

Proposition 5.4 *Consider a fixed hedgeable T_0-claim X and its corresponding φ. Also fix a real number (maturity) T_1. Then the hedging portfolio can be composed entirely of bonds with maturities greater than T_1 if and only if the following condition holds for every $t \le T_0$.*

$$\lim_{y\to\infty} e^{y\Delta(t,T_1)} \left(\varphi^c(t,y) - \varphi^c(t,\infty)\right) = 0. \tag{59}$$

Proof The h-component of a portfolio consists entirely of bonds with maturities greater than T_1 if and only if the measure $G(t, dT)$ has its support in $[T_1, \infty[$, which is equivalent to the property that G^Δ has its support in $[\Delta(t, T_1), \infty[$. Thus we can rewrite (58) as

$$\int_{\Delta(t,T_1)}^{\infty} e^{-yu} G^\Delta(t, du) = \varphi^c(t, y) - \varphi^c(t, \infty),$$

which, after the change of variables $v = u - \Delta(t, T_1)$, becomes

$$\int_0^\infty e^{-y[v+\Delta(t,T_1)]} G^{\Delta,T_1}(t, dv) = \varphi^c(t, y) - \varphi^c(t, \infty),$$

where G^{Δ,T_1} is the translation of G^Δ. Thus we obtain

$$\int_0^\infty e^{-yv} G^{\Delta,T_1}(t, dv) = e^{y\Delta(t,T_1)}(\varphi^c(t, y) - \varphi^c(t, \infty)),$$

and the result follows. ∎

5.2 The case of a finite mark space

In the case when the mark space E is finite, we can write the forward rate dynamics as

$$df(t, T) = \alpha(t, T)dt + \sigma(t, T)d\tilde{W}_t + \sum_{i=1}^{n} \delta_i(t, T)dN_t^i \tag{60}$$

where $N^1, \cdots, N^n$ are counting processes with predictable intensity processes $\lambda^1, \cdots, \lambda^n$. The process $\tilde{W}$ is supposed to be m-dimensional standard Wiener, so $\sigma(t, T)$ is an m-dimensional (row) vector process. In this case it is reasonable to look for a hedging portfolio with the h-component instantaneously consisting of $n + m$ bonds with different maturities $T_1, \cdots, T_n$, $T_{n+1}, \cdots, T_{n+m}$ (i.e. $h(t, dT)$ is a discrete measure concentrated in these points), and the hedging equation can be written in the following matrix form (where m may be equal to zero).

$$A(t, T_1, \cdots, T_{n+m}) \begin{bmatrix} G_t^1 \\ \vdots \\ G_t^{n+m} \end{bmatrix} = \begin{bmatrix} \gamma_t \\ \varphi(t) \end{bmatrix} \tag{61}$$

where

$$\gamma_t = \begin{bmatrix} \gamma_t^1 \\ \vdots \\ \gamma_t^m \end{bmatrix}, \qquad \varphi(t) = \begin{bmatrix} \varphi(t, 1) \\ \vdots \\ \varphi(t, n) \end{bmatrix}, \tag{62}$$

$$A(t, T_1, \cdots, T_{n+m}) = \begin{bmatrix} S_m(t,T_1) & \cdots & S_m(t,T_{n+m}) \\ \vdots & \vdots & \vdots \\ S_1(t,T_1) & \cdots & S_1(t,T_{n+m}) \\ e^{D_1(t,T_1)}-1 & \cdots & e^{D_1(t,T_{n+m})}-1 \\ \vdots & \vdots & \vdots \\ e^{D_n(t,T_1)}-1 & \cdots & e^{D_n(t,T_{n+m})}-1 \end{bmatrix}, \qquad (63)$$

$$S_i(t,T_j) = -\int_t^{T_j} \sigma_i(t,s)\,ds\,, \qquad D_i(t,T_j) = -\int_t^{T_j} \delta_i(t,s)\,ds. \qquad (64)$$

Here the (γ, φ)-process, as usual, comes from the martingale representation theorem, with γ^i as the integrand corresponding to $\tilde{W}^i$ and $\varphi(i)$ as the integrand corresponding to the N^i-process. The process G^i is (see comment after (49)) the discounted amount invested in the portfolio corresponding to the bonds with maturity T_i.

The main problem in this section is to give conditions that guarantee completeness of the bond market. In concrete terms this means that we want to give conditions on the forward rate dynamics implying the existence of maturities $T_1, \cdots, T_{n+m}$ such that the matrix $A(t, T_1, \cdots, T_{n+m})$ is invertible. From a practical point of view it would be particularly pleasing if these maturities can be chosen in such a way that they stay fixed when the time t is running. Intuitively, it is also natural to expect that the maturities can be chosen arbitrarily, as long as they are distinct from one another.

The main result in this section says that, given smoothness of S and D in the maturity variable T, we can choose maturities almost arbitrarily. If, furthermore the volatilities are deterministic and S and D are also smooth in the t-variable, then the maturities can be chosen fixed over time, i.e. maturities do not change with the running time t.

We start with a general mathematical observation in the following

Proposition 5.5 *Let* $f_1, \cdots, f_M$ *be a set of real-valued functions such that*

(i) *For each i the function f_i is real-valued analytic, i.e. it can be extended to a holomorphic function in the complex plane.*

(ii) *The functions $f_1, \cdots, f_M$ are linearly independent.*

For each choice of reals $T_1, \cdots, T_M$ consider the matrix B defined by

$$B(T_1, \cdots, T_M) = \{f_i(T_j)\}_{i,j}. \tag{65}$$

Then, given any finite interval $[I_L, I_R]$ of a positive length, we can choose $T_1, \cdots, T_M$ in $[I_L, I_R]$ such that B is invertible. Furthermore, apart from a finite set of points, we can choose $T_1, \cdots, T_M$ arbitrarily in $[I_L, I_R]$ as long as they are distinct.

Proof. We fix the interval $[I_L, I_R]$ and prove the result by induction on the number of functions. For $M = 1$ the statement is obviously true, since by analyticity the function f_1 can have at most finitely many zeroes on a compact set. Suppose therefore that the statement is true for $M = n-1$, and consider the matrix function $B(t)$ defined by

$$B(t) = \begin{bmatrix} f_1(t) & f_1(T_2) & \cdots & f_1(T_n) \\ f_2(t) & f_2(T_2) & \cdots & f_2(T_n) \\ \vdots & \vdots & & \vdots \\ f_n(t) & f_n(T_2) & \cdots & f_n(T_n) \end{bmatrix} \tag{66}$$

where, by the induction hypothesis, we have chosen $T_2, \cdots, T_n$ in such a way that all $(n-1)$-dimensional quadratic submatrices of the last $n-1$ columns are invertible. Our task is now to prove that we can choose a point t such that $B(t)$ is invertible and to do this we consider the determinant $\det B(t)$. Expanding $\det B(t)$ along the first column we see that

$$\det B(t) = \sum_{i=1}^{n} a_i f_i(t) \tag{67}$$

where the a_i's are subdeterminants of the last $n-1$ columns and hence (by the induction hypothesis) nonzero. Thus we see from (67) that $\det B(t)$ is an analytic function and, because of the assumed linear independence, it is not identically equal to zero. Thus it has at most finitely many zeroes in the interval $[I_L, I_R]$. If we choose T_1 as any number in $[I_L, I_R]$, except for the finite set of "forbidden" values, we get the result. ∎

Applying this result to the bond market situation we have

Theorem 5.6 *Assume that*

(i) *For each ω, t all functions $\delta_i(t,T)$ and $\sigma_j(t,T)$ are analytic in the T-variable.*

(ii) *For each ω, t the following functions of the argument T are linearly independent:*

$$e^{D_i(t,T)} - 1, \qquad S_j(t,T), \qquad i = 1, \cdots, n,\ j = 1, \cdots, m. \tag{68}$$

Then the market is complete. Furthermore, for each t we can choose the distinct bond maturities arbitrarily, apart from a finite number of values on every compact interval. If all functions above are deterministic and analytic also in the t-variable, then the maturities can be chosen to be the same for every t.

Proof. The main part of the statement follows immediately from Proposition 5.5. The last statement follows from the fact that, if we fix the maturities at $t = 0$ such that the corresponding $\det B(t) \neq 0$, then, again by the assumed analyticity in the t-variable, $\det B(t)$ is zero only for finitely many t-values. Furthermore, in the replicating portfolio we are integrating compensated Poisson processes having intensities, so the strategies can be chosen arbitrarily on the zero set of B, since this (deterministic) set has Lebesgue measure zero, while outside this set they have to satisfy the system (61). ∎

As an easy corollary we immediately have the following extension of a result of Shirakawa (see [28]). Note that we allow for more than one Wiener process, whereas the proof in [28] depends critically on an assumption of only one Wiener process. In addition, in [28] the maturities of the bonds in the hedging portfolio cannot be chosen freely, and the maturities also vary with running time t. In contrast, we can prespecify arbitrary maturities (as long as they are distinct) and these maturities are allowed to stay fixed as t varies. For practical purposes this is extremely important, since in real life we only have access to a finite set of maturities for traded bonds.

Corollary 5.7 *Assume that the forward rate volatilities have the form*

$$\begin{cases} \sigma_j(t,T) = q_{j-1}(T-t), & j = 1, \cdots, m, \\ \delta_i(t,T) = \tau_i, & i = 1, \cdots, n, \end{cases} \tag{69}$$

where $\tau_1, \cdots, \tau_n$ are constants and $q_{j-1}(s)$ is a polynomial of degree $j-1$ with a non-vanishing leading term. Then the market is complete. Furthermore, the maturities can be chosen arbitrarily.

Proof. Follows immediately from Theorem 5.6. ∎

The next result and its proof explain Shirakawa's idea of using the Vandermonde matrix to construct the "basic bonds".

Corollary 5.8 *Let* $m = 0$ *and* $\delta_i(t,T) = \tau_i\delta(T-t)$ *where* δ *is a strictly positive function and* τ_i *are distinct non-zero constants. Then the market is complete.*

Proof. One can always choose a number $a > 0$ and a monotone sequence of u_k such that

$$\int_0^{u_k} \delta(s)ds = ka, \qquad k = 1, \dots n.$$

Take maturities $T_k = t + u_k$. Since $D_i(t, T_k) = ka\tau_i$ we have, putting $\gamma_i = e^{a\tau_i}$, that

$$\det A(t, T_1, \dots, T_n) = \det\,(\gamma_i^k - 1) \neq 0.$$

Indeed, the linear dependence condition can be written as

$$f(\gamma_i) := \sum_{k=1}^{n} \alpha_k \gamma_i^k - \sum_{k=1}^{n} \alpha_k = 0, \quad i = 1, ..., n,$$

with a nontrivial vector α; this is impossible: since also $f(1) = 0$ the coefficients of the polynomial $f(\gamma)$ of degree n must be equal to zero. ∎

Remark 5.9 There is an important practical as well as mathematical difference between the situation when the maturities of bonds in hedging portfolios depend or do not depend on the current time t. In the former case the portfolio contains *only instantaneously* a finite set of bonds ("basic bonds" at t) but when t varies, then the union of these sets of securities may happen to be infinite and even non-countable, and hence one *can not* apply the classical theory of stochastic integration. As can be seen from Corollary 5.8, the system of "basic bonds" constructed in [28] depends unfortunately on t. We note again that in our results above we may, in fact, chose maturities which stay fixed during the entire trading period.

6 Affine term structures

As soon as one moves from abstract theory to practical applications, and in particular to algorithms which have to be executed in real time on a computer, the need emerges of easily manageable analytical formulas. In the case of interest rate derivatives one is particularly fortunate if the models possess a so-called **affine term structure**.

In this section the starting point is that we take as given the dynamics of the short rate. The notations are a bit different from those of the others sections and we omit certain somewhat boring mathematical details, such as e.g. technical conditions ensuring that solutions to certain equations below actually exist and have desirable properties (integrability etc.).

Definition 6.1 *An interest rate model is said to have an* **affine term structure** *if bond prices can be described as*

$$p(t,T) = F(t,r_t,T), \tag{70}$$

where

$$\log F(t,r,T) = A(t,T) - B(t,T)r, \tag{71}$$

and where A and B are deterministic functions. We sometimes use the notation

$$F(t,r,T) = F^T(t,r).$$

A model exhibiting an affine term structure occurs naturally only in a Markovian environment and so the starting point in this section is that we consider the dynamics of the short rate of interest given *a priori* as a Markov process. Furthermore, we choose to specify the r-dynamics directly under the martingale measure Q.

Assumption 6.2 *We assume that under Q all bounded discounted price processes are martingales, the short rate is assumed to be the solution of a stochastic differential equation of the form*

$$dr_t = a(t,r_t)dt + b(t,r_t)d\tilde{W}_t + \int_E q(t,r_t,x)\mu(dt,dx), \tag{72}$$

where $a(t,r), b(t,r)$, and $q(t,r,x)$ are given deterministic functions. The process $\tilde{W}$ is Q-Wiener and μ has a predictable Q-intensity

$$\lambda(\omega,t,dx) = \lambda(t,r_{t-},dx), \tag{73}$$

where $\lambda(t,r,dx)$ is a deterministic measure for each t and r.

The main problem here is that of finding sufficient conditions on a, b, q, and λ for the existence of an affine term structure. We start by presenting the fundamental partial differential-difference equation in this context concerning the pricing of simple claims in a general Markovian setting.

Proposition 6.3 *Suppose that the short rate is given by (72) and consider, for a fixed T, any bounded (discounted) contingent claim X, to be paid at T, of the form*

$$X = \Phi(r_T). \tag{74}$$

Then the arbitrage-free price process $\pi(t; X)$ of this asset is given by

$$\pi(t; X) = F(t, r_t), \tag{75}$$

where F is a (sufficiently regular) function which is the a solution of the Cauchy problem

$$\begin{cases} \frac{\partial F}{\partial t}(t,r) + \mathcal{A}F(t,r) - rF(t,r) &= 0, \\ F(T,r) &= \Phi(r), \end{cases} \tag{76}$$

with

$$\begin{aligned} \mathcal{A}F(t,r) &= a(t,r)\frac{\partial F}{\partial r}(t,r) + \frac{1}{2}b^2(t,r)\frac{\partial^2 F}{\partial r^2}(t,r) \\ &+ \int_E \{F(t, r + q(t,r,x)) - F(t,r)\}\ \lambda(t,r,dx). \end{aligned} \tag{77}$$

Proof. By the Itô formula we have the following representation:

$$F(t,r_t)\exp\left\{-\int_0^t r_s ds\right\} = F_0 + \int_0^t \frac{\partial F(s,r_s)}{\partial r}\exp\left\{-\int_0^s r_u du\right\} b(s,r_s)d\tilde{W}_s$$

$$+\int_0^t\int_E \{F(s, r_{s-} + q(s, r_{s-}, x)) - F(s, r_{s-})\}\exp\left\{-\int_0^s r_u du\right\}\tilde{\mu}(ds,dx)$$

where $F_0 = F(0, r_0)$ and $\tilde{\mu}(ds, dx) = \mu(ds, dx) - \lambda(s, r_{s-}, dx)ds$. The right-hand side of this representation defines a local martingale. By Assumption 6.2 it is in fact a true martingale so, using the boundary condition, we see that it is the discounted price process of the contingent claim X and (75) follows. ∎

Notice that, in general,

$$\pi(t; X) = E^Q\left[\Phi(r_T)\exp\left\{-\int_t^T r_s ds\right\} | \mathcal{F}_t\right], \tag{78}$$

and, because of the Markovian setting, we have in fact (75) with

$$F(t,r) = E^Q\left[\Phi(r_T)\exp\left\{-\int_t^T r_s ds\right\}|r_t = r\right]. \tag{79}$$

Since $\mathcal{A}$ is easily seen to be the infinitesimal operator of r, the relation (76) is nothing but the Kolmogorov backward equation.

Corollary 6.4 *Given the short rate dynamics (72) – (73), bond prices are given by (70), where*

$$\begin{cases} \frac{\partial F^T}{\partial t}(t,r) + \mathcal{A}F^T(t,r) - rF^T(t,r) &= 0. \\ \\ F^T(T,r) &= 1. \end{cases} \tag{80}$$

We now turn to the existence of the affine term structure. The assertion below is an extension of a result by Duffie ([13], see also [8]).

Proposition 6.5 *Suppose that the r-dynamics under Q is given by (72) and the model parameters $a, b, q,$ and λ have the following structure*

$$\begin{aligned} a(t,r) &= \alpha_1(t) + \alpha_2(t)r, \\ b(t,r) &= \sqrt{\beta_1(t) + \beta_2(t)r}, \\ q(t,r,x) &= q(t,x), \\ \lambda(t,r,dx) &= l_1(t,dx) + l_2(t,dx)r, \end{aligned} \tag{81}$$

Suppose that the functions $A(.,T)$ and $B(.,T)$ on $[0,T]$ solve the following system of ODE's

$$\begin{aligned} \frac{\partial B}{\partial t}(t,T) + \alpha_2(t)B(t,T) - \tfrac{1}{2}\beta_2(t)B^2(t,T) + \Psi_2(t,B(t,T)) &= -1, \\ B(T,T) &= 0, \end{aligned} \tag{82}$$

$$\begin{aligned} \frac{\partial A}{\partial t}(t,T) + \alpha_1(t)B(t,T) + \tfrac{1}{2}\beta_1(t)B^2(t,T) + \Psi_1(t,B(t,T)) &= 0, \\ A(T,T) &= 0, \end{aligned} \tag{83}$$

where

$$\Psi_i(t,y) = \int_E \left\{1 - e^{-yq(t,x)}\right\} l_i(t,dx), \quad i = 1,2. \tag{84}$$

Then the model has an affine term structure of the form (70) – (71).

We see that, for fixed T, the system (82) – (83) has a nice recursive structure (and, in particular, when $l_2 = 0$ (82) is a Riccati-type equation for B). Given a solution to (82), we can then easily determine A by a simple integration.

Notice also that one needs some caution to ensure λ to be positive. This is the case if $l_1 \geq 0$ and $l_2 = 0$ or $l_1 \geq 0$, $l_2 \geq 0$, and the process r is positive (recall that, in principle, we admit negative values of interest rates).

Remark 6.6 Notice that the class of models satisfying (81) generalizes some well-known term structure models such as Vasiček [29], Cox–Ingersoll–Ross [9], Ho–Lee [20], Hull–White [21].

Remark 6.7 If a model possesses an affine term structure, then, always under the assumption 6.2 and that of Proposition 6.5, the forward rate dynamics are easily obtained as

$$\begin{aligned} df(t,T) &= \{B_{t,T}(t,T)r_t - A_{t,T}(t,T) + B_T(t,T)a(t,r)\}dt \\ &\quad + B_T(t,T)b(t,r)d\tilde{W}_t + \textstyle\int_E B_T(t,T)q(t,x)\mu(dt,dx) \end{aligned} \tag{85}$$

where $B_T(t,T)$ is the partial derivative with respect to T and $B_{t,T}$ the partial with respect to t and T.

Remark 6.8 We close this section by pointing out that, if a model possesses an affine term structure, then (see Remark 6.7) the forward rate dynamics satisfy (85) from which it is immediately seen that assumption 5.1, in particular the decomposition property (51), is satisfied with $c_t(x) = -q(t,x)$ and $\delta_0(t,T) = B_T(t,T)$. This implies that for the affine term structure models the hedging problem can be approached by means of a Laplace transform inversion (see section 5.1), which makes this problem considerably easier compared to the general setting of section 4.2.

References

[1] Artzner, P. & Delbaen, F. (1989) Term structure of interest rates. *Advances in Applied Mathematics* **10**, 95-129.

[2] Babbs, S. & Webber, N. (1993) A theory of the term structure with an official short rate. Working paper, University of Warwick.

[3] Björk, T. (1995) On the term structure of discontinuous interest rates. *Surveys in Industrial and Applied Mathematics* **2** No.4, 626-657.

[4] Björk, T. & Di Masi, G. & Kabanov, Y. & Runggaldier, W. (1995) Towards a general theory of bond markets. To appear.

[5] Björk, T. & Kabanov, Y. & Runggaldier, W. (1995) Bond markets where prices are driven by a general marked point process. Working paper No 67, Department of Finance. Stockholm School of Economics.

[6] Brace, A. & Musiela M. (1994) A multifactor Gauss Markov implementation of Heath, Jarrow, and Morton. *Mathematical Finance* **4**, 3, 259-283.

[7] Brémaud, P. (1981) *Point Processes and Queues: Martingale Dynamics.* Springer-Verlag, Berlin.

[8] Brown, R.H. & Schaefer, S.M. (1994) Interest rate volatility and the shape of the term structure. *Phil. Trans. R. Soc. Lond. A* **347**, 563-576.

[9] Cox, J. & Ingersoll, J. & Ross, S. (1985) A theory of the term structure of interest rates. *Econometrica* **53**, 385-408.

[10] Dana, R.-A. & Jeanblanc-Picqué M. (1994) *Marchés Financiers en Temps Continu. Valorisation et Equilibre.* Economica, Paris.

[11] Delbaen, F. & Schachermayer, W. (1994) A general version of the fundamental theorem on asset pricing. *Mathematishe Annalen* **300**, 463-520.

[12] Dellacherie, C & Meyer, P-A. (1972) *Probabilités et Potentiel.* Hermann, Paris.

[13] Duffie, D. (1992) *Dynamic Asset Pricing Theory.* Princeton Univ. Press.

[14] El Karoui, N. & Myneni, R. & Viswanathan, R. (1992) Arbitrage pricing and hedging of interest rate claims with state variables. I, II. Preprints, Laboratoire de Probabilités, Paris 6.

[15] Elliott, R.J. (1982) *Stochastic Calculus and Applications.* Springer-Verlag, Berlin.

[16] Geman, H. & El Karoui, N & Rochet, J-C. (1995) Changes of numéraire, changes of probability measure and option pricing. *Journal of Applied Probability* **32**, 443-458.

[17] Harrison, J.M. & Kreps, D. (1979) Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory* **20**, 381-408.

[18] Harrison, J.M. & Pliska, S. (1981) Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes & Appl.* **11**, 215-260.

[19] Heath, D. & Jarrow, R. & Morton, A. (1992) Bond pricing and the term structure of interest rates. *Econometrica* **60** No.1, 77-106.

[20] Ho, T. & Lee, S. (1986) Term structure movements and pricing interest rate contingent claims. *Journal of Finance* **41**, 1011-1029.

[21] Hull, J & White, A. (1990) Pricing interest-rate-derivative securities. *The Review of Financial Studies* **3**, 573-592.

[22] Jacod, J. & Shiryaev A.N. (1987) *Limit Theorems for Stochastic Processes.* Springer-Verlag, Berlin.

[23] Jarrow, R. & Madan D. (1995) Option pricing using the term structure of interest rates to hedge systematic discontinuities in asset returns. *Mathematical Finance* **5**, 311-336.

[24] Jarrow, R. & Madan D. (1995) Valuing and hedging contingent claims on semimartingales. Preprint.

[25] Lindberg, H. & Orszag, M. & Perraudin, W. (1995) Yield curves with jump short rates. Preprint.

[26] Protter, P. (1990) *Stochastic integration and Differential Equations.* Springer-Verlag, Berlin.

[27] Rogers, L.C.G. (1995) What is a good model of interest rates. Preprint.

[28] Shirakawa, H. (1991) Interest rate option pricing with Poisson–Gaussian forward rate curve processes. *Mathematical Finance* **1**, 77-94.

[29] Vasiček, O. (1977) An equilibrium characterization of the term structure. *Journal of Financial Economics* **5**, 177-188.