An Axiomatization of the Euclidean Compromise Solution

by

M. Voorneveld[*], A. van den Nouweland[†], and R. McLean[‡]

SSE/EFI Working Paper Series in Economics and Finance, No. 703; This version: October 13, 2008.

Abstract

The utopia point of a multicriteria optimization problem is the vector that specifies for each criterion the most favourable among the feasible values. The Euclidean compromise solution in multicriteria optimization is a solution concept that assigns to a feasible set the alternative with minimal Euclidean distance to the utopia point. The purpose of this paper is to provide a characterization of the Euclidean compromise solution. Consistency plays a crucial role in our approach.

[*]Department of Economics, Stockholm School of Economics, Stockholm, SE-11383 Sweden. E-mail: mark.voorneveld@hhs.se.

[†]Department of Economics, University of Oregon, Eugene, OR 97403-1285, USA. E-mail: annev@uoregon.edu.

[‡]Department of Economics, Rutgers University, New Brunswick, NJ, 08901-1248, USA. E-mail: rpmclean@rci.rutgers.edu.

1. Introduction

Multicriteria optimization extends optimization theory by permitting several — possibly conflicting — objective functions, which are to be 'optimized' simultaneously. By now an important branch of Operations Research (see Steuer *et al.*, 1996), it ranges from highly verbal approaches like Larichev and Moshkovich (1997) to highly mathematical approaches like Sawaragi *et al.* (1985), and is known by various other names, including Pareto optimization, vector optimization, efficient optimization, and multiobjective optimization. Formally, a multicriteria optimization problem can be formulated as

$$\begin{aligned} &\text{Optimize} \quad f_1(x), \ldots, f_n(x) \\ &\text{subject to} \quad x \in X, \end{aligned} \tag{1.1}$$

where X denotes the feasible set of alternatives and n is the number of separate objective functions $f_k : X \to \mathbb{R}$ $(k = 1, \ldots, n)$.

The simultaneous optimization of multiple objective functions suggests the question: what does it mean to optimize, i.e., *what is a good outcome*? Different answers to this question lead to different ways of solving multicriteria optimization problems. For detailed descriptions and good introductions to the area, see White (1982), Yu (1985), and Zeleny (1982).

Yu (1973) introduced compromise solutions, based on the idea of finding a feasible point that is as close as possible to an ideal outcome. Zeleny (1976, p. 174) even states this informally as an 'axiom of choice':

> "Alternatives that are closer to the ideal are preferred to those that are farther away. To be as close as possible to the perceived ideal is the rationale of human choice."

The ideal point, or utopia point, specifies for each objective function separately the optimal feasible value. Being "close to" a point, of course, requires the specification of a metric. Yu (1973) concentrates on distance functions defined by ℓ_p-norms, but possible extensions include the use of different norms (cf. Gearhart, 1979) or penalty functions (cf. White, 1984).

Bouyssou *et al.* (1993) observe that within multicriteria decision making '[a] systematic axiomatic analysis of decision procedures and algorithms is yet to be carried out'. Yu (1973, 1985) and Freimer and Yu (1976) already indicate several properties of compromise solutions. In this paper we concentrate on the Euclidean compromise solution, selecting the feasible point that minimizes the Euclidean distance to the utopia point. We study the properties of this solution and provide several axiomatic characterizations: the Euclidean compromise solution is shown to be the unique solution concept satisfying several of the properties on a domain of multicriteria optimization problems.

This paper contributes to the economic literature on consistency of solution concepts. Many characterizations of concepts in areas like cooperative and non-cooperative game theory, abstract economies, bargaining, and matching theory rely on a consistency property (see Thomson, 2006, for a detailed overview). Loosely speaking, consistency entails the following: consider a domain of problems $\mathcal{P}$ and a solution concept φ that assigns a payoff vector to each problem in $\mathcal{P}$. Consider a problem $P \in \mathcal{P}$ with a set N of economic agents and let M be a subset of N. Give agents outside M their payoff according to φ in P and consider an appropriately defined "reduced problem" $P_M^{\varphi} \in \mathcal{P}$ for the remaining members – those in M. The solution concept φ is consistent if it does not involve a sudden change of plans: The prescribed allocation to each member of M in the reduced problem P_M^{φ} is the same as that in the original problem P when φ is used to determine allocations in both problems. Lensberg (1988) axiomatizes the Nash bargaining solution using consistency, calling it multilateral stability. The current paper uses essentially the same axiom.

The set-up of the paper is as follows. Section 2 contains definitions and preliminary results. The domain of choice sets and the Euclidean compromise solution are defined in Section 3. Section 4 contains all the properties as well as the main results; the Euclidean compromise solution is shown to be the *unique* solution concept satisfying various sets of the properties. All proofs are in Section 5. Section 6 addresses the logical independence of the axioms. We conclude in Section 7, which contains remarks on related literature.

2. Preliminaries

Throughout this paper, N will denote a non-empty, finite set of positive integers ($N \subseteq \mathbb{N}$, $N \neq \emptyset$, N finite). Let $\mathbb{R}^N$ denote the $|N|$-dimensional Euclidean space with axes indexed by the elements of N. As usual, $\mathbb{R}^N_+$, $\mathbb{R}^N_{++}$, and $\mathbb{R}^N_-$ will denote, respectively, the nonnegative, positive and nonpositive orthants of $\mathbb{R}^N$. For vectors $x, y \in \mathbb{R}^N$, write $x \geqq y$ if $x - y \in \mathbb{R}^N_+$, $x > y$ if $x - y \in \mathbb{R}^N_{++}$, and $x \geq y$ if $x - y \in \mathbb{R}^N_+$ and $x \neq y$. Relations $\leqq$, $\leq$, and $<$ are defined analogously. If $x \in \mathbb{R}^N$ and $I \subseteq N$, we denote $x_I = (x_i)_{i \in I}$. For two sets $X, Y \subseteq \mathbb{R}^N$, define $X + Y = \{x + y \mid x \in X, y \in Y\}$. If $x \in \mathbb{R}^N$, we will abuse notation slightly and sometimes write $x + Y$ instead of $\{x\} + Y$.

Let $S \subseteq \mathbb{R}^N$. A point $x \in S$ is *Pareto optimal in S* if there is no $y \in S$ such that $y \geq x$. The set of Pareto optimal points of S is denoted by

$$PO(S) = \{x \in S \mid \text{if } y \in \mathbb{R}^N \text{ and } y \geq x, \text{ then } y \notin S\}.$$

The *comprehensive hull* of S is the set

$$comp(S) = \{y \in \mathbb{R}^N \mid y \leqq x \text{ for some } x \in S\}.$$

The inner product of two vectors $x, y \in \mathbb{R}^N$ is denoted by $\langle x, y \rangle = \sum_{i \in N} x_i y_i$ and the Euclidean norm of $x \in \mathbb{R}^N$ is $\|x\| = \sqrt{\langle x, x \rangle}$. The (closed) ball centered at $x \in \mathbb{R}^N$ with radius $r > 0$ is denoted $B(x, r)$:

$$B(x, r) = \{y \in \mathbb{R}^N \mid \|y - x\| \leqq r\}.$$

Remark 1. *Let* $y \in B(x, r)$ *with* $\|y - x\| = r$. *We often use the fact that*

$$\{z \in \mathbb{R}^N \mid \langle x - y, z \rangle = \langle x - y, y \rangle\}$$

is the unique hyperplane supporting the ball $B(x, r)$ *at the point* y.

For $x, y \in \mathbb{R}^N$, define the vector obtained by coordinatewise multiplication $x * y \in \mathbb{R}^N$ by $(x * y)_i = x_i y_i$ for each $i \in N$. For a set $S \subseteq \mathbb{R}^N$ and an $x \in \mathbb{R}^N$, $x * S = \{x * s \mid s \in S\}$. For $x \in \mathbb{R}^N_{++}$, define the vector obtained by coordinate-wise reciprocals $x^{-1} \in \mathbb{R}^N$ by $(x^{-1})_i = \frac{1}{x_i}$ for all $i \in N$.

For a normal $h \in \mathbb{R}^N$ and a number $a \in \mathbb{R}$, the hyperplane $H(h, a)$ and corresponding halfspace $H^-(h, a)$ are defined as follows:

$$\begin{aligned} H(h, a) &= \{x \in \mathbb{R}^N \mid \langle h, x \rangle = a\}, \\ H^-(h, a) &= \{x \in \mathbb{R}^N \mid \langle h, x \rangle \leqq a\}. \end{aligned}$$

Remark 2. *If* $h, b \in \mathbb{R}^N_{++}$ *and* $a \in \mathbb{R}$, *then it is straightforward to verify that* $b * H^-(h, a) = H^-(h * b^{-1}, a)$.

Let $|N| \geqq 2$ and consider a coordinate $i \in N$. The function that projects each $x \in \mathbb{R}^N$ onto $\mathbb{R}^{N \setminus i}$ by omitting the coordinate indexed by i is denoted by p_{-i}.[1] If $S \subseteq \mathbb{R}^N$, then

$$p_{-i}(S) := \{p_{-i}(s) \mid s \in S\} \subseteq \mathbb{R}^{N \setminus i}.$$

3. The Euclidean compromise solution

We identify alternatives with their evaluations according to pertinent criteria. Hence, an alternative is a vector $x \in \mathbb{R}^N$, where the coordinate x_k ($k \in N$) indicates how alternative x is evaluated according to criterion k. It is assumed throughout that larger values are preferred to smaller values for each criterion.

[1] Throughout this paper, we will write $N \setminus i$ instead of the technically correct $N \setminus \{i\}$ as we believe this does not introduce confusion, but makes things look less cluttered.

The Euclidean compromise solution assigns to a feasible set of alternatives the alternative with minimal Euclidean distance to the utopia point. The feasible sets are those that are expressible as the comprehensive hull of a nonempty, compact, and convex set. Formally, define

$$\Sigma^N = \{S \subseteq \mathbb{R}^N \mid S = comp(C) \text{ for some nonempty, compact, convex } C \subseteq \mathbb{R}^N\}.$$

The collection of all choice sets is denoted

$$\Sigma = \cup_{N \subseteq \mathbb{N},\ N \neq \emptyset,\ N \text{ finite}} \Sigma^N.$$

The *utopia point* $u(S)$ of $S \in \Sigma^N$ is the point in $\mathbb{R}^N$ that specifies for each criterion separately the highest achievable value:

$$u_i(S) = \max_{s \in S} s_i \text{ for each } i \in N.$$

Because $S = comp(C)$ for some nonempty, compact, convex $C \subseteq \mathbb{R}^N$, the utopia point is well-defined. Indeed, it is straightforward to prove that $u(S) = u(C)$. The choice sets with utopia point equal to the zero vector will play an important role in our axiomatization.

$$\Sigma_0^N = \{S \in \Sigma^N \mid u(S) = 0\},$$

$$\Sigma_0 = \cup_{N \subseteq \mathbb{N},\ N \neq \emptyset,\ N \text{ finite}} \Sigma_0^N.$$

A *solution* on Σ is a function φ on Σ that assigns to each choice set $S \in \Sigma$ a feasible point $\varphi(S) \in S$. The *Euclidean compromise solution* or Yu solution (cf. Yu, 1973) is the solution Y that assigns to each $S \in \Sigma$ the feasible point closest to the utopia point $u(S)$, i.e.,

$$Y(S) = \arg\min_{s \in S} \|u(S) - s\|.$$

Geometrically, $Y(S)$ is the projection of the point $u(S)$ onto the set S. The function Y is well-defined. This can be seen as follows. If $u(S) \in S$, then $Y(S) =$

$u(S)$. Otherwise, let $y \in S$. Since the point in S that is closest to $u(S)$ cannot be further away than y is, to find $Y(S)$, it suffices to find the point s in $S \cap B(u(S), \|u(S) - y\|)$ that minimizes $\|u(S) - s\|$. Such a point exists and is unique because $S \cap B(u(S), \|u(S) - y\|)$ is a nonempty, compact, and convex set and, moreover, $s \mapsto \|u(S) - s\|$ is a continuous function that is strictly convex.

4. Axiomatization of the Euclidean compromise solution

We start this section by introducing several properties of solution concepts. Let φ be a solution on Σ and consider the following axioms.

> **Pareto Optimality (PO)**: $\varphi(S) \in PO(S)$ for all $S \in \Sigma$.
>
> **Translation Invariance (T.INV)**: If $S \in \Sigma^N$ and $x \in \mathbb{R}^N$, then $\varphi(S + x) = \varphi(S) + x$.
>
> **Symmetry (SYM)**: If $S \in \Sigma^N$ and if $\pi(S) = S$ for each permutation π of N, then $\varphi_i(S) = \varphi_j(S)$ for all $i, j \in N$.
>
> **u-Independence of Irrelevant Alternatives (u-IIA)**: Suppose that $S, T \in \Sigma^N$ with $S \subseteq T$ and $u(S) = u(T)$. Then $\varphi(T) \in S$ implies $\varphi(S) = \varphi(T)$.

The preceeding four axioms can be found in Yu (1973, 1985) and Freimer and Yu (1976). The axioms PO, T.INV, and SYM are obvious translations of Nash's axioms from bargaining theory to the choice problem framework. Furthermore, u-IIA is the natural adaptation of Nash's original IIA axiom for bargaining problems to the choice framework with the disagreement point d replaced by the utopia point $u(S)$.

Other important properties in the game theoretic literature on bargaining (cf. Nash (1950) and Roth (1985)) are proportionality properties such as scale covariance. Conley et al. (2008) exploit the duality between bargaining problems and multi-criteria optimization problems and they formulate a proportional losses axiom for multi-criteria optimization problems that is inspired by scale covariance.

Proportional Losses (P.LOSS): Suppose $S \in \Sigma^N$ is such that $PO(S) = H(h, a) \cap [u(S) - \mathbb{R}^N_+]$ for some $h \in \mathbb{R}^N_{++}$ and $a \in \mathbb{R}$. Then for any $\lambda \in \mathbb{R}^N_{++}$ and all $i, j \in N$

$$\lambda_i[u_i(\lambda * S) - \varphi_i(\lambda * S)][u_j(S) - \varphi_j(S)] = \lambda_j[u_j(\lambda * S) - \varphi_j(\lambda * S)][u_i(S) - \varphi_i(S)].$$

The proportional losses axiom indicates how a solution reacts to rescaling the coordinates of choice sets whose Pareto frontier is part of a hyperplane with a positive normal. If such a choice set S is rescaled by a $\lambda \in \mathbb{R}^N_{++}$ with $\lambda_i/\lambda_j = 2$, then according to P.LOSS,

$$\frac{u_i(\lambda * S) - \varphi_i(\lambda * S)}{u_j(\lambda * S) - \varphi_j(\lambda * S)} = \frac{1}{2}\left[\frac{u_i(S) - \varphi_i(S)}{u_j(S) - \varphi_j(S)}\right]$$

Hence, the *loss relative to the utopia point* measured in criterion i relative to that in criterion j in the re-scaled choice set $\lambda * S$ should be half the *loss relative to the utopia point* measured in criterion i relative to that in criterion j in the original choice set S. Therefore, if the unit of measurement of an objective doubles, then the relative loss in this objective as measured in the new units halves compared to that as measured in the old units. A weaker form of P.LOSS is

Scaling (SCA): Suppose $S \in \Sigma^N$ is of the form $S = \{x \in \mathbb{R}^N_- \mid \sum_{i \in N} x_i \leqq a\}$ for some $a \in \mathbb{R}$, $a < 0$. Then for any $\lambda \in \mathbb{R}^N_{++}$ and all $i, j \in N$

$$\lambda_i \varphi_j(S) \varphi_i(\lambda * S) = \lambda_j \varphi_i(S) \varphi_j(\lambda * S).$$

The scaling axiom imposes more restrictions on the choice sets than P.LOSS and is a weaker axiom: P.LOSS implies SCA. To see this suppose that $a \in \mathbb{R}$, $a < 0$ and $S = \{x \in \mathbb{R}^N_- \mid \sum_{i \in N} x_i \leqq a\}$. Then

$$PO(S) = \{x \in \mathbb{R}^n_- \mid \sum_{i \in N} x_i = a\} = H(h, a) \cap [-\mathbb{R}^N_+],$$

where $h = (1, .., 1) \in \mathbb{R}^N_{++}$. Since $u_i(S) = 0$ for each $i \in N$, it follows that S satisfies the conditions in P.LOSS. The conclusion now follows from the observation that for any $\lambda \in \mathbb{R}^N_{++}$ it holds that $u_i(\lambda * S) = 0$ for each $i \in N$.

Central to our axiomatization of the Euclidean compromise solution is a consistency axiom that to the best of our knowledge has not been considered by any other authors before in a setting of multi-criteria optimization problems.

u-Consistency (u-CONS): Let $S \in \Sigma^N$ and $I \subseteq N$, and define $S_I^\varphi \in \Sigma^I$ by

$$S_I^\varphi := \{s \in \mathbb{R}^I \mid (s, \varphi_{N \setminus I}(S)) \in S\}.$$

If $u_i(S_I^\varphi) = u_i(S)$ for each $i \in I$, then $\varphi_i(S_I^\varphi) = \varphi_i(S)$ for each $i \in I$.

In the statement of u-CONS, the $|I|$-coordinate choice set S_I^φ is the reduced problem alluded to in the introduction. It is the set of feasible criterion values for the subset of criteria I after the criteria outside I have been fixed at their values according to φ. Part (e) of Lemma 1 (which appears in Section 5) guarantees that $S_I^\varphi \in \Sigma$, so that we can apply φ to the reduced choice set. Suppose that the utopia levels of the criteria in I are the same in the reduced choice set S_I^φ as in the original choice set S. Then u-consistency requires that the solution prescribes the same values for the criteria in I in both choice sets. A weaker version of u-CONS that applies only to subsets of N obtained by deleting exactly one player, is

Weak u-Consistency (W.u-CONS): Let $S \in \Sigma^N$ and $i \in N$, and define $S_{-i}^\varphi \in \Sigma^{N \setminus i}$ by

$$S_{-i}^\varphi := \{s \in \mathbb{R}^{N \setminus i} \mid (s, \varphi_i(S)) \in S\}.$$

If $u(S_{-i}^\varphi) = p_{-i}(u(S))$, then $\varphi(S_{-i}^\varphi) = p_{-i}(\varphi(S))$.

We can now state the first result of this paper.

Theorem 1. *The Euclidean compromise solution is the unique solution on* Σ *satisfying PO, T.INV, SYM,* u*-IIA, SCA, and* u*-CONS.*

Theorem 1 differs from the axiomatization of the Euclidean compromise solution in Conley *et al.* (2008) in two respects. Theorem 1 uses SCA instead of the stronger P.LOSS and, more importantly, it uses u-CONS instead of continuity (with respect to the Hausdorff metric). Consequently, the axiomatizations are quite different: consistency links the solutions for feasible sets for different player sets, whereas continuity links the solutions for feasible sets for a fixed player set.

An alternative axiomatization is possible using a variant of the scaling axiom which we state next.

Symmetric Scaling (S.SCA): Suppose $S \in \Sigma^N$ is of the form $S = \{x \in \mathbb{R}^N_- \mid \sum_{i \in N} x_i \leqq a\}$ for some $a \in \mathbb{R}$, $a < 0$. Then for any $\lambda \in \mathbb{R}^N_{++}$ and all $i, j \in N$

$$\lambda_i \varphi_i(\lambda * S) = \lambda_j \varphi_j(\lambda * S).$$

SCA and S.SCA are not logically nested but are equivalent in the presence of SYM (see Lemma 2 in Section 5). Note that S.SCA implies that $\varphi_i(S) = \varphi_j(S)$ for the special set S in the statement of the axiom (simply let $\lambda = (1, ..., 1)$) and this is just the right amount of symmetry needed to characterize the compromise solution. In particular, S.SCA can be used in place of SYM and SCA in the statement of Theorem 1 to obtain the next result.

Theorem 2. *The Euclidean compromise solution is the unique solution on* Σ *satisfying PO, T.INV,* u*-IIA, S.SCA, and* u*-CONS.*

An axiomatization different from that presented in Theorems 1 and 2 is also possible using the following variation on the theme of consistency.

> **Projection (PROJ)**: Suppose that $S \in \Sigma^N_0$. If for some $i \in N$ it holds that $s_i = 0$ for all $s \in PO(S)$, then $\varphi_i(S) = 0$ and $\varphi(p_{-i}(S)) = p_{-i}(\varphi(S))$.

The projection axiom requires that φ satisfy a consistency-like property when restricted to a special class of choice sets in which criterion i attains the same value at any Pareto optimal alternative. For such choice sets, the solution prescribes alternatives in which criterion i is maintained at its constant Pareto-optimal level, whereas the levels of the other criteria can be found from the lower-dimensional choice set that is obtained by disregarding criterion i. The very mild partial Pareto optimality requirement embodied in PROJ on a very restricted class of problems is enough to characterize the Euclidean compromise solution without requiring PO as an explicit axiom.

Theorem 3. *The Euclidean compromise solution is the unique solution on Σ satisfying T.INV, SYM, u-IIA, SCA, and PROJ.*

Finally, we can replace SYM and SCA in Theorem 3 with S.SCA and obtain the following axiomatically parsimonious characterization of the Euclidean compromise solution.

Theorem 4. *The Euclidean compromise solution is the unique solution on Σ satisfying T.INV, u-IIA, S.SCA, and PROJ.*

5. Proofs of Theorems 1, 2, 3, and 4

The proofs of the four theorems are split up into a sequence of partial results in order to make the proofs more accessible.

We start by proving two lemmas. The first indicates that Σ is closed under projections, that Pareto optima and utopia vectors are in a sense robust against projections, and that Σ is closed under rescaling of its coordinates or reduction. Mostly, the proofs of these statements are trivial exercises; we will only provide the proof of parts (b) and (e).

Lemma 1. *Suppose that* $|N| \geqq 2$, $S \in \Sigma^N$, $i \in N$, *and* $\lambda \in \mathbb{R}^N_{++}$. *The following claims hold:*

(a) $p_{-i}(S) \in \Sigma^{N \setminus i}$.

(b) *If* $PO(S) \subseteq \{x \in \mathbb{R}^N \mid x_i = 0\}$, *then* $p_{-i}(PO(S)) = PO(p_{-i}(S))$.

(c) $p_{-i}(u(S)) = u(p_{-i}(S))$.

(d) $\lambda {*} S \in \Sigma^N$.

(e) *If* $I \subseteq N$, $I \neq \emptyset$, *and* $s \in S$, *then* $\{y \in \mathbb{R}^I \mid (y, s_{N\setminus I}) \in S\} \in \Sigma^I$.

Proof. To prove **part (b)**, assume that $PO(S) \subseteq \{x \in \mathbb{R}^N \mid x_i = 0\}$.

First, let $v \in p_{-i}(PO(S))$. Then there exists a $\widetilde{v} \in PO(S)$ such that $p_{-i}(\widetilde{v}) = v$. Suppose $v \notin PO(p_{-i}(S))$. Then $w \geq v$ for some $w \in p_{-i}(S)$. Let $\widetilde{w} \in S$ be such that $p_{-i}(\widetilde{w}) = w$. Since $S \in \Sigma$, there exists an $\widetilde{x} \in PO(S)$ such that $\widetilde{x} \geqq \widetilde{w}$. Then $\widetilde{v}, \widetilde{x} \in PO(S)$ implies $\widetilde{v}_i = \widetilde{x}_i = 0$ and we also know that $p_{-i}(\widetilde{x}) \geqq p_{-i}(\widetilde{w}) = w \geq v = p_{-i}(\widetilde{v})$. So $\widetilde{x} \geq \widetilde{v}$, contradicting $\widetilde{v} \in PO(S)$. Hence, $v \in PO(p_{-i}(S))$ and we conclude that $p_{-i}(PO(S)) \subseteq PO(p_{-i}(S))$.

Now, let $v \in PO(p_{-i}(S))$. Then there exists a $\widetilde{v} \in S$ such that $p_{-i}(\widetilde{v}) = v$. Since $S \in \Sigma$, there exists a $\widetilde{w} \in PO(S)$ such that $\widetilde{w} \geqq \widetilde{v}$. Then $p_{-i}(\widetilde{w}) \in p_{-i}(S)$ and $p_{-i}(\widetilde{w}) \geqq p_{-i}(\widetilde{v}) = v \in PO(p_{-i}(S))$, so the weak inequality between $p_{-i}(\widetilde{w})$ and $p_{-i}(\widetilde{v})$ must be an equality. Since $p_{-i}(\widetilde{w}) \in p_{-i}(PO(S))$, it follows that $v = p_{-i}(\widetilde{v}) = p_{-i}(\widetilde{w}) \in p_{-i}(PO(S))$ and we conclude that $p_{-i}(PO(S)) \supseteq PO(p_{-i}(S))$.

To prove **part (e)**, let $I \subseteq N$, $I \neq \emptyset$, and $s \in S$. Because $S \in \Sigma^N$, there is a nonempty, compact, convex set $C \subseteq \mathbb{R}^N$ such that $S = \mathit{comp}(C)$. Note that the set $\widetilde{C} = \{c \in C \mid c_{N\setminus I} \geqq s_{N\setminus I}\} \subseteq \mathbb{R}^N$ is nonempty, compact, and convex as well. These properties are not lost under a projection p_{-j} for any $j \in N\setminus I$. Therefore, writing $N\setminus I = \{j(1), \ldots, j(m)\}$, we find that the set $\widehat{C} =$

$p_{-j(m)} \circ \cdots \circ p_{-j(1)}(\widetilde{C}) \subseteq \mathbb{R}^I$, which is obtained from $\widetilde{C}$ by succesive projections with respect to the coordinates in $N \backslash I$, is a nonempty, compact, convex set. Noting that $\{y \in \mathbb{R}^I \mid (y, s_{N\backslash I}) \in S\} = comp(\widehat{C}) \subseteq \mathbb{R}^I$, it follows that $\{y \in \mathbb{R}^I \mid (y, s_{N\backslash I}) \in S\} \in \Sigma^I$. ■

In the following lemma we explore logical dependencies between the various axioms.

Lemma 2. *Let φ be a solution on Σ. The following claims hold:*

(a) *If φ satisfies PO, u-IIA, and W.u-CONS, then φ satisfies PROJ.*

(b) *Suppose that φ satisfies SYM. Then φ satisfies SCA if and only if φ satisfies S.SCA.*

Proof. To prove **part (a)**, let $S \in \Sigma_0^N$ and $i \in N$ and suppose that $s_i = 0$ for all $s \in PO(S)$. Note that $\varphi_i(S) = 0$ by PO. It remains to show that $\varphi(p_{-i}(S)) = p_{-i}(\varphi(S))$. First, note that

$$S^{\varphi}_{-i} = \{s \in \mathbb{R}^{N\backslash i} \mid (s, \varphi_i(S)) \in S\} = \{s \in \mathbb{R}^{N\backslash i} \mid (s, 0) \in S\}.$$

Hence, $u(S^{\varphi}_{-i}) = p_{-i}(u(S))$. Also, by Lemma 1 (c), it holds that $p_{-i}(u(S)) = u(p_{-i}(S))$. Therefore, we know that

$$u(S^{\varphi}_{-i}) = u(p_{-i}(S)).$$

By definition of S^{φ}_{-i} we know that

$$S^{\varphi}_{-i} \subseteq p_{-i}(S).$$

Next, we show that $PO(p_{-i}(S)) \subseteq S^{\varphi}_{-i}$ and that this implies $\varphi(p_{-i}(S)) \in S^{\varphi}_{-i}$. Choose $v \in PO(p_{-i}(S))$. Then there exists a $\tilde{v} \in S$ such that $p_{-i}(\tilde{v}) = v$. In addition, there exists a $\tilde{w} \in PO(S)$ such that $\tilde{w} \geqq \tilde{v}$, from which it follows that

$p_{-i}(\tilde{w}) \in p_{-i}(S)$ and $p_{-i}(\tilde{w}) \geqq p_{-i}(\tilde{v}) = v$. Since $v \in PO(p_{-i}(S))$, we conclude that $p_{-i}(\tilde{w}) = v$. Since $s_i = 0$ for all $s \in PO(S)$, it follows that $\tilde{w}_i = 0$ and therefore $(v, 0) \in S$ and $v \in S^{\varphi}_{-i}$. This proves that $PO(p_{-i}(S)) \subseteq S^{\varphi}_{-i}$. We can now apply PO and conclude that $\varphi(p_{-i}(S)) \in PO(p_{-i}(S))$ from which it follows that

$$\varphi(p_{-i}(S)) \in S^{\varphi}_{-i}.$$

Summarizing, we have $u(S^{\varphi}_{-i}) = u(p_{-i}(S))$, $S^{\varphi}_{-i} \subseteq p_{-i}(S)$, and $\varphi(p_{-i}(S)) \in S^{\varphi}_{-i}$. Applying u-IIA, it follows that

$$\varphi(S^{\varphi}_{-i}) = \varphi(p_{-i}(S)).$$

Using $u(S^{\varphi}_{-i}) = p_{-i}(u(S))$ and applying W.u-CONS, we conclude that

$$\varphi(S^{\varphi}_{-i}) = p_{-i}(\varphi(S)).$$

Therefore, $\varphi(p_{-i}(S)) = p_{-i}(\varphi(S))$ and the proof of part (a) is complete.

To prove **part (b)**, suppose that φ satisfies SYM. Define $S = \{x \in \mathbb{R}^N_- \mid \sum_{i \in N} x_i \leqq a\}$ with $a < 0$ and choose $\lambda \in \mathbb{R}^N_{++}$. Applying SYM to the set S, it follows that $\varphi(S) = \mu(1, ..., 1)$ for some real number μ. Furthermore, $\varphi(S) \in S$ since φ is a solution. Observing that $0 \notin S$ (since $a < 0$), we deduce that $\mu < 0$, and consequently, that $\varphi_i(S) = \mu \neq 0$ for all $i \in N$. For each i, j, it follows that

$$\lambda_i \varphi_j(S) \varphi_i(\lambda * S) = \lambda_j \varphi_i(S) \varphi_j(\lambda * S)$$

if and only if

$$\lambda_i \varphi_i(\lambda * S) = \lambda_j \varphi_j(\lambda * S).$$

This proves that φ satisfies SCA if and only if φ satisfies S.SCA, and the proof is complete. ■

We are now set to prove that the Euclidean compromise solution satisfies all the axioms introduced in Section 4.

Proposition 3. *The Euclidean compromise solution satisfies PO, T.INV, SYM, u-IIA, P.LOSS, SCA, S.SCA, u-CONS, W.u-CONS, and PROJ.*

Proof. It is straightforward to verify that the Euclidean compromise solution satisfies PO, T.INV, SYM and u-IIA. Conley *et al.* (2008) show that the Euclidean compromise solution satisfies P.LOSS, from which it follows that it also satisfies the weaker SCA. Since SCA and SYM are satisfied, it follows from Lemma 2(b) that S.SCA is satisfied.

To see that the Euclidean compromise solution Y satisfies u-CONS, let $S \in \Sigma^N$ and $I \subseteq N$ and suppose that $u_i(S_I^Y) = u_i(S)$ for each $i \in I$. It follows that $(u(S_I^Y), u_{N\setminus I}(S)) = u(S)$. Also, by definition of S_I^Y, it holds that $t \in S_I^Y$ if and only if $(t, Y_{N\setminus I}(S)) \in S$. Using this, we deduce that $s \in S$ solves $\min_{s\in S} \|u(S) - s\|$ if and only if s_I solves $\min_{t\in S_I^Y} \|(u(S_I^Y), u_{N\setminus I}(S)) - (t, Y_{N\setminus I}(S))\|$. However, $Y(S_I^Y) = \arg\min_{t\in S_I^Y} \|u(S_I^Y) - t\| = \arg\min_{t\in S_I^Y} \|(u(S_I^Y), u_{N\setminus I}(S)) - (t, Y_{N\setminus I}(S))\|$. This shows that $Y_i(S_I^Y) = Y_i(S)$ for each $i \in I$.

Since u-CONS implies the weaker W.u-CONS, we can now can deduce from Lemma 2 that the Euclidean compromise solution satisfies PROJ. ■

Proposition 3 and Lemma 2(a) imply that Theorem 1 is an immediate consequence of Theorem 3 and that Theorem 2 is an immediate consequence of Theorem 4. In addition, SYM and SCA imply S.SCA as a consequence of Lemma 2(b), so Theorem 3 is an immediate consequence of Theorem 4. Hence, the remainder of this section is devoted to proving Theorem 4. We begin by proving that two of the four properties in that theorem imply that a solution concept selects a Pareto optimal feasible point for certain choice sets.

Lemma 4. *Let φ be a solution on Σ that satisfies u-IIA and S.SCA. Suppose that $|N| \geqq 2$, $h \in \mathbb{R}^N_{++}$, $a \in \mathbb{R}$, $a < 0$, and*

$$A = \{x \in \mathbb{R}^N \mid x \leqq 0 \text{ and } \langle h, x\rangle \leqq a\}.$$

Then $\varphi(A) \in PO(A)$.

Proof. Let $B = h * A$. Remark 2 implies that

$$B = \{x \in \mathbb{R}^N \mid x \leqq 0 \text{ and } \sum_{i \in N} x_i \leqq a\}.$$

Because $A = h^{-1} * B$, it follows from S.SCA applied to φ (with h^{-1} in the role of λ and B in the role of S) that

$$\forall i, j \in N : \varphi_i(A) h_j = \varphi_j(A) h_i.$$

Since $0 \notin A$, it follows that $\varphi(A) < 0$ implying that

$$\varphi(A) = \mu h \text{ for some } \mu \in \mathbb{R},\ \mu < 0. \tag{5.1}$$

Suppose that $\varphi(A) \notin PO(A)$, which implies that $\sum_{i \in N} h_i \varphi_i(A) < a$. We will derive a contradiction. Fix some $i \in N$. Using (5.1) and $h \in \mathbb{R}^N_{++}$, we deduce that $\frac{a - \sum_{j \in N} h_j \varphi_j(A)}{h_i \varphi_i(A)} < 0$. Choose $c \in (0, 1)$ such that $c > 1 + \frac{a - \sum_{j \in N} h_j \varphi_j(A)}{h_i \varphi_i(A)}$. Define $\lambda \in \mathbb{R}^N_{++}$ by $\lambda_j := 1$ for all $j \neq i$ and $\lambda_i := \frac{1}{c}$. Consider the choice set $C = \lambda * A$. Notice that $u(C) = \lambda * u(A) = 0$. By Remark 2,

$$C = \{x \in \mathbb{R}^N \mid x \leqq 0 \text{ and } \sum_{j \in N} \frac{h_j}{\lambda_j} x_j \leqq a\}.$$

Because $\lambda \geq (1, \ldots, 1)$, it easily follows that $C \subseteq A$. Also,

$$\begin{aligned}
\sum_{j \in N} \frac{h_j}{\lambda_j} \varphi_j(A) &= \sum_{j \in N,\, j \neq i} h_j\, \varphi_j(A) + c\, h_i\, \varphi_i(A) \\
&< \sum_{j \in N,\, j \neq i} h_j\, \varphi_j(A) + \left(1 + \frac{a - \sum_{j \in N} h_j \varphi_j(A)}{h_i \varphi_i(A)}\right) h_i\, \varphi_i(A) \\
&= a,
\end{aligned}$$

which proves that $\varphi(A) \in C$. Consequently, u-IIA of φ and (5.1) imply that

$$\varphi(C) = \varphi(A) = \mu h. \tag{5.2}$$

On the other hand, notice that $(h^{-1} * \lambda) * B = (h^{-1} * \lambda) * (h * A) = \lambda * A = C$. Let $j \in N$, $j \neq i$. We conclude that

$$\frac{h_i}{h_j} = \frac{\varphi_i(A)}{\varphi_j(A)} = \frac{\varphi_i(C)}{\varphi_j(C)} = \frac{\frac{\lambda_j}{h_j}}{\frac{\lambda_i}{h_i}} = c\frac{h_i}{h_j}$$

where the first two equalities follow from (5.2), the third from $(h^{-1} * \lambda) * B = C$ and S.SCA applied to φ (with $h^{-1} * \lambda$ in the role of λ and B in the role of S), and the fourth from the definition of λ. This is impossible since $c \neq 1$ and we conclude that $\varphi(A) \in PO(A)$. ■

Lemma 4 shows that Pareto optimality on a small set of multicriteria optimization problems is implied by other axioms. This is reminiscent of a result of Roth (1977), where he shows that the axiom of Pareto optimality in Nash's (1950) formulation of the bargaining problem is implied by a number of other axioms.

We now proceed to study choice sets in which the utopia outcome equals the zero vector and is actually feasible. If this is the case, the utopia outcome is selected by a solution concept satisfying PROJ.

Proposition 5. *Let φ be a solution concept on Σ that satisfies PROJ. Let $S \in \Sigma_0$ be such that $u(S) \in S$. Then $\varphi(S) = u(S) = Y(S)$.*

Proof. Obviously, $u(S) \in S$ implies that $Y(S) = u(S)$. We proceed by showing that $\varphi(S) = u(S)$. Since $u(S) \geqq s$ for each $s \in S$, $u(S) \in S$ implies $PO(S) = \{u(S)\}$.

Let N be such that $S \in \Sigma_0^N$ and discern two cases:

Case 1: If $|N| \geqq 2$, then $PO(S) = \{u(S)\} = \{0\}$ and PROJ of φ imply that $\varphi_i(S) = 0$ for each $i \in N$, so that $\varphi(S) = 0 = u(S)$.

Case 2: If $|N| = 1$, consider the set $T = S \times \{0\} = \{(s, 0) \in \mathbb{R}^2 \mid s \in S\} \in \Sigma^2$. Then $u(S) = 0 \in S$ implies that $u(T) = (u(S), 0) = (0, 0) \in T$ and $PO(T) = \{(0, 0)\}$. It follows that $\varphi(S) = p_{-2}(\varphi(T)) = p_{-2}((0, 0)) = 0 = u(S)$, where the first equality holds by applying PROJ with respect to the second coordinate of T, and the second equality holds since $\varphi(T) = u(T) = (0, 0)$ by case 1. Hence $\varphi(S) = u(S)$, as was to be shown. ■

In the next Proposition we show that in choice sets with utopia point zero and a Euclidean compromise solution which is smaller in each coordinate than the utopia point, every solution concept satisfying two of the four properties in Theorem 4 coincides with the Euclidean compromise solution.

Proposition 6. *Let φ be a solution concept on Σ that satisfies u-IIA and S.SCA. Suppose that $S \in \Sigma_0$ satisfies $Y(S) < u(S)$. Then $\varphi(S) = Y(S)$.*

Proof. Let N be such that $S \in \Sigma_0^N$. Because $u(S) = 0$, we know that $Y(S) < u(S) = 0$. Note that this implies that $|N| \geqq 2$. By definition of $Y(S)$, the choice set S and the ball $B(0, \|Y(S)\|)$ around the utopia point $u(S) = 0$ with radius $\|Y(S)\|$ have only the point $Y(S)$ in common. By the separating hyperplane theorem, there exists a hyperplane that separates the ball $B(0, \|Y(S)\|)$ and S, supporting the ball at $Y(S)$. By Remark 1, this is the hyperplane $H(h, a)$ with

$$h = u(S) - Y(S) = -Y(S) > 0 \text{ and } a = -\|h\|^2 < 0.$$

The choice set S lies in the halfspace $H^-(h, a) = \{x \in \mathbb{R}^N \mid \langle h, x\rangle \leqq a\}$. The choice set

$$A = \{x \in \mathbb{R}^N \mid x \leqq 0 \text{ and } \langle h, x\rangle \leqq a\}$$

satisfies

$$S \subseteq A \text{ and } u(S) = u(A) = 0.$$

By Remark 2,

$$B := h * A = \{x \in \mathbb{R}^N \mid x \leqq 0 \text{ and } \sum_{i \in N} x_i \leqq a\}.$$

Notice that $u(B) = h * u(A) = h * 0 = 0 \notin B$, because $a < 0$. Applying S.SCA with $S = B, \lambda = h^{-1} \in R^N_{++}$, and $A = h^{-1} * B$, it follows that

$$\forall i, j \in N: \quad \varphi_i(A)h_j = \varphi_j(A)h_i.$$

Applying Lemma 4 to the choice set A implies that $\langle h, \varphi(A)\rangle = a$, from which we conclude that

$$\varphi(A) = \frac{a}{||h||^2}h = -h = Y(S).$$

Since $S \subseteq A$, $u(S) = u(A) = 0$, and $\varphi(A) = Y(S) \in S$, it follows from u-IIA of φ that $\varphi(S) = \varphi(A) = Y(S)$. ■

We proceed by considering choice sets in Σ_0 for which the Euclidean compromise solution has some, but not all, coordinates equal to the corresponding coordinates of the utopia point. On such choice sets, solution concepts satisfying u-IIA, PROJ, and S.SCA coincide with the Euclidean compromise solution.

Proposition 7. *Let φ be a solution concept on Σ that satisfies u-IIA, PROJ, and S.SCA. Suppose that $S \in \Sigma_0$ satisfies $Y(S) \leq u(S)$, but not $Y(S) < u(S)$. Then $\varphi(S) = Y(S)$.*

Proof. Let N be such that $S \in \Sigma_0^N$. Note that $Y(S) \leq u(S)$, but not $Y(S) < u(S)$, implies that $|N| \geqq 2$. As in the proof of Proposition 6, we deduce that the unique tangent hyperplane $H(h, a)$ separating the sets S and $B(0, \|Y(S)\|)$ has normal $h = -Y(S)$ and $a = -\|Y(S)\|^2$. Define

$$T = \{x \in \mathbb{R}^N \mid x \leqq 0 \text{ and } \langle h, x\rangle \leqq a\} \in \Sigma^N.$$

Then

$$S \subseteq T,\ u(S) = u(T) = 0, \text{ and } Y(S) = Y(T). \tag{5.3}$$

The equality $Y(S) = Y(T)$ follows from the fact that by construction the ball $B(0, \|Y(S)\|)$ and T have exactly the point $Y(S)$ in common. It suffices to prove that

$$\varphi(T) = Y(T), \tag{5.4}$$

since (5.3), (5.4), and u-IIA of φ then imply $\varphi(S) = \varphi(T) = Y(T) = Y(S)$, which was to be shown.

By assumption, the set

$$I = \{i \in N \mid Y_i(S) = u_i(S)\} = \{i \in N \mid Y_i(T) = u_i(T)\} = \{i \in N \mid h_i = 0\}$$

is nonempty. It follows that $T = \{x \in \mathbb{R}^N \mid x \leqq 0 \text{ and } \sum_{j \in N \setminus I} h_j x_j \leqq a\}$, so that

$$\forall i \in I: \quad PO(T) \subseteq \{x \in \mathbb{R}^N \mid x_i = 0\}. \tag{5.5}$$

By (5.5), PO of Y, and PROJ of φ, we know

$$\forall i \in I : \varphi_i(T) = Y_i(T) = 0. \tag{5.6}$$

Lemma 1 (b) and (c) and PROJ of Y imply that for each $i \in I$

$$\begin{aligned} p_{-i}(PO(T)) &= PO(p_{-i}(T)), \\ p_{-i}(u(T)) &= u(p_{-i}(T)), \\ p_{-i}(Y(T)) &= Y(p_{-i}(T)). \end{aligned}$$

Whereas the set T has $|I|$ coordinates i for which $Y_i(T) = u_i(T)$, the choice set $p_{-i}(T)$ has only $|I| - 1$ such coordinates when $i \in I$. Repeated application of projection reduces this number to zero: Write $I = \{i(1), \ldots, i(m)\}$ and take

$$V = p_{-i(m)} \circ \cdots \circ p_{-i(1)}(T),$$

the choice set in $\Sigma_0^{N \setminus I}$ obtained from T by successive projection with respect to the coordinates in I. Then the set of coordinates j for which $Y_j(V) = u_j(V)$ is empty, so that $Y(V) < u(V)$. Since the Euclidean compromise solution selects the utopia point in one-dimensional choice sets, this implies that $V \in \Sigma_0^{N \setminus I}$ must be a choice set of dimension greater than or equal to two. Proposition 6 and PROJ of φ and Y imply

$$p_{-i(m)} \circ \cdots \circ p_{-i(1)}(Y(T)) = Y(V) = \varphi(V) = p_{-i(m)} \circ \cdots \circ p_{-i(1)}(\varphi(T)). \tag{5.7}$$

Equality (5.6) indicates that $Y_i(T) = \varphi_i(T)$ if $i \in I$ and equality (5.7) indicates that $Y_i(T) = \varphi_i(T)$ if $i \notin I$, which proves (5.4). ■

Now we merely have to combine the results obtained.

Proof of Theorem 4. Y satisfies T.INV, u-IIA, S.SCA, and PROJ by Proposition 3. Let φ be a solution concept on Σ that also satisfies these four properties. Let $S \in \Sigma$ and let $T = -u(S) + S \in \Sigma_0$. By T.INV of Y and φ, it suffices to show that $\varphi(T) = Y(T)$. If $u(T) \in T$, this follows from Proposition 5. If $Y(T) < u(T)$, it follows from Proposition 6. In all other cases, it follows from Proposition 7. ■

6. Independence of the axioms

Theorem 4 provides the most parsimonious axiomatization of the Yu solution. In this section, we establish the logical independence of the properties used to axiomatize the Euclidean compromise solution in Theorem 4. To accomplish this, we construct four alternative solution concepts, each of which violates exactly one of the four axioms T.INV, u-IIA, S.SCA, and PROJ. Since it is mostly straightforward to check that the solution concepts that we provide satisfy or violate certain axioms, we will not go into details. However, all proofs can be obtained from the authors upon request.

Example 1. *Minimization of a weighted generalization of the Yu solution yields a concept that violates S.SCA. Let $\{w_i\}_{i \in \mathbb{N}}$ be a sequence of positive numbers with $w_j \neq w_k$ for some $j, k \in \mathbb{N}$. Define a solution φ^1 on Σ by taking*

$$\forall S \in \Sigma^N : \varphi^1(S) = \arg\min_{x \in S} \sum_{i \in N} w_i [x_i - u_i(S)]^2 \ .$$

The solution concept φ^1 satisfies u-IIA, T.INV, and PROJ, but not S.SCA.

In the constructions of the remaining examples, we will make use of an auxiliary solution ψ on Σ defined as follows:

$$\forall S \in \Sigma^N : \quad \psi(S) = \arg\min_{s \in S} \left(\sum_{i \in N} (u_i(S) - s_i)^4 \right)^{1/4} .$$

Note that ψ satisfies u-IIA, T.INV, and PROJ.

Example 2. *Define φ^2 on Σ by taking*

$$\forall S \in \Sigma : \quad \varphi^2(S) = \begin{cases} Y(S) & \text{if } Y(S) < u(S), \\ \psi(S) & \text{otherwise.} \end{cases}$$

Solution concept φ^2 satisfies u-IIA, T.INV, and S.SCA, but not PROJ.

Example 3. *Define φ^3 on Σ by*

$$\forall S \in \Sigma : \quad \varphi^3(S) = \begin{cases} Y(S) & \text{if } S \in \Sigma_0, \\ \psi(S) & \text{otherwise.} \end{cases}$$

Solution concept φ^3 satisfies u-IIA, PROJ, and S.SCA, but not T.INV.

Example 4. *For $|N| \geqq 2$ and $a \in \mathbb{R}$, $a < 0$, we define $\widetilde{\Sigma}^{N,a} = \{x \in \mathbb{R}^N \mid x \leqq 0$ and $\sum_{i \in N} x_i \leqq a\}$ and $\widetilde{\Sigma} = \cup_{a \in \mathbb{R},\ a<0} \cup_{N \subseteq \mathbb{N},\ N \text{ finite},\ |N| \geqq 2}\ \widetilde{\Sigma}^{N,a}$. A choice set $S \in \widetilde{\Sigma}$ has utopia point $u(S) = 0$.*

Let $S \in \Sigma^N$ with $|N| \geqq 2$ and $I = \{i \in N \mid x_i = y_i$ for all $x, y \in PO(S)\}$ the collection of coordinates for which all Pareto optimal outcomes of S achieve the same value. Notice that $I = N$ if and only if $u(S) \in S$. If $u(S) \notin S$, define $p_{-I}(S)$ to be the set obtained from S by projecting away all coordinates in I. Part (a) of Lemma 1 implies that $p_{-I}(S) \in \Sigma^{N \setminus I}$ is indeed a choice set.

We say that a set $S \in \Sigma$ reduces to a set in $\widetilde{\Sigma}$ if $u(S) \notin S$, and there is a rescaling vector $\lambda \in \mathbb{R}^{N \setminus I}_{++}$ of the coordinates of $p_{-I}(-u(S) + S) \in \Sigma_0^{N \setminus I}$ such that $\lambda * p_{-I}(-u(S) + S)$ is in $\widetilde{\Sigma}$. Informally, S reduces to a set in $\widetilde{\Sigma}$ if $u(S) \notin S$ and, *moreover, after translation to a set with utopia point* 0*, projection, and rescaling of its coordinates,* S *yields a set in* $\widetilde{\Sigma}$*.*

Define φ^4 *on* Σ *by*

$$\forall S \in \Sigma : \quad \varphi^4(S) = \begin{cases} Y(S) & \text{if } S \text{ reduces to a set in } \widetilde{\Sigma}, \\ \psi(S) & \text{otherwise.} \end{cases}$$

Solution concept φ^4 *satisfies T.INV, PROJ, and S.SCA, but not* u*-IIA.*

7. Concluding remarks

Bouyssou *et al.* (1993) promote an axiomatic approach to the study of decision procedures in multicriteria optimization. In Theorems 1 through 4, we provide four axiomatic characterizations of the Euclidean compromise solution. Most of the properties that we use in these axiomatizations, namely PO, u-IIA, SYM, T.INV, u-CONS, and PROJ, are shared by a larger class of compromise solutions. For example, suppose that $f : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing and strictly convex and for each N and each $S \in \Sigma^N$, let

$$\psi(S) = \arg\min_{x \in S} \sum_{i \in N} f(u_i(S) - x_i).$$

Then ψ defines a solution on Σ satisfying PO, u-IIA, SYM, T.INV, u-CONS, and PROJ. Axioms of "scale covariance" type play an important role in the literature on bargaining and the axioms P.LOSS, SCA, and S.SCA are proportionality properties specific to the Euclidean compromise solution.

Consistency properties like u-CONS and PROJ play an essential role in our characterizations of the Euclidean compromise solution. Such properties allow us to, under certain circumstances, reduce the feasible set of alternatives to one with lower dimension. A very different characterization, relying on a continuity axiom instead of a consistency property, is given by Conley *et al.* (2008). They exploit an interesting duality between the compromise approach in multicriteria optimization and the game theoretic approach to bargaining. The compromise approach entails formulating a desirable, ideal point (the utopia point) and then 'working your way down' to a feasible solution as close as possible to the ideal. The bargaining approach entails formulating a typically undesirable disagreement point and then 'working your way up' to a feasible point dominating the disagreement outcome. Mixtures of the two approaches, like the Kalai-Smorodinsky (1975) solution, exist as well.

Rubinstein and Zhou (1999) characterize the solution concept that assigns to each choice set the point closest to an exogenously given and fixed reference point, rather than the utopia point, which varies as a function of the choice set. Their axiomatization involves a symmetry condition and independence of irrelevant alternatives. Whereas the symmetry condition in Section 4, taken from Yu (1973), requires symmetry only in the line through the origin with equal coordinates, the symmetry condition of Rubinstein and Zhou (1999) applies to choice sets that are symmetric with respect to *any* line through the reference point.

Pfingsten and Wagener (2003) also consider solution concepts defined in terms of optimal distances from a reference point. Unlike the approach we have taken in this paper, their reference point is explicitly assumed to be exogenous, as in Rubinstein and Zhou (1999). In addition, Pfingsten and Wagener (2003) restrict the class of solution concepts to those optimizing a distance function and they employ an axiomatic approach to single out a particular distance function. This makes their approach very different from ours, in that we derive the existence of a distance function from properties that do not make any reference to distance.

The domain of our solution concepts is the collection of all sets that can be expressed as the comprehensive hull of a nonempty, compact, and convex subset of a finite-dimensional Euclidean space. Other authors (e.g., Conley et al. (2008) and Yu (1973)) have considered the domain of nonempty, compact, and convex subsets of finite-dimensional Euclidean spaces. We believe that, with appropriate modifications of the statements of our axioms, we can prove some of our results for this domain and we leave this as a topic for future research.

In Yu (1973), a nonsymmetric generalization of the compromise solution was proposed and this weighted solution was characterized in Conley *et al.* (2008). A consistency-based axiomatization of this weighted extension is also possible using the methods developed in the current paper and we will pursue this in future work.

References

[1] Bouyssou, D., P. Perny, M. Pirlot, and P. Vincke (1993): "A manifesto for the new MCDA era", *Journal of Multi-Criteria Decision Analysis*, 2, 125-127.

[2] Conley, J., R. McLean, and S.Wilkie (2008): "Axiomatic foundations for compromise theory: the duality of bargaining theory and multi-objective programming", mimeo.

[3] Freimer, M. and P.L. Yu (1976): "Some new results on compromise solutions for group decision problems", *Management Science*, 22, 688-693.

[4] Gearhart, W.B. (1979): "Compromise solutions and estimation of the non-inferior set", *Journal of Optimization Theory and Applications*, 28, 29-47.

[5] Kalai, E. and M. Smorodinsky (1975): "Other solutions to Nash's bargaining problem", *Econometrica*, 43, 513-518.

[6] Larichev, O.I. and H.M. Moshkovich (1997): *Verbal decision analysis for unstructured problems*, Boston/Dordrecht/London: Kluwer Academic Publishers.

[7] Lensberg, T. (1988): "Stability and the Nash solution," *Journal of Economic Theory*, 45, 330-341.

[8] Nash, J.F. (1950): "The bargaining problem", *Econometrica*, 18, 155-162.

[9] Pfingsten, A. and A. Wagener (2003): "Bargaining solutions as social compromises", *Theory and Decision*, 55, 359-389.

[10] Roth, A.E. (1977): "Individual rationality and Nash's solution to the bargaining problem", *Mathematics of Operations Research*, 2, 64-65.

[11] Roth, A.E. (1985): *Game-theoretic models of bargaining*, Cambridge: Cambridge University Press.

[12] Rubinstein, A and L. Zhou (1999): "Choice problems with a 'reference' point", *Mathematical Social Sciences*, 17, 205-209.

[13] Sawaragi, Y. H. Nakayama and T. Tanino (1985): *Theory of multiobjective optimization*, New York: Academic Press.

[14] Steuer, R. E., L.R. Gardiner, and J. Gray (1996): "A bibliographic survey of the activities and international nature of multiple criteria decision making", *Journal of Multi-Criteria Decision Analysis*, 5, 195-217.

[15] Thomson, W. (2006): *Consistent Allocation Rules*, unpublished manuscript.

[16] White, D.J. (1982): *Optimality and efficiency*, New York: John Wiley and Sons.

[17] White, D.J. (1984): "Multiple objective programming and penalty functions", *Journal of Optimization Theory and Applications*, 43, 583-599.

[18] Yu, P.L. (1973): "A class of solutions for group decision problems", *Management Science*, 19, 936-946.

[19] Yu, P.L. (1985): *Multiple-criteria decision making: Concepts, techniques, and extensions*, New York: Plenum Pub. Corp.

[20] Zeleny, M. (1976): "The theory of the displaced ideal", in *Multi-Criteria Decision Making*, Zeleny M. (ed.), Berlin: Springer Verlag.

[21] Zeleny, M. (1982): *Multiple criteria decision making*, New York: McGraw-Hill.