Probabilistic choice in games: Properties of Rosenthal's t-solutions*

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Abstract

The t-solutions introduced in R. W. Rosenthal (1989, Int. J. Game Theory 18, 273-292) are quantal response equilibria based on the linear probability model. Choice probabilities in t-solutions are related to the determination of leveling taxes in taxation problems. The set of t-solutions coincides with the set of Nash equilibria of a game with quadratic control costs. Evaluating the set of t-solutions for increasing values of t yields that players become increasingly capable of iteratively eliminating never-best replies and eventually only play rationalizable actions with positive probability. These features are not shared by logit quantal response equilibria. Moreover, there exists a path of t-solutions linking uniform randomization to Nash equilibrium.

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1 Introduction

The literature on quantal response equilibria (QRE) provides a set of solution concepts for boundedly rational players in noncooperative games by replacing expected utility maximization with probabilistic choice models. In these probabilistic choice models, players may choose suboptimal strategies, but at least play "better" responses with probabilities not lower than "worse" responses. The QRE studied in most detail use the logit choice model, well-known from pioneering contributions of Nobel laureate McFadden (1974) and formulated in a game-theoretic framework by McKelvey and Palfrey (1995). Numerous experimental studies (cf. Camerer, 2003, Goeree and Holt, 2001) indicate that QRE have substantial descriptive power.

The current paper focuses on one of the earliest contributions — if not *the* earliest contribution — to the literature on QRE: the t-solutions of Rosenthal (1989). Instead of using the logit choice model, it is based on the linear probability model (cf. Ben-Akiva and Lerman, 1985, Section 4.2), where choice probabilities are essentially linear in expected payoff differences: there is a parameter $t \geq 0$ (hence the nomenclature "t-solutions") such that the difference between the probabilities with which a player chooses any pair of pure strategies is equal to t times their expected payoff difference. Since probability differences are bounded in absolute value by one, this is impossible for large values of t, so the actual definition, provided and discussed in Section 2, requires a slight modification.

The main results are the following.

- The determination of a player's choice probabilities given mixed strategies of the others is related to the computation of leveling taxes in taxation problems (Young, 1987) or — equivalently — the constrained equal losses solution in bankruptcy problems (Aumann and Maschler, 1985). This perhaps surprising connection is discussed in Section 3.

- Theorem 4.2 generalizes an unproven statement in Rosenthal (1989, p. 292) about the connection between t-solutions and equilibria of games with control costs: the set of t-solutions equals the set of Nash equilibria of a game with quadratic control costs. This provides a microeconomic foundation for the probabilistic choice model as rational behavior for decision makers who have to make some effort (incur costs) to implement their strategic choices. A similar derivation of logit QRE using an entropic control cost function is provided in Mattsson and Weibull (2002).

- As a consequence of these results, we obtain two proofs (Remarks 3.4 and 4.1) of the existence of t-solutions; a matter of interest, because Rosenthal's proof is not entirely correct (see Remark 2.3). The proofs use distinct economically relevant notions: taxation and control costs. Therefore, we believe it is of interest to provide both proof methods.

- Evaluating the set of t-solutions for increasing values of t, one finds (Theorem 5.1) that players become increasingly capable of iteratively eliminating never-best replies (in particular strictly dominated actions) and that players eventually only choose rationalizable actions with positive probability. These properties discern the linear probability QRE from the logit QRE, where *all* actions are chosen with strictly positive probability.

- Theorem 5.2 indicates that there is a continuous way to walk from low-rationality t-solutions to Nash equilibrium behavior as t approaches infinity: there is a path of t-solutions linking uniform randomization at $t = 0$ to Nash equilibrium behavior in the limit as $t \to \infty$.

Concluding remarks are provided in Section 6.

2 Notation and preliminaries

A (finite strategic) **game** is a tuple $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$, where N is a nonempty, finite set of players, each player $i \in N$ has a nonempty, finite set of pure strategies (or actions) A_i and payoff/utility function $u_i : \times_{j \in N} A_j \to \mathbb{R}$.

For $i \in N$, $A_{-i} = \times_{j \in N \setminus \{i\}} A_j$ denotes the set of pure strategy profiles of the remaining players. Payoffs are extended to mixed strategies in the usual way. The set of mixed strategies of player $i \in N$ is denoted by

$$\Delta(A_i) = \{\alpha_i : A_i \to \mathbb{R}_+ \mid \sum_{a_i \in A_i} \alpha_i(a_i) = 1\}.$$

The mixed strategy space of G is denoted by $\Delta = \times_{i\in N}\Delta(A_i)$. As usual, (α_i, α_{-i}) is the profile of mixed strategies where player $i \in N$ plays $\alpha_i \in \Delta(A_i)$ and his opponents play according to the strategy profile $\alpha_{-i} = (\alpha_j)_{j\in N\setminus\{i\}} \in \times_{j\in N\setminus\{i\}} \Delta(A_j)$. The set of Nash equilibria of the game G is denoted by

$$NE(G) = \{\alpha \in \Delta \mid \forall i \in N : u_i(\alpha) = \max_{\beta_i\in\Delta(A_i)} u_i(\beta_i, \alpha_{-i})\}.$$

Rosenthal's linear probability QRE are defined as follows:

Definition 2.1 (Rosenthal, 1989, p. 276) Let $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a game and let $t \in [0, \infty)$. A strategy profile $\alpha \in \Delta$ is a ***t*-solution** of G if for each player $i \in N$ and all $a_i, a_j \in A_i$:

$$\alpha_i(a_i) > 0 \Rightarrow \alpha_i(a_i) - \alpha_i(a_j) \leq t(u_i(a_i, \alpha_{-i}) - u_i(a_j, \alpha_{-i})).$$

The set of t-solutions of G is denoted by $S(t, G) \subseteq \Delta$, or, if no confusion arises, by $S(t)$. $\triangleleft$

Some comments on Rosenthal's definition: Let $t \in [0, \infty)$, let $\alpha \in \Delta$ be a t-solution of G and let $a_i, a_j \in A_i$ be two pure strategies of player $i \in N$. Notice:

- If $u_i(a_i, \alpha_{-i}) \geq u_i(a_j, \alpha_{-i})$, then $\alpha_i(a_i) \geq \alpha_i(a_j)$. This is clear if $\alpha_i(a_j) = 0$. If $\alpha_i(a_j) > 0$, it follows from the definition of a t-solution:

  $$\alpha_i(a_j) - \alpha_i(a_i) \leq t(u_i(a_j, \alpha_{-i}) - u_i(a_i, \alpha_{-i})) \leq 0.$$

  The last inequality holds because $t \geq 0$ and $u_i(a_i, \alpha_{-i}) \geq u_i(a_j, \alpha_{-i})$. This means in particular that a player in a t-solution chooses pure strategies having equal expected payoff with equal probability and that the probabilities are weakly increasing with expected payoffs: as stated in the introduction, "better" responses are chosen with a probability not less than "worse" responses.

- If both a_i and a_j are chosen with positive probability ($\alpha_i(a_i) > 0$ and $\alpha_i(a_j) > 0$), then Definition 2.1 implies that

  $$\alpha_i(a_i) - \alpha_i(a_j) \leq t(u_i(a_i, \alpha_{-i}) - u_i(a_j, \alpha_{-i}))$$

and

$$\alpha_i(a_j) - \alpha_i(a_i) \leq t(u_i(a_j, \alpha_{-i}) - u_i(a_i, \alpha_{-i})),$$

so

$$\alpha_i(a_i) - \alpha_i(a_j) = t(u_i(a_i, \alpha_{-i}) - u_i(a_j, \alpha_{-i})),$$

the difference in the probabilities is linear in the expected payoff difference. Since probability differences are bounded in absolute value by one, this linearity requirement is infeasible for large values of t, motivating Rosenthal's use of a weak inequality in Definition 2.1: linearity holds if both actions are chosen with positive probability, but — in accordance with the previous point — actions with low expected payoff are chosen with probability zero.

- For $t = 0$, the unique t-solution is the mixed strategy combination in which each $i \in N$ chooses all pure strategies with equal probability $1/ \mid A_i \mid$.

- Since t-solutions are defined in terms of expected payoff *differences*, adding a constant to each payoff function does not affect the set of t-solutions. This allows us, for instance, to assume without loss of generality that payoffs are nonnegative, as is done in Theorem 3.2 and Remark 3.3.

These properties are used often in the remainder of the text. In addition to many examples, Rosenthal (1989) provides the following results:

Proposition 2.2 *Let $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a game.*

(a) Existence: For each $t \in [0, \infty)$, $S(t) \neq \emptyset$.

(b) Upper semicontinuity: The t-solution correspondence $S : [0, \infty) \twoheadrightarrow \Delta$ is upper semicontinuous.

(c) Convergence to Nash equilibrium: Let $(t_n)_{n\in\mathbb{N}}$ be an increasing, unbounded sequence in $[0, \infty)$ and let $\alpha^n \in S(t_n)$ for all $n \in \mathbb{N}$. If $\alpha^n \to \alpha$, then α is a Nash equilibrium of G.

Intuitively, the parameter t is an indication of rationality: at $t = 0$, players disregard payoffs and randomize uniformly over their pure strategies, while t-solutions approximate Nash equilibrium behavior for large values of t. This intuition is made more precise in Section 5, where we show that *(i)* higher values of t correspond with higher levels of iterated elimination of never-best replies and *(ii)* there is a path of t-solutions linking uniform randomization at $t = 0$ to Nash equilibrium behavior as t goes to infinity.

Remark 2.3 The existence proof of Rosenthal (1989, p. 288) is not entirely correct: he provides an algorithm to define a function from and to the mixed strategy space of the game. The interested reader may check that the claim that this algorithm stops after finitely many steps is not correct: the algorithm may cycle. Indeed, it does so in the simple 2×1 bimatrix game with payoff matrices:

$$\begin{bmatrix} 1, 1 \\ 1, 1 \end{bmatrix}$$

The result, however, remains true. Two different existence proofs will be provided, both having economic appeal: the first is related to taxation (Remark 3.4), the second to control costs (Remark 4.1). $\triangleleft$

3 t-Solutions and leveling tax

There is an extensive economic/game-theoretic literature on taxation and bankruptcy problems; the reader is referred to Thomson (2003) for a recent survey and Young (1987) and Aumann and Maschler (1985) for pathbreaking studies. In a taxation problem (Young, 1987), a tax $T \geq 0$ has to be paid by drawing from the gross income $x_i \geq 0$ of individuals $i = 1, \ldots, n$, where $T \leq \sum_{i=1}^{n} x_i$: the tax is feasible. A well-known taxation rule is the socalled **leveling tax**: in a taxation problem with tax T and income vector $x \in \mathbb{R}^n_+$, each i pays a tax defined by

$$\max\{0, x_i - \lambda\},$$

where $\lambda \geq 0$ is such that $\sum_{i=1}^{n} \max\{0, x_i - \lambda\} = T$. Similarly, in a bankruptcy problem (Aumann and Maschler, 1985), an estate $E \geq 0$ is to be divided among $n \in \mathbb{N}$ claimants with claims $c_1, \ldots, c_n \geq 0$, where $E \leq \sum_{i=1}^n c_i$. The **constrained equal losses rule** in a bankruptcy problem with estate E and claims vector $c \in \mathbb{R}_+^n$ assigns to claimant i an amount

$$\max\{0, c_i - \lambda\},$$

where $\lambda \geq 0$ is such that $\sum_{i=1}^n \max\{0, c_i - \lambda\} = E$.

Determination of the choice probabilities in t-solutions is closely related to the computation of leveling taxes in taxation problems or — equivalently — constrained equal losses rules in bankruptcy problems. Theorem 3.1 indicates that t-solutions are fixed points of a certain function. In Theorem 3.2, this function is related to the leveling tax rule.

Theorem 3.1 *Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a game.*

(a) For each $\alpha \in \Delta, t \in [0, \infty)$, and $i \in N$, there is a unique $\mu_i(t, \alpha) > 0$ such that

$$\sum_{a_i \in A_i} \max\left\{0, \mu_i(t, \alpha) - t\left(\max_{a_k \in A_i} u_i(a_k, \alpha_{-i}) - u_i(a_i, \alpha_{-i})\right)\right\} = 1.$$

(b) For each $\alpha \in \Delta, t \in [0, \infty)$, let $\tau(t, \alpha) \in \Delta$ be the strategy profile defined for each $i \in N$ and $a_i \in A_i$ by

$$\tau_i(t, \alpha)(a_i) = \max\left\{0, \mu_i(t, \alpha) - t\left(\max_{a_k \in A_i} u_i(a_k, \alpha_{-i}) - u_i(a_i, \alpha_{-i})\right)\right\}. \tag{1}$$

Then $\alpha \in \Delta$ is a t-solution if and only if $\tau(t, \alpha) = \alpha$.

Proof.

(a) Let $\alpha \in \Delta, t \in [0, \infty)$, and $i \in N$. The function $T : [0, \infty) \to [0, \infty)$ defined for each $\mu \in [0, \infty)$ by

$$T(\mu) = \sum_{a_i \in A_i} \max\left\{0, \mu - t\left(\max_{a_k \in A_i} u_i(a_k, \alpha_{-i}) - u_i(a_i, \alpha_{-i})\right)\right\}$$

is the composition of continuous functions, hence continuous. Since

$$\max\left\{0, \mu - t\left(\max_{a_k \in A_i} u_i(a_k, \alpha_{-i}) - u_i(a_i, \alpha_{-i})\right)\right\} = \mu$$

if a_i is a best response to α_{-i}, i.e., if $u_i(a_i, \alpha_{-i}) = \max_{a_k \in A_i} u_i(a_k, \alpha_{-i})$, the function T is strictly increasing. Moreover, $T(0) = 0$ and $T(\mu) \to \infty$ as $\mu \to \infty$. By the intermediate value theorem there exists a $\mu_i(t, \alpha) > 0$ with $T(\mu_i(t, \alpha)) = 1$. Since T is strictly increasing, this number is unique.

(b) The strategy profile $\tau(t, \alpha)$ is well-defined by the above. If $\tau(t, \alpha) = \alpha$, then $\alpha \in S(t)$: for each $i \in N$ and $a_i, a_j \in A_i$, if $\alpha_i(a_i) > 0$, then

$$\begin{aligned}
\alpha_i(a_i) - \alpha_i(a_j) &= \tau_i(t, \alpha)(a_i) - \tau_i(t, \alpha)(a_j) \\
&= \mu_i(t, \alpha) - t\left(\max_{a_k \in A_i} u_i(a_k, \alpha_{-i}) - u_i(a_i, \alpha_{-i})\right) \\
&- \max\left\{0, \mu_i(t, \alpha) - t\left(\max_{a_k \in A_i} u_i(a_k, \alpha_{-i}) - u_i(a_j, \alpha_{-i})\right)\right\} \\
&\leq \mu_i(t, \alpha) - t\left(\max_{a_k \in A_i} u_i(a_k, \alpha_{-i}) - u_i(a_i, \alpha_{-i})\right) \\
&- \left(\mu_i(t, \alpha) - t\left(\max_{a_k \in A_i} u_i(a_k, \alpha_{-i}) - u_i(a_j, \alpha_{-i})\right)\right) \\
&= t(u_i(a_i, \alpha_{-i}) - u_i(a_j, \alpha_{-i})),
\end{aligned}$$

in correspondence with Definition 2.1.

Conversely, let $\alpha \in S(t)$. We know that player i chooses all best responses to α_{-i} with equal, positive probability, say $\mu_i > 0$. Fix a best response $a_j \in A_i$ against α_{-i}:

$$u_i(a_j, \alpha_{-i}) = \max_{a_k \in A_i} u_i(a_k, \alpha_{-i}) \text{ and } \alpha_i(a_j) = \mu_i > 0. \tag{2}$$

Let $a_i \in A_i$.

- If $\alpha_i(a_i) > 0$, then by definition of a t-solution and (2):

  $$\alpha_i(a_j) - \alpha_i(a_i) = \mu_i - \alpha_i(a_i) = t(u_i(a_j, \alpha_{-i}) - u_i(a_i, \alpha_{-i})),$$

  which together with $\alpha_i(a_i) > 0$ and (2) yields

  $$\alpha_i(a_i) = \max\left\{0, \mu_i - t\left(\max_{a_k \in A_i} u_i(a_k, \alpha_{-i}) - u_i(a_i, \alpha_{-i})\right)\right\}.$$

- If $\alpha_i(a_i) = 0$, then by definition of a t-solution:

  $$\begin{aligned}
  0 &= \alpha_i(a_i) \\
  &\geq \alpha_i(a_j) - t(u_i(a_j, \alpha_{-i}) - u_i(a_i, \alpha_{-i})) \\
  &= \mu_i - t\left(\max_{a_k \in A_i} u_i(a_k, \alpha_{-i}) - u_i(a_i, \alpha_{-i})\right),
  \end{aligned}$$

so

$$\alpha_i(a_i) = \max\left\{0, \mu_i - t\left(\max_{a_k \in A_i} u_i(a_k, \alpha_{-i}) - u_i(a_i, \alpha_{-i})\right)\right\}.$$

Hence for all $a_i \in A_i$:

$$\alpha_i(a_i) = \max\left\{0, \mu_i - t\left(\max_{a_k \in A_i} u_i(a_k, \alpha_{-i}) - u_i(a_i, \alpha_{-i})\right)\right\}$$

and $\sum_{a_i \in A_i} \alpha_i(a_i) = 1$, which given (a) implies that $\mu_i = \mu_i(t, \alpha)$. Conclude that $\alpha_i(a_i) = \tau_i(t, \alpha)(a_i)$, showing that $\alpha = \tau(t, \alpha)$. □

The fact that every $\alpha \in S(t)$ is a fixed point of the function $\tau(t, \cdot) : \Delta \to \Delta$ allows a nice, perhaps somewhat surprising link to the literature on taxation and bankruptcy problems:

Theorem 3.2 *Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a game, $\alpha \in \Delta$, and $t \in (0, \infty)$. For each $i \in N$, assume without loss of generality that*

$$u_i \geq 0 \text{ and } \sum_{a_i \in A_i} t u_i(a_i, \alpha_{-i}) \geq 1.$$

The strategy $\tau_i(t, \alpha) \in \Delta(A_i)$ of player $i \in N$ coincides with the vector of leveling taxes of the taxation problem with tax $T = 1$ and gross income vector $(t u_i(a_i, \alpha_{-i}))_{a_i \in A_i} \in \mathbb{R}_+^{|A_i|}$.

Proof. By assumption, the gross income vector is nonnegative and its coordinates add up to at least $T = 1$, so the taxation problem is well-defined. The leveling tax rule in this problem associates with every gross income $t u_i(a_i, \alpha_{-i})$ a tax equal to $\max\{0, t u_i(a_i, \alpha_{-i}) - \lambda\}$, where $\lambda \geq 0$ is such that $\sum_{a_i \in A_i} \max\{0, t u_i(a_i, \alpha_{-i}) - \lambda\} = 1$. Notice from (1) that for each $a_i \in A_i$:

$$\tau_i(t, \alpha)(a_i) = \max\left\{0, t u_i(a_i, \alpha_{-i}) - t \max_{a_k \in A_i} u_i(a_k, \alpha_{-i}) + \mu_i(t, \alpha)\right\}. \quad (3)$$

Since $\sum_{a_i \in A_i} \tau_i(t, \alpha)(a_i) = 1 \leq \sum_{a_i \in A_i} t u_i(a_i, \alpha_{-i})$ by assumption, it follows from $t u_i(a_i, \alpha_{-i}) \geq 0$ for all $a_i \in A_i$ and (3) that the number $\lambda := t \max_{a_k \in A_i} u_i(a_k, \alpha_{-i}) - \mu_i(t, \alpha)$ is nonnegative. Rewriting (3) yields

$$\tau_i(t, \alpha)(a_i) = \max\{0, t u_i(a_i, \alpha_{-i}) - \lambda\},$$

where $\lambda \geq 0$ is such that $\sum_{a_i \in A_i} \tau_i(t, \alpha)(a_i) = \sum_{a_i \in A_i} \max\{0, t u_i(a_i, \alpha_{-i}) - \lambda\} = 1$, showing that $\tau_i(t, \alpha)$ is indeed the stated vector of leveling taxes. □

Remark 3.3 Similarly: let $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a game, $t \in [0,\infty)$, and $\alpha \in \Delta$. Assume without loss of generality that $u_i \geq 0$ for all $i \in N$. The strategy $\tau_i(t,\alpha)$ of player $i \in N$ coincides with the vector of leveling taxes of the taxation problem with tax $T = 1$ and gross income vector $(u_i(a_i, \alpha_{-i}) + 1)_{a_i \in A_i} \in \mathbb{R}_+^{|A_i|}$. $\triangleleft$

The relations between our probabilistic choice model and the computation of leveling taxes/constrained equal losses in the theorem and remark above are more than just a mathematical curiosity. The problems are related in a very intuitive way that is perhaps most explicit in terms of the bankruptcy problem: in QRE, each player divides a probability mass (estate) of one over his pure strategies, where pure strategies with higher expected payoffs exert a higher claim, reflecting the intuition that "better" responses are chosen with a probability not lower than "worse" responses.

Remark 3.4 Using the connection in Theorem 3.2 and Remark 3.3 with leveling taxes, which are continuous in the gross income vector (Young, 1987, Thomson, 2003) and the continuity of the gross income vector in t and α, it follows that the function $\tau : [0,\infty) \times \Delta \to \Delta$ defined in Theorem 3.1 is continuous. By Brouwer's fixed point theorem, for every $t \in [0,\infty)$ there is an $\alpha \in \Delta$ with $\tau(t,\alpha) = \alpha$, so $S(t) \neq \emptyset$, proving Proposition 2.2(a). We also find an alternative proof of upper semicontinuity of the t-solution correspondence S, since its graph

$$\begin{aligned}\mathrm{graph}(S) &= \{(t,\alpha) \in [0,\infty) \times \Delta : \alpha \in S(t)\} \\ &= \{(t,\alpha) \in [0,\infty) \times \Delta : \tau(t,\alpha) = \alpha\} \\ &= \{(t,\alpha) \in [0,\infty) \times \Delta : \tau(t,\alpha) - \alpha = 0\}\end{aligned}$$

is the pre-image of the closed set $\{0\}$ under the continuous function $(t,\alpha) \mapsto \tau(t,\alpha) - \alpha$, hence closed. $\triangleleft$

4 Control costs

Rosenthal (1989, p. 292) states[1] — without proof — that "every t-solution is a special case of an equilibrium of a game with control costs", i.e., if a strategy profile is a t-solution,

[1] He credits the result to Eric van Damme. The result has, to my knowledge, corroborated by Eric van Damme and Jacob Goeree, not appeared in print.

it is also a Nash equilibrium of a game with suitably chosen control-cost function (see Van Damme, 1991, Ch. 4, for a detailed discussion of the control cost approach to equilibrium refinements). This section makes Rosenthal's statement more precise and also proves the converse: a strategy profile is a t-solution if and only if it is a Nash equilibrium of a game with quadratic control costs.

Recall that for $t = 0$, the unique t-solution is the strategy profile in which each player randomizes uniformly over his pure strategies. Next, fix $t \in (0, \infty)$. Suppose that in order to implement a mixed strategy $\alpha_i \in \Delta(A_i)$, player $i \in N$ incurs a control cost equal to

$$\frac{1}{2t} \sum_{a_i \in A_i} (\alpha_i(a_i) - 1/ \mid A_i \mid)^2.$$

This means that strategies further away in terms of Euclidean distance from the equiprobable mixture $(1/ \mid A_i \mid, \ldots, 1/ \mid A_i \mid)$ incur larger costs and the parameter t is a scaling parameter. This transforms the payoff function of player $i \in N$ to

$$\alpha \mapsto u_i(\alpha) - \frac{1}{2t} \sum_{a_i \in A_i} (\alpha_i(a_i) - 1/ \mid A_i \mid)^2. \tag{4}$$

Remark 4.1 Due to the concavity of this payoff function in player i's own strategy, a direct application of Kakutani's fixed point theorem to the best-response correspondences in the game with control costs gives that this game has a Nash equilibrium (Glicksberg, 1952). The following theorem shows that the nonempty set of Nash equilibria of the control-cost game coincides with the set of t-solutions $S(t, G)$, yielding a second proof of the existence of t-solutions (Prop. 2.2(a)). $\triangleleft$

Theorem 4.2 *Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a game and let $t \in (0, \infty)$. A strategy profile $\alpha^* \in \Delta$ is a t-solution of G if and only if it is a Nash equilibrium of the game with control costs with payoffs as in (4).*

Proof. By definition, $\alpha^* \in \Delta$ is a Nash equilibrium of the control cost game if and only if for every $i \in N$, $\alpha_i^* \in \Delta(A_i)$ is a best response to α_{-i}^*, i.e., α_i^* solves

$$[P] \qquad \begin{cases} \max & u_i(\alpha_i, \alpha_{-i}^*) - \frac{1}{2t} \sum_{a_i \in A_i} (\alpha_i(a_i) - 1/ \mid A_i \mid)^2 \\ & \\ \text{s.t.} & \forall a_i \in A_i : \alpha_i(a_i) \geq 0, \\ & \sum_{a_i \in A_i} \alpha_i(a_i) = 1. \end{cases}$$

This is a concave quadratic optimization problem with linear constraints, so the Karush-Kuhn-Tucker conditions are necessary and sufficient for a maximum: $\alpha_i^* \in \Delta(A_i)$ solves [P] if and only if there exist Lagrange multipliers $\lambda(a_i) \geq 0$ associated with the inequality constraints $\alpha_i(a_i) \geq 0$ and $\mu \in \mathbb{R}$ associated with the equality constraint $\sum_{a_i \in A_i} \alpha_i(a_i) = 1$ such that for each $a_i \in A_i$:

$$\begin{aligned} u_i(a_i, \alpha_{-i}^*) - \frac{1}{t}(\alpha_i^*(a_i) - 1/\mid A_i \mid) + \lambda(a_i) - \mu &= 0, & (5) \\ \lambda(a_i)\alpha_i^*(a_i) &= 0. & (6) \end{aligned}$$

Condition (5) is the first order condition obtained from differentiating the Lagrange function

$$\begin{aligned} (\alpha_i, (\lambda(a_i))_{a_i \in A_i}, \mu) \quad \mapsto \quad & u_i(\alpha_i, \alpha_{-i}^*) - \frac{1}{2t} \sum_{a_i \in A_i} (\alpha_i(a_i) - 1/\mid A_i \mid)^2 \\ + \quad & \sum_{a_i \in A_i} \lambda(a_i)\alpha_i(a_i) + \mu(1 - \sum_{a_i \in A_i} \alpha_i(a_i)) \end{aligned}$$

with respect to $\alpha_i(a_i)$ and condition (6) is the complementary slackness condition. So two things remain to be shown:

STEP 1: If for each $i \in N$ there exist Lagrange multipliers $\lambda(a_i) \geq 0$ for each $a_i \in A_i$ and $\mu \in \mathbb{R}$ such that (5) and (6) hold, then α^* is a t-solution.

Indeed, assume such Lagrange multipliers exist. Let $i \in N$ and $a_i, a_j \in A_i$. If $\alpha_i^*(a_i) > 0$, then $\lambda(a_i) = 0$ by (6). Equating the first order conditions in (5) for a_i and a_j and substituting $\lambda(a_i) = 0$, we find

$$\alpha_i^*(a_i) - \alpha_i^*(a_j) = t(u_i(a_i, \alpha_{-i}^*) - u_i(a_j, \alpha_{-i}^*)) - t\lambda(a_j) \leq t(u_i(a_i, \alpha_{-i}^*) - u_i(a_j, \alpha_{-i}^*)),$$

where the inequality follows from $t > 0$ and $\lambda(a_j) \geq 0$. By Definition 2.1, α^* is a t-solution.

STEP 2: If $\alpha^* \in \Delta$ is a t-solution, there exist, for each player $i \in N$, Lagrange multipliers $\lambda(a_i) \geq 0$ for each $a_i \in A_i$ and $\mu \in \mathbb{R}$ such that (5) and (6) hold.

Let $\alpha^* \in S(t, G)$ and $i \in N$. Choose $a_j \in A_i$ with $\alpha_i^*(a_j) > 0$ and define

$$\mu = u_i(a_j, \alpha_{-i}^*) - \frac{1}{t}(\alpha_i^*(a_j) - 1/\mid A_i \mid) \in \mathbb{R}. \tag{7}$$

Then μ is unambiguously defined: for all $a_i, a_j \in A_i$, if both $\alpha_i^*(a_i) > 0$ and $\alpha_i^*(a_j) > 0$, we know by definition of a t-solution that

$$\alpha_i^*(a_i) - \alpha_i^*(a_j) = t(u_i(a_i, \alpha_{-i}^*) - u_i(a_j, \alpha_{-i}^*)),$$

so

$$u_i(a_i, \alpha^*_{-i}) - \frac{1}{t}(\alpha^*_i(a_i) - 1/\mid A_i \mid) = u_i(a_j, \alpha^*_{-i}) - \frac{1}{t}(\alpha^*_i(a_j) - 1/\mid A_i \mid) = \mu. \tag{8}$$

For each $a_i \in A_i$, define

$$\lambda(a_i) = \begin{cases} 0 & \text{if } \alpha^*_i(a_i) > 0, \\ \frac{1}{t}(\alpha^*_i(a_i) - 1/\mid A_i \mid) - u_i(a_i, \alpha^*_{-i}) + \mu & \text{if } \alpha^*_i(a_i) = 0. \end{cases} \tag{9}$$

For each $a_i \in A_i$, the complementary slackness condition (6) follows from (9). Also the first order condition (5) is satisfied: simply discern between $a_i \in A_i$ with $\alpha^*_i(a_i) = 0$ and $\alpha^*_i(a_i) > 0$ and substitute (8) and (9). It remains to show that the Lagrange multiplier $\lambda(a_i)$ is nonnegative if $\alpha^*_i(a_i) = 0$. So assume $\alpha^*_i(a_i) = 0$. Choose $a_j \in A_i$ such that $\alpha^*_i(a_j) > 0$. By definition of a t-solution:

$$\alpha^*_i(a_j) - \alpha^*_i(a_i) \leq t(u_i(a_j, \alpha^*_{-i}) - u_i(a_i, \alpha^*_{-i})). \tag{10}$$

Substituting (7) in (9) and using (10) yields

$$\begin{aligned} \lambda(a_i) &= \frac{1}{t}(\alpha^*_i(a_i) - 1/\mid A_i \mid) - u_i(a_i, \alpha^*_{-i}) \\ &- \frac{1}{t}(\alpha^*_i(a_j) - 1/\mid A_i \mid) + u_i(a_j, \alpha^*_{-i}) \\ &= (u_i(a_j, \alpha^*_{-i}) - u_i(a_i, \alpha^*_{-i})) - \frac{1}{t}(\alpha^*_i(a_j) - \alpha^*_i(a_i)) \\ &\geq 0. \end{aligned}$$

This proves the existence of the desired Lagrange multipliers. □

Theorem 4.2 indicates a close relation with Mattsson and Weibull (2002) who prove that logit QRE coincide with the Nash equilibria of a game with control costs of the relative-entropy form.

5 Towards rational behavior

The parameter t essentially measures rationality: at $t = 0$, players disregard payoffs and choose by uniformly randomizing over their pure strategies, while for $t \to \infty$ the solutions converge to Nash equilibrium (Prop. 2.2(c)). We show in Theorem 5.1 that — along the way — players become increasingly capable of iteratively eliminating never-best replies (in particular strictly dominated actions) and that player eventually only choose rationalizable actions with positive probability. This distinguishes the QRE based on the linear probability model from the logit QRE: probabilities in the logit QRE are strictly positive, since the exponential function takes values in $(0, \infty)$. Hence the logit QRE cannot properly explain behavior where certain actions are played with zero probability, except in the limiting case of full rationality, where the logit QRE select Nash equilibria (McKelvey and Palfrey, 1995, Thm. 2).

Subsection 5.2 shows that there is a continuous link of t-solutions connecting low-rationality behavior starting with uniform randomization to Nash equilibrium behavior as t approaches infinity.

5.1. Iterated elimination and rationalizability

Recall that an action $a_j \in A_i$ of player $i \in N$ is a **never-best reply** if, regardless of the behavior of the remaining players, there is always an action of i that is better than a_j: $\max_{b_i \in A_i} u_i(b_i, \alpha_{-i}) - u_i(a_j, \alpha_{-i}) > 0$ for all $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(A_j)$. The next result indicates that for sufficiently large values of t, never-best replies are played with zero probability in a t-solution. Extending this result implies that for ever larger values of t, the players are capable of successively eliminating higher levels of never-best replies and that eventually the only actions that are chosen with positive probability in a t-solution must be rationalizable:

Theorem 5.1 *Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a game.*

(a) Let $i \in N$ and $a_j \in A_i$. If a_j is a never-best reply, then there is a $T \in (0, \infty)$ such that for all $t \geq T$ and all $\alpha \in S(t)$, $\alpha_i(a_j) = 0$.

(b) Inductively, let $G^1 = G$ and for all $n \in \mathbb{N}$, let G^{n+1} be the game obtained from G^n by deleting its never-best replies. There is an increasing sequence $(T_n)_{n \in \mathbb{N}}$ in $(0, \infty)$ such that for all $n \in \mathbb{N}$ and all $t \geq T_n$: if player i's action a_j is a never-best reply in G^n, then for all $\alpha \in S(t), \alpha_i(a_j) = 0$.

(c) *For $t \in (0,\infty)$ sufficiently large, all actions chosen with positive probability in a t-solution are rationalizable.*

Proof. *(a):* If a_j is a never-best reply of $i \in N$, it follows by continuity of the function $\alpha_{-i} \mapsto \max_{b_i \in A_i} u_i(b_i, \alpha_{-i}) - u_i(a_j, \alpha_{-i})$ and compactness of $\times_{j \in N \setminus \{i\}} \Delta(A_j)$ that

$$\varepsilon(i, a_j) := \min_{\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(A_j)} \left(\max_{b_i \in A_i} u_i(b_i, \alpha_{-i}) - u_i(a_j, \alpha_{-i}) \right) > 0 \tag{11}$$

and $T = 2/\varepsilon(i, a_j) > 0$ are well-defined. Then for all $t \geq T$ and all $\alpha \in S(t), \alpha_i(a_j) = 0$. Suppose, to the contrary, that for some $t \geq T$ and $\alpha \in S(t), \alpha_i(a_j) > 0$. Let $a_i \in A_i$ be a best reply to α_{-i}. Since a_j is a never-best reply and choice probabilities in a t-solution are weakly increasing in expected payoffs:

$$u_i(a_i, \alpha_{-i}) - u_i(a_j, \alpha_{-i}) = \max_{b_i \in A_i} u_i(b_i, \alpha_{-i}) - u_i(a_j, \alpha_{-i}) \geq \varepsilon(i, a_j) \text{ and } \alpha_i(a_i) > 0.$$

By definition of a t-solution:

$$\begin{aligned} \alpha_i(a_i) - \alpha_i(a_j) &= t(u_i(a_i, \alpha_{-i}) - u_i(a_j, \alpha_{-i})) \\ &\geq t\varepsilon(i, a_j) \\ &\geq T\varepsilon(i, a_j) \\ &= 2, \end{aligned} \tag{12}$$

which is impossible, since $\alpha_i(a_i)$ and $\alpha_i(a_j)$ are probabilities.

(b): We prove that for $t \in (0,\infty)$ sufficiently large, the set of t-solutions of $G = G^1$ coincides — up to zero probability assigned to omitted actions — with the set of t-solutions of the game G^2 obtained from $G = G^1$ by eliminating its never-best replies. Since the game G is arbitrary, repeated application of this result to $G^2, G^3, \ldots$ implies that consecutively increasing the parameter t makes the players capable of more and more steps of iterative elimination of never-best replies, proving *(b)*.

To avoid trivialities, assume some player has a never-best reply. For each $i \in N$ and each never-best reply $a_j \in A_i$, define $\varepsilon(i, a_j)$ as in (11). Let $\varepsilon = \min\{\varepsilon(i, a_j)\} > 0$ be the minimum of these numbers and $T = 2/\varepsilon > 0$.

Let $t \geq T$ and $\alpha \in S(t, G^1)$. Since never-best replies in α are chosen with probability zero by *(a)*, it follows that α (after omitting zero coordinates associated with never-best replies) also is an element of $S(t, G^2)$. The converse is a bit harder. Let $\alpha \in S(t, G^2)$. With a minor abuse of notation, we consider α to be a mixed strategy in the original game $G = G^1$ by assigning probability zero to the remaining actions. We prove that $\alpha \in S(t, G^1)$. It suffices to show for arbitrary $i \in N, a_i \in A_i$, and never-best reply $a_j \in A_i$: if $\alpha_i(a_i) > 0$, then

$$\alpha_i(a_i) = \alpha_i(a_i) - \alpha_i(a_j) \leq t(u_i(a_i, \alpha_{-i}) - u_i(a_j, \alpha_{-i})). \tag{13}$$

The remaining conditions for a t-solution in G^1 are automatically satisfied, since $\alpha \in S(t, G^2)$ and never-best replies in G^1 are assigned probability zero.

First, assume that a_i is a best reply to α_{-i}. Then $\alpha_i(a_i) > 0$ and, similar to (12):

$$t(u_i(a_i, \alpha_{-i}) - u_i(a_j, \alpha_{-i})) \geq 2. \tag{14}$$

Since $\alpha_i(a_i) = \alpha_i(a_i) - \alpha_i(a_j) \in (0, 1]$, this proves (13). Finally, consider any other $b_i \in A_i$ with $\alpha_i(b_i) > 0$. By definition of a t-solution:

$$\alpha_i(b_i) - \alpha_i(a_i) = t(u_i(b_i, \alpha_{-i}) - u_i(a_i, \alpha_{-i})) \tag{15}$$

To see that

$$\alpha_i(b_i) = \alpha_i(b_i) - \alpha_i(a_j) \leq t(u_i(b_i, \alpha_{-i}) - u_i(a_j, \alpha_{-i})),$$

notice that

$$\begin{aligned} t(u_i(b_i, \alpha_{-i}) - u_i(a_j, \alpha_{-i})) &= t(u_i(a_i, \alpha_{-i}) - u_i(a_j, \alpha_{-i})) \\ &+ t(u_i(b_i, \alpha_{-i}) - u_i(a_i, \alpha_{-i})) \\ &\geq 2 + \alpha_i(b_i) - \alpha_i(a_i) \\ &\geq \alpha_i(b_i), \end{aligned}$$

where the first inequality follows from (14) and (15).

(c): Iterated elimination of never-best replies requires at most $\sum_{i \in N}(|A_i| - 1)$ rounds, so, by *(b)*, for sufficiently large t only actions that survive this process are played with positive probability in a t-solution. By Bernheim (1984, p. 1025), these are exactly the rationalizable ones. $\square$

Not *all* rationalizable actions are necessarily played with positive probability in t-solutions for large values of t. Possibly the simplest example is the 2×2 bimatrix game below.

	L	R
T	$1, 1$	$0, 0$
B	$0, 0$	$0, 0$

Since T and L are weakly dominant strategies, it follows that $\alpha_1(T) \geq \alpha_1(B)$ and $\alpha_2(L) \geq \alpha_2(R)$ for all t-solutions $\alpha \in \Delta$. Indeed, simple calculations indicate that $S(t)$ consists of the single vector $\alpha \in \Delta$ where $\alpha_1(T) = \alpha_2(L) = 1/(2-t)$ if $t \in [0, 1)$ and $\alpha_1(T) = \alpha_2(L) = 1$ otherwise. Conclude that for large values of t, only T and L have positive support, although both B and R are rationalizable: (B, R) is even a Nash equilibrium.

A stronger statement[2] than Theorem 5.1, namely that only actions with positive support in a Nash equilibrium will be part of a t-solution for large values of t, is not correct. Consider the 2×2 bimatrix game below.

	L	R
T	$1, 0$	$0, 0$
B	$0, 1$	$1, 0$

Its set of Nash equilibria equals $\{\alpha \in \Delta \mid \alpha_1(T) = 1, \alpha_2(L) \in [1/2, 1]\}$ and for every $t \in [0, \infty)$, its set of t-solutions equals

$$\{\alpha \in \Delta \mid \alpha_1(T) = (2 + 2t^2)/(4 + 2t^2), \alpha_2(L) = (t^2 + t + 2)/(4 + 2t^2)\},$$

so B is chosen with positive probability in all t-solutions, while it is chosen with probability zero in the Nash equilibria of the game. Moreover, this example (as well as the game of Rosenthal, 1989, p. 285) shows that a weakly dominated strategy like R can remain in the support of t-solutions. Consequently, there is no obvious relation between t-solutions or Nash equilibria that are limits of t-solutions on the one hand and refinements of the Nash equilibrium concept like perfect and proper equilibria on the other hand.

5.2. A path to Nash equilibrium

The purpose of this subsection is to show that the graph of the t-solution correspondence contains a path linking uniform randomization at $t = 0$ to Nash equilib-

[2] I am grateful to Andrés Perea for raising this issue.

rium behavior in the limit as $t \to \infty$. To make this statement precise, fix a game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ and parametrize the t-variable as follows. The function

$$f : [0,1) \to [0,\infty) \text{ with } f(\lambda) = \lambda/(1-\lambda) \text{ for each } \lambda \in [0,1) \tag{16}$$

is a strictly increasing bijection: instead of letting t range from 0 to infinity, we can let λ range from 0 to 1. By Proposition 2.2(c), the t-solutions approximate a subset of the Nash equilibria of G as $t \to \infty$, or, equivalently, as $\lambda \to 1$. So define the correspondence $\widetilde{S} : [0,1] \twoheadrightarrow \Delta$ by

$$\widetilde{S}(\lambda) = \begin{cases} S(\frac{\lambda}{1-\lambda}) & \text{if } \lambda \in [0,1), \\ NE(G) & \text{if } \lambda = 1. \end{cases} \tag{17}$$

We prove:

Theorem 5.2 *For every game G, define $\widetilde{S}$ as in (17). There is a continuous function $\gamma : [0,1] \to \text{graph}(\widetilde{S})$, i.e., a path, such that $\gamma(0) \in \text{graph}(\widetilde{S}) \cap (\{0\} \times \Delta)$ and $\gamma(1) \in \text{graph}(\widetilde{S}) \cap (\{1\} \times \Delta)$.*

By definition of $\widetilde{S}$,

$$\text{graph}(\widetilde{S}) \cap (\{0\} \times \Delta) = \{0\} \times S(0)$$

consists of the single vector $(0, \epsilon)$, where $\epsilon \in \Delta$ is the strategy profile in which each player randomizes uniformly over all pure strategies and

$$\text{graph}(\widetilde{S}) \cap (\{1\} \times \Delta) = \{1\} \times NE(G).$$

Hence Theorem 5.2 indeed gives us the desired path of t-solutions linking uniform randomization at $t = 0$ to Nash equilibrium behavior. The proof is inspired by Herings' (2000) proof of the feasibility of the linear tracing procedure and uses results from real algebraic geometry which can be found in clear presentations of Blume and Zame (1994) and Schanuel, Simon, and Zame (1991). Notice that

$$\begin{aligned}
\text{graph}(\widetilde{S}) &= \{(\lambda,\alpha)\in[0,1]\times\Delta \mid \alpha\in\widetilde{S}(\lambda)\} \\
&= \{(\lambda,\alpha)\in[0,1)\times\Delta \mid \alpha\in S(\lambda/(1-\lambda))\} \qquad (18) \\
&\cup \; (\{1\}\times NE(G)) \\
&= \{(\lambda,\alpha)\in[0,1)\times\Delta \mid \forall i\in N, \forall a_i,a_j\in A_i : \alpha_i(a_i)>0 \Rightarrow \qquad (19) \\
&\quad (1-\lambda)(\alpha_i(a_i)-\alpha_i(a_j)) \le \lambda(u_i(a_i,\alpha_{-i})-u_i(a_j,\alpha_{-i}))\} \\
&\cup \; (\{1\}\times NE(G)).
\end{aligned}$$

The set of Nash equilibria of G is semi-algebraic (Blume and Zame, 1994, p. 789). Rewriting the condition $(\lambda,\alpha)\in[0,1)\times\Delta$ in terms of (linear) equalities and inequalities and combining this with (19), it follows that the set in (18) can be defined in terms of a first-order formula over the real variables λ and $(\alpha_i(a_i))_{i\in N, a_i\in A_i}$. By the Tarski-Seidenberg theorem (cf. Blume and Zame, 1994, p. 787), it is semi-algebraic. As the union of two semi-algebraic sets, graph($\widetilde{S}$) is semi-algebraic: it can be described by a finite number of polynomial (in)equalities and all its components, i.e., maximally connected subsets, are path-connected (cf. Schanuel, Simon, and Zame, 1991). Hence, it suffices to show that graph($\widetilde{S}$) contains a component that intersects both $\{0\}\times\Delta$ and $\{1\}\times\Delta$. We use a special case of a result by Mas-Colell (1974); see Herings (2000, Thm. 3.2):

Theorem 5.3 *Let S be a nonempty, convex, compact subset of $\mathbb{R}^m$ and let $\varphi:[0,1]\times S \twoheadrightarrow S$ be an upper semicontinuous correspondence. Then the set $F=\{(\lambda,x)\in[0,1]\times S \mid x\in\varphi(\lambda,x)\}$ contains a connected set F^c such that $(\{0\}\times S)\cap F^c\neq\emptyset$ and $(\{1\}\times S)\cap F^c\neq\emptyset$.*

Define the correspondence $\varphi:[0,1]\times\Delta\twoheadrightarrow\Delta$ for each $(\lambda,\alpha')\in[0,1]\times\Delta$ by

$$\varphi(\lambda,\alpha')=\widetilde{S}(\lambda).$$

Its value is independent of α'. The correspondence φ is upper semicontinuous, since its graph

$$\begin{aligned}
\text{graph}(\varphi) &= \{(\lambda,\alpha',\alpha)\in[0,1)\times\Delta\times\Delta \mid \alpha\in S(\lambda/(1-\lambda))\} \\
&\cup \; (\{1\}\times\Delta\times NE(G))
\end{aligned}$$

is closed: consider a sequence $(\lambda^n, \alpha'^n, \alpha^n)$ in graph(φ) converging to $(\lambda, \alpha', \alpha) \in [0,1] \times \Delta \times \Delta$. If $\lambda = 1$, then either $\lambda^n = 1$ for all sufficiently large n or there is a subsequence of $(\lambda^n)_{n\in\mathbb{N}}$ with $\lambda^n \neq 1$. In the first case, a tail of the sequence $(\lambda^n, \alpha'^n, \alpha^n)$ lies in the closed set $\{1\} \times \Delta \times NE(G)$, so

$$(\lambda, \alpha', \alpha) \in \{1\} \times \Delta \times NE(G) \subset \text{graph}(\varphi). \tag{20}$$

In the second case, the subsequence converges to one, so $\lambda^n/(1-\lambda^n) \to \infty$. With Proposition 2.2(c), this implies that $\alpha^n \in S(\lambda^n/(1-\lambda^n))$ converges to a Nash equilibrium, so that again (20) holds. Similarly, if $\lambda \in [0,1)$, continuity of the function f in (16) and S having a closed graph (Proposition 2.2(b)) implies

$$(\lambda, \alpha', \alpha) \in \{\lambda\} \times \Delta \times S(\lambda/(1-\lambda)) \subset \text{graph}(\varphi).$$

By Theorem 5.3, the set

$$\begin{aligned} F &= \{(\lambda, \alpha) \in [0,1] \times \Delta \mid \alpha \in \varphi(\lambda, \alpha)\} \\ &= \{(\lambda, \alpha) \in [0,1] \times \Delta \mid \alpha \in \widetilde{S}(\lambda)\} \\ &= \text{graph}(\widetilde{S}) \end{aligned}$$

contains a connected component F^c intersecting both $\{0\} \times \Delta$ and $\{1\} \times \Delta$, finishing the proof of Theorem 5.2.

6 Concluding remarks

In this paper we investigated properties of the QRE based on the linear probability model: the t-solutions of Rosenthal (1989). There are several directions for further research. We briefly mention two.

The present paper was of a theoretical nature; a more experimentally oriented paper could investigate — for instance — the empirical content of Theorem 5.1: in contrast with logit QRE, the t-solutions exhibit higher levels of iterated elimination of never-best replies and eventually rationalizable behavior due to increases in the model's parameter. This may imply better predictive power than logit QRE in certain dominance solvable games or other games where actions are chosen with small or zero probability. More generally, it would be interesting to compute the t-solution correspondence for different economically relevant games. The examples of Rosenthal (1989) and those in Section 5 involve only small games; already in these games, the computations can become tedious. The reader is referred to Rosenthal (1989, Remark 6.1, p. 290 - 291) for additional remarks on computational aspects.

Section 5.2 proves existence of a path of t-solutions linking uniform randomization to Nash equilibrium in all games. An interesting open question is whether, in correspondence with a result of McKelvey and Palfrey (1995) for logit QRE, such paths select a unique equilibrium in generic games. This is a challenging problem: the graph of the t-solution correspondence is typically not a manifold (see the examples of Rosenthal, 1989), so usual techniques from differential topology cannot be used. The next step would be to use real algebraic geometry, but so far, the problem defies my attacks.

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