A Note on the Pricing of Real Estate Index Linked Swaps

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Abstract

In this paper we discuss the pricing of commercial real estate index linked swaps (CREILS). This particular pricing problem has been studied by Buttimer *et al.* (1997) in a previous paper. We show that their results are only approximately correct and that the true theoretical price of the swap is in fact equal to zero. This result is shown to hold regardless of the specific model chosen for the index process, the dividend process, and the interest rate term structure. We provide an intuitive economic argument as well as a full mathematical proof of our result. In particular we show that the nonzero result in the previous paper is due to two specific numerical approximations introduced in that paper, and we discuss these approximation errors from a theoretical as well as from a numerical point of view.

Key words: Real estate; index linked swaps; arbitrage
JEL Classification Numbers: G13, G20; L85.

1 Introduction

The object under study in the present paper is a commercial real estate index linked swap (CREILS). The basic construction of such a swap is that the appreciation and yield of a given real estate index is swapped, quarterly, against the three months spot LIBOR. In an interesting paper previously published in this journal, Buttimer *et al.* (1997) presented a two-state model for pricing securities dependent upon a real estate index as well as upon an interest rate, and the model was then used to calculate the arbitrage free value of a CREILS. For this concrete application, the authors in Buttimer *et al.* (1997) used a numerical method based upon replacing their original continuous time model by a bivariate binomial tree, and it was found that, for a notional amount of $10,000,000, the value (to the receiver of the swap) was around $50. Buttimer *et al.* (1997) then proceed to discuss the sensitivity of their numerical results to changes in volatilities, correlations and the initial term structure.

The object of the present paper is to show that the results in Buttimer *et al.* (1997) are only approximatively true, in the sense that the arbitrage free theoretical value of the CREILS is in fact exactly equal to zero. More precisely we carry out the following program.

- In Section 2 we present the institutional setup of the swap.
- We begin the theoretical analysis in Section 3 where we give a simple verbal arbitrage argument showing that the theoretical value of the swap in fact equals zero.
- In Section 4 we add to the verbal discussion in the previous section by presenting a very general mathematical framework for the swap along the following lines.
  - The real estate index is allowed to be a general (semimartingale) process with the only requirement that it should be possible to view it as the price of a traded asset.
  - The income (dividend) process associated to the index is allowed to be completely general.
  - The interest rate model is allowed to be completely general.
  - We assume absence of arbitrage.

  This framework is considerably more general than that of Buttimer *et al.* (1997) where the index is assumed to be lognormal with a constant dividend yield, and where the interest rate structure is given by a CIR short rate model.
- Within the above framework, and using the standard (martingale) machinery of arbitrage theory, we prove formally that the arbitrage free value of the swap is exactly equal to zero.

- In Section 5 we discuss why the pricing results in Buttimer *et al.* (1997) differ (although not much) from the correct value zero. We show that the reasons for the nonzero computational results in Buttimer *et al.* (1997) are due to two specific approximation errors introduced in the numerical calculations. We discuss these errors from a theoretical as well as numerical a perspective.

2 Institutional setup

In this section we give a description of the institutional setup of a commercial real estate index linked swap (CREILS). We follow Buttimer *et al.* (1997).

- The swap is assumed to be active over a prespecified time period. This period is subdivided by equidistant time points $t_0 < t_1 < \ldots < t_n$, and we denote by Δ the length of an elementary time interval, i.e. $\Delta = t_{k+1} - t_k$. In a typical example the length $t_n - t_0$ of the total time period could be five years, whereas the length Δ of the elementary time interval would be three months.
- One leg of the CREILS is based upon a real estate index, henceforth denoted by I_t. This index varies stochastically over time and it also carries with it a (possibly stochastic) income (dividend).
- The other leg of the CREILS is based upon a market rate, such as the spot LIBOR rate, over the elementary time intervals.
- The CREILS is a sum of individual "swaplets", where the individual swaplet is active over an elementary interval $[t_{k-1}, t_k]$.
- At the end of each elementary interval $[t_{k-1}, t_k]$ the CREILS **receiver** will have the following cash flows from the swaplet active over the interval:
  - A cash inflow consisting of appreciation of the index plus all income generated by the index over the interval $[t_{k-1}, t_k]$.
  - A cash outflow equal to the spot LIBOR, plus a given spread δ, for the period $[t_{k-1}, t_k]$, acting on the ingoing index $I_{t_{k-1}}$.
- The CREILS **payer** will have the same cash flows with opposite signs.
- The CREILS would in real life be operating on a *notional amount*, rather than directly on the index value. This however is only a scaling factor, and without loss of generality we disregard this (or rather set it equal to one).

We assume that we are standing at time t, and that $t \leq t_0$. Our problem is to find the arbitrage free value, at time t, of the CREILS.

A typical (see Buttimer *et al.*, 1997) value of the spread δ could be $\delta = 0.00125\%$. For the rest of the paper we will follow Buttimer *et al.* (1997) in assuming that there is no spread, i.e. we assume that $\delta = 0$.

The main object of the present paper is to show that, regardless of any specific assumptions about the dynamics of the index, the income process, or the interest rate model, the arbitrage free value of the CREILS is in fact equal to zero.

For this strong result to hold, we will however need the following important assumption.

Assumption 2.1 *We assume that the real estate index process I_t can be treated as the price process of a traded asset with a certain associated dividend (income) process.*

The practical relevance of this assumption can of course be questioned. The assumption is however in complete agreement with Buttimer *et al.* (1997), who in fact assume Geometrical Brownian Motion for the index, and model the income process as a constant dividend yield.

3 Verbal discussion

We begin our analysis by giving an simple verbal arbitrage argument, which shows that the value at an arbitrary time $t \leq t_0$ of the CREILS equals zero. Let us thus consider a trading strategy starting at time t and ending at t_n. The strategy consists of the following simple scheme which is repeated at each elementary time period $[t_{k-1}, t_k]$, for $k = 1, \ldots, n$.

- At time t_{k-1}, borrow the sum $I_{t_{k-1}}$ over the period $[t_{k-1}, t_k]$ at the spot LIBOR $L = L(t_{k-1}, t_k)$. Use all the borrowed money to buy one unit of the index.
- All the income generated by the index holdings during the interval $[t_{k-1}, t_k]$ is invested in the bank.
- At t_k sell the index to obtain I_{t_k}. Repay the loan, i.e. the principal $I_{t_{k-1}}$ plus the accrued interest $\Delta \cdot L \cdot I_{t_{k-1}}$ where $L = L(t_{k-1}, t_k)$ is the spot LIBOR for $[t_{k-1}, t_k]$. Collect the income that was generated and invested during the elementary period.

The net result of this strategy is that we obtain the following cash flow at each t_k for $k = 1, \ldots, n$.

- Plus: I_{t_k} (selling the index).
- Minus: $I_{t_{k-1}}$ (repayment of the principal of the loan).
- Plus: the value, at t_k, of the invested income during the period $[t_{k-1}, t_k]$.
- Minus: LIBOR on the borrowed capital during the period, i.e. $\Delta \cdot L \cdot I_{t_{k-1}}$.

We have thus exactly replicated the cash flow of the receiver of a CREILS. Since the strategy is self financing and the initial cost of setting up the strategy is zero, the arbitrage free value of the strategy, and hence that of the CREILS, has to equal zero.

4 Formal analysis

In this section we present a formal mathematical proof of our claim that the arbitrage free price of the CREILS is zero. The reasons for including this "extra" proof are as follows.

- It highlights the logical structure of the argument and shows more precisely where the various assumptions are needed.
- By presenting a formalized argument we can more easily compare our calculations to the computations made in Buttimer *et al.* (1997). In particular we will see that certain approximation errors are in fact introduced into the computations in Buttimer *et al.* (1997) and we will be able to study the relative importance of these numerical errors.

4.1 The mathematical model

Our chosen framework is a very general one (see Bjrk *et al.*, 1999; Harrison *et al.*, 1981; Musiela *et al.*, 1997). We consider a financial market living on a stochastic basis (filtered probability space) $(\Omega, \mathcal{F}, \mathbf{F}, P)$ where $\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}$. Here the measure P is interpreted as the objective probability measure, whereas the σ-algebra $\mathcal{F}_t$ formalizes the idea of the information available to the agents in the economy at time t. We assume that the basis carries the following basic financial objects:

- An index process I_t. As a notational convention we consider the index process *ex dividend*.
- A cumulative dividend (income) process D_t. The interpretation is that if you hold the index over an infinitesimal interval $(t, t + dt]$ then you will receive the amount $dD_t = D_{t+dt} - D_t$ in dividend payments. Put in other words; over the interval $(s, t]$ the holder of the index will receive the (undiscounted) amount $D_t - D_s$.
- A short rate process r_t.
- A liquid bond market (at any time) for bonds of all possible maturities. The market price at time t for a zero coupon bond maturing at T is denoted by $p(t, T)$.
- A money market account process denoted by B_t, where by definition

$$dB_t = r_t B_t dt.$$

Note that we make no assumptions whatsoever about any specific dynamical structure of the index, the short rate, or the dividend process. For example; we do not assume that the processes above are driven by Wiener processes or that they are Markov processes (they are allowed to be arbitrary semimartingales). Our setup is thus extremely general and in particular it includes the model considered in Buttimer *et al.* (1997). In that paper, I_t is assumed to be Geometrical Brownian Motion, the dividend process is assumed to be of the form

$$dD_t = \eta I_t dt,$$

(i.e. a constant dividend yield η), and the short rate process is assumed to be of Cox-Ingersoll-Ross type. The one assumption we make is that the market is free of arbitrage possibilities in the sense that there exists an equivalent martingale measure $Q \sim P$. We recall (see [1]) the following standard properties of Q.

- The normalized gains process G_t^B, defined by

$$G_t^B = \frac{I_t}{B_t} + \int_0^t \frac{1}{B_s} dD_s. \tag{1}$$

  is a Q-martingale.

- Bond prices are given by the expression

$$p(t,T) = E^Q \left[e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right]. \tag{2}$$

- For any contingent claim X, payed out at time T, the corresponding arbitrage free price process $\Pi(t;X)$ is given by the "risk neutral valuation formula"

$$\Pi(t;X) = E^Q \left[e^{-\int_t^T r_s ds} \cdot X \middle| \mathcal{F}_t \right]. \tag{3}$$

4.2 Pricing

We now proceed to price the CREILS within the above framework, and by convention this is done from the point of view of the receiver. Denoting the arbitrage free value (always for the receiver) at time t of the CREILS by $\Pi(t;CREILS)$ we have

$$\Pi(t;CREILS) = \sum_{k=1}^{n} \Pi(t;X_k)$$

where X_k denotes the net payments, at time t_k, to the CREILS receiver. We now go on to compute $\Pi(t;X_k)$ and by the "risk neutral valuation formula" (3) we have

$$\Pi(t;X_k) = E^Q \left[e^{-\int_t^{t_k} r_s ds} \cdot X_k \middle| \mathcal{F}_t \right].$$

Now, from the definition of the CREILS, it follows that X_k is given by

$$X_k = I_{t_k} - I_{t_{k-1}} + \int_{t_{k-1}}^{t_k} e^{\int_s^{t_k} r_u du} dD_s - \Delta L(t_{k-1}, t_k) I_{t_{k-1}}. \tag{4}$$

In this expression, the first term $I_{t_k} - I_{t_{k-1}}$ equals the appreciation of the index.

The integral term represents the total value, at time t_k, of all dividends generated by the index during the interval $(t_{k-1}, t_k]$. By convention all dividends are being invested in the bank account until time t_k, so the dividend dD_s generated during the small time interval $(s, s + ds]$ will, at time t_k have grown to

$$e^{\int_s^{t_k} r_u du} dD_s.$$

The integral term is thus the total value at time t_k of the entire income stream generated during the interval.

The third term obviously represents the cash outflow, which by definition is the spot LIBOR operating on $I_{t_{k-1}}$.

Remark 4.1 Note the reinvestment of the dividends into the bank account. This is of course an institutional assumption, and to a certain extent it is crucial to our results below. The exact logical situation is as follows.

- For our results below to hold it is essential that dividends either are payed out directly at the time they are generated by the index, (i.e. the CREILS receiver obtains the amount dD_s at time s) or are being reinvested in a traded asset and payed out at time t_k.
- Exactly **which** asset that is used for reinvesting the dividends is, from the point of view of our calculations, irrelevant. We have by convention chosen the bank account, but the dividends could in fact be invested in any traded asset (or reinvested in the index) without affecting our results.

We now go on to compute $\Pi(t; X_k)$ and to this end we recall that the spot LIBOR $L(t_{k-1}, t_k)$ for the period $[t_{k-1}, t_k]$ is given by the relation

$$p(t_{k-1}, t_k) = \frac{1}{1 + \Delta L(t_{k-1}, t_k)},$$

so in particular we have

$$\Delta L(t_{k-1}, t_k) = \frac{1}{p(t_{k-1}, t_k)} - 1.$$

We can thus write X_k as

$$\begin{aligned} X_k &= I_{t_k} - I_{t_{k-1}} + \int_{t_{k-1}}^{t_k} e^{\int_s^{t_k} r_u du} dD_s - \left(\frac{1}{p(t_{k-1}, t_k)} - 1 \right) I_{t_{k-1}} \\ &= I_{t_k} - \frac{I_{t_{k-1}}}{p(t_{k-1}, t_k)} + \int_{t_{k-1}}^{t_k} e^{\int_s^{t_k} r_u du} dD_s, \end{aligned}$$

and we obtain

$$\Pi(t; X_k) = E^Q \left[e^{-\int_t^{t_k} r_s ds} I_{t_k} \middle| \mathcal{F}_t \right]$$

$$
\begin{aligned}
&- \quad E^Q\left[e^{-\int_t^{t_k} r_s ds} \frac{I_{t_{k-1}}}{p(t_{k-1}, t_k)} \middle| \mathcal{F}_t \right] \\
&+ \quad E^Q\left[e^{-\int_t^{t_k} r_s ds} \int_{t_{k-1}}^{t_k} e^{\int_s^{t_k} r_u du} dD_s \middle| \mathcal{F}_t \right].
\end{aligned}
$$

In this expression we can, by iterated conditional expectation, write the second term as

$$
E^Q\left[e^{-\int_t^{t_k} r_s ds} \frac{I_{t_{k-1}}}{p(t_{k-1}, t_k)} \middle| \mathcal{F}_t \right] = E^Q\left[E^Q\left[e^{-\int_t^{t_k} r_s ds} \frac{I_{t_{k-1}}}{p(t_{k-1}, t_k)} \middle| \mathcal{F}_{t_{k-1}} \right] \middle| \mathcal{F}_t \right]
$$

The inner expectation can now be simplified as

$$
\begin{aligned}
&E^Q\left[e^{-\int_t^{t_k} r_s ds} \frac{I_{t_{k-1}}}{p(t_{k-1}, t_k)} \middle| \mathcal{F}_{t_{k-1}} \right] = e^{-\int_t^{t_{k-1}} r_s ds} \frac{I_{t_{k-1}}}{p(t_{k-1}, t_k)} E^Q\left[e^{-\int_{t_{k-1}}^{t_k} r_s ds} \middle| \mathcal{F}_{t_{k-1}} \right] \\
&= e^{-\int_t^{t_{k-1}} r_s ds} \frac{I_{t_{k-1}}}{p(t_{k-1}, t_k)} \cdot p(t_{k-1}, t_k) = e^{-\int_t^{t_{k-1}} r_s ds} I_{t_{k-1}},
\end{aligned}
$$

where we have used (2) together with the fact that the objects

$$
I_{t_{k-1}}, \quad p(t_{k-1}, t_k), \quad e^{-\int_t^{t_{k-1}}}
$$

are known at t_{k-1} and can thus be brought outside the conditional expectation.

The third term in the expression for $\Pi\left(t; X_k\right)$ can be written

$$
E^Q\left[e^{-\int_t^{t_k} r_s ds} \int_{t_{k-1}}^{t_k} e^{\int_s^{t_k} r_u du} dD_s \middle| \mathcal{F}_t \right] = E^Q\left[\int_{t_{k-1}}^{t_k} e^{-\int_t^{s} r_u du} dD_s \middle| \mathcal{F}_t \right].
$$

Collecting our results we finally obtain

$$
\begin{aligned}
\Pi\left(t; X_k\right) &= E^Q\left[e^{-\int_t^{t_k} r_s ds} I_{t_k} \middle| \mathcal{F}_t \right] - E^Q\left[e^{-\int_t^{t_{k-1}} r_s ds} I_{t_{k-1}} \middle| \mathcal{F}_t \right] \\
&+ E^Q\left[\int_{t_{k-1}}^{t_k} e^{-\int_t^{s} r_u du} dD_s \middle| \mathcal{F}_t \right] \\
&= B_t \cdot E^Q\left[\frac{I_{t_k}}{B_{t_k}} - \frac{I_{t_{k-1}}}{B_{t_{k-1}}} + \int_{t_{k-1}}^{t_k} \frac{1}{B_s} dD_s \middle| \mathcal{F}_t \right] \\
&= B_t \cdot E^Q\left[G_{t_k}^B - G_{t_{k-1}}^B \middle| \mathcal{F}_t \right] = 0.
\end{aligned}
$$

In the last equality we have used the fact that the normalized gains process G_t^B defined in (1) is a martingale under Q.

Since this holds for every k and the receiver value of the CREILS is given by

$$
\Pi\left(t; CREILS\right) = \sum_{k=1}^{n} \Pi\left(t; X_k\right),
$$

we have thus formally proved the following proposition, which is the main result of this paper.

Proposition 4.1 *At any time $t \leq t_0$ the arbitrage free price of the CREILS is zero, i.e.*

$$\Pi\left(t; CREILS\right) = 0.$$

Remark 4.2 In this section we have formally worked within a continuous time model. We note, however, that the results above are true also in a discrete time framework and that, in particular, Proposition 4.1 remains unchanged. The proofs remain the same, the difference simply being that all integrals are interpreted as sums.

5 The effects of approximation errors

As we have seen above, the theoretical value of the swap equals zero regardless of the model under consideration. In Buttimer *et al.* (1997), however, the authors computed a numerical (receiver) value of $50 on a notional amount of $10,000,000 and even if, in relative terms, this is very close to the true value it may still be interesting to see exactly where the numerical errors were introduced in Buttimer *et al.* (1997).

A closer look at Buttimer *et al.* (1997) reveals that, compared to their original model, the following three approximations were made by the authors:

- The original continuous time model was approximated by a discrete time tree model.
- The dividend process was approximated.
- There was an approximation made in the computation of the bond prices and hence of the LIBOR rates.

It is clear that numerical computations will in general produce approximation errors that cause estimated and theoretical values to diverge. However, as explained below the discrete time method employed by Buttimer *et al.* (1997) should still produce the zero result. The reason is that it is a pure arbitrage argument that holds also in a correctly specified discrete time model.

Even so, Buttimer *et al.* (1997) find a value that differs from zero. The basic reason for this appears to be that simple interest and dividend rates are approximated by continuous rates. We estimate that the errors introduced by these approximations are in the order of 10^{-5} to 10^{-6}, which is in line with the reported results. The side that is long in the real estate swap gains from an increase in the absolute difference between the short rate and the dividend yield.

Once simple rates are replaced by continuous rates, other parameters may also come into play though it is not obvious what the net effect will be. In Buttimer *et al.* (1997) it is for instance reported that the variance of the index and the short rate both have an impact on the value of the real estate swap. The interpretation given in the main text is that a higher volatility increases the value of the swap to the side that is long in the corresponding asset, while the data given (eg. table III) present an opposite result. Understanding these results would require precise knowledge of the numerical implementation and we therefore do not pursue the issue further.

5.1 The discrete time approximation

Within a continuous time model, a discrete time approximation can be introduced in essentially two ways:

- The continuous time **valuation equation**, such as a pricing partial differential equation, may be replaced by a discrete time approximation, such as a finite difference scheme. This can be done as a purely numerical approximation, and it is not necessarily the case that the numerical approximation scheme has an economic interpretation as an arbitrage free discrete time financial model. Thus; in such a case the numerical result is typically not an exact arbitrage free price, neither in the original continuous time model, nor in any discrete time financial model.

- The continuous time **model** for the evolution of asset prices may be replaced by a discrete time model. The binomial model of Cox *et al.* (1979), and its extension to the bivariate binomial model used by Buttimer *et al.* (1997), is an example of this procedure. In this approach one requires that the discrete time model should be arbitrage free, and also that it should converge to the continuous time model as the length of the time step goes to zero. As a consequence, in this approach the numerical prices produced are arbitrage free within the discrete time model. They are, however, not necessarily arbitrage free w.r.t the continuous time model, but they are (hopefully good) approximations to the continuous time arbitrage free prices.

In Buttimer *et al.* (1997) the approach taken is clearly the latter one, i.e. the intention is is to approximate the original continuous time model by an arbitrage free discrete time model. In doing so, an approximation error will typically be introduced, but for the CREILS the situation is in fact different:

> Since, by the results of Sections 3 and 4, the arbitrage free value of a CREILS equals zero **regardless of the model**, this should also hold for any arbitrage free discrete time approximation of the original model. In other words, the result that the real estate swap has value zero should remain intact.

As noted in Remark 4.2, the result of Proposition 4.1, that the arbitrage free value of the CREILS equals zero, remains true also in any arbitrage free discrete time model. However, for the sake of completeness we now also give an independent proof of this discrete time result within the framework of Buttimer *et al.* (1997).

Let us thus consider the pricing of a swaplet in the discrete model from the viewpoint of the side that is long in the real estate index. We stand at time t_{k-1} and consider the payoff at time t_k with time step $\Delta = t_k - t_{k-1}$. We denote by R_k the dividends paid out at time t_k, and by L the LIBOR rate for the period. The value of the swaplet X_k at time t_k is then given by the expression

$$X_k = I_{t_k} - I_{t_{k-1}} + R_k - \Delta \cdot L \cdot I_{t_{k-1}}.$$

From equation (6) in Buttimer *et al.* (1997) it is clear that the short rate is quoted as a continously compounded rate. We denote the level of this short rate at t_{k-1} by r, and hence the price at t_{k-1} of a zero coupon bond maturing at t_k is given by

$$p(t_{k-1}, t_k) = e^{-r\Delta}.$$

To ensure that the discrete time model is arbitrage free it is necessary and sufficient that the discounted gain process of the index is a Q-martingale, i.e. that

$$e^{-r\Delta} E^Q[I_{t_k} + R_k | \mathcal{F}_{t_{k-1}}] = I_{t_{k-1}}$$

Furthermore, the (simple) LIBOR rates are given by the standard definition

$$\begin{aligned} L(t_{k-1}, t_k) &= \frac{1/p(t_{k-1}, t_k) - 1}{\Delta} \\ &= \frac{1/e^{-r\Delta} - 1}{\Delta} = \frac{e^{r\Delta} - 1}{\Delta}. \end{aligned} \tag{5}$$

The arbitrage free value of the swaplet at t_{k-1} is given by

$$\Pi\left(t_{k-1}; X_k\right) = e^{-r\Delta} E^Q\left[X_k | \mathcal{F}_{t_{k-1}}\right]$$

and we have the simple calculation

$$\begin{aligned} E^Q\left[X_k | \mathcal{F}_{t_{k-1}}\right] &= E^Q\left[I_{t_k} + R_k | \mathcal{F}_{t_{k-1}}\right] - I_{t_{k-1}} - \Delta \cdot L \cdot I_{t_{k-1}} \\ &= I_{t_{k-1}}\left[e^{r\Delta} - 1 - (e^{r\Delta} - 1)\right] \\ &= 0. \end{aligned}$$

From this it follows that the value of the swap is indeed equal to zero in a correctly specified discrete model. Thus the discrete time approximation in Buttimer *et al.* (1997) should not *per se* introduce an approximation error for the swap. However, in Buttimer *et al.* (1997) two further approximations are introduced, which cause the calculated price to differ from zero, and we now go on to discuss these.

5.2 The dividend approximation

From the discussion in Buttimer *et al.* (1997) and especially equation (2) it follows that,

$$E^Q\left[I_{t_k} | \mathcal{F}_{t_{k-1}}\right] = I_{t_{k-1}} e^{(r-\eta)\Delta}, \tag{6}$$

which indicates that the dividend yield is (implicitly) quoted as a *continuously compounded* rate. However, according to equation (9) in Buttimer *et al.* (1997), the dividend in the discrete model paid out at time t_k is set to a fraction of the index at time t_{k-1}:

$$R_k^s = \Delta\eta I_{t_{k-1}}. \tag{7}$$

In this expression the dividend yield η is thus quoted on a *simple* basis, (hence the superscript) which is not consistent with the continuously compounded interpretation of η in (6) above. In fact it follows directly from (6)-(7) that the expected payoff at time t_k discounted back to t_{k-1} is:

$$e^{-r\Delta}E^Q\left[I_{t_k} + R_k^s \middle| \mathcal{F}_{t_{k-1}}\right] = e^{-r\Delta}I_{t_{k-1}}\left[e^{(r-\eta)\Delta} + \eta\Delta\right].$$

Thus the discounted gain process is not a martingale under Q, which means that the model is not arbitrage free. In more concrete terms: if one insists on quoting η as continuously compounded as in equation (6), then one has to use this convention consistently and replace (7) by the continuously compounded counterpart

$$R_k = e^{r\Delta}I_{t_{k-1}}\left[1 - e^{-\eta\Delta}\right]. \tag{8}$$

With this expression for R_k, the no arbitrage condition

$$e^{-r\Delta}E^Q\left[I_{t_k} + R_k \middle| \mathcal{F}_{t_{k-1}}\right] = I_{t_{k-1}},$$

is indeed satisfied, and the precise interpretation of (8) is that the amount $\left(1 - e^{-\eta\Delta}\right) I_{t_{k-1}}$ is put into the bank account at t_{k-1}, and payed out at t_k.

What the authors do in Buttimer *et al.* (1997) is thus to approximate R in (8) with R^s in (7). The approximation error can easily be calculated, and is as follows for the benchmark values in Buttimer *et al.* (1997):

$$\begin{aligned}
R_k^s - R_k &= \eta\Delta I_{t_{k-1}} + e^{(r-\eta)\Delta}I_{t_{k-1}} - e^{r\Delta}I_{t_{k-1}} \\
&= I_{t_{k-1}}\left[0.04 \cdot 0.25 + e^{(0.05-0.04)\cdot 0.25} - e^{0.05\cdot 0.25}\right] \\
&= -I_{t_{k-1}} \cdot 7.53 \times 10^{-5}.
\end{aligned}$$

Thus Buttimer *et al.* (1997) somewhat underestimate the dividends.

We end this section by noting that in the continuous model, the discounted expected dividends over the interval $[t_{k-1}, t_k]$ at time t_{k-1} can easily be computed as

$$\begin{aligned}
E^Q\left[\int_{t_{k-1}}^{t_k} e^{-\int_{t_{k-1}}^{u} r_s ds}\eta I_u du \middle| \mathcal{F}_{t_{k-1}}\right] &= I_{t_{k-1}}\int_{t_{k-1}}^{t_k}\eta e^{-\eta(u-t_{k-1})}du \\
&= I_{t_{k-1}}\left[-e^{-\int_{t_{k-1}}^{u}\eta ds}\right]_{t_{k-1}}^{t_k} \\
&= I_{t_{k-1}}\left[1 - e^{-\eta\Delta}\right]
\end{aligned}$$

If paid out at t_k this amount will in the discrete model grow with interest to,

$$R_k = e^{r\Delta} I_{t_{k-1}} \left[1 - e^{-\eta\Delta}\right],$$

which is exactly equal to the expression in (8). With this method the discounted gain process is of course a Q-martingale:

$$e^{-r\Delta} E[I_{t_k} + R_k] = I_{t_{k-1}} e^{-\eta\Delta} + I_{t_{k-1}} \left[1 - e^{-\eta\Delta}\right] = I_{t_{k-1}}.$$

It is interesting to note that,

$$e^{-\eta\Delta} = 1 - \eta\Delta + \frac{(\eta\Delta)^2}{2} - ...$$

Therefore a first order Taylor expansion gives,

$$I_{t_{k-1}} \left[1 - e^{-\eta\Delta}\right] \approx I_{t_{k-1}} \eta\Delta.$$

This is the term used by Buttimer *et al.* (1997) and given in (7), apart from the interest factor.

5.3 The interest rate approximation

In Buttimer *et al.* (1997), the LIBOR rate is set equal to the spot rate, i.e.

$$\widehat{L}(t_{k-1}, t_k) = r.$$

Now, as noted earlier, r is quoted as continuously compounded, while the LIBOR rate should be a simple rate, so this is again an inconsistency introduced in Buttimer *et al.* (1997). The correct expression for the simple LIBOR rate is given in (5) and it is easily seen that

$$\widehat{L}(t_{k-1}, t_k) = r < \frac{e^{r\Delta} - 1}{\Delta} = L(t_{k-1}, t_k),$$

so Buttimer *et al.* (1997) underestimate the LIBOR rate and thus overestimate the value of the swap for the side that is long in the real estate index. For the benchmark values used by Buttimer *et al.* (1997) we have that,

$$\begin{aligned} I_{t_{k-1}} \left[(e^{r\Delta} - 1) - r\Delta\right] &= \left(e^{0.05 \cdot 0.25} - 1\right) - 0.05 \cdot 0.25 \\ &= I_{t_{k-1}} \cdot 7.85 \times 10^{-5}. \end{aligned}$$

This is the magnitude of the overestimate of the value of the swap due to the interest rate approximation.

5.4 Combined effect

With the approximations introduced in Buttimer *et al.* (1997) we have,

$$\begin{aligned} E\left[X_k | \mathcal{F}_{t_{k-1}}\right] &= I_{t_{k-1}} e^{(r-\eta)\Delta} + I_{t_{k-1}} \eta\Delta - I_{t_{k-1}} + \Delta \cdot r \cdot I_{t_{k-1}} \\ &= I_{t_{k-1}} \left[e^{(r-\eta)\Delta} - (1 + (r - \eta)\Delta)\right]. \end{aligned}$$

The net effect is roughly that continuous and simple interest rates are confused.

With the benchmark values in Buttimer *et al.* (1997) the following is obtained,

$$\begin{aligned} E\left[X_k|\mathcal{F}_{t_{k-1}}\right] &= I_{t_{k-1}}\left[e^{(0.05-0.04)\cdot 0.25} - (1 + (0.05 - 0.04) \cdot 0.25)\right] \\ &= I_{t_{k-1}} \cdot 3.13 \times 10^{-6}. \end{aligned}$$

This is of the same order of magnitude as the numbers reported in Buttimer *et al.* (1997).

6 Conclusion

We studied the pricing of a real estate index linked swap and found that in a very general setting, its arbitrage free price is exactly equal to zero. This sharpens the result of Buttimer *et al.* (1997) who, under specific assumptions, found a result close to zero using numerical methods.

References

[1] Björk T. (1999). *Arbitrage Theory in Continuous Time*, Oxford University Press.

[2] Cox, J. S. & Ross S. & Rubinstein, M. (1979), Option Pricing: A Simplified Approach, *Journal of Financial Economics*, **7**, 229-264.

[3] Buttimer, R.J. & Kau, J.B. & Slawson, V.C. (1997), A Model for Pricing Securities Dependent upon a Real Estate Index. *Journal of Housing Economics*, **6**, 16-30.

[4] Harrison, J. & Pliska, S. (1981). Martingales and Stochastic Integrals in the Theory of Continuous Trading. *Stochastic Processes & Applications* **11**, 215–260.

[5] Musiela, M. & Rutkowski, M. (1997) *Martingale Methods in Financial Modelling*, Springer-Verlag.