The cutting power of preparation

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Abstract

In a strategic game, a curb set [Basu and Weibull, Econ. Letters 36 (1991) 141] is a product set of pure strategies containing *all* best responses to every possible belief restricted to this set. Prep sets [Voorneveld, Games Econ. Behav. 48 (2004) 403] relax this condition by only requiring the presence of *at least one* best response to such a belief. The purpose of this paper is to provide economically interesting classes of games in which minimal prep sets give sharp predictions, whereas in relevant subclasses of these games, minimal curb sets have no cutting power whatsoever and simply consist of the entire strategy space. These classes include potential games, congestion games with player-specific payoffs, and supermodular games.

JEL classification: C72

Keywords: curb sets, prep sets, potential games, congestion games, supermodular games.

SSE/EFI Working paper series in economics and finance, no. 583,
February 1, 2005

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1. Introduction

Among the set-valued coarsenings of the Nash equilibrium concept, the minimal curb sets of Basu and Weibull (1991) received a lot of attention. In a strategic game, a curb set ('curb' is short for 'closed under rational behavior') is a product set of pure strategies containing *all* best responses to every possible belief restricted to this set. A curb set is minimal if it does not properly contain another curb set.

It is within these minimal curb sets that many intuitive models of strategic adjustment settle down (Young, 1993, Hurkens, 1995, Kosfeld et al., 2002). Moreover, in several classes of relevant games, minimal curb sets yield nice results. For instance: *(i)* in games with costly pre-play communication, minimal curb sets lead to the most preferred outcome for the players with the ability to communicate (Hurkens, 1996), *(ii)* in extensive form games with perfect information and finite horizon, the minimal curb set is unique and contains all subgame perfect Nash equilibria (Pruzhansky, 2003).

Voorneveld (2004, 2005) relaxes the constraint on curb sets by studying product sets of pure strategies containing *at least one* best response to every belief restricted to this set. This makes the players 'prepared' against such beliefs, in the sense that their component of the product set always contains a best reply, without insisting on being exhaustive. This excludes, as in curb sets, the possibility that a player may feel regret due to being recommended a set of strategies that is too small. Sets with this property are called prep sets ('prep' is short for preparation); again, a prep set is minimal if it does not properly contain another prep set.

In addition to proving a very general existence result for minimal prep sets, Voorneveld (2004, 2005) contains a detailed comparison with other game-theoretic solutions. In particular, it is established that minimal curb sets and minimal prep sets coincide in generic finite games, regardless of whether one uses a topological or a measure-theoretic definition of genericity. This result is of interest in its own right, but recall that many classes of games have additional structure, making them nongeneric. Examples include zero-sum games, the (reduced) normal form of extensive form games, or generally any class of finite games where preferences are defined over an outcome space with a cardinality smaller than that of the pure strategy space.

The purpose of this paper is to provide economically interesting classes of games in which minimal prep sets give sharp predictions, whereas in relevant subclasses of these games, minimal curb sets have no cutting power whatsoever and simply consist of the entire strategy space.

In particular, Proposition 3.1 provides sufficient conditions under which minimal prep sets and pure Nash equilibria coincide. This result is illustrated by means of a classical pure saddle-point theorem of Shapley (1964) for zero-sum games. Next, the result is applied to three classes of games that together cover a large range of economic applications:

The first class consists of potential games, in particular the most general class of potential games of Monderer and Shapley (1996) and the best-response potential games of Voorneveld (2000). These potential games include applications to, for instance, congestion games (Rosenthal, 1973), oligopoly models (Slade, 1994), coalition formation (Slikker, 2001), and the financing of public goods (Koster et al., 2003). On the other hand, in the subclass of so-called minority games, minimal curb sets have no cutting power: they select the entire pure strategy space. These minority games model situations where players strive to be in the most exclusive of two groups, for one of many possible reasons: standing out from the crowd might give status; one would prefer to choose the less crowded of two roads to work; if demand for a good is larger than supply, one would rather be a supplier, etc. See Moro (2003) for an introduction to minority games and Challet et al. (2004) for a book containing many of the path-breaking papers and applications to phenomena in financial markets.

The second class consists of the congestion games of Quint and Shubik (1994), which typically are not potential games. Nevertheless, they include minority games, once again providing a subclass where minimal curb sets have no cutting power.

The third and final class consists of supermodular games, games where the best-response correspondences have certain monotonicity properties (Topkis, 1978). Milgrom and Roberts (1990) and Topkis (1998) provide numerous applications, including search models, facility location, arms races, and oligopoly models. Again, we provide a simple example of a subclass of these supermodular games where minimal curb sets have no cutting power.

This is by no means meant as a critique against minimal curb sets, nor do we think that minimal prep sets 'dominate' minimal curb sets. It is too much to hope for that a single solution concept works well in all games. Our aim is simply to complement the mainly theoretic study of minimal prep sets conducted so far with an illustration of its appeal in classes of games that cover many economic situations.

The material is organized as follows. Section 2 contains preliminary definitions. In Section 3 we provide sufficient conditions for the collection of minimal prep sets and the collection of pure Nash equilibria to coincide and give a first illustration of the result using a well-known result concerning pure saddle-points by Shapley (1964). Applications to potential games, the congestion games of Quint and Shubik (1994), and supermodular games follow in Sections 4 to 6.

2. Preliminaries

A strategic game is a tuple $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$, where N is a nonempty, finite set of players, each player $i \in N$ has a nonempty set of pure strategies (or actions) A_i and a von Neumann-Morgenstern utility function $u_i : \times_{j\in N} A_j \to \mathbb{R}$. Write $A = \times_{i\in N} A_i$ and for each $i \in N$, $A_{-i} = \times_{j\in N\setminus\{i\}} A_j$.

Payoffs are extended to mixed strategies in the usual way. Assuming each A_i to be a topological space, $\Delta(A_i)$ denotes the set of Borel probability measures over A_i. Using a common, minor abuse of notation, α_{-i} denotes both an element of $\times_{j\in N\setminus\{i\}} \Delta(A_j)$ specifying a profile of mixed strategies of the opponents of player $i \in N$, and the probability measure it induces over the set A_{-i} of pure strategy profiles of his opponents. Beliefs of player i take the form of such a mixed strategy profile. Similarly, if $B_i \subseteq A_i$ is a Borel set, then $\Delta(B_i)$ denotes the set of Borel probability measures with support in B_i:

$$\Delta(B_i) = \{\alpha_i \in \Delta(A_i) \mid \alpha_i(B_i) = 1\}.$$

As usual, (a_i, α_{-i}) is the profile of strategies where player $i \in N$ plays $a_i \in A_i$ and his opponents play according to the mixed strategy profile $\alpha_{-i} = (\alpha_j)_{j\in N\setminus\{i\}} \in \times_{j\in N\setminus\{i\}} \Delta(A_j)$.

Let Γ denote the set of all games satisfying the following simple assumption on the players' utility functions: for each player $i \in N$, for each $a_i \in A_i$ and each $\alpha_{-i} \in \times_{j\in N\setminus\{i\}} \Delta(A_j)$, the expected payoff $u_i(a_i, \alpha_{-i}) = \int_{A_{-i}} u_i(a_i, a_{-i})\, \mathrm{d}\alpha_{-i}$ is well-defined and finite. The set Γ contains, in particular, all (mixed extensions of) finite strategic games.

Let $i \in N$ and let $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(A_j)$ be a belief of player i. The set

$$BR_i(\alpha_{-i}) = \{a_i \in A_i \mid \forall b_i \in A_i : u_i(a_i, \alpha_{-i}) \geq u_i(b_i, \alpha_{-i})\}$$

is the set of pure best responses of player i against α_{-i}.

We recall the definitions of minimal curb sets (Basu and Weibull, 1991) and minimal prep sets (Voorneveld, 2004, 2005).

Definition 2.1 Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$. A **curb set** is a product set $X = \times_{i \in N} X_i$, where

- for each $i \in N$, $X_i \subseteq A_i$ is a nonempty, compact set of pure strategies;
- for each $i \in N$ and each belief α_{-i} of player i with support in X_{-i}, the set X_i contains all best responses of player i against his belief:

$$\forall i \in N, \forall \alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(X_j) : \quad BR_i(\alpha_{-i}) \subseteq X_i.$$

A curb set X is **minimal** if no prep set is a proper subset of X. Similarly, a **prep set** is a product set $X = \times_{i \in N} X_i$, where

- for each $i \in N$, $X_i \subseteq A_i$ is a nonempty, compact set of pure strategies;
- for each $i \in N$ and each belief α_{-i} of player i with support in X_{-i}, the set X_i contains at least one best response of player i against his belief:

$$\forall i \in N, \forall \alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(X_j) : \quad BR_i(\alpha_{-i}) \cap X_i \neq \emptyset.$$

A prep set X is **minimal** if no prep set is a proper subset of X. $\triangleleft$

3. Sufficient conditions for coincidence

In this section, we show that some simple conditions are sufficient for the collection of minimal prep sets and the collection of pure Nash equilibria to coincide in a class of games. This statement is intuitively clear, but since we are comparing set-valued solutions with point-valued solutions, let us define the coincidence formally: in a game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$, the collection of minimal prep sets and the collection of pure Nash equilibria coincide if:

- for each minimal prep set $X = \times_{i \in N} X_i$ of G and each player $i \in N$, there is a pure strategy $a_i \in A_i$ such that $X_i = \{a_i\}$ and $a = (a_i)_{i \in N}$ is a pure Nash equilibrium, and conversely:

- for each pure Nash equilibrium $a = (a_i)_{i \in N} \in A$, the product set $\times_{i \in N} \{a_i\}$ is a minimal prep set.

If $\Gamma' \subseteq \Gamma$ is a class of games, we say that the collection of minimal prep sets and the collection of pure Nash equilibria coincide on Γ' if they coincide for every game $G \in \Gamma'$.

For a game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$, we will sometimes wish to restrict players' pure strategies to a product set $B = \times_{i \in N} B_i \subseteq A$. The game's payoffs are trivially obtained by restricting the payoff functions $(u_i)_{i \in N}$ to B. With a slight abuse of notation (letting the domain of payoffs be implicit), this game is denoted by $\langle N, (B_i)_{i \in N}, (u_i)_{i \in N} \rangle$.

Let us formulate the conditions under which we will establish coincidence. A class of games $\Gamma' \subseteq \Gamma$:

- is **closed w.r.t. subgames** if for each game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma'$ and every nonempty product set $B = \times_{i \in N} B_i \subseteq A$ of compact action sets $B_i \subseteq A_i$, also $\langle N, (B_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma'$;

- is **closed w.r.t. minimal prep sets** if for each game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma'$ and every minimal prep set $B = \times_{i \in N} B_i \subseteq A$, also $\langle N, (B_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma'$;

- has the **pure Nash property** if every game $G \in \Gamma'$ has a pure Nash equilibrium.

Clearly, if Γ' is closed w.r.t. subgames, it is closed w.r.t. minimal prep sets. Apart from that, the properties are logically independent: the set of matrix games (i.e., finite, two-player zero-sum games) is closed w.r.t. subgames and in particular w.r.t. minimal prep sets, but does not have the pure Nash property. The set of best-response potential games with a finite pure strategy space (see Section 4.1) has the pure Nash property and is closed w.r.t. minimal prep sets, but is not closed w.r.t. subgames.

Proposition 3.1 *If* $\Gamma' \subseteq \Gamma$ *has the pure Nash property and is closed w.r.t. minimal prep sets, or — more strongly — w.r.t. subgames, then the set of pure Nash equilibria and the collection of minimal prep sets coincide on* Γ'*.*

Proof. Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma'$. By the pure Nash property, G has a pure Nash equilibrium $a \in A$. By definition, $\times_{i \in N}\{a_i\}$ is a minimal prep set. Conversely, let $B = \times_{i \in N} B_i$ be a minimal prep set of G. Since Γ' is closed w.r.t. minimal prep sets, also $\langle N, (B_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma'$. By the pure Nash property, $\langle N, (B_i)_{i \in N}, (u_i)_{i \in N} \rangle$ has a pure Nash equilibrium $b = (b_i)_{i \in N} \in B$ and hence minimal prep set $\times_{i \in N}\{b_i\}$. Since B is a prep set of G, it follows that $\times_{i \in N}\{b_i\}$ is a minimal prep set of the original game G. Since $\times_{i \in N}\{b_i\} \subseteq B$ and B is a minimal prep set of G, it follows that $\times_{i \in N}\{b_i\} = B$: the minimal prep set B corresponds with a pure Nash equilibrium. $\square$

Proposition 3.1 is intuitively most appealing if it applies to a class of games which is closed w.r.t. subgames, which is regularly the case if the games are defined by common types of strategic interactions: subgames of zero-sum games are zero-sum, subgames of congestion games, where players choose among different roads/facilities, are congestion games, etc. Some interesting classes of games with the pure Nash property, however, are not closed w.r.t. subgames, even though they are closed w.r.t. minimal prep sets. Specific examples include the best-response potential games in Section 4.1 and games with strategic complementarities in Section 6.2.

A pure saddle-point theorem of Shapley (1964, pp. 6-7) serves as a first illustration of the use of Proposition 3.1. Other economically relevant applications are provided in later sections.

Proposition 3.2 *Let* Γ' *be the set of finite two-person zero-sum games*

$$G = \langle \{1, 2\}, (A_1, A_2), (u_1, -u_1) \rangle$$

in which each 2×2 *subgame (a subgame in which both players have exactly two pure strategies) has a pure saddle point/Nash equilibrium*[2]*. For every game* $G \in \Gamma'$*, the set of pure saddle points/Nash equilibria and the collection of minimal prep sets coincide.*

[2] This assumption vacuously holds for finite two-person zero-sum games in which some player has only one pure strategy: there are no 2×2 subgames! Hence, such games are included in Γ'.

Proof. Let $G \in \Gamma'$ and let H be a subgame of G. Since every 2×2 subgame of H is a 2×2 subgame of G, it follows that all 2×2 subgames of H have a pure saddle point. Conclude that Γ' is closed w.r.t. subgames. Moreover, Γ' has the pure Nash property by Thm. 2.1 of Shapley (1964). The result now follows from Proposition 3.1. □

4. Potential games

4.1. Generalized ordinal and best-response potential games

Monderer and Shapley (1996) define four classes of potential games, in increasing order of generality: exact, weighted, ordinal, and generalized ordinal potential games. These games have applications to, for instance, congestion models (Rosenthal, 1973) and oligopoly models (Slade, 1994). All finite potential games in Monderer and Shapley (1996) have the finite improvement property: start with an arbitrary strategy profile. Each time, let a player that can benefit from unilateral deviation switch to a better strategy. Under the finite improvement property, this process eventually ends, obviously in a Nash equilibrium. Voorneveld (2000) introduces best-response potential games that allow infinite improvement paths by imposing restrictions only on paths in which players that can improve actually deviate to a best response. These games include the best-response potential games of Morris and Ui (2004, p. 264, after Def. 6). Formally, a game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is

- a **generalized ordinal potential game** if there is a function $P : A \to \mathbb{R}$ such that, for each player $i \in N$, each strategy profile $a_{-i} \in A_{-i}$ of his fellow players, and each pair of strategies $a_i, b_i \in A_i$:

$$u_i(a_i, a_{-i}) - u_i(b_i, a_{-i}) > 0 \Rightarrow P(a_i, a_{-i}) - P(b_i, a_{-i}) > 0. \tag{1}$$

- a **best-response potential game** if there is a function $P : A \to \mathbb{R}$ such that, for each player $i \in N$ and each strategy profile $a_{-i} \in A_{-i}$ of his fellow players:

$$\arg\max_{a_i \in A_i} u_i(a_i, a_{-i}) = \arg\max_{a_i \in A_i} P(a_i, a_{-i}). \tag{2}$$

The function P is called a **(generalized ordinal or best-response) potential**.

There is no logical dependence between (1) and (2): Examples 4.1 and 4.2 in Voorneveld (2000) indicate that there are generalized ordinal potential games which are not best-response potential games, and conversely, that there are best-response potential games which are not generalized ordinal potential games.

Recall that a function $P : A \to \mathbb{R}$ on a topological space A is upper semicontinuous (u.s.c.) if its upper contour sets are closed:

$$\forall r \in \mathbb{R} : \{a \in A \mid P(a) \geq r\} \text{ is closed.}$$

Proposition 4.1 *Let Γ^1 (Γ^2) be the set of games with compact strategy spaces and an upper semicontinuous generalized ordinal (best-response) potential. For any $G \in \Gamma^1 \cup \Gamma^2$, the set of pure Nash equilibria and the collection of minimal prep sets coincide.*

Proof.

Γ^1 IS CLOSED W.R.T. SUBGAMES: Simply restrict the domain of the potential function.

Γ^2 IS CLOSED W.R.T. MINIMAL PREP SETS: Let $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle \in \Gamma^2$ have u.s.c. best-response potential P and assume $B = \times_{i\in N} B_i$ is a minimal prep set of G. For each player $i \in N$ and each strategy profile $a_{-i} \in \times_{j\in N\setminus\{i\}} B_j$,

$$\emptyset \neq \arg\max_{a_i\in A_i} u_i(a_i, a_{-i}) \cap B_i = \arg\max_{a_i\in A_i} P(a_i, a_{-i}) \cap B_i,$$

where the inequality follows by definition of a prep set and the equality follows from (2). Hence, also the game $H = \langle N, (B_i)_{i\in N}, (u_i)_{i\in N}\rangle$ is a best-response potential game, whose best response potential is obtained from P by restricting its domain. Its strategy spaces $(B_i)_{i\in N}$ are compact by definition of a prep set and P remains u.s.c. in the subspace topology. Conclude that $H \in \Gamma^2$.

Γ^1 AND Γ^2 HAVE THE PURE NASH PROPERTY: Since A is compact in the product topology and each $G \in \Gamma^1 \cup \Gamma^2$ has a continuous potential P, the potential achieves a maximum. By (1) or (2), such a maximum is a pure Nash equilibrium.

The result now follows from Proposition 3.1. □

Some remarks:

	L	M	R
T	$2,2$	$1,0$	$0,1$
B	$0,0$	$0,1$	$1,0$

	L	M	R
T	4	3	0
B	0	2	1

Figure 1: Best-response potential games: not closed w.r.t. subgames.

(i) Endowing A with the discrete topology, the conclusion of Proposition 4.1 applies in particular to finite generalized ordinal/best-response potential games.
(ii) As opposed to the set of generalized ordinal potential games, the set of best-response potential games is not closed w.r.t. subgames: The two-player game to the left in Figure 1 has a best-response potential (to the right). The subgame with action space $\{T, B\} \times \{M, R\}$ is not a best-response potential game: such a potential would have to satisfy

$$P(T, M) < P(T, R) < P(B, R) < P(B, M) < P(T, M),$$

a contradiction.
(iii) The assumption that the game needs to have an u.s.c. potential is not an innocuous one. Voorneveld (1997, p. 167-168) gives an example of an ordinal potential game with compact strategy spaces and continuous payoff functions for which no potential achieves a maximum and which, consequently, has no u.s.c. potential.
(iv) The conclusion of Proposition 4.1 does not hold for the pseudo-potential games recently introduced by Dubey et al. (2004). Formally, a game $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ is a pseudo-potential game if there is a function $P : A \to \mathbb{R}$ such that, for each player $i \in N$ and each strategy profile $a_{-i} \in A_{-i}$ of his fellow players:

$$\arg\max_{a_i\in A_i} u_i(a_i, a_{-i}) \supseteq \arg\max_{a_i\in A_i} P(a_i, a_{-i}).$$

The two-player game to the left in Figure 2 has a pseudo-potential (to the right). But its pure Nash equilibria and minimal prep sets do not coincide, since $\{T, B\} \times \{A, B\}$ is a minimal prep set. The game has neither a generalized ordinal nor a best-response potential function P, which by definition would have to satisfy:

$$P(T, A) < P(T, B) < P(B, B) < P(B, A) < P(T, A),$$

a contradiction.

	A	B	C	D
T	$1,-1$	$-1,1$	$1,1$	$-2,-2$
B	$-1,1$	$1,-1$	$-2,-2$	$1,1$

	A	B	C	D
T	1	0	2	0
B	0	1	0	2

Figure 2: A pseudo-potential game

4.2. Minority games

Above, we saw that in large classes of potential games, minimal prep sets have substantial cutting power, yielding equilibrium predictions. On the other hand, in economically relevant subclasses of these games, minimal curb sets have no cutting power whatsoever. As an example, this section considers so-called minority games, a type of congestion problems introduced by Challet and Zhang (1997) and inspired by the El Farol Bar problem of Arthur (1994). See Moro (2003) for an introductory overview, Challet et al. (2004) for a book containing many of the path-breaking papers within the physics literature and applications to phenomena in financial markets, and Coolen (2005) for a thorough mathematical treatment.

Minority games study congestion problems where players aim to avoid crowds and prefer choosing the minority alternative. They have an odd number of players: $N = \{1, \ldots, 2k+1\}$ for some $k \in \mathbb{N}$. Each player $i \in N$ chooses among two actions: $A_i = \{-1, +1\}$ for all $i \in N$. Associated with each action $s \in \{-1, +1\}$, there is a function

$$f_s : \{1, \ldots, 2k+1\} \to \mathbb{R},$$

where for each $m \in \{1, \ldots, 2k+1\}$, $f_s(m) \in \mathbb{R}$ indicates the utility/payoff to a player choosing s if the total number of players choosing s equals m. The payoff/utility function $u_i : \times_{j \in N} A_j \to \mathbb{R}$ of player $i \in N$ is then naturally defined as follows:

$$\forall a = (a_j)_{j \in N} \in \times_{j \in N} A_j : u_i(a) = f_{a_i}(\mid \{j \in N : a_j = a_i\} \mid).$$

Characteristic for a minority game is that unilateral deviation from a majority to a minority pays off:

$$\forall s, t \in \{-1, +1\}, s \neq t, \forall m \in \{k+2, \ldots, 2k+1\} : f_s(m) < f_t(2k+2-m). \qquad (3)$$

Example 4.2 Challett and Zhang (1997, p. 408), who introduce minority games, initially assign payoff one to each member of the minority, and payoff zero to each member of a majority:

$$f_{-1}(m) = f_{+1}(m) = \begin{cases} 1 & \text{if } m \in \{1, \ldots, k\}, \\ 0 & \text{if } m \in \{k+1, \ldots, 2k+1\}. \end{cases} \tag{4}$$

In a variant (Challet and Zhang, 1997, p. 411), they suggest payoffs giving zero reward to majority members and positive payoffs to minorities, favoring small ones:

$$f_{-1}(m) = f_{+1}(m) = \begin{cases} \mid N \mid /m - 2 & \text{if } m \in \{1, \ldots, k\}, \\ 0 & \text{if } m \in \{k+1, \ldots, 2k+1\}. \end{cases} \tag{5}$$

Given an action profile $a = (a_i)_{i \in N}$, the minority alternative is -1 if $\sum_{i \in N} a_i > 0$ and $+1$ if $\sum_{i \in N} a_i < 0$. Other frequently occurring payoff functions (Moro, 2003) assign to player i a payoff given by $-a_i\, g(\sum_{j \in N} a_j)$, where g is an odd function, i.e., $g(x) = -g(-x)$, with $g(x) > 0$ if $x > 0$. In particular, common examples include

$$g(x) = x / \mid N \mid \ \text{ and } g(x) = sgn(x), \tag{6}$$

where the sign function is defined as:

$$sgn(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ +1 & \text{if } x > 0. \end{cases}$$

In our notation:

$$f_{-1}(m) = f_{+1}(m) = g(2(k - m) + 1). \tag{7}$$

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As is seen from the examples, the payoff functions to the two alternatives are traditionally assumed to be identical: $f_{-1} = f_{+1}$. We relax this assumption by only requiring

$$f_{-1}(k+1) = f_{+1}(k+1). \tag{8}$$

Proposition 4.3 *In a minority game G (in particular, under assumptions (3) and (8)) with* $2k+1$ *players, the following holds:*

(a) *G is a potential game, so its pure Nash equilibria and minimal prep sets coincide;*

(b) *A pure strategy profile $a = (a_i)_{i \in N}$ is a pure Nash equilibrium if and only if there is an alternative $s \in \{-1, +1\}$ such that exactly k players choose s;*

(c) *The unique minimal curb set of G is the entire pure strategy space.*

Proof. **(a)** G is a congestion game as in Rosenthal (1973) and hence a (finite exact) potential game (Monderer and Shapley, 1996, Thm. 3.1). By Proposition 4.1, its pure Nash equilibria and minimal prep sets coincide.

(b) Fix a pure strategy profile. The number of players is odd, so some option $s \in \{-1, +1\}$ is chosen by a majority of at least $k+1$ players. If the majority has $k+2$ or more players, (3) implies that a majority member can unilaterally deviate to the other option and achieve a strictly higher payoff. Thus, the strategy profiles in Proposition 4.3(b) are the only candidates for pure Nash equilibria. They are indeed equilibria: by (3), a minority member never has an incentive to deviate and join a majority. Next, let $s \in \{-1, +1\}$ be the alternative chosen by the $k+1$ majority members. If a majority member deviates to $t \neq s$, his payoff changes from $f_s(k+1)$ to $f_t(k+1)$. By (8), these payoffs are the same. Conclude: the profiles in Proposition 4.3(b) are indeed the game's pure Nash equilibria.

(c) Let $B = \times_{i \in N} B_i$ be a minimal curb set of G. There is a player $i \in N$ with $B_i = \{-1, +1\}$. Otherwise, all components of the minimal curb set would be singleton sets. By definition of a curb set, the unique pure strategy profile in B would then have to be a pure Nash equilibrium, as characterized in Proposition 4.3(b). But in a pure Nash equilibrium, by (8), the minority members are indifferent between staying and switching: both -1 and $+1$ are best replies to the action profile of the remaining players, which consequently must be included in their component of the curb set. This contradicts that all components of the minimal curb set are singletons. Conclude: there is an $i \in N$ with $B_i = \{-1, +1\}$.

We proceed by showing that $B_j = \{-1, +1\}$ for all $j \in N$. Suppose, to the contrary, that there is a player $j \in N$ with B_j being a singleton set, w.l.o.g. $B_j = \{-1\}$. By definition of a curb set, there is no element of B_{-j} against which $+1$ is a best reply. Since $+1$ is a best reply to any profile in which at most k players choose $+1$, no such profiles can be included in B_{-j}: in every element of B_{-j}, at least $k+1$ players choose $+1$, in particular in every profile where player i, who had $B_i = \{-1, +1\}$, chooses -1. Hence, in every element of $\times_{\ell \in N \setminus \{i,j\}} B_\ell$, at least $k+1$ players choose $+1$: there are at least $k+1$ players in $N \setminus \{i,j\}$ whose component of the minimal curb set B equals $\{+1\}$.

Let $b_{-i} \in B_{-i}$ and let $\ell \geq k+1$ be the number of players $j \in N \setminus \{i\}$ with $b_j = +1$. Player i's payoff is $f_{+1}(\ell+1)$ or $f_{-1}(2k-\ell+1)$ for actions $+1$ and -1, respectively. By (3) with $m = \ell+1$, player i's unique best reply to any $b_{-i} \in B_{-i}$ is to choose -1:

$$f_{+1}(\ell+1) < f_{-1}(2k-\ell+1),$$

Hence, the same holds for every belief $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(B_j)$: action $+1$ can be omitted from his component B_i of the curb set, contradicting the assumption that B is a minimal curb set. Conclude: there is no player $j \in N$ with B_j being a singleton set, proving Proposition 4.3(c). □

5. The Quint-Shubik congestion model

Monderer and Shapley (1996, Thm 3.2) show that every finite exact potential game is isomorphic to a congestion game as defined in Rosenthal (1973). In these games, players choose subsets of facilities from a common pool. The payoff associated with each facility is a function only of the number of players using it. Quint and Shubik (1994) and — as a special case — Milchtaich (1996) considered a different class of congestion games by also allowing payoffs to be player-dependent. In general, these games do not admit a potential function, but nevertheless have pure Nash equilibria. The notation in this section follows the overview article on congestion models by Voorneveld et al. (2000).

Quint and Shubik (1994) consider finite games $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ satisfying the following three properties:

(QS1) There is a nonempty, finite set F of facilities such that $A_i \subseteq F$ for all $i \in N$.

By (QS1), an action of a player is to choose a facility from a collection F, possibly subject to feasibility constraints: players may not have access to all elements of F.

Let $a \in A$, $f \in F$. Denote by $n_f(a)$ the number of players choosing facility f in action profile a, i.e., $n_f(a) = | \{i \in N : a_i = f\} |$.

(QS2) For each player $i \in N$ and all pure-strategy profiles $a, b \in A$ with $a_i = b_i = f$: if $n_f(a) = n_f(b)$, then $u_i(a) = u_i(b)$.

By (QS2), the utility of player i depends only on the number of users of the facility that i has chosen.

(QS3) For each player $i \in N$, each pure strategy profile $a \in A$, each player $j \in N \setminus \{i\}$ with $a_j = a_i$, and each alternative action choice $b_j \in A_j \setminus \{a_j\}$ of this player: $u_i(a_j, a_{-j}) \leq u_i(b_j, a_{-j})$.

Property (QS3) models the congestion: a player is not harmed if an other user of the same facility switches to a different one. The benefit to each player from choosing a facility is weakly decreasing in the total number of users.

Proposition 5.1 *Let Γ' denote the set of Quint-Shubik congestion games, i.e., the set of strategic games satisfying (QS1) to (QS3). For each game $G \in \Gamma'$, the set of pure Nash equilibria and the collection of minimal prep sets coincide.*

Proof. Quint and Shubik (1994, Thm. 3) prove that Γ' has the pure Nash property. Property (QS1) allows us to restrict the set of facilities from which players make their choices, so Γ' is closed w.r.t. subgames and in particular w.r.t. minimal prep sets. The result follows from Proposition 3.1. □

Remark 5.2 Milchtaich (1996) allows no restrictions on access to facilities: he assumes (QS2), (QS3), and $A_i = F$ instead of the weaker assumption (QS1). Hence, his class of games is not closed w.r.t. subgames or minimal prep sets. Since they are special cases of the Quint-Shubik congestion games, we can nevertheless conclude that minimal prep sets and pure Nash equilibria coincide. ◁

The Quint-Shubik congestion games contain numerous minority games, including all our explicit examples with payoffs as defined in (4), (5), and (7) with g as in (6). Conclude that also here, there is a relevant subclass in which minimal curb sets have no cutting power.

6. Strategic complementarities

Well-known existence results for Nash equilibria in supermodular games or games with strategic complementarities rely on monotonicity properties of the best-reply correspondence. The theory was initiated by Topkis (1978) and has been successfully applied to a wide range of economic models; the reader is referred to, for instance, Milgrom and Roberts (1990) or the book of Topkis (1998). This section relies on a general existence result by Zhou (1994).

6.1. Lattices and order

A **partially ordered set** $(S, \leq)$ is a set S with a binary relation $\leq$ which is reflexive, antisymmetric, and transitive. Consider a subset T of S. An element $x \in S$ is a lower bound of T if $x \leq y$ for all $y \in T$ and an upper bound of T if $y \leq x$ for all $y \in T$. If it exists, the least upper bound of T is called the supremum $\sup(T)$ of T in S and the greatest lower bound of T is called the infimum $\inf(T)$ of T in S. A **lattice** is a partially ordered set $(S, \leq)$ that contains the infimum $x \wedge y = \inf\{x, y\}$ and supremum $x \vee y = \sup\{x, y\}$ of each pair of elements $x, y \in S$. The lattice is **complete** if, for all nonempty subsets $T \subseteq S$: $\inf(T) \in S$ and $\sup(T) \in S$.

$(T, \leq)$ is a **sublattice** of lattice $(S, \leq)$ if $T \subseteq S$ is closed under $\wedge$ and $\vee$, i.e., if it is a lattice with the same join and meet relations as S. As above, this sublattice is **complete** if, for all nonempty subsets $U \subseteq T$: $\inf(U) \in T$ and $\sup(U) \in T$.

An **interval** $[x, y]$ in $(S, \leq)$ is the set $\{z \in S \mid x \leq z \leq y\}$. For $x \in S$, we denote $(-\infty, x] = \{z \in S \mid z \leq x\}$ and $[x, \infty) = \{z \in S \mid x \leq z\}$. The **interval topology** on a lattice $(S, \leq)$ is the topology for which all closed sets are intersections or finite unions of intervals of the form S, $(-\infty, x]$, and $[x, \infty)$, where $x \in S$. By the Frink-Birkhoff theorem, a lattice is complete if and only if it is compact in its interval topology. Hence, any sublattice of a complete lattice is complete if and only if it is closed in its interval topology.

A note of caution: a subset of $(S, \leq)$ that is a complete lattice in its own right may not be a complete sublattice of $(S, \leq)$. Milgrom and Roberts (1990, p. 1260) give enlightening examples. For instance, the set $T = [0, 1) \cup \{2\}$ is a complete lattice under its standard order. In this case $\sup[0,1) = 2 \in T$. It is not a complete sublattice of $[0,2]$, where $\sup[0,1) = 1 \notin T$.

Consider a lattice $(S, \leq)$. A correspondence $\varphi : S \twoheadrightarrow S$ is **ascending** if, for all $x, y \in S$ with $x \leq y$, all $s \in \varphi(s)$ and $t \in \varphi(y)$, it is true that $s \wedge t \in \varphi(x), s \vee t \in \varphi(y)$.

6.2. Games with ascending best replies

As stated, well-known existence results for Nash equilibria rely on the best-response correspondence being ascending. This can be derived from other assumptions under names like supermodularity/strategic complementarity/increasing differences, but the key to the result is always the monotonicity of best replies. Therefore, we state our result in terms of ascending best replies and refer to for instance Zhou (1994, p. 299) for a readable account on how to achieve it from other conditions.

We use a general existence result by Zhou (1994, Thm. 2) with the only modification that we assume all action sets A_i to be linearly ordered, rather than just a lattice. In most applications (see Milgrom and Roberts, 1990, or Topkis, 1998), this assumption is satisfied. Often, for instance, A_i is a set of real numbers with its usual order.

Proposition 6.1 *Consider the set $\Gamma' \subseteq \Gamma$ of all strategic games $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ where for each player $i \in N$:*

- *there is a linear order $\leq_i$ on A_i such that A_i is compact in a topology τ_i equal to or finer than the interval topology.*
- *u_i is upper semicontinuous on A_i in the topology τ_i.*

Moreover, let $A = \times_{i\in N} A_i$ be the direct product compact lattice of all the A_i and assume that the best response correspondence $BR : A \twoheadrightarrow A$ is ascending (w.r.t. the product order on A). For every game $G \in \Gamma'$ the collection of minimal prep sets and pure Nash equilibria coincide.

Proof.

Γ' HAS THE PURE NASH PROPERTY: All games in Γ' satisfy the lattice and upper semicontinuity properties of Zhou (1994, p. 298, lines 1-5 of Section 3) and the best-response correspondence is ascending. Hence, by Zhou (1994, p. 299, proof of Thm. 2), every game in Γ' has a pure Nash equilibrium.

Γ' IS CLOSED W.R.T. MINIMAL PREP SETS: Let $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle \in \Gamma'$ and let $B = \times_{i\in N} B_i \subseteq A$ be a minimal prep set of G. To show: $H = \langle N, (B_i)_{i\in N}, (u_i)_{i\in N}\rangle \in \Gamma'$.

(i) Restricting the linear order $\leq_i$ on A_i to B_i, we see that $(B_i, \leq_i)$ is linearly ordered.

(ii) By definition of a minimal prep set, B_i is compact in τ_i, which is equal to or finer than the interval topology on A_i. Hence, the same holds for the topology restricted to B_i, the usual subspace topology.

(iii) Since u_i is upper semicontinuous in the topology τ_i on A_i, it remains so on B_i.

(iv) Since B is a minimal prep set, the best-response correspondence $BR_i(\cdot \mid H)$ of the subgame H is given by

$$BR_i(\cdot \mid H) = B_i \cap BR_i(\cdot \mid G),$$

the — by definition of a minimal prep set — nonempty intersection of the best-response correspondence of the original game and i's component B_i of the minimal prep set. Since $BR_i(\cdot \mid G)$ is ascending and B_i is a lattice given its linear order $\leq_i$, it follows that $BR_i(\cdot \mid H)$ is ascending.

Combining these observations, one concludes that $\langle N, (B_i)_{i\in N}, (u_i)_{i\in N}\rangle \in \Gamma'$, i.e., Γ' is closed w.r.t. minimal prep sets. The result now follows from Proposition 3.1. □

The set Γ' in Theorem 6.1 is not closed w.r.t. subgames: in the two-player game in Figure 3, each player's action 0 is strictly dominant. Hence, the best-response correspondences

	0	1	2
0	$2,2$	$2,0$	$2,0$
1	$0,2$	$1,-1$	$-1,1$
2	$0,2$	$-1,1$	$1,-1$

Figure 3: Supermodular games: not closed w.r.t. subgames

are constant and in particular ascending in the usual order on $\{0,1,2\}$: the game belongs to Γ'. But the subgame on $\{1,2\}\times\{1,2\}$ is not in Γ': it has no pure Nash equilibrium.

Theorem 6.1 does not hold if the assumption that each action set A_i is linearly ordered is relaxed to assuming that there is an order $\leq_i$ on A_i such that $(A_i, \leq_i)$ is a complete lattice: the associated class of games is not closed w.r.t. minimal prep sets. Consider the two-player game in Figure 4. Define, for each player $i = 1, 2$ the partial order $\leq_i$ on

	0	1	2	3
0	$2,2$	$0,0$	$0,0$	$0,0$
1	$0,0$	$1,0$	$0,1$	$0,0$
2	$0,0$	$0,1$	$1,0$	$0,0$
3	$0,0$	$0,0$	$0,0$	$2,2$

Figure 4: Supermodular games: non-linear orders

$A_i = \{0, 1, 2, 3\}$ with $0 \leq_i 1 \leq_i 3$ and $0 \leq_i 2 \leq_i 3$, but which does not compare 1 and 2. Then $(A_i, \leq_i)$ is a complete lattice. The players' best-response correspondences are:

$$BR_1(a_2) = \{a_2\} \text{ for all } a_2 \in \{0,1,2,3\} \text{ and } BR_2(a_1) = \begin{cases} \{0\} & \text{if } a_1 = 0, \\ \{2\} & \text{if } a_1 = 1, \\ \{1\} & \text{if } a_1 = 2, \\ \{3\} & \text{if } a_1 = 3. \end{cases}$$

Hence, the best-response correspondence $BR = BR_1 \times BR_2$ is ascending w.r.t. the product order on A. The product set $\{1, 2\} \times \{1, 2\}$ is a minimal prep set of the game. In the subgame restricted to these action profiles, we still have that the best response correspondence is (trivially) ascending with respect to the product order induded by the restriction of $\leq_i$ to $\{1, 2\}$. But $(\{1, 2\}, \leq_i)$ is not a lattice: $1 \wedge 2$ and $1 \vee 2$ do not exist. Notice, indeed, that this subgame does not have a pure Nash equilibrium.

6.3. A class of coordination games

By Proposition 6.1, minimal prep sets have substantial cutting power in a very general class of supermodular games. Just as for potential games, however, one can easily construct plausible subclasses of such games where minimal curb sets have no cutting power. We give a simple example.

Consider a two-player coordination game where the players find each other if they choose close-by alternatives. Formally, consider the game $G = \langle \{1,2\}, (A_1, A_2), (u_1, u_2) \rangle$ where

$$A_1 = A_2 = \{0, 1, \ldots, k\} \text{ for some } k \in \mathbb{N} \tag{9}$$

and for each pair of alternatives $(a_1, a_2) \in A_1 \times A_2$:

$$u_1(a_1, a_2) = u_2(a_1, a_2) = \begin{cases} 1 & \text{if } |a_1 - a_2| \leq 1, \\ 0 & \text{otherwise,} \end{cases} \tag{10}$$

i.e., each player chooses one of the locations $0, \ldots, k$ and they are rewarded ('find each other') if they choose neighboring locations.

Proposition 6.2 *In a two-player coordination game* $G = \langle \{1,2\}, (A_1, A_2), (u_1, u_2) \rangle$ *as in (9) and (10), the following hold:*

(a) *the collections of pure Nash equilibria and minimal prep sets coincide;*

(b) *the collection of pure Nash equilibria is*

$$\{(a_1, a_2) \in A_1 \times A_2 : |a_1 - a_2| \leq 1\};$$

(c) *the unique (hence minimal) curb set is* $A_1 \times A_2$.

Proof. **(a)** Endowing the action space $A_i = \{0, 1, \ldots, k\}$ of player $i \in \{1, 2\}$ with its standard order, the game is easily seen to belong to the class of games with ascending best responses in Proposition 6.1, so that pure Nash equilibria and minimal prep sets indeed coincide.
(b) Follows easily from (10).
(c) Let $X = X_1 \times X_2$ be a curb set of G. Fix a player $i \in \{1, 2\}$. By (10), it follows that if $a_i \in X_i$, then $\{a_i - 1, a_i, a_i + 1\} \cap \{0, 1, \ldots, k\} \subseteq X_j$ for $j \neq i$: player j's component of the curb set contains not only a_i, but also the neighboring actions. The only set with this property is $A_1 \times A_2$, finishing the proof. □

Acknowledgments

Mark Voorneveld thanks the Netherlands Organization for Scientific Research (NWO) and the Wallander/Hedelius Foundation for financial support during parts of the research.

References

Arthur, W.B., 1994. Inductive reasoning and bounded rationality. Amer. Econ. Rev. 84, 406-411.

Basu, K., Weibull, J.W., 1991. Strategy subsets closed under rational behavior. Econ. Letters 36, 141-146.

Challet, D., Marsili, M., and Zhang, Y.-C., 2004. Minority games: Interacting agents in financial markets. Oxford University Press.

Challet, D., Zhang, Y.-C., 1997. Emergence of cooperation and organization in an evolutionary game. Physica A 246, 407-418.

Coolen, A.C.C., 2005. The mathematical theory of minority games: Statistical mechanics of interacting agents. Oxford University Press.

Dubey, P., Haimanko, O., Zapechelnyuk, A., 2004. Strategic complements and substitutes, and potential games. Games Econ. Behav., to appear.

Hurkens, S., 1995. Learning by forgetful players. Games Econ. Behav. 11, 304-329.

Hurkens, S., 1996. Multi-sided pre-play communication by burning money. J. Econ. Theory 69, 186-197.

Kosfeld, M., Droste, E., Voorneveld, M., 2002. A myopic adjustment process leading to best-reply matching. Games Econ. Behav. 40, 270-298.

Koster, M., Reijnierse, H., Voorneveld, M., 2003. Voluntary contribution to multiple public projects. J. Public Econ. Theory 5, 25-50.

Milchtaich, I., 1996. Congestion games with player-specific payoff functions. Games Econ. Behav. 13, 111-124.

Milgrom, P., Roberts, J., 1990. Rationalizability, learning, and equilibrium in games with strategic complementarities. Econometrica 58, 1255-1277.

Monderer, D., Shapley, L.S., 1996. Potential games. Games Econ. Behav. 14, 124-143.

Moro, E., 2003. The minority game: an introductory guide. In: Korutcheva, A., Cuerno, R. (Eds.), Advances in condensed matter and statistical physics. Nova Science Publishers, New York.

Morris, S., Ui, T., 2004. Best response equivalence. Games Econ. Behav. 49, 260-287.

Pruzhansky, V., 2003. On finding curb sets in extensive games. Int. J. Game Theory 32, 205-210.

Quint, T., Shubik, M., 1994. A model of migration. Cowles Foundation Discussion Papers, No. 1088, Yale University.

Rosenthal, R.W., 1973. A class of games possessing pure-strategy Nash equilibria. Int. J. Game Theory 2, 65-67.

Shapley, L.S., 1964. Some topics in two-person games. In: Dresher, M., Shapley, L.S., Tucker, A.W. (Eds.), Advances in game theory, Ann. of Math. Stud., Vol. 52, pp. 1-28.

Slade, M.E., 1994. What does an oligopoly maximize? J. Ind. Econ. 57, 45-62.

Slikker, M., 2001. Coalition formation and potential games. Games Econ. Behav. 37, 436-448.

Topkis, D.M., 1978. Minimizing a submodular function on a lattice. Oper. Res. 26, 305-321.

Topkis, D.M., 1998. Supermodularity and complementarity. Princeton, NJ: Princeton University Press.

Voorneveld, M., 1997. Equilibria and approximate equilibria in infinite potential games. Econ. Letters 56, 163-169.

Voorneveld, M., 2000. Best-response potential games. Econ. Letters 66, 289-295.

Voorneveld, M., 2004. Preparation. Games Econ. Behav. 48, 403-414.

Voorneveld, M., 2005. Persistent retracts and preparation. Games Econ. Behav. 51, to appear.

Voorneveld M., Borm P., van Megen F., Tijs S., and Facchini G., 2000. Congestion games and potentials reconsidered. Int. Game Theory Rev. 1, 283-299.

Young, H.P., 1993. The evolution of conventions. Econometrica 61, 57-84.

Zhou, L., 1994. The set of Nash equilibria of a supermodular game is a complete lattice. Games Econ. Behav. 7, 295-300.