Evolutionary dynamics may eliminate all strategies used in correlated equilibrium

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Abstract

In (Viossat, 2006, "The replicator dynamics does not lead to correlated equilibria", forthcoming in *Games and Economic Behavior*), it was shown that the replicator dynamics may eliminate all pure strategies used in correlated equilibrium, so that only strategies that do not take part in any correlated equilibrium remain. Here, we generalize this result by showing that it holds for an open set of games, and for many other dynamics, including the best-response dynamics, the Brown-von Neumann-Nash dynamics and any monotonic or weakly sign-preserving dynamics satisfying some standard regularity conditions. For the replicator dynamics and the best-response dynamics, elimination of all strategies used in correlated equilibrium is shown to be robust to the addition of mixed strategies as new pure strategies.

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1 Introduction

A major goal of evolutionary game theory is to clarify the connection between the outcome of simple adaptive processes modelling the evolution of behavior in populations of boundedly rational agents and equilibrium concepts. For a fairly wide class of dynamics, it has been found that, if a solution converges to a point and if initially all pure strategies are played with positive probability, then this point is a Nash equilibrium. Similarly, weak dynamic stability (Lyapunov stability) implies Nash equilibrium behavior (Weibull, 1995). On the other hand, evolutionary dynamics need not lead to Nash equilibria. For instance, in a version of the child game Rock-Paper-Scissors, the replicator dynamics does not converge to the unique Nash equilibrium, but cycles outward towards the boundary of the state space (Zeeman, 1980; Hofbauer and Sigmund, 1998). Non-convergence to Nash equilibria is a universal phenomenon: for any dynamics satisfying some minimal adaptivity and regularity conditions, there are games with a unique Nash equilibrium and such that, for an open set of initial conditions, the solution does not converge to the equilibrium but cycles (Hofbauer and Swinkels, unpublished; Hofbauer and Sigmund, 1998, section 8.6)[1,2].

There are several ways to try to find nonetheless a general connection between evolutionary dynamics and equilibria. A first possibility is to replace convergence to the set of Nash equilibria by some weaker connection, like convergence in time-average or simply a connection between strategies that survive and strategies that are played in equilibrium. The latter works for games with few strategies: under the single-population replicator dynamics, for any 3×3 symmetric game and any interior initial condition, all strategies that do not belong to the support of any Nash equilibrium are eliminated. This follows from Bomze's (1983) classification of the replicator dynamics' phase portraits in 3×3 symmetric games. The same result holds for the best-response dynamics (Viossat, 2005, chapter 9a).

Another possibility is to replace Nash equilibrium by a weaker concept. The main current candidate would probably be correlated equilibrium. Indeed, recent articles surveyed by Hart (2005) show that simple adaptive pro-

[1]This holds for every myopic adjustment dynamics (Swinkels, 1993) whose vector field depends smoothly on the payoff matrix of the game.

[2]We are interested here in evolutionary dynamics, but we stress that other types of adaptive processes more readily lead to approximate Nash equilibrium behavior. See e.g. Young (2004) and Foster and Young (2006).

cesses converge to the set of correlated equilibria, at least in a time-average sense. Though these processes are not evolutionary dynamics, this suggest that correlated equilibrium might be better related to the outcome of evolutionary dynamics than Nash equilibrium.

Again, in small dimension, some positive results can be obtained. For instance, in 3×3 symmetric games, under the single- *or two-population* replicator dynamics and for any interior initial condition, all strategies that do not belong to the support of any correlated equilibrium are eliminated (Viossat, 2005, chapter 9b).[3] The same result holds for the best-response dynamics, and for any convex monotonic dynamics, in the sense of Hofbauer and Weibull (1996).

However, for games with more strategies, even such a weak connection between correlated equilibrium and the outcome of evolutionary dynamics cannot be found. Indeed, for the single-population replicator dynamics, there are 4×4 symmetric games for which, for an open set of initial conditions, all strategies belonging to the support of at least one correlated equilibrium are eliminated; thus, only strategies that do not take part in any correlated equilibrium remain (Viossat, 2006). It follows that no kind of time-average of the replicator dynamics converges to the set of correlated equilibria.

The purpose of this article is to show that elimination of all strategies used in correlated equilibrium is a robust phenomenon; that is, it does not only occur for the replicator dynamics and very specific games, but for many dynamics and for an open set of games.

The games we study are 4×4 symmetric games built by adding a strategy to an outward-cycling Rock-Paper-Scissors game and assuming some payoff inequalities. Under the replicator dynamics and the best-response dynamics, the attractor of the underlying Rock-Paper-Scissors game is asymptotically stable in the augmented, 4×4 game. This is in spite of the fact that, for an open set of such games, the unique strategy belonging to the support of a correlated equilibrium is the added, fourth strategy. It follows that, for the replicator dynamics and the best-response dynamics, there is an open set of games for which, for an open set of initial conditions, all strategies used in correlated equilibrium are eliminated. This is also true of (i) a family of dynamics including the Brown-von Neumann-Nash dynamics; (ii) any mono-

[3]The reason why this does not follow from the above stated result on Nash equilibrium is that we now consider two-population dynamics, so that the state space has no longer dimension 2 but 4.

tonic or weakly sign-preserving dynamics in which no new strategy arises by mutation, and whose vector field depends continuously on the payoffs. Our proofs for these dynamics apply to a smaller open set of games than for the replicator dynamics and the best-response dynamics.

An issue is that our results might rest on the implicit assumption that agents play pure strategies.[4] To address this issue, we consider the following model, taken from Hofbauer and Sigmund (1998, section 7.2): there is a basic strategic situation, modelled by a finite normal form game, called the base game. The population is divided in a finite number of types of agents, and each type plays a pure or mixed strategy of the base game. Selection acts on types. Thus, the true game, i.e., the game in which selection operates, is the game whose pure strategies are the types and whose payoffs are induced by the base game payoffs. We assume that every pure strategy of the base game is played by one type of agent, but that otherwise we do not know which types of agents are present, nor the number of types. The question is whether we can nonetheless be sure that, in the true game, all strategies used in correlated equilibrium are eliminated for an open set of initial conditions, or whether this depends on the types that are present. We show that, at least for the replicator dynamics and the best-response dynamics, elimination of all strategies used in correlated equilibrium is robust in this sense.

The remainder of this article is organized as follows. The games studied throughout are introduced in section 2. Section 3 and the appendix show that, under the replicator dynamics, elimination of all pure strategies used in correlated equilibrium occurs for an open set of games (section 3). The same result is then shown to hold for the best-response dynamics (section 4), a family of dynamics including the Brown-von Neumann-Nash dynamics (section 5) and for monotonic or weakly sign-preserving dynamics (section 6). Section 7 studies the robustness of these results when agents can play mixed strategies. Finally, section 8 concludes.

We first introduce the framework and the notations.

Framework and notations. We study single-population dynamics in two-player, finite symmetric games. The set of pure strategies is $I = \{1, 2, .., N\}$ and S_N denotes the simplex of mixed strategies (henceforth, "the simplex"). Its vertices $\mathbf{e}_i$, $1 \leq i \leq N$, correspond to the pure strategies of the game. We denote by $x_i(t)$ the proportion of the population playing strategy i

[4] Note that in Hart and Mas-Collel's (2003) model, agents play mixed strategies.

at time t and by $\mathbf{x}(t) = (x_1(t), ..., x_N(t)) \in S_N$ the population profile (or mean strategy). We study its evolution under dynamics of type $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{U})$, where $\mathbf{U} = (u_{ij})_{1 \leq i,j \leq 4}$ is the payoff matrix of the game. We often skip the indication of time. For every $\mathbf{x}$ in S_N, the probability distribution on $I \times I$ induced by $\mathbf{x}$ is denoted by $\mathbf{x} \otimes \mathbf{x}$.

We assume known the definition of a correlated equilibrium distribution (Aumann, 1974) and, with a slight abuse of vocabulary, we write throughout correlated equilibrium for correlated equilibrium distribution. A pure strategy i is *used in correlated equilibrium* if there exists a correlated equilibrium μ under which strategy i has positive marginal probability (since the game is symmetric, whether we restrict attention to symmetric correlated equilibria or not is irrelevant; see footnote 2 in (Viossat, 2006)). Finally, the pure strategy i is *eliminated* (for a given solution $\mathbf{x}(\cdot)$ of a given dynamics) if $x_i(t) \to 0$ as $t \to +\infty$.

2 A family of games with a unique correlated equilibrium

A RPS (Rock-Paper-Scissors) game is a 3×3 symmetric game

$$\begin{pmatrix} a_1 & b_2 & c_3 \\ c_1 & a_2 & b_3 \\ b_1 & c_2 & a_3 \end{pmatrix} \tag{1}$$

in which the second strategy (Paper) beats the first (Rock), the third (Scissors) beats the second, and the first beats the third. That is,

$$b_i < a_i < c_i \text{ for } i = 1, 2, 3. \tag{2}$$

For $i = 1, 2, 3$, let

$$\alpha_i = a_i - b_i \, , \, \beta_i = c_i - a_i. \tag{3}$$

Every RPS game has a unique Nash equilibrium: $(\hat{\mathbf{n}}, \hat{\mathbf{n}})$, where

$$\hat{\mathbf{n}} = \frac{1}{\Sigma}(\alpha_2\alpha_3 + \alpha_3\beta_2 + \alpha_2\beta_3, \alpha_1\alpha_3 + \alpha_1\beta_3 + \beta_3\beta_1, \alpha_1\alpha_2 + \alpha_2\beta_1 + \beta_1\beta_2), \tag{4}$$

with $\Sigma > 0$ such that $\hat{\mathbf{n}} \in S_3$ (see Zeeman, 1980; Gaunersdorfer and Hofbauer, 1995, or Hofbauer and Sigmund, 1998). It was shown in (Viossat, 2006) that $\hat{\mathbf{n}} \otimes \hat{\mathbf{n}}$ is actually the unique correlated equilibrium of the game.

We say that a RPS game is *outward cycling* if

$$\prod_{i=1}^{3} \alpha_i > \prod_{i=1}^{3} \beta_i. \tag{5}$$

In that case, under the replicator dynamics, the unique Nash equilibrium is dynamically unstable, and for every initial condition different from the Nash equilibrium, the solution of the replicator dynamics converges to the boundary of the simplex (Zeeman, 1980).

The games we consider are 4×4 symmetric games built by adding a strategy to an outward cycling RPS game. That is, letting

$$\mathbf{U} = \left(\begin{array}{ccc|c} a_1 & b_2 & c_3 & d_1 \\ c_1 & a_2 & b_3 & d_2 \\ b_1 & c_2 & a_3 & d_3 \\ \hline f_1 & f_2 & f_3 & a_4 \end{array} \right) \tag{6}$$

denote the payoff matrix of the row player, and defining α_i and β_i as in (3) for $i = 1, 2, 3$, we assume that α_i and β_i are positive and that equation (5) is satisfied.

In addition, we assume that near the vertices $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of the simplex S_4, strategy 4 earns strictly less than the mean payoff. That is,

$$f_i < a_i \text{ for } i = 1, 2, 3. \tag{7}$$

In particular, there is a best-response cycle from $\mathbf{e}_1$ to $\mathbf{e}_2$ to $\mathbf{e}_3$ and back to $\mathbf{e}_1$.

We now state our first results. Consider a 4×4 symmetric game satisfying (2). Let $\mathbf{n} = (\hat{n}_1, \hat{n}_2, \hat{n}_3, 0)$ where $\hat{\mathbf{n}} = (\hat{n}_1, \hat{n}_2, \hat{n}_3)$, defined in (4), is the Nash equilibrium of the underlying RPS game.[5]

Proposition 1 *If $d_1 = d_2 = d_3 < a_4$ and $\boldsymbol{e}_4 \cdot \boldsymbol{U n} > \boldsymbol{n} \cdot \boldsymbol{U n}$ then the game has a unique correlated equilibrium: $\boldsymbol{e}_4 \otimes \boldsymbol{e}_4$.*

Proof. The proof is a straightforward extension of the proof of proposition 1 in (Viossat, 2006) and so we omit it. ■

Proposition 2 (Viossat, 2005a, proposition 1) *The set of finite games with a unique correlated equilibrium is open.*

[5] Instead of $\mathbf{n}$, the notation $\mathbf{n}_{123}$ was used in (Viossat, 2006).

A simple example of a game satisfying conditions (2), (5) and (7), and the conditions of proposition 1 is:

Example 3

$$\mathbf{U}_\alpha = \left(\begin{array}{ccc|c} 0 & -1 & \epsilon & -\alpha \\ \epsilon & 0 & -1 & -\alpha \\ -1 & \epsilon & 0 & -\alpha \\ \hline \frac{-1+\epsilon}{3} + \alpha & \frac{-1+\epsilon}{3} + \alpha & \frac{-1+\epsilon}{3} + \alpha & 0 \end{array}\right) \tag{8}$$

with ϵ in $]0, 1[$, and $0 < \alpha < (1-\epsilon)/3$.

It follows from example 3 and propositions 1 and 2 that:

Corollary 2.1 *There exists an open set of* 4×4 *symmetric games satisfying (2), (5) and (7) and with* $\boldsymbol{e}_4 \otimes \boldsymbol{e}_4$ *as unique correlated equilibrium.*

We can now precise the outline of the article. The next two sections show that, for any game satisfying (2), (5) and (7), and for an open set of initial conditions, the replicator dynamics and the best-response dynamics eliminate strategy 4. Sections (5) and (6) deal with the Brown-von Neumann-Nash dynamics and with any monotonic or weakly sign-preserving dynamics satisfying some standard regularity conditions; elimination of strategy 4, for an open set of initial conditions, is shown to occur in any game in a neighborhood of (8), provided that α is small enough for the Brown-von Neumann-Nash dynamics, and that ϵ is small enough for monotonic of weakly sign-preserving dynamics.

3 Replicator dynamics

The replicator dynamics (Taylor and Jonker, 1978) is given by

$$\dot{x}_i(t) = x_i(t)\left[(\mathbf{U}\mathbf{x}(t))_i - \mathbf{x}(t)\cdot \mathbf{U}\mathbf{x}(t)\right].$$

Its behavior in example 3 was studied in (Viossat, 2006). It was shown that, under the replicator dynamics and for α small enough, even though strategy 4 is the unique strategy used in correlated equilibrium, this strategy is eliminated for an open set of initial conditions. The purpose of this section is to give a more general proof of this fact and to show that the same result holds for an open set of games. We first need a definition:

Definition Let C be a closed subset of S_4. The set C is *asymptotically stable* if it is:

(a) invariant: $\mathbf{x}(0) \in C \Rightarrow (\forall t \in \mathbb{R}, \mathbf{x}(t) \in C)$

(b) Lyapunov stable: for every neighborhood N_1 of C, there exists a neighborhood N_2 of C such that, for every initial solution $\mathbf{x}(0)$ in N_2, $\mathbf{x}(t) \in N_1$ for all $t \geq 0$.

(c) locally attracting: there exists a neighborhood N of C such that, for every initial condition $\mathbf{x}(0)$ in N, $\min_{c \in C} ||\mathbf{x}(t) - c|| \to_{t \to +\infty} 0$ (where $|| \cdot ||$ is any norm on $\mathbb{R}^I$).

Let

$$\Gamma = \{\mathbf{x} \in S_4, x_4 = 0 \text{ and } x_1 x_2 x_3 = 0\}. \tag{9}$$

Proposition 4 *For every 4×4 symmetric game (6) satisfying (2), (5) and (7), the set Γ is asymptotically stable.*

Proof. The intuition is that, due to (5), the solution spirals outward as long as x_4 is low, and that, due to (7), if the initial condition is close to Γ, x_4 will never increase substantially and will eventually decrease to 0. The formal proof consists in checking that the stability criteria for heteroclinic cycles developed by Hofbauer (1994) are satisfied. As these criteria will be introduced in section 6, the proof is postponed and given in the appendix. ■

Together with corollary 2.1, proposition 4 implies that there exists an open set of 4×4 symmetric games for which, from an open set of initial conditions, the unique strategy used in correlated equilibrium is eliminated.

Note that not all games satisfying (2), (5) and (7) have $\mathbf{e}_4 \otimes \mathbf{e}_4$ as unique correlated equilibrium or even unique Nash equilibrium. Actually, proposition 4 provides an example of a family of games with a common attractor but very different sets of Nash equilibria[6]. The point is that this attractor, Γ, is bounded away from the set of equilibria and that its asymptotic stability only depends on the payoffs in its neighbourhood. This explains why the

[6] For instance, assuming throughout that (2), (5) and (7) are satisfied: if the f_i are low enough, then the Nash equilibrium of the underlying RPS game induces a Nash equilibrium of (6). If $d_1 = d_2 = d_3 > a_4$ and if the f_i are high enough, then there is a unique symmetric Nash equilibrium, which is a convex combination of $\mathbf{e}_4$ and of the Nash equilibrium of the underlying RPS game, and there is an infinity of asymmetric Nash equilibria [to see that there is a unique symmetric Nash equilibrium, mimick the proof of proposition 3 in (Viossat, 2006)]. If $d_i < a_4$ for all i and if d_2 and d_3 are low enough (with respect to d_1 and f_1), then there is a Nash equilibrium in the interior of the edge $[\mathbf{e}_1, \mathbf{e}_4]$, etc.

stability of Γ is in a large part unrelated to the structure of the equilibrium set.

4 Best-response dynamics

4.1 Main result

The best-response dynamics (Gilboa and Matsui, 1991; Matsui, 1992) is given by the differential inclusion:

$$\dot{\mathbf{x}}(t) \in BR(\mathbf{x}(t)) - \mathbf{x}(t), \tag{10}$$

where $BR(\mathbf{x})$ is the set of best responses to $\mathbf{x}$:

$$BR(\mathbf{x}) = \{\mathbf{y} \in S_N : \mathbf{y} \cdot \mathbf{U}\mathbf{x} = \max_{\mathbf{z} \in S_N} \mathbf{z} \cdot \mathbf{U}\mathbf{x}\}.$$

A solution $\mathbf{x}(\cdot)$ of the best-response dynamics is an absolutely continuous function satisfying (10) for almost every t. In general, there might be several solutions starting from the same initial condition. However, for the games and the initial conditions that we will consider, there is a unique solution starting from each initial condition.[7]

Consider a 4×4 symmetric game with payoff matrix (6) satisfying (2), (5) and (7). Let

$$V(\mathbf{x}) := \max_{1 \leq i \leq 3} \left[(\mathbf{U}\mathbf{x})_i - \sum_{1 \leq i \leq 4} a_i x_i \right] \text{ and } W(\mathbf{x}) := \max(x_4, |V(\mathbf{x})|). \tag{11}$$

The set

$$ST := \{\mathbf{x} \in S_4 : W(\mathbf{x}) = 0\} \tag{12}$$

is a triangle, which, following Gaunersdorfer and Hofbauer (1995), we call the Shapley triangle. Gaunersdorfer and Hofbauer (1995) show that in the underlying RPS game, this triangle attracts all solutions of (10) except the one starting and remaining at the Nash equilibrium. Here, we show that in the full 4×4 game, this triangle still attracts all solutions from an open set of initial conditions.

[7] We focus on forward time and so never study whether a solution is uniquely defined in backward time.

Proposition 5 *Consider a* 4×4 *symmetric game satisfying (2), (5) and (7). If strategy* 4 *is not a best response to* $\boldsymbol{x}(0)$ *and if strategies* 1, 2 *and* 3 *are not all best responses to* $\boldsymbol{x}(0)$*, then for all* $t \geq 0$, $\boldsymbol{x}(t)$ *is uniquely defined, and* $\boldsymbol{x}(t)$ *converges to the Shapley triangle (12) as* $t \to +\infty$. [8]

Proof. We begin with a lemma, which is the continuous time version of the improvement principle of Monderer and Sela (1997):

Lemma 4.1 (Improvement principle) *Let* $t_1 < t_2$*, let* $\boldsymbol{b}$ *be a best response to* $\boldsymbol{x}(t_1)$ *and let* $\boldsymbol{b}' \in S_4$*. Assume that* $\dot{\boldsymbol{x}} = \boldsymbol{b} - \boldsymbol{x}$ *(hence the solution points towards* $\boldsymbol{b}$*) for all* t *in* $]t_1, t_2[$*. If* $\boldsymbol{b}'$ *is a best response to* $\boldsymbol{x}(t_2)$ *then* $\boldsymbol{b}' \cdot \boldsymbol{U}\boldsymbol{b} \geq \boldsymbol{b} \cdot \boldsymbol{U}\boldsymbol{b}$*, with strict inequality if* $\boldsymbol{b}'$ *is not a best response to* $\boldsymbol{x}(t_1)$*.*

Proof of lemma 4.1. Between t_1 and t_2, the solution points towards b. Therefore, there exists λ in $]0, 1[$ such that

$$\mathbf{x}(t_2) = \lambda \mathbf{x}(t_1) + (1 - \lambda)\mathbf{b}. \tag{13}$$

If $\mathbf{b}'$ is a best response to $\mathbf{x}(t_2)$ then $(\mathbf{b}' - \mathbf{b}) \cdot \mathbf{U}\mathbf{x}(t_2) \geq 0$ so that, substituting the right-hand-side of (13) for $\mathbf{x}(t_2)$, we get:

$$(1 - \lambda)(\mathbf{b}' - \mathbf{b}) \cdot \mathbf{U}\mathbf{b} \geq \lambda(\mathbf{b} - \mathbf{b}') \cdot \mathbf{U}\mathbf{x}(t_1). \tag{14}$$

Since $\mathbf{b}$ is a best response to $\mathbf{x}(t_1)$, the right hand side of (14) is nonnegative, and positive if $\mathbf{b}'$ is not a best response to $\mathbf{x}(t_1)$. The result follows. ■

Now fix a solution $\mathbf{x}(\cdot)$ with initial condition satisfying the conditions of proposition 5. Using lemma 4.1, it is easy to see that, at least for some time, the solution $\mathbf{x}(t)$ is uniquely defined and has the following behavior: assume for concreteness that at time t, strategy 1 is the unique best response to $\mathbf{x}(t)$; the solution will then point towards $\mathbf{e}_1$ till some time $t' > t$ when some other pure strategy becomes a best response. Due to the improvement principle (lemma 4.1), this strategy can only be strategy 2. Thus, the solution must then point towards the edge $[\mathbf{e}_1, \mathbf{e}_2]$. Since strategy 2 strictly dominates strategy 1 in the game restricted to $\{1, 2\} \times \{1, 2\}$, it follows that, immediately after time T, strategy 2 becomes the unique best response; therefore

[8] Note that for any game in a neighborhood of game (8), the former condition implies the latter: if strategy 4 is not a best-response to $\mathbf{x} \in S_4$, then strategy 1, 2, and 3 are not all best-responses to $\mathbf{x}$. This is because in game (8), strategy 4 earns a strictly higher payoff than $(1/3, 1/3, 1/3, 0)$. Proposition 5 applies to much more general games, in which the first condition need not imply the second.

the solution will actually point towards $\mathbf{e}_2$. The solution keeps pointing towards $\mathbf{e}_2$ till some other pure strategy becomes a best response; due to the improvement principle, this strategy must be strategy 3. The solution then changes direction, and points towards $\mathbf{e}_3$ till 1 becomes a best response again, and so on.

To show that this behavior continues for ever, it suffices to show that the times at which the direction of the trajectory changes do not accumulate. This is the object of the following claim, which will be proved in the end:

Claim 4.2 *The time length between two successive times when the direction of $\boldsymbol{x}(t)$ changes is bounded away from zero.*

Now recall the definition of the functions V and W in (11), and let $v(t) = V(\mathbf{x}(t))$, $w(t) = W(\mathbf{x}(t))$. When $\mathbf{x}(t)$ points towards $\mathbf{e}_i$ (with $i \in \{1,2,3\}$), we have:

$$\dot{v} = (\mathbf{U}\dot{\mathbf{x}})_i - \sum_{1\leq j\leq 4} a_j \dot{x}_j = (\mathbf{U}(\mathbf{e}_i - \mathbf{x}))_i - \left(a_i - \sum_{1\leq j\leq 4} a_j x_j\right) = -v \qquad (15)$$

and we also have $\dot{x}_4 = -x_4$; therefore $\dot{w} = -w$. Since for almost time t, $\mathbf{x}(t)$ points towards $\mathbf{e}_1$, $\mathbf{e}_2$ or $\mathbf{e}_3$, it follows that $\dot{w}(t) = -w(t)$ holds for almost all t, hence that $w(t)$ decreases exponentially to 0. Therefore, $\mathbf{x}(t)$ converges to the Shapley triangle.

To complete the proof, we still need to prove claim 4.2:

Proof of claim 4.2: In what follows $i \in \{1,2,3\}$ and $i+1$ is counted modulo 3. Fix an initial condition and let

$$g(t) := \max_{1\leq i,j\leq 3} [(\mathbf{U}\mathbf{x}(t))_i - (\mathbf{U}\mathbf{x}(t))_j]$$

denote the maximum difference between the payoffs of strategies in {1,2,3}. This may be seen as a measure of the distance between $\mathbf{x}(t)$ and the set of points $\mathbf{x}$ such that $(\mathbf{U}\mathbf{x})_1 = (\mathbf{U}\mathbf{x})_2 = (\mathbf{U}\mathbf{x})_3$. Let t_i^k denote the k^{th} time at which strategy i becomes a best response and choose i such that $t_i^k < t_{i+1}^k$. Let $\mathbf{x} = \mathbf{x}(t_i^k)$, $g = g(t_i^k)$ and $\mathbf{x}' = \mathbf{x}(t_{i+1}^k)$, $g' = g(t_{i+1}^k)$. We now compute g' as a function of g.

Between t_i^k and t_{i+1}^k, the solution points towards $\mathbf{e}_i$. Therefore, there exists λ in $]0,1[$ such that

$$\mathbf{x}' = \lambda \mathbf{e}_i + (1-\lambda)\mathbf{x}. \qquad (16)$$

Furthermore, by definition of t_i^k and t_{i+1}^k,

$$(\mathbf{Ux})_{i-1} = (\mathbf{Ux})_i = (\mathbf{Ux})_{i+1} + g \tag{17}$$

and

$$(\mathbf{Ux}')_i = (\mathbf{Ux}')_{i+1} = (\mathbf{Ux}')_{i-1} + g'. \tag{18}$$

Using in this order (18), (16) and (17), we get:

$$\begin{aligned} 0 = (\mathbf{Ux}')_{i+1} - (\mathbf{Ux}')_i &= (\mathbf{e}_{i+1} - \mathbf{e}_i) \cdot \mathbf{Ux}' \\ &= (\mathbf{e}_{i+1} - \mathbf{e}_i) \cdot \mathbf{U}(\lambda \mathbf{e}_i + (1-\lambda)\mathbf{x}) \\ &= \lambda(c_i - a_i) - (1-\lambda) g \end{aligned}$$

and

$$g' = (\mathbf{e}_i - \mathbf{e}_{i-1}) \cdot \mathbf{Ux}' = (\mathbf{e}_i - \mathbf{e}_{i-1}) \cdot \mathbf{U}(\lambda \mathbf{e}_i + (1-\lambda)\mathbf{x}) = \lambda(a_i - b_i).$$

Solving for g' we get $g'/g = \alpha_i/(g + \beta_i)$ with, as defined in (3), $\alpha_i = a_i - b_i$ and $\beta_i = c_i - a_i$. Iterating this argument, we obtain the return map:

$$g(t_i^{k+1}) = \frac{\alpha_1 \alpha_2 \alpha_3}{\beta_1 \beta_2 \beta_3 + g(t_k^i)(\alpha_1 \alpha_2 + \alpha_1 \beta_3 + \beta_2 \beta_3)} g(t_i^k).$$

Since, by (5), $\alpha_1 \alpha_2 \alpha_3 > \beta_1 \beta_2 \beta_3$, it follows that for small $g(t_i^k)$, we have $g(t_i^{k+1}) > g(t_i^k)$; therefore $g(t_i^k)$ is bounded away from zero. Now, since $(\mathbf{Ux}(t))_i - (\mathbf{Ux}(t))_{i+1}$ decreases from $g(t_i^k)$ to 0 between t_i^k and t_{i+1}^k, and since the speed at which this quantity varies is bounded, it follows that $t_{i+1}^k - t_i^k$ is bounded away from zero too. That is, the time length between two successive times at which the direction of $\mathbf{x}(t)$ changes is bounded away from zero. This proves claim 4.2 and completes the proof of proposition 5. ■

Together with corollary 2.1, proposition 5 implies that there exists an open set of 4×4 symmetric games for which the unique strategy used in correlated equilibrium is strategy 4, but, from an open set of initial conditions, the solution $\mathbf{x}(\cdot)$ of (10) is uniquely defined and $x_4(t) \to 0$ as $t \to +\infty$.

4.2 Remarks

This section is independent of the next sections and may be skipped.

Remark 1. For every $\eta > 0$, we may set the parameters of (6) so that the set $\{\mathbf{x} \in S_4 : \mathbf{e}_4 \in BR(\mathbf{x}) \text{ or } \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \subseteq BR(\mathbf{x})\}$ has Lebesgue measure less than η. In this sense, the basin of attraction of the Shapley triangle may be made arbitrarily large.

Remark 2. When $d_1 = d_2 = d_3$ the behavior of the best-response dynamics in the 4×4 game (6) can be precisely related to its behavior in the underlying RPS game (1), which was fully analyzed by Gaunersdorfer and Hofbauer (1995). Indeed, for $\mathbf{x} \neq \mathbf{e}_4$ and i in $\{1, 2, 3\}$, let $\hat{x}_i$ denote the proportion of the population playing strategy i relative to the proportion of the population playing strategy 1, 2 or 3:

$$\hat{x}_i = \frac{x_i}{x_1 + x_2 + x_3}$$

Let $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \hat{x}_3) \in S_3$. Since $d_1 = d_2 = d_3$, it follows that, provided that 4 is not a best response to $\mathbf{x}$, the strategies i in $\{1, 2, 3\}$ which are best responses to $\mathbf{x}$ are exactly those which are best responses to $\hat{\mathbf{x}}$ in the underlying RPS game. This implies that, up to a change of velocity, $\hat{\mathbf{x}}$ follows the best-response dynamics in the underlying RPS game. More precisely, straightforward computations show that if 4 is not a best response to $\mathbf{x}$ then:

$$\dot{\mathbf{x}} \in BR(\mathbf{x}) - \mathbf{x} \Rightarrow (1 - x_4)\dot{\hat{\mathbf{x}}} \in BR(\hat{\mathbf{x}}) - \hat{\mathbf{x}}.$$

A similar result holds for the replicator dynamics (Viossat, 2006, lemma 5.1).

Remark 3. The proof of proposition 5 uses condition (7), i.e., $f_i < a_i$ for $i = 1, 2, 3$. Since $(\mathbf{Ux})_4 - \sum_{1\leq i\leq 4} a_i x_i = \sum_{1\leq i\leq 3}(f_i - a_i)x_i$ is linear in $\mathbf{x}$, condition (7) means that $(\mathbf{Ux})_4 - \sum_{1\leq i\leq 4} a_i x_i$ is negative on the face of the simplex spanned by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. If instead of requiring (7), we only require that $(\mathbf{Ux})_4 - \sum_{1\leq i\leq 4} a_i x_i$ be negative on the Shapley triangle:

$$W(\mathbf{x}) = 0 \Rightarrow (\mathbf{Ux})_4 - \sum_{1\leq i\leq 4} a_i x_i < 0. \tag{19}$$

then proposition 5 does not hold.[9] However, strategy 4 is still eliminated from an open set of initial conditions:

[9] For instance, if $f_1 > a_1$, $f_3 < a_3$ and $d_1 = d_2 = d_3 < a_4$, then from every initial condition sufficiently close to the mixed strategy $\mathbf{x} \in [\mathbf{e}_3, \mathbf{e}_4]$ to which strategies 1 and 4 are both best responses, including initial conditions to which strategy 1 is the unique best response, every solution of (10) converges to $\mathbf{e}_4$.

Proposition 6 *If (19) holds, then there exists $\gamma > 0$ such that from every initial condition in $N_\gamma := \{\boldsymbol{x} \in S_4 : W(\boldsymbol{x}) < \gamma\}$, there is a unique solution to (10), and it converges to the Shapley triangle.*

Proof. If we can find $\gamma > 0$ such that in N_γ strategy 4 is never a best response, then the proof of proposition 5 implies that, as long as $\mathbf{x}(t) \in N_\gamma$, the solution is unique and $W(\mathbf{x}(t))$ decreases exponentially. The later implies that N_γ is forward invariant and that W goes to zero, hence the result.

Now, in light of the definition (11) of W, (19) means that on the Shapley triangle, strategy 4 is never a best response. Therefore, if (19) holds, then there exists an open neighborhood Ω of the Shapley triangle on which 4 is not a best response. Since W is positive on the compact set $S_4 \backslash \Omega$, it follows that $\gamma := \min_{\mathbf{x} \in S_4 \backslash \Omega} W(\mathbf{x})$ is positive. Furthermore, the definition of γ implies that $N_\gamma \subseteq \Omega$; hence, in N_γ, strategy 4 is never a best response and the result follows. ■

Following Gaunersdorfer and Hofbauer (1995), it is interesting to compare the behavior of the best-response dynamics and of the time-average of the replicator dynamics. If $f_i > a_i$ for some $i \in \{1, 2, 3\}$, then under the replicator dynamics, the set Γ is not stable[10]. But

Proposition 7 *If (19) holds, then under the replicator dynamics, Γ attracts an open set of orbits, along which the time-average converges to the Shapley triangle.*

Proof. This follows from proposition 3.1 of Brannath (1994). A sketch of proof in this particular case is given in (Viossat, 2005b, chapter 10, part B, second appendix) ■

5 Brown-von Neumann-Nash dynamics

The Brown-von Neumann-Nash dynamics (henceforth BNN) is given by:

$$\dot{x}_i = k_i(\mathbf{x}) - x_i \sum_{j \in I} k_j(\mathbf{x}) \tag{20}$$

where

$$k_i(\mathbf{x}) := \max(0, (\mathbf{U}\mathbf{x})_i - \mathbf{x} \cdot \mathbf{U}\mathbf{x}) \tag{21}$$

[10] Neither asymptotically stable nor Lyapunov stable.

is the excess payoff of strategy i over the average payoff. As in the best-response dynamics, strategies that are initially absent may appear, the proportion of every strategy earning less than average decreases and the rest-points are exactly the Nash equilibria of the game.[11] Furthermore, since the right-hand side of (20) is Lipschitz continuous, BNN has a unique solution from each initial condition. We refer to (Hofbauer, 2000; Berger and Hofbauer, 2006) and references therein for a motivation of and results on BNN.

Let G_0 denote the game (8) with $\alpha = 0$. Recall that $\mathbf{U}_0$ denote its payoff matrix. The mixed strategy corresponding to the Nash equilibrium of the underlying RPS game is $\mathbf{n} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)$ and it may be shown that the set of symmetric Nash equilibria of G_0 is the segment $E_0 = [\mathbf{n}, \mathbf{e}_4]$. That is, $(\mathbf{p}, \mathbf{p})$ is a Nash equilibrium if and only if $\mathbf{p}$ is a convex combination of $\mathbf{n}$ and $\mathbf{e}_4$.[12] This section is devoted to a proof of the following proposition:

Proposition 8 *If C is a closed subset of S_4 disjoint from E_0, then there exists a neighborhood of G_0 such that, for every game in this neighborhood and every initial condition in C, $x_4(t) \to 0$ as $t \to +\infty$.*

Propositions 1 and 2 imply that any neighborhood of the game G_0 contains an open set of games for which the unique correlated equilibrium is $\mathbf{e}_4 \otimes \mathbf{e}_4$. Together with proposition 9, this implies that there exists an open set of games for which, under BNN, the unique strategy played in correlated equilibrium is eliminated from an open set of initial conditions.

The essence of the proof of proposition 9 is to show that, for games close to G_0, there is a "tube" surrounding E_0 such that: (i) the tube repels solutions coming from outside; (ii) outside of the tube, strategy 4 earns less than average, hence x_4 decreases. We first show that in G_0 the segment E_0 is locally repelling.

The function

$$V_0(\mathbf{x}) := \frac{1}{2}\sum_{i\in I} k_i^2 = \frac{1}{2}\sum_{i\in I} \left[\max\left(0, (\mathbf{U}_0\mathbf{x})_i - \mathbf{x}\cdot\mathbf{U}_0\mathbf{x}\right)\right]^2$$

is continuous, nonnegative and equals 0 exactly on the symmetric Nash equilibria, i.e. on E_0, so that $V_0(\mathbf{x})$ may be seen as a distance from $\mathbf{x}$ to E_0. Fix an initial condition and let $v_0(t) := V_0(\mathbf{x}(t))$.

[11] The *symmetric* Nash equilibria, for the single-population version presented here.

[12] The game G_0 has other, asymmetric equilibria, but they will play no role.

Lemma 5.1 *There exists an open neighborhood N_{eq} of E_0 such that, under BNN in the game G_0, $\dot{v}_0(t) > 0$ whenever $\boldsymbol{x}(t) \in N_{eq} \backslash E_0$.*

Proof. It is easily checked that:

$$\mathbf{n} \cdot \mathbf{U}_0 \mathbf{x} = \mathbf{e}_4 \cdot \mathbf{U}_0 \mathbf{x} \quad \forall \mathbf{x} \in S_4 \tag{22}$$

(that is, $\mathbf{n}$ and $\mathbf{e}_4$ always earn the same payoff) and

$$(\mathbf{x} - \mathbf{x}') \cdot \mathbf{U}_0 \mathbf{e}_4 = (\mathbf{x} - \mathbf{x}') \cdot \mathbf{U}_0 \mathbf{n} = 0 \quad \forall \mathbf{x} \in S_4, \forall \mathbf{x}' \in S_4 \tag{23}$$

(that is, against $\mathbf{e}_4$ [resp. $\mathbf{n}$], all strategies earn the same payoff). Furthermore, as follows from lemma 4.1 in (Viossat, 2006), for every $\mathbf{p}$ in E_0 and every $\mathbf{x} \notin E_0$,

$$(\mathbf{x} - \mathbf{p}) \cdot \mathbf{U}_0 \mathbf{x} = (\mathbf{x} - \mathbf{p}) \cdot \mathbf{U}_0 (\mathbf{x} - \mathbf{p}) = \frac{1-\epsilon}{2} \sum_{1 \leq i \leq 3} \left(x_i - \frac{1 - x_4}{3} \right)^2 > 0. \tag{24}$$

Hofbauer (2000) shows that the function v_0 satisfies

$$\dot{v}_0 = \bar{k}^2 \left[(\mathbf{q} - \mathbf{x}) \cdot \mathbf{U}_0 (\mathbf{q} - \mathbf{x}) - (\mathbf{q} - \mathbf{x}) \cdot \mathbf{U}_0 \mathbf{x} \right] \tag{25}$$

with $\mathbf{x} = \mathbf{x}(t)$, $\bar{k} = \sum_i k_i$ and $q_i = k_i / \bar{k}$. It follows from equation (23) that if $\mathbf{p} \in E_0$, then against $\mathbf{p}$ all strategies earn the same payoff. Therefore, the second term $(\mathbf{q} - \mathbf{x}) \cdot \mathbf{U}_0 \mathbf{x}$ goes to 0 as $\mathbf{x}$ approaches E_0. Thus, to prove lemma 5.1, it suffices to show that as $\mathbf{x}$ approaches E_0, the first term $(\mathbf{q} - \mathbf{x}) \cdot \mathbf{U}_0 (\mathbf{q} - \mathbf{x})$ is positive and bounded away from 0. But for $\mathbf{x} \notin E_0$,

$$\min_{1 \leq i \leq 3} (\mathbf{U}_0 \mathbf{x})_i \leq \mathbf{n} \cdot \mathbf{U}_0 \mathbf{x} = (\mathbf{U}_0 \mathbf{x})_4 < \mathbf{x} \cdot \mathbf{U}_0 \mathbf{x} \tag{26}$$

(the first inequality holds because $\mathbf{n}$ is a convex combination of $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, the equality follows from (22) and the strict inequality from (24) applied to $\mathbf{p} = \mathbf{e}_4$). It follows from $(\mathbf{U}_0 \mathbf{x})_4 < \mathbf{x} \cdot \mathbf{U}_0 \mathbf{x}$ that $k_4 = 0$ hence $q_4 = 0$; similarly, it follows from $\min_{1 \leq i \leq 3} (\mathbf{U}_0 \mathbf{x})_i < \mathbf{x} \cdot \mathbf{U}_0 \mathbf{x}$ that $q_i = 0$ for some i in $\{1, 2, 3\}$. Together with (24), this implies that for every $\mathbf{p}$ in E_0,

$$(\mathbf{q} - \mathbf{p}) \cdot \mathbf{U}_0 (\mathbf{q} - \mathbf{p}) = \frac{1-\epsilon}{2} \sum_{1 \leq i \leq 3} \left(q_i - \frac{1}{3} \right)^2 \geq \frac{1-\epsilon}{18}.$$

This completes the proof. ■

We now prove proposition 9. Consider first the BNN dynamics in the game G_0. Recall lemma 5.1 and let

$$0 < \delta < \min_{\mathbf{x} \in S_4 \backslash N_{eq}} V_0(x) \tag{27}$$

(the latter is positive because V_0 is positive on $S_4 \backslash E_0$, hence on $S_4 \backslash N_{eq}$, and because $S_4 \backslash N_{eq}$ is compact). Note that if $V_0(\mathbf{x}) \leq \delta$ then $\mathbf{x} \in N_{eq}$. Therefore it follows from lemma 5.1 and $\delta > 0$ that

$$v_0(t) = \delta \Rightarrow \dot{v}_0(t) > 0. \tag{28}$$

Let

$$C_\delta := \{\mathbf{x} \in S_4 : V_0(\mathbf{x}) \geq \delta\}.$$

Since $\delta > 0$, the sets C_δ and E_0 are disjoint. Therefore, by (24) applied to $\mathbf{p} = \mathbf{e}_4$,

$$\mathbf{x} \in C_\delta \Rightarrow (\mathbf{U}_0\mathbf{x})_4 - \mathbf{x} \cdot \mathbf{U}_0\mathbf{x} < 0 \tag{29}$$

so that x_4 decreases strictly as long as $\mathbf{x} \in C_\delta$ and $x_4 > 0$. Since, by (28), the set C_δ is forward invariant, it follows that for any initial condition in C_δ, strategy 4 is eliminated.

Now let $\nabla V_0(\mathbf{x}) = (\partial V_0 / \partial x_i)_{1 \leq i \leq n}(\mathbf{x})$ denote the gradient of V_0 at $\mathbf{x}$. It is easy to see that V_0 is C^1. Therefore, it follows from (28), $\dot{v}_0(t) = \nabla V_0(\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t)$ and compactness of $\{\mathbf{x} \in S_4 : V_0(\mathbf{x}) = \delta\}$ that

$$\exists \gamma > 0, [v_0(t) = \delta \Rightarrow \dot{v}_0(t) \geq \gamma > 0]. \tag{30}$$

Similarly, since C_δ is compact, it follows from (29) that there exists $\gamma' > 0$ such that

$$\mathbf{x} \in C_\delta \Rightarrow (\mathbf{U}_0\mathbf{x})_4 - \mathbf{x} \cdot \mathbf{U}_0\mathbf{x} \leq -\gamma' < 0. \tag{31}$$

Since $\dot{\mathbf{x}}$ is Lipschitz in the payoff matrix, it follows from (30) that for $\mathbf{U}$ close enough to $\mathbf{U}_0$, we still have $v_0(t) = \delta \Rightarrow \dot{v}_0 > 0$ under the perturbed dynamics. Similarly, due to (31), we still have $\mathbf{x} \in C_\delta \Rightarrow (\mathbf{U}\mathbf{x})_4 - \mathbf{x} \cdot \mathbf{U}\mathbf{x} < 0$. Therefore, the above reasoning applies and for every initial condition in C_δ, strategy 4 is eliminated.

Note that δ can be chosen arbitrarily small (see (27)). Therefore, to complete the proof of proposition 9, it suffices to show that if C is a compact set disjoint from E_0 then, for δ sufficiently small, $C \subset C_\delta$. But since V_0 is positive on $S_4 \backslash E_0$, and since C is compact and disjoint from E_0, it follows that there exists $\delta' > 0$ such that, for all $\mathbf{x}$ in C, $V_0(\mathbf{x}) \geq \delta'$; hence, for all $\delta \leq \delta'$, $C \subset C_\delta$. This completes the proof. ■

Hofbauer (2000, section 6) considers a generalization of the BNN dynamics:

$$\dot{x}_i = f(k_i) - x_i \sum_{j=1}^{n} f(k_j) \tag{32}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function with $f(0) = 0$ and $f(u) > 0$ for $u > 0$, and where k_i is defined as in (21). The results of this section generalize straightforwardly to any such dynamics:

Proposition 9 *Consider a dynamics of type (32). If C is a closed subset of S_4 disjoint from E_0, then there exists a neighborhood of G_0 such that, for every game in this neighborhood and every initial condition in C, $x_4(t) \to 0$ as $t \to +\infty$.*

Proof. Replace $V_0(\mathbf{x})$ by $W_0(\mathbf{x}) := \sum_i F(k_i(\mathbf{x}))$, where F is an anti-derivative of f, and replace k_i by $f(k_i)$. Let $\bar{f} = \sum_i f(k_i)$, $\tilde{f}_i = f(k_i)/\bar{f}$, and $\tilde{\mathbf{f}} = \left(\tilde{f}_i\right)_{1\leq i \leq N}$. Finally, let $w_0(t) = W_0(\mathbf{x}(t))$. As shown by Hofbauer (2000),

$$\dot{w}_0 = \bar{f}^2 \left[(\tilde{\mathbf{f}} - \mathbf{x}) \cdot \mathbf{U}_0(\tilde{\mathbf{f}} - \mathbf{x}) - (\tilde{\mathbf{f}} - \mathbf{x}) \cdot \mathbf{U}_0 \mathbf{x}\right]$$

which is the analogue of (25). Then apply exactly the same proof as for BNN. ■

6 Monotonic and weakly sign-preserving dynamics

Consider a dynamics of the form

$$\dot{x}_i = x_i g_i(\mathbf{x}) \tag{33}$$

where the C^1 functions g_i have the property that $\sum_{i \in I} x_i g_i(\mathbf{x}) = 0$ for all $\mathbf{x}$ in S_4, so that the simplex S_4 and its boundary faces are invariant.

Such a dynamics is *monotonic* if the growth rates of the different strategies are ranked according to their payoffs[13]:

$$g_i(\mathbf{x}) > g_j(\mathbf{x}) \Leftrightarrow (\mathbf{Ux})_i > (\mathbf{Ux})_j \quad \forall i \in I, \forall j \in I.$$

A dynamics of type (33) is *weakly sign-preserving* (WSP) (Ritzberger and Weibull, 1995) if whenever a strategy earns below average, its growth rate is negative:

$$[(\mathbf{Ux})_i < \mathbf{x} \cdot \mathbf{Ux}] \Rightarrow g_i(\mathbf{x}) < 0.$$

Finally,[14] dynamics of type (33) implicitly depend on the payoff matrix $\mathbf{U}$. Thus, a more correct writing of (33) would be:

$$\dot{x}_i = x_i g_i(\mathbf{x}, \mathbf{U})$$

where $\mathbf{U} \in \mathbb{R}^{N \times N}$. Such a dynamics *depends continuously on the payoff matrix* if, for every i in I, the functions g_i are defined for an open set of payoff matrices and depend continuously on $\mathbf{U}$. A prime example of a dynamics of type (33) which is monotonic, WSP, and depends continuously on the payoff matrix is the replicator dynamics.

We now state the result: fix a monotonic or WSP dynamics (33) that depends continuously on the payoff matrix.

Proposition 10 *For every α in $]0, 1/3[$, there exists $\epsilon > 0$ such that for every game in the neighborhood of (8), the set Γ defined by (9) is asymptotically stable.*

Together with propositions 1 and 2 and example 3, this implies that there exists an open set of games for which $\mathbf{e}_4 \otimes \mathbf{e}_4$ is the unique correlated equilibrium but strategy 4 is eliminated for an open set of initial conditions.
Proof. For every monotonic or WSP dynamics (33), and for every game in the neighborhood of (8), the set Γ is an heteroclinic cycle. That is, a set consisting of saddle rest points and of the saddle orbits connecting these rest points. Thus we may use the asymptotic stability's criteria for heteroclinic

[13]This property goes under various names in the literature: *relative monotonicity* in (Nachbar, 1990), *order-compatibility* of pre-dynamics in (Friedman, 1991), *monotonicity* in (Samuelson and Zhang, 1992), which we follow, and *payoff monotonicity* in (Hofbauer and Weibull, 1996).

[14]Instead of dynamics of type (33), Ritzberger and Weibull (1995) consider dynamics of the more general type $\dot{x}_i = h_i(\mathbf{x})$, that need not leave the faces of the simplex positively invariant. Thus, we only consider a subclass of their WSP dynamics.

cycles developed by Hofbauer (1994) (a more accessible reference for this result is theorem 17.5.1 in (Hofbauer and Sigmund, 1998)). Specifically, associate with the heteroclinic cycle Γ its so-called characteristic matrix. That is, the 3×4 matrix whose entry in row i and column j is $g_j(\mathbf{e}_i)$ (for $i \neq j$, this is the eigenvalue in the direction of $\mathbf{e}_j$ of the linearization of the vector field at $\mathbf{e}_i$):

	1	2	3	4
$\mathbf{e}_1$	0	$g_2(\mathbf{e}_1)$	$g_3(\mathbf{e}_1)$	$g_4(\mathbf{e}_1)$
$\mathbf{e}_2$	$g_1(\mathbf{e}_2)$	0	$g_3(\mathbf{e}_2)$	$g_4(\mathbf{e}_2)$
$\mathbf{e}_3$	$g_1(\mathbf{e}_3)$	$g_2(\mathbf{e}_3)$	0	$g_4(\mathbf{e}_3)$

(note that $g_i(\mathbf{e}_i) = 0$ because $\mathbf{e}_i$ is a rest point of (33)).

Call $\mathbf{C}$ this matrix. If $\mathbf{p}$ is a real vector, let $\mathbf{p} < 0$ (resp. $\mathbf{p} > 0$) mean that all coordinates of $\mathbf{p}$ are negative (resp. positive). Hofbauer (1994) shows that if the following conditions are satisfied:

$$\Gamma \text{ is asymptotically stable within the boundary of } S_4 \text{ [15]} \tag{34}$$

$$\text{There exists a vector } \mathbf{p} \text{ in } \mathbb{R}^4 \text{ such that } \mathbf{p} > 0 \text{ and } \mathbf{Cp} < 0 \tag{35}$$

then Γ is asymptotically stable. Therefore, to prove proposition 10, it is enough to show that for every α in $]0, 1/3[$, there exists $\epsilon > 0$ such that, for every game in the neighborhood of (8), conditions (34) and (35) are satisfied. We begin with a lemma. In the remainder of this section, $i \in \{1, 2, 3\}$ and $i-1$ and $i+1$ are counted modulo 3.

Lemma 6.1 *For every $0 < \alpha < 1/3$, there exists $\epsilon > 0$ such that in the game (8) and for every i in $\{1, 2, 3\}$,*

$$g_4(\boldsymbol{e}_i) < 0 \ \ \textit{and} \ \ 0 < g_{i+1}(\boldsymbol{e}_i) < -g_{i-1}(\boldsymbol{e}_i). \tag{36}$$

Proof of lemma 6.1 for monotonic dynamics. For $\epsilon > 0$, at the vertex $\mathbf{e}_i$, the payoff of strategy 4 (resp. $i+1$) is strictly smaller (greater) than the payoff of strategy i. Since the growth rate of strategy i at $\mathbf{e}_i$ is 0, this implies by monotonicity $g_4(\mathbf{e}_i) < 0$ (resp. $g_{i+1}(\mathbf{e}_i) > 0$). It remains to show that $g_{i+1}(\mathbf{e}_i) < -g_{i-1}(\mathbf{e}_i)$. For $\epsilon = 0$, we have: $(\mathbf{Ue}_i)_i = (\mathbf{Ue}_i)_{i+1} > (\mathbf{Ue}_i)_{i-1}$ so that $0 = g_{i+1}(\mathbf{e}_i) > g_{i-1}(\mathbf{e}_i)$. Therefore $g_{i+1}(\mathbf{e}_i) < -g_{i-1}(\mathbf{e}_i)$ and since the dynamics depends continuously on the payoff matrix, this still holds for

[15] That is, for each proper face (subsimplex) F of S_4, if $\Gamma \bigcap F$ is nonempty, then it is asymptotically stable for the dynamics restricted to F.

small positive ϵ. ■

Proof of lemma 6.1 for WSP dynamics. For concreteness, set $i = 2$. At $\mathbf{e}_2$, strategy 4 earns less than average. Therefore $g_4(\mathbf{e}_2) < 0$. Now consider the case $\epsilon = 0$: at every point $\mathbf{x}$ in the (relative) interior of the edge $[\mathbf{e}_1, \mathbf{e}_2]$, strategy 3 earns strictly less than average hence its growth rate is negative. By continuity at $\mathbf{e}_2$ this implies $g_3(\mathbf{e}_2) \leq 0$. Since at $\mathbf{e}_2$, strategy 1 earns strictly less than average, it follows that $g_1(\mathbf{e}_2) < 0$, hence $g_3(\mathbf{e}_2) < -g_1(\mathbf{e}_2)$. Since the dynamics depends continuously on the payoff matrix, this still holds for small positive ϵ.

To establish (36), it suffices to show that $g_3(\mathbf{e}_2)$ is positive for every sufficiently small positive ϵ. Let $\epsilon > 0$. If $\lambda > 0$ is sufficiently small then, for all $\mu > 0$ small enough, the unique strategy which earns weakly above average at $\mathbf{x} = (\lambda\mu, 1 - \mu - \lambda\mu, \mu, 0)$ is strategy 3, hence $g_i(\mathbf{x}) < 0$ for $i \neq 3$. Since $\sum_{1 \leq i \leq 4} x_i g_i(\mathbf{x}) = 0$, it follows that $x_1 g_1(\mathbf{x}) + x_3 g_3(\mathbf{x}) > 0$, hence $\lambda\mu g_1(\mathbf{x}) + \mu g_3(\mathbf{x}) > 0$, hence $g_3(\mathbf{x}) > -\lambda g_1(\mathbf{x})$. By letting μ go to zero, we obtain $g_3(\mathbf{e}_2) \geq -\lambda g_1(\mathbf{e}_2) > 0$ ($g_1(\mathbf{e}_2) < 0$ was proved in the previous paragraph). ■

We now prove proposition 10. Fix α and ϵ as in lemma 6.1. Note that since the dynamics we consider depends continuously on the payoff matrix, there exists a neighborhood of the game (8) in which the strict inequalities (36) still hold. Thus, to prove proposition 10, it suffices to show that (36) implies (34) and (35). Fix a game for which (36) holds.

Proof that condition (35) holds. It follows from (36) that $g_4(\mathbf{e}_i)$ is negative for all i in $\{1, 2, 3\}$. This implies that (35) holds (fix $p_1 = p_2 = p_3 = -1$ and take a very high p_4).

Proof that condition (34) holds. To prove (34), i.e. asymptotic stability of Γ on the boundary, we use again characteristic matrices. Let $\hat{\mathbf{C}}$ denote the 3×3 matrix obtained from $\mathbf{C}$ by eliminating the fourth column. This corresponds to the characteristic matrix of Γ, when viewed as an heteroclinic cycle of the underlying 3×3 RPS game. In this RPS game, the set Γ is trivially asymptotically stable on the relative boundary of S_3 (Γ *is* the relative boundary!). Furthermore, for $\hat{\mathbf{p}} = (1/3, 1/3, 1/3) > 0$, the second inequation in (36) implies that $\hat{\mathbf{C}}\hat{\mathbf{p}} < 0$. Therefore, it follows from theorem 1 of Hofbauer (1994) that, in the 4×4 initial game, Γ is asymptotically stable on the face spanned by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Asymptotic stability on the face spanned by $\mathbf{e}_i, \mathbf{e}_{i+1}, \mathbf{e}_4$ is easy. This concludes the proof. ■

7 Robustness to the addition of mixed strategies as new pure strategies

We showed that for many dynamics, there exists an open set of symmetric 4×4 games for which, from an open set of initial conditions, the unique strategy used in correlated equilibrium is eliminated. Since we might not want to rule out the possibility that individuals use mixed strategies, and that mixed strategies be heritable, it is important to check whether our results change if we explicitly introduce mixed strategies as new pure strategies of the game. As explained in the introduction, the paradigm is the following: there is an underlying normal-form game, called the base game, and a finite number of types of agents. Each type plays a pure or mixed strategy of the base game. We assume that each pure strategy of the base game is played (as a pure strategy) by at least one type of agent, but otherwise we make no assumptions on the agents' types. The question is whether we can nonetheless be sure that, for an open set of initial conditions, all strategies used in correlated equilibrium are eliminated. This section shows that the answer is positive, at least for the best-response dynamics and for the replicator dynamics. We first need some notations and vocabulary.

Let G be a finite game with strategy set $I = \{1, ..., N\}$ and payoff matrix $\mathbf{U}$. A finite game G' is said to be *built on G by adding mixed strategies as new pure strategies* if:

First, letting $I' = \{1, ..., N, N+1, ..., N'\}$ be the set of pure strategies of G' and $\mathbf{U}'$ its payoff matrix, we may associate to each pure strategy i in I' a mixed strategy $\mathbf{p}^i$ in S_N in such a way that:

$$\forall i \in I', \forall j \in I', \mathbf{e}'_i \cdot \mathbf{U}'\mathbf{e}'_j = \mathbf{p}^i \cdot \mathbf{U}\mathbf{p}^j \tag{37}$$

where $\mathbf{e}'_i$ is the unit vector in $S_{N'}$ corresponding to the pure strategy i.

Second, if $1 \leq i \leq N$, the pure strategy i in the game G' corresponds to the pure strategy i in the base game G:

$$1 \leq i \leq N \Rightarrow \mathbf{p}^i = \mathbf{e}_i. \tag{38}$$

If $\mu' = (\mu(k,l))_{1\leq k,l\leq N'}$ is a probability distribution over $I' \times I'$, then it induces the probability distribution μ on $I \times I$ given by:

$$\mu(i,j) = \sum_{1\leq k,l\leq N'} \mu'(k,l) p_i^k p_j^l \qquad \forall (i,j) \in I \times I.$$

It follows from a version of the revelation principle (see Myerson, 1994) that, if G' is built on G by adding mixed strategies as new pure strategies, then for any correlated equilibrium μ' of G', the induced probability distribution on $I \times I$ is a correlated equilibrium of G. Thus, if G is a 4×4 symmetric game with $\mathbf{e}_4 \otimes \mathbf{e}_4$ as unique correlated equilibrium, then μ' is a correlated equilibrium of G' if and only if, for every k, l in I' such that $\mu'(k,l)$ is positive, $\mathbf{p}^k = \mathbf{p}^l = \mathbf{e}_4$. Thus, the unique strategy of G used in correlated equilibria of G' is strategy 4. We show below that:

Proposition 11 *For the replicator dynamics and for the best-response dynamics, there exists an open set of* 4×4 *symmetric games such that, for any game G in this set:*

(i) $\boldsymbol{e}_4 \otimes \boldsymbol{e}_4$ *is the unique correlated equilibrium of G*

(ii) For any game G' built on G by adding mixed strategies as new pure strategies and for an open set of initial conditions, every pure strategy k in I' such that $p_4^k > 0$ *is eliminated.*

(the open set of initial conditions in property (ii) is a subset of $S_{N'}$, the simplex of mixed strategies of G', and may depend on G')[16,17]

For the best-response dynamics, proposition 11 follows from an easy and very general result: for any finite game and in a sense made precise in the next section, adding mixed strategies as new pure strategies does not modify the behavior of the best-response dynamics.

[16]Our insistence on elimination of strategies k with $p_4^k > 0$, and not only of strategies k with $\mathbf{p}^k = \mathbf{e}_4$, stems from the following consideration: an observer who could only see which actions are taken by the agents, but not which pure or mixed strategies these actions come from, could not be sure that strategies k with $\mathbf{p}^k = \mathbf{e}_4$ have been eliminated unless he never observes action 4. This requires that all types of agents whose associated mixed strategy puts positive probability on action 4 be eliminated.

[17]An observer cannot determine the current population profile in $S_{N'}$ from the actions of the agents, but only the point in S_N induced by the population profile in $S_{N'}$. So ideally, we would like to show that there is an open subset of S_N such that, for any interior initial condition in $S_{N'}$ inducing a point in this subset, all strategies used in correlated equilibrium are eliminated. As should be clear from the next section, this is easy for the best-response dynamics, but we do not know whether this holds for the replicator dynamics.

7.1 Proof for the best-response dynamics

Let G be a finite game and let G' be a finite game built on G by adding mixed strategies of G as new pure strategies. We want to relate the behavior of the best-response dynamics in the game G' to its behavior in the base game G. For this purpose, associate to each mixed strategy $\mathbf{x}'$ in $S_{N'}$ the induced mixed strategy $\mathbf{x}$ in S_N defined by:

$$\mathbf{x} := \sum_{k=1}^{N'} x'_k \mathbf{p}^k. \tag{39}$$

Let $\mathbf{x}'(\cdot)$ be a solution of the best-response dynamics in G' and $\mathbf{x}(\cdot)$ the induced mapping from $\mathbb{R}_+$ to S_N.

Proposition 12 *$\boldsymbol{x}(\cdot)$ is a solution of the best-response dynamics in G.*

Proof. For almost all $t \geq 0$, there exists a vector $\mathbf{b}' \in BR(\mathbf{x}'(t))$ such that $\dot{\mathbf{x}}'(t) = \mathbf{b}' - \mathbf{x}'(t)$. Let $\mathbf{b} := \sum_{k \in I'} b'_k \mathbf{p}^k \in S_N$. It follows from (39) that:

$$\dot{\mathbf{x}}(t) = \sum_{k=1}^{N'} \dot{\mathbf{x}}'_k \mathbf{p}^k = \sum_{k=1}^{N'} (b'_k - x'_k)\mathbf{p}^k = \mathbf{b} - \mathbf{x}(t). \tag{40}$$

Furthermore, since $\mathbf{b}'$ is a best response to $\mathbf{x}'(t)$ it follows from (37) and (38) that $\mathbf{b}$ is a best response to $\mathbf{x}(t)$ (otherwise, letting $i \in \{1, ..., N\}$ be a best response to $\mathbf{x}$, we have: $\mathbf{b}' \cdot \mathbf{U}'\mathbf{x}' = \mathbf{b} \cdot \mathbf{U}\mathbf{x} < \mathbf{e}_i \cdot \mathbf{U}\mathbf{x} = \mathbf{e}'_i \cdot \mathbf{U}'\mathbf{x}'$, hence $\mathbf{b}'$ is not a best response to $\mathbf{x}'$, a contradiction). Together with (40), this implies that, for almost all t, $\dot{\mathbf{x}} \in BR(\mathbf{x}) - \mathbf{x}$. The result follows. ■

Assume that G is a 4×4 symmetric game satisfying conditions (2), (5) and (7) and let $\mathbf{x}'(0)$ be an initial condition in G' to which strategy 4 is not a best response and to which strategies 1, 2 and 3 are not all best responses. Note that there is an open set of such initial conditions. It follows from propositions 12 and 5 that $x_4(t) = \sum_{k=1}^{N'} x'_k(t) p_4^k \to 0$ as $t \to +\infty$. This implies that, for every k in I' with $p_4^k > 0$, $x'_k(t) \to 0$. Together with corollary 2.1, this proves proposition 11 for the best-response dynamics.

7.2 Proof for the replicator dynamics

Recall that G_0 denote the game of example 3 with $\alpha = 0$ and $\mathbf{U}_0$ its payoff matrix. Note that in a game built on G_0 by adding mixed strategies as new pure strategies, the heteroclinic cycle $\Gamma' : \mathbf{e}'_1 \to \mathbf{e}'_2 \to \mathbf{e}'_3 \to \mathbf{e}'_1$ need not be asymptotically stable. Indeed (letting $i \in \{1, 2, 3\}$ and counting $i + 1$ modulo 3), some of the added mixed strategies might be better responses to $\mathbf{e}_i$ than $\mathbf{e}_i$ (e.g. $(1 - \lambda)\mathbf{e}_{i+1} + \lambda\mathbf{e}_4$, with λ small); this leads to instability near $\mathbf{e}_i$. Nevertheless, we will show that every game close enough to G_0 satisfies property (ii) of proposition 11. Since, as already mentioned, every neighborhood of G_4 contains an open set of games with $\mathbf{e}_4 \otimes \mathbf{e}_4$ as unique correlated equilibrium, this implies that, for the replicator dynamics, elimination of all strategies used in correlated equilibrium is indeed robust to the addition of mixed strategies as new pure strategies, in the sense of proposition 11.

Before formally proving that games close to G_0 satisfy (ii), we provide the intuition: for a game G close to G_0, the set Γ defined in (9) is an attractor, close to which strategy 4 earns less than average. Now consider a game G' built on G by adding mixed strategies as new pure strategies, and a solution of the replicator dynamics in G': (a) as long as the share of strategies $k \geq 4$ remains low, the solution remains close to the base-game attractor; (b) as long as the solution is close to the base-game attractor, strategy 4 earns less than average and its share decreases; (c) as long as the share of strategy 4 does not increase, the share of strategies $k \geq 5$ remains low, moreover, if the share of strategy x_4 decreases, so does, on average over time, the share of each added mixed strategy in which strategy 4 is played with positive probability. The latter follows from a basic property of the replicator dynamics[18] and requires that the share of the added strategies $k \geq 5$ be initially low with respect to the minimal share of the strategies played in the base-game attractor (1, 2 and 3). Putting (a), (b) and (c) together gives the result. Now to the details:

As in section 5, let E_0 denote the convex hull of $\mathbf{n} = (1/3, 1/3, 1/3, 0)$ and $\mathbf{e}_4$. For $\mathbf{x}$ in $S_4 \backslash \{\mathbf{e}_4\}$, let

$$V(\mathbf{x}) := 3\frac{(x_1 x_2 x_3)^{1/3}}{x_1 + x_2 + x_3}.$$

The function V takes its maximal value 1 on $E_0 \backslash \{\mathbf{e}_4\}$ and its minimal value 0 on the set $\{\mathbf{x} \in S_4 \backslash \{\mathbf{e}_4\} : x_1 x_2 x_3 = 0\}$. Fix δ in $]0, 1[$. If $V(\mathbf{x}) \leq \delta$ then $\mathbf{x} \notin E_0$, hence it follows from (26) that, at $\mathbf{x}$, strategy 4 earns strictly less than average. Together with a compactness argument, this implies that there

[18] Namely, the fact that the replicator dynamics is convex monotonic in the sense of Hofbauer and Weibull (1996); loosely said, this amounts to not giving an advantage to mixed strategies over pure ones.

exists $\gamma_1 > 0$ such that:

$$V(\mathbf{x}) \leq \delta \Rightarrow [(\mathbf{U}_0\mathbf{x})_4 - \mathbf{x}\cdot\mathbf{U}_0\mathbf{x} \leq -\gamma_1]\,. \tag{41}$$

Furthermore, it is shown in (Viossat, 2006) that in G_0, under the replicator dynamics, the function $V(\mathbf{x})$ decreases strictly along interior trajectories (except those starting in E_0). More precisely, for every interior initial condition $\mathbf{x}(0) \notin E_0$ and every $t \in \mathbb{R}$, the function $v_0(t) := V(\mathbf{x}(t))$ satisfies $\dot{v}_0(t) < 0$. Together with the compactness of $\{\mathbf{x} \in S_4 \backslash \{\mathbf{e}_4\}, V(\mathbf{x}) = \delta\}$, this implies that there exists $\gamma_2 > 0$ such that

$$v_0(t) = \delta \Rightarrow \dot{v}_0(t) \leq -\gamma_2. \tag{42}$$

Fix a 4×4 matrix $\mathbf{U}$ and a solution $\mathbf{x}(\cdot)$ of the replicator dynamics with payoff matrix $\mathbf{U}$, with $\mathbf{x}(0) \neq \mathbf{e}_4$. Let $v(t) := V(\mathbf{x}(t))$. Thus, the difference between v_0 and v is that the solution $\mathbf{x}(\cdot)$ intervening in the definition of v is a solution of the replicator dynamics for the payoff matrix $\mathbf{U}$ and not for $\mathbf{U}_0$. Since $(\mathbf{U}\mathbf{x})_4 - \mathbf{x}\cdot\mathbf{U}\mathbf{x}$ and $\dot{\mathbf{x}}$ are Lipschitz in $\mathbf{U}$, it follows from (41) and (42) that there exists $\gamma > 0$ such that, if $||\mathbf{U} - \mathbf{U}_0|| < \gamma$:

$$V(\mathbf{x}) \leq \delta \Rightarrow [(\mathbf{U}\mathbf{x})_4 - \mathbf{x}\cdot\mathbf{U}\mathbf{x} \leq -\gamma] \tag{43}$$

and

$$v(t) = \delta \Rightarrow \dot{v}(t) \leq -\gamma. \tag{44}$$

Fix a game G with payoff matrix $\mathbf{U}$ such that $||\mathbf{U} - \mathbf{U}_0|| < \gamma$. Let G' be a game built on G by adding mixed strategies of G as new pure strategies, and let $\mathbf{U}'$ be its payoff matrix. For $\mathbf{x}'$ in $S_{N'}$ such that $x'_1 + x'_2 + x'_3 > 0$, let

$$V'(\mathbf{x}') := 3\frac{(x'_1 x'_2 x'_3)^{1/3}}{x'_1 + x'_2 + x'_3}.$$

Consider a solution $\mathbf{x}'(\cdot)$ of the replicator dynamics in G' (with $\sum_{1\leq i\leq 3} x'_i(0) > 0$) and let $v'(t) = V'(\mathbf{x}'(t))$. On the face of $S_{N'}$ spanned by the strategies of the original game:

$$\{\mathbf{x} \in S_{N'} : \sum_{1\leq i\leq 4} x'_i = 1\},$$

the replicator dynamics behaves just as in the base-game. Therefore, (43) and (44) imply trivially that:

$$\left[\sum_{1\leq i\leq 4} x'_i = 1 \text{ and } V'(\mathbf{x}) \leq \delta\right] \Rightarrow [(\mathbf{U}'\mathbf{x}')_4 - \mathbf{x}'\cdot\mathbf{U}'\mathbf{x}' \leq -\gamma] \tag{45}$$

and

$$\left[\sum_{1\leq i\leq 4} x'_i = 1 \text{ and } v'(t)=\delta\right] \Rightarrow \dot{v}'(t) \leq -\gamma. \tag{46}$$

Now define $\bar{\mathbf{x}}' \in S_{N'}$ as the projection of $\mathbf{x}$ on the face of $S_{N'}$ spanned by the strategies of the original game. That is,

$$\bar{\mathbf{x}}'_i = \frac{x'_i}{\sum_{1\leq j\leq 4} x'_j} \text{ if } 1\leq i\leq 4, \text{ and } \bar{\mathbf{x}}'_i = 0 \text{ otherwise.}$$

Note that $V'(\mathbf{x}') = V(\mathbf{x})$. Furthermore, a simple computation shows that

$$\max_{1\leq i\leq N'} |x'_i - \bar{x}'_i| \leq N' \max_{5\leq k\leq N'} x'_k.$$

Therefore, since $(\mathbf{U}'\mathbf{x}')_4 - \mathbf{x}'\cdot\mathbf{U}'\mathbf{x}'$ and the vector field $\dot{\mathbf{x}}'$ are Lipschitz in $\mathbf{x}'$, it follows from (45) and (46) that there exist positive constants η and γ' such that

$$\left[\max_{5\leq k\leq N'} x'_k \leq \eta \text{ and } V'(\mathbf{x}') \leq \delta\right] \Rightarrow (\mathbf{U}'\mathbf{x}')_4 - \mathbf{x}'\cdot\mathbf{U}'\mathbf{x}' \leq -\gamma' \tag{47}$$

and

$$\left[\max_{5\leq k\leq N'} x'_k \leq \eta \text{ and } v'(t)=\delta\right] \Rightarrow \dot{v}'(t) \leq -\gamma'. \tag{48}$$

Fix $\mathbf{y}' \in S_{N'}$ such that

$$\sum_{1\leq i\leq 4} y'_i = 1, V(\mathbf{y}') < \delta \text{ and } C := \min_{1\leq i\leq 3} y'_i > 0.$$

There exists an open neighborhood Ω of $\mathbf{y}$ in $S_{N'}$ such that

$$\forall \mathbf{x}' \in \Omega, \left[\min_{1\leq i\leq 3} x'_i > C/2,\ \max_{5\leq k\leq N'} x'_k < C\eta/2, \text{ and } V'(\mathbf{x}') < \delta\right].$$

Consider an interior solution $\mathbf{x}'(\cdot)$ of the replicator dynamics in G' with initial condition in Ω. Recall that $\mathbf{p}^k$ denote the mixed strategy of G associated with the pure strategy k of G'. To prove proposition 11 for the replicator dynamics, it suffices to show that:

Proposition 13 *For all k in $\{4, ..., N'\}$ such that $p^k_4 > 0$, $x'_k(t) \to_{t\to+\infty} 0$.*

Proof. We begin with two lemmas:

Lemma 7.1 *Let $T > 0$ and $k \in \{5, ..., N'\}$. If $x'_4(T) \leq x'_4(0)$ then $x'_k(T) < \eta$.*

Proof. By construction of G', strategy $k \in I'$ earns the same payoff as the mixed strategy $\sum_{1 \leq i \leq 4} p_i^k \mathbf{e}'_i$:

$$(\mathbf{U}'\mathbf{x}')_k = \sum_{1 \leq i \leq 4} p_i^k (\mathbf{U}'\mathbf{x}')_i \qquad \forall \mathbf{x}' \in S'_N.$$

Therefore, it follows from the definition of the replicator dynamics that:

$$\frac{\dot{x}'_k}{x'_k} = \sum_{1 \leq i \leq 4} p_i^k \frac{\dot{x}'_i}{x'_i}.$$

Integrating between 0 and T and taking the exponential of both members leads to:

$$x'_k(T) = x'_k(0) \prod_{1 \leq i \leq 4} \left(\frac{x'_i(T)}{x'_i(0)} \right)^{p_i^k}. \tag{49}$$

Noting that for $1 \leq i \leq 3$, we have $x'_i(T) \leq 1$ and $1 \leq 1/x'_i(0) \leq 2/C$, we get:

$$\prod_{1 \leq i \leq 3} \left(\frac{x'_i(T)}{x'_i(0)} \right)^{p_i^k} \leq \prod_{1 \leq i \leq 3} \left(\frac{2}{C} \right)^{p_i^k} = \left(\frac{2}{C} \right)^{1 - p_4^k} \leq \frac{2}{C}. \tag{50}$$

Since furthermore $x'_k(0) < C\eta/2$, we obtain from (49) and (50):

$$x'_k(T) < \frac{C\eta}{2} \frac{2}{C} \left(\frac{x'_4(T)}{x'_4(0)} \right)^{p_4^k} = \eta \left(\frac{x'_4(T)}{x'_4(0)} \right)^{p_4^k}. \tag{51}$$

The result follows. ■

Lemma 7.2 *For all $t > 0$, $\max_{k \in \{5,...,N'\}} x'_k(t) < \eta$ and $v'(t) < \delta$.*

Proof. Otherwise there is a first time $T > 0$ such that $\max_{k \in \{5,...,N'\}} x'_k(T) = \eta$ or $v'(T) = \delta$ (or both). It follows from (47) and the definition of the replicator dynamics that if $0 \leq t \leq T$ then $\dot{x}'_4(t) \leq -\gamma' < 0$. Therefore $x'_4(T) \leq x'_4(0)$. By lemma 7.1, this implies that $\max_{k \in \{5,...,N'\}} x'_k(T) < \eta$. Therefore, $v'(T) = \delta$. Due to (48), this implies that $\dot{v}'(T) < 0$. Therefore, there exists a time T_1 with $0 < T_1 < T$ such that $v'(T_1) > \delta$, hence a time T_2 with $0 < T_2 < T_1 < T$ such $v'(T_2) = \delta$, contradicting the minimality of T. ■

We now conclude: it follows from lemma 7.2, equation (47) and the definition of the replicator dynamics that for all $t \geq 0$, $x'_4(t) \leq \exp(-\gamma' t) x'_4(0)$. By (51) this implies that for every k in $\{5, ..., N'\}$,

$$\forall t \geq 0, x_k(t) < \eta \exp(-\gamma' p_4^k t).$$

Therefore, if $p_4^k > 0$ then $x'_k(t) \to 0$ as $t \to +\infty$. ■

8 Discussion

We showed that elimination of all strategies used in correlated equilibrium is a robust phenomenon, in that it occurs for many dynamics, an open set of games and an open set of initial conditions. Furthermore, at least for some of the leading dynamics, the results are robust to the addition of mixed strategies as new pure strategies. It is relatively easy to prove some other forms of robustness, e.g. robustness to discretization or perturbation of the vector field (Viossat, 2005, chapter 10a). Furthermore, under the replicator dynamics, the best-response dynamics or the Brown-von Neumann-Nash dynamics, for appropriate values of the payoffs in game (8), the basin of attraction of the Nash equilibrium can be made arbitrarily small, and the minimal distance from the attractor on the face $x_4 = 0$ to the basin of attraction of the Nash equilibrium much larger than the minimal distance from the Nash equilibrium to the basin of attraction of the attractor on the face $x_4 = 0$. It follows that this attractor would be stochastically stable in a model à la Kandori, Mailath and Rob (1993).[19,20] These results show a sharp difference between evolutionary dynamics and "adaptive heuristics" such as no-regret dynamics (Hart and Mas-Collel, 2003; Hart, 2005) or hypothesis testing (Young, 2004, chapter 8).

Some limitations of our results should however be stressed. First, our results hold only for single-population dynamics. Of course, they imply that

[19]The (unperturbed) dynamics used by Kandori, Mailath and Rob (1993) is a discrete-time version of the best-response dynamics, but it could easily be replaced by a discrete-time version of another dynamics.

[20]For BNN, it is not known whether in a RPS game, there is a unique limit cycle or if there are several; but there is an asymptotically stable annulus (Berger and Hofbauer, 2006). So for BNN, what we mean by "attractor on the face $x_4 = 0$ is this asymptotically stable annulus.

for *some* games and *some* interior initial conditions, two-population dynamics eliminate all strategies used in correlated equilibrium[21]; but maybe not for an open set of games nor for an open set of initial conditions.

Second, the monotonic and weakly sign-preserving dynamics of section 6 are non-innovative: strategies initially absent do not appear. This has the effect that, even when focusing on interior initial conditions, the growth of the share of the population playing strategy i is limited by the current value of this share. This is appropriate if we assume that agents have to meet an agent playing strategy i to become aware of the possibility of playing strategy i; but in general, as discussed by e.g. Swinkels (1993, p.459), this seems more appropriate in biology than in economics. While our results hold also for some important innovative dynamics, such as the best-response dynamics and a family of dynamics including the Brown-von Neumann-Nash dynamics, more general results would be welcome.

Third, in the games we considered, the unique correlated equilibrium is a strict Nash equilibrium, and is thus asymptotically stable under most reasonable dynamics, including all those we studied. Thus, even though the unique strategy used in correlated equilibrium is eliminated for many initial conditions, there is still an important connection between correlated equilibrium and the outcome of evolutionary dynamics.

For Nash equilibrium, these three limitations can be overcome, at least partially: there are wide classes of multi-population innovative dynamics for which there exists an open set of games such that, for an open set of initial conditions, all strategies belonging to the support of at least one Nash equilibrium are eliminated (Viossat, 2005, chapter 11). Moreover, for the single-population replicator dynamics or the single-population best-response dynamics, there are games for which, for almost all initial conditions, all strategies used in Nash equilibrium are eliminated (Viossat, 2005, chapter 12). Whether these results extend to correlated equilibrium is an open question.

[21]This is because for symmetric two-player games with symmetric initial conditions, two-population dynamics reduce to single-population dynamics, at least for the replicator dynamics, the best-response dynamics and the Brown-von-Neumann Nash dynamics.

A Proof of proposition 4

We provide two proofs, as they provide different insights. The first one, in the spirit of section 6, consists in checking that the sufficient conditions for asymptotic stability of heteroclinic cycles given by Hofbauer (1994) are satisfied. The second proof, in the spirit of the proof of proposition 4 in (Viossat, 2006) exhibits an average Lyapunov function.[22] In both proofs, $i \in \{1, 2, 3\}$ and $i+1$ and $i-1$ are counted modulo 3.

Proof 1. We use the tools introduced at the beginning of the proof of proposition 10 (up to lemma 6.1). The heteroclinic cycle Γ is asymptotically stable on the boundary of S_4: asymptotic stability on the face spanned by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ follows from (5), as shown by Zeeman (1980); asymptotic stability on the face spanned by $\mathbf{e}_i, \mathbf{e}_{i+1}, \mathbf{e}_4$ is easy. Furthermore, under the replicator dynamics, the characteristic matrix $\mathbf{C}$ of Γ has the sign structure:

	1	2	3	4
e_1	0	−	+	−
e_2	+	0	−	−
e_3	−	+	0	−

.

It follows that there exists a vector $\mathbf{p}$ in $\mathbb{R}^4$ such that $\mathbf{p} > 0$ and $\mathbf{Cp} < 0$ (fix $p_1 = p_2 = p_3 = 1$ and take a very high p_4). By theorem 1 of Hofbauer (1994), this implies that Γ is asymptotically stable. ■

Proof 2 (sketch). Applying lemma 7 from Zeeman (1980) in the spirit of (Hofbauer and Sigmund, 1998, proof of theorem 7.7.2), we may assume without loss of generality that there exists a positive constant c such that $b_i - a_{i+1} = c$ for $i = 1, 2, 3$. Let $\mathbf{p} \in S_3$ denote the Nash equilibrium of the underlying RPS game and let $V(\mathbf{x}) = \prod_{1 \le i \le 3} \hat{\mathbf{x}}_i^{p_i}$ (where $\hat{\mathbf{x}}_i = x_i/(1 - x_4)$). The function $\dot{V}/V$ extends to a continuous function on S_4 which is strictly negative on Γ (more precisely, if $x_4 = 0$, then $\dot{V} = -cV \sum_{1 \le i \le 3}(x_i - p_i)^2$; see Hofbauer and Sigmund, 1998, proof of theorem 7.7.2). This implies that V decreases exponentially in the neighborhood of Γ. The only difference with the proof of proposition 4 in (Viossat, 2006) is then that $W(\mathbf{x}) = \max(x_4, V(\mathbf{x}))$ is no longer a local Lyapunov function (because x_4 need not decrease everywhere in the neighbourhood of Γ) but only a local *average*

[22] Our first proof relies on theorem 1 of Hofbauer (1994), which itself relies on the construction of an (average) Lyapunov function; however this average Lyapunov function is not explicit.

Lyapunov function (it decreases in average over an approximate cycle). We only give the heuristic argument: there exists a neighbourhood N_i of e_i in which strategy 4 earns strictly less than the mean payoff, so that x_4 decreases. As long as $\mathbf{x}(t)$ is close enough to Γ, $V(\mathbf{x}(t))$ decreases and the solution describes a cycling movement from N_1 to N_2 to N_3 and back to N_1. During this (approximate) cycle, most of the time[23] is spent in the N_i, so that x_4 decreases over the cycle. This allows to show that for every $\delta > 0$, there exists $\delta' > 0$ such that if $W(\mathbf{x}(0)) \leq \delta'$ then $W(\mathbf{x}(t)) \leq \delta$ for all $t \geq 0$ and $W(\mathbf{x}(t)) \to 0$ as $t \to +\infty$.

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