Working Papers

Institute of Mathematical Economics

394

September 2007

Optimal Consumption Choice with Intolerance for Declining Standard of Living

Frank Riedel

IMW · Bielefeld University
Postfach 100131
33501 Bielefeld · Germany

email: imw@wiwi.uni-bielefeld.de
http://www.wiwi.uni-bielefeld.de/~imw/Papers/showpaper.php?394
ISSN: 0931-6558

Optimal Consumption Choice with Intolerance for Declining Standard of Living

By Frank Riedel

Institute of Mathematical Economics (IMW)
Bielefeld University
Germany
friedel@uni-bielefeld.de

Abstract

Duesenberry introduced the notion of a ratchet investor who does not tolerate any decline in her consumption rate. We connect the demand behavior of such an agent to the behavior of standard time–additive agents. A ratchet investor demands the running maximum of the optimal plan a conventional time–additive investor with lower initial wealth would choose.

JEL subject classification. D91
Key words and phrases. Intertemporal Consumption Choice, Habit Formation, Non-Time Separable Utility

Introduction

Intertemporal preferences are fundamental for the microeconomic theory of intertemporal consumption and investment, the theory of financial markets and asset pricing, as well as macroeconomic growth and business cycle theory. The assumptions imposed on preferences have a decisive impact, of course, on the shape of these theories.

Conventionally, temporal economic models are built on time–additive expected utility. There, the utility of a multiperiod consumption plan is given by the expected discounted sum of period utilities, and the period utility depends only on the consumption of that period. In continuous time, the period utility is a function of the rate of consumption, that is the infinitesimal consumption per unit time.

Time–additivity is, of course, a strong assumption. In particular, it excludes any path dependence of utility from consumption and does not allow to model phenomena like habit formation. Moreover, models based on time–additive preferences lead to results which are hard to reconcile with the data on prices and consumption behavior. Consumption is much too volatile in time–additive models, and the equity premium too high, the short interest rate too low, to cite just a few 'puzzles' derived from time–additive models.

Several other utility functionals have been proposed to overcome these weaknesses of time–additive utility. A prominent class form the habit formation preferences (see (Constantinides 1990) and (Sundaresan 1989)) where period utility is a function of current consumption and an index of past consumption. Intuitively, the index of past consumption represents a floor for future consumption rates. This idea is pushed to its extreme by (Dybvig 1995) where the investor does not accept any decline in his consumption rate. The investor keeps the time–additive utility functional to evaluate consumption plans as long as these are nondecreasing, while assigning a value of negative infinity to all other consumption plans. Dybvig derives an explicit solution when investors have constant relative risk aversion and can invest in a complete financial market driven by Brownian motion. He also dicusses extensions to multiple goods, intolerance beyond some rate of decline and portfolio constraints.

This paper studies the utility maximization problem of the investor with Dybvig's preferences, or the ratchet investor, as we will call him here. We extend Dybvig's analysis in several directions. First, the problem is solved for all separable felicity functions explicitly. This is remarkable because in non time–additive models closed-form solutions are usually available only for restricted classes of preferences, as constant relative risk aversion, e.g. As an important new insight, the result shows that the consumption plan of the ratchet investor is the running maximum of the consumption plan of a corresponding time–additive investor. The ratchet investor derives therefore his demand from equating period marginal utility and current price as the time–additive investor does, but uses ratchet and pawl to avoid any decline in his consumption.

Second, we derive this result in a more general stochastic framework. Dybvig considers a complete financial market driven by Brownian motion. Thus,

log–returns of assets are assumed to be normally distributed. We drop this assumption while keeping the convenient homogeneous Markovian structure. We only assume independent and stationary increments for the stochastic process which describes the underlying risk.

A minor improvement is that this paper removes Dybvig's assumption that the minimal level of consumption be strictly positive.

Our method of proof does not rely on dynamic programming. Instead, we use the usual concavity argument from demand theory as well as the regularity of consumption paths required by the ratchet investor. Essentially, the proof rests on two integrations by parts, made possible by the fact that consumption plans are nondecreasing, and the calculation of some expected values where the Markovian structure plays a role. In this sense, the present approach delivers a more elementary proof of the result.

The next section describes the model, states the main result, and considers some case studies— the deterministic case, a model with Poisson jumps, and Brownian motion. In Section 2, we study conditions under which the candidate solutions have finite prices, an assumption made in the main theorem. Section 3 contains the proof of the main result and the final section discusses some possible extensions and concludes.

1 Model and Result

We consider an investor who chooses a rate of consumption $(c_t)_{t\geq 0}$ within an infinite horizon. Following the approach initiated by Dybvig (1995), we assume that the investor does not accept a decline in his period rate of consumption. A simple way to model such preferences is given by

$$V(c) = \begin{cases} \mathbb{E}\int_0^\infty e^{-\delta t}u(c_t)dt & \text{if } c \text{ nondecreasing and } c_0 \geq c_{0-} \\ -\infty & \text{else} \end{cases}.$$

Thus, the investor uses the time–additive expected utility function with discount factor $\delta > 0$ as long as the additional requirement of monotonicity is satisfied. $c_{0-} \geq 0$ is the minimal level of consumption required by the consumer. The expectation is taken with respect to a probability measure $\mathbb{P}$ of a suitable filtered probability space $\left(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}\right)$.

We will assume that the the felicity function u is strictly increasing, concave, and continuously differentiable with a strictly decreasing derivative u' satisfying $u'(\infty) = 0$. The strictly decreasing inverse of u' is denoted by i. Note that we do not need to assume that marginal felicity at zero is infinite. The important case of felicity functions with constant absolute risk aversion is thus included.

The investor is endowed with initial capital $w > 0$ which he uses to buy a consumption plan[1] $(c_t)_{t\geq 0}$ on a complete market with Arrow–Debreu price functional

$$\Psi(c) = \mathbb{E}\int_0^\infty \psi_t c_t dt\,.$$

[1] Consumption plans are adapted, nonnegative processes.

Moreover, we assume that the Arrow–Debreu price density ψ has the following structure:

$$\psi_t = \exp\left(-rt - \theta Z_t - \pi(-\theta)t\right) .$$

Here, $r > 0$ is the interest rate, θ is the market price of risk. Z is a Markov process with stationary and independent increments starting in $Z_0 = 0$. The function π is the Laplace exponent which is given by $\mathbb{E}\exp(\xi Z_1) = \exp(\pi(\xi))$.

Such a structure arises canonically from a financial market driven by the Markov process Z[2]. Dybvig uses the widespread Samuelson–Merton model of the asset market where Z is a Brownian motion. The model presented here is more general in that we drop the assumption of normally distributed increments while keeping the convenient Markovian structure. Our setup includes, beyond the Brownian model, also the case of jump processes like the Poisson process (cf. Section 1.1).

In the following, we study the investor's maximization problem:

$$\text{maximize } V(c) \text{ subject to } \Psi(c) \leq w\,. \tag{1}$$

Since the budget set is empty when the perpetuity value of the minimal consumption plan $c_t \equiv c_{0-}$ exceeds the initial capital, we assume throughout that

$$w > \frac{c_{0-}}{r}.$$

Before the solution to the utility maximization problem is presented, it may be useful to recall the optimal consumption plan when the consumption rate is not constrained to be nondecreasing. Time–additive investors just equate marginal felicity and current price at time t. For a suitable Lagrange parameter $K > 0$, the optimal consumption plan $m(K)$ is given by

$$e^{-\delta t}u'(m(K)_t) = K\psi_t\,, \tag{2}$$

if $e^{-\delta t}u'(0) \geq K\psi_t$ and $m(K)_t = 0$ otherwise.

The following theorem provides a complete solution to the investor's problem.

Theorem 1.1 *For a positive constant K, let $m(K)_t$ be the optimal consumption plan of an unconstrained time–additive investor with Lagrange parameter K, that is $m(K)$ solves (2). Denote by $c(K)$ the maximum of the minimum consumption level c_{0-} and the running maximum of the time–additive investor's optimal plan,*

$$c(K)_t = \max\left\{c_{0-},\ \sup_{0\leq s\leq t} m(K)_s\right\} .$$

$c(K)$ solves (1) for initial capital $w = \Psi(c(K))$, as long as $\Psi\left(c(K)\right) < \infty$.

[2] On the relation between dynamically complete financial markets and Arrow–Debreu price densities in continuous time, see (Cox and Huang 1989), (Karatzas, Lehoczky, and Shreve 1987) for the Brownian framework and (Back 1991) for general processes.

The proof of the theorem is given in Section 3. Here, we discuss and interpret the solution.

The above theorem provides a complete and explicit solution. In order to find a solution for given initial capital w, one has to compute the prices $\Psi(c(K))$ for all parameters K (which play the role of Lagrange multipliers). These prices are a decreasing function of K and it remains to determine the value of K which matches the initial capital. Of course, this method requires that the prices of the candidates $c(K)$ are finite. In general, assumptions on the parameters of the problem are required to ensure this. This question is studied in detail in Section 2.

The main insight provided by the above theorem is that consumption rate of a ratchet investor is the running maximum of the consumption rate a suitable time–additive investor would choose. In fact, with time–additive utilities, the solution is given by equating period marginal felicity and price, that is, by $m(K)$. The ratchet investor copies the behavior of the time–additive type while introducing the ratchet — his consumption rate cannot decline. It increases whenever the time–additive type reaches a new running maximum in his consumption rate.

Since the consumption rate of the time–additive investor is either $m(K)_t = i\left(K\psi_t e^{\delta t}\right)$ or zero, and i is strictly decreasing, the optimal plan of the ratchet type can be written as

$$c(K)_t = \max\left\{c_{0-}, i\left(K \inf_{0\le s\le t} \psi_s e^{\delta s}\right)\right\} \tag{3}$$

$$= \max\left\{c_{0-}, i\left(e^{-\sup_{0\le s\le t}(\theta Z_s + (r-\delta+\pi(-\theta)))s}\right)\right\}. \tag{4}$$

The consumption behavior of the ratchet type is thus determined by the running maximum of the Arrow–Debreu price process ψ adjusted by the time preferences of the investor. The consumption rate increases when the ratio of the price process and time preference $\psi_t/e^{-\delta t}$ reaches a new minimum. This implies, in particular, that the times when the consumption rate increases do not depend on the investor's risk attitude, but only on price level and time preference. All investors with the same time preference increase their demand at the same points in time.

The second formula (4) shows that the running maximum of the Markov process $X_t := \theta Z_t + (r - \delta + \pi(-\theta))\, t$ determines the optimal behavior of the ratchet investor. As a caveat, we emphasize that the Markovian assumption plays an important role here. Whether the investor optimally increases his rate of consumption forever or not, depends, of course, on the expected future evolution of the price process. Due to the Markov assumption, the current value of the price process is a sufficient statistic for that decision and it sufficies therefore to base one's behavior on this one variable.

The nature of the underlying risk structure of the financial market leads to several possible consumption patterns as is illustrated in the following case studies.

1.1 Case Studies

1.1.1 Deterministic Case

Our setup includes the deterministic case ($\theta = 0$). There, the time–additive investor's optimal consumption plan is given by

$$e^{-\delta t} u'(m(K)_t) = Ke^{-rt}$$

for a suitable Lagrange multiplier K, or, equivalently,

$$m(K)_t = i\left(Ke^{(\delta - r)t}\right),$$

where i denotes the decreasing inverse of marginal felicity u'.

This leads to two cases. When the discount factor is greater than the interest rate, $\delta > r$, the time–additive agent exhibits a decreasing consumption pattern. Accordingly, the corresponding consumption rate of the ratchet investor is constant over time, $c(K)_t = \max\{c_{0-}, m(K)_0\}$. When, instead, the investor is relatively patient, $\delta \leq r$, then $m(K)$ is nondecreasing. Thus, the monotonicity constraint of the ratchet investor does not bind, and both types exhibit the same consumption behavior:

$$c(K)_t = \max\{c_{0-}, m(K)_t\}.$$

In order to gain some intuition, it might be useful to consider a numerical example. Suppose the lifetime income of the investor is 4 million dollars, the interest rate is $r = 5\%$, the personal discount factor is $\delta = 20\%$, and the investor has constant relative risk aversion 1 (log-investor). The time–additive investor consumes at a rate $m_t = 800000 * \exp(-0.15t)$. Thus, due to his impatience, he spends a lot of money in his early years. By contrast, the ratchet investor exhibits a constant consumption rate $c_t = 200000$. The ratchet investor is thus much more cautious in his early years, and after 9.2 years, the ratchet investor can afford a higher consumption rate than the time–additive investor. After 30 years, poor time–additive investor consumes at a rate of about 8900 dollars, while his counterpart still enjoys his 200000 dollars.

1.1.2 Poisson Jumps

Next, we consider a world in which positive shocks of a fixed size occur at unpredictable random times. Such a scenario is well described by a a financial market whose risk structure is given by a Poisson process Z. The Laplace exponent of a Poisson process with intensity λ is given by $\pi(\xi) = \lambda\left(e^{\xi} - 1\right)$. A Poisson process jumps by one at independent exponentially distributed random times. For a positive market price of risk $\theta > 0$, this means that the price process has negative jumps when the Poisson process jumps.

In this case, the relevant process $X_t = \theta Z_t + (r - \delta + \pi(-\theta))\, t$ is nondecreasing when $r + \pi(-\theta) \geq \delta$, and so is the optimal rate $m(K)$ of the time–additive investor. The monotonicity constraint does not bind, therefore, and the ratchet

investor exhibits the same consumption behavior as the time–additive type — the consumption rate steadily increases and jumps by a certain size when a price shock occurs.

In the case $r + \pi(-\theta) < \delta$ the process X has a continuous negative drift and jumps upwards whenever the Poisson process jumps. If this jump is large enough to induce a new running maximum of that process, the ratchet investor reacts with a discontinuous upward increase of his consumption rate, while his consumption rate is constant otherwise.

1.1.3 Brownian Motion

When Z is a Brownian motion, the consumption rate of the ratchet investor is given by the running maximum of a Brownian motion with drift $r-\delta+\frac{\theta^2}{2}$. This is a continuous, yet not absolutely continuous, increasing process. The lack of absolute continuity should come as no surprise here. Indeed, the time–additive investor's consumption rate in this model is a monotone function of Brownian motion with drift and, therefore, nowhere differentiable.

Continuing the numerical example from above, it can be shown that the time–additive investor still starts with a consumption rate of 800000 dollars which decreases almost surely to zero, whereas the ratchet investor has a lower initial consumption rate due to the risky investment opportunity set. For a market price of risk $\theta = 30\%$, the initial consumption rate is around 156000 dollars.

2 Finiteness of Prices

In this section, we provide general conditions under which the price of the candidate solution is finite. To this end, it is useful to introduce the equivalent martingale measure $\mathbb{P}^*$. Its density with respect to $\mathbb{P}$ on $\mathcal{F}_t$ is given by $d_t = e^{rt}\psi_t$. The price of a consumption plan is equal to its expected present value under $\mathbb{P}^*$, that is[3]

$$\Psi(c) = \mathbb{E}^* \int_0^\infty e^{-rt} c_t dt\,.$$

Note that the discount factor e^{-rt} is proportional to the density of an exponentially distributed random variable τ with parameter r. With this auxiliary random variable τ, independent of c, the price of a consumption plan can be written as $\Psi(c) = \frac{1}{r}\mathbb{E}^* c_\tau$.

We have the following

Lemma 2.1 *The price of the consumption plan $c(K)$ is finite iff one of the*

[3] compare, e.g., (Duffie 1992, Chapter 9.E).

following conditions holds true:

$$\mathbb{E}^* \int_0^\infty e^{-rt} \max\left\{c_{0-}, i\left(Ke^{-\sup_{0\le s\le t} X_s}\right)\right\} dt < \infty \tag{5}$$

$$\int_0^\infty \max\left\{c_{0-}, i\left(Ke^{-\xi}\right)\right\} G(d\xi) < \infty\,, \tag{6}$$

where G is the distribution function of the random variable $\sup_{0\le s\le\tau} X_s$ under P^* and τ an independent exponentially distributed random variable with parameter r.

The process X is smaller for larger values of the discount factor δ. The first characterization (5) implies therefore that if prices are finite for some δ, then they are finite for all larger values $\delta' > \delta$. As plausible in infinite horizon models, the discount factor must be large enough to ensure wellposedness of the problem.

The second characterization (6) is very useful in situations when the distribution function of the running maximum of X stopped at an independent exponential time is known. This is trivially the case when X is nonincreasing or when X is a deterministic drift, that is $X_t = At$ for some constant A. When X has no upward jumps, the running maximum process is still a continuous process. This continuity, the Markov property and the lack of memory of the exponential law allow to identify the distribution of that maximum as exponential, see (Bertoin 1996, Chapter 7).

Theorem 2.2 *The price of the candidate solutions $c(K)$ is finite for all $K > 0$ in the following classes of models:*

1. *X is nonincreasing;*
2. *X has no upward jumps and the agent has constant absolute risk aversion.*
3. *X has no upward jumps, the agent has constant relative risk aversion α, and*
$$\delta > \delta^* := (1-\alpha)(r + \pi(-\theta)) + \alpha\pi\left(\frac{(1-\alpha)\theta}{\alpha}\right) \tag{7}$$
holds true. Condition (7) holds true for $\alpha > 1$.

Proof : When X is nonincreasing, the candidate $c(K)$ is constant, and its price is thus $\Psi(c(K)) = \frac{c(K)_0}{r}$.

From now on, assume that X is not a nonincreasing process and has no upward jumps. In this case, an important theorem from the theory of Markov processes (Bertoin 1996, Chapter 7, Corollary 2) tells us that the distribution G of the running maximum of X stopped at an independent exponential time τ is exponential. Its parameter is the unique positive solution k of $\pi^*(k) = r$, where

π^* is the Laplace exponent of X under the equivalent martingale measure $\mathbb{P}^*$. The Laplace exponent π^* can be calculated as follows:

$$\begin{aligned}\mathbb{E}^* e^{\xi X_1} &= \mathbb{E} e^{\xi X_1 - \theta Z_1 - \pi(-\theta)} \\ &= \mathbb{E} e^{(\xi-1)\theta Z_1 + \xi(r-\delta) + (\xi-1)\pi(-\theta)} \\ &= \mathbb{E} e^{\pi((\xi-1)\theta) + \xi(r-\delta) + (\xi-1)\pi(-\theta)}\,.\end{aligned}$$

We obtain therefore $\pi^*(\xi) = \pi((\xi-1)\theta) + \xi(r-\delta) + (\xi-1)\pi(-\theta)$.

Consider next the case of constant absolute risk aversion α, that is $u(x) = -\frac{1}{\alpha}e^{-\alpha x}$, and $i(x) = \max\left\{0, -\frac{1}{\alpha}\log(x)\right\}$. Applying (6), we obtain that the price of $c(K)$ is finite iff

$$\int_0^\infty \max\left\{c_{0-}, \frac{1}{\alpha}\left(\xi - \log K\right)\right\} k e^{-kx} dx < \infty\,,$$

which is obviously the case.

With constant relative risk aversion α, one has $u(x) = \frac{x^{1-\alpha}}{1-\alpha}$, and $i(x) = x^{-\frac{1}{\alpha}}$. It is easy to see from (6) that the price of $c(K)$ is finite iff $\int e^{\frac{\xi}{\alpha}} G(d\xi) < \infty$, that is, iff $\alpha^{-1} < k$. Since π^* is convex and increasing when positive, this is equivalent to $\pi^*(\alpha^{-1}) < \pi^*(k) = r$, or (7).

Finally, note that for $\alpha > 1$, the convexity of π^* yields $(1-\alpha)\pi(-\theta) + \alpha\pi\left(\frac{(1-\alpha)\theta}{\alpha}\right) \leq \pi^*(0) = 0$, and one obtains $\delta^* \leq (1-\alpha)r < 0$. Hence, the condition $\delta > \delta^*$ is always satisfied. □

It may be interesting to note that the critical value δ^* in the case of constant relative risk aversion is the same as for time–additive utility functions (see (Merton 1990, Section 4.6)) as well as for Hindy–Huang–Kreps preferences, see (Bank and Riedel 2001, Theorem 4.9). For the Brownian case, we recover, of course, Dybvig's condition (7)[4].

3 Proof of the Main Theorem

The proof is relatively straightforward in that it requires only partial integrations and the calculation of expected values. Here is an outline. It suffices, of course, to consider nondecreasing consumption plans only because all other plans lead to negative infinite utility. In a first step, we use a partial integration to show that the price of a nondecreasing consumption plan can be written as

$$\Psi(c) = \frac{1}{r}\left(\mathbb{E}\int_0^\infty \psi_t\, dc_t + c_{0-}\right)\,. \tag{8}$$

Now let $c(K)$ be our candidate solution and take another nondecreasing consumption plan c with $\Psi(c) \leq \Psi(c(K))$. Concavity of the felicity function u

[4]Dybvig expresses the condition as a critical value for the parameter $1-\alpha$. In terms of the discount factor, this is equivalent to (7), compare p.295 in (Dybvig 1995).

implies that

$$V(c(K)) - V(c) \geq \mathbb{E} \int_0^\infty e^{-\delta t} u'(c(K)_t) \left(c(K)_t - c_t\right) dt \,,$$

and partial integration leads to

$$V(c(K)) - V(c) \geq \mathbb{E} \int_0^\infty \mathbb{E}\left[\int_s^\infty e^{-\delta t} u'(c(K)_t) dt | \mathcal{F}_s\right] \left(dc(K)_s - dc_s\right) . \quad (9)$$

Then, we show that the definition of $c(K)$ leads to the following inequality for all (stopping) times s and some constant L:

$$\mathbb{E}\left[\int_s^\infty e^{-\delta t} u'(c(K)_t) dt | \mathcal{F}_s\right] \leq L\psi_s \,. \quad (10)$$

It is furthermore shown that equality holds true whenever $c(K)$ has a point of increase in s, that is $dc(K)_s > 0$. By plugging (10) into (9), one obtains

$$V(c(K)) - V(c) \geq L\mathbb{E} \int_0^\infty \psi_s \left(dc(K)_s - dc_s\right) ,$$

and (8) yields then

$$V(c(K)) - V(c) \geq \frac{L}{r} \left(\Psi(c(K)) - \Psi(c)\right) \geq 0 \,,$$

and the proof is done.

The remainder of this section is devoted to the proofs of (8), (9), and (10).

Proof of (8) As a preparation, note that the conditional expectation of the price process is

$$\mathbb{E}\left[\psi_t | \mathcal{F}_s\right] = \psi_s e^{-r(t-s)} \,. \quad (11)$$

Let c be a nondecreasing consumption plan with a finite price, $\Psi(c) < \infty$, and $c_0 \geq c_{0-}$. Fubini's theorem or partial integration shows that for $0 < T < \infty$

$$\int_0^T \psi_t c_t dt = \int_0^T \psi_t \left(\int_{[0,t]} dc_s + c_{0-}\right) dt \quad (12)$$

$$= \int_0^T \int_s^T \psi_t dt dc_s + c_{0-} \int_0^T \psi_t dt \,. \quad (13)$$

Taking expectations, we get with the help of (11)

$$\mathbb{E} \int_0^T \psi_t c_t dt = \mathbb{E} \int_0^T \mathbb{E}\left[\int_s^T \psi_t dt | \mathcal{F}_s\right] dc_s + c_{0-} \int_0^T e^{-rt} dt \,,$$

where we may take the conditional expectation under the integral on the right side because c is an adapted process (cf. (Jacod and Shiryaev 1987, Lemma I.3.12)).

Due to (11), we have

$$\mathbb{E}\left[\int_s^T \psi_t dt|\mathcal{F}_s\right] = \psi_s \int_s^T e^{-r(t-s)} dt\,,$$

and

$$\mathbb{E}\int_0^T \psi_t c_t dt = \mathbb{E}\int_0^T \psi_s \int_s^T e^{-r(t-s)} dt dc_s + c_{0-}\int_0^T e^{-rt} dt$$

follows. The desired relation (8) is obtained by letting $T \to \infty$ and using monotone convergence.

Proof of (9) Let c be a nondecreasing consumption plan with $\Psi(c) \leq \Psi(c(K))$. Since the felicity function is concave, we have

$$u(c(K)_t) - u(c_t) \geq u'(c(K)_t)\,(c(K)_t - c_t)\ ,$$

and thus

$$V(c(K)) - V(c) \geq \mathbb{E}\int_0^\infty e^{-\delta t} u'(c(K)_t)\,(c(K)_t - c_t)\,dt\,.$$

Another partial integration as in (12) and (13) above yields

$$V(c(K)) - V(c) \geq \mathbb{E}\int_0^\infty \mathbb{E}\left[\int_s^\infty e^{-\delta t} u'(c(K)_t) dt|\mathcal{F}_s\right](dc(K)_s - dc_s)\ .$$

This is (9).

Proof of (10) Since

$$c(K)_t = \max\left\{c_{0-}, i\left(Ke^{-\sup_{0\leq u\leq t} X_u}\right)\right\}$$

for $X_t = \theta Z_t + (r - \delta + \pi(-\theta))\,t$ (cf. (4)), we have for all stopping times $s \leq t$

$$u'(c(K)_t) \leq K\exp\left(-\sup_{0\leq u\leq t} X_u\right) \leq K\exp\left(-\sup_{s\leq u\leq t} X_u\right)\ .$$

Moreover, equality holds true when $c(K)$ has a point of increase in s because in this case we have

$$c(K)_t = \max\left\{c(K)_s, i\left(Ke^{-\sup_{s\leq u\leq t} X_u}\right)\right\}\ .$$

We thus obtain

$$\begin{aligned}\mathbb{E}\left[\int_s^\infty e^{-\delta t}u'(c(K)_t)dt|\mathcal{F}_s\right] &\leq \mathbb{E}\left[\int_s^\infty e^{-\delta t}K\exp\left(-\sup_{s\leq u\leq t}X_u\right)dt|\mathcal{F}_s\right]\\ &= K\exp(-X_s-\delta s)\mathbb{E}\left[\int_s^\infty e^{-\delta(t-s)}\exp\left(-\sup_{s\leq u\leq t}(X_u-X_s)\right)dt|\mathcal{F}_s\right]\\ &= K\psi_s\mathbb{E}\left[\int_s^\infty e^{-\delta(t-s)}\exp\left(-\sup_{s\leq u\leq t}(X_u-X_s)\right)dt|\mathcal{F}_s\right],\end{aligned}$$

with equality when $c(K)$ has a point of increase in s. The Markov property yields now that

$$\mathbb{E}\left[\int_s^\infty e^{-\delta(t-s)}\exp\left(-\sup_{s\leq u\leq t}(X_u-X_s)\right)dt|\mathcal{F}_s\right] = \mathbb{E}\int_0^\infty e^{-\delta t-\sup_{0\leq u\leq t}X_t}dt\,,$$

and we finally get (10) with

$$L := K\,\mathbb{E}\int_0^\infty e^{-\delta t}\exp\left(-\sup_{0\leq u\leq t}X_t\right)dt\,.$$

4 Conclusion

Several extensions of the model have already been discussed in (Dybvig 1995). For example, one could weaken the requirement of nondecreasing consumption plans to intolerance beyond some positive rate of decline D, that is $\frac{dc_t}{c_t} \geq -D\,dt$. When the investor has constant relative risk aversion, this problem can be transformed to the one solved above by a simple change of variables. In a similar spirit, one could require that the consumption plan does not decrease faster than at a certain speed, e.g. $dc_t \geq -D\,dt$ for a positive constant D. A similar change of variables reduces this problem to the original one when the investor has constant absolute risk aversion. Also the case of multiple goods is easily solved as long as the felicity function is additive across goods. In this case, the investor distributes his wealth optimally across goods and solves the corresponding optimization problem separately for every good.

A more difficult task is to drop the Markovian assumption. In this case, the main theorem above does not apply, and it might be difficult to obtain explicit solutions at all. Still, as the proof of the main theorem shows, something can be said about optimal solutions. The careful reader will note that the proof can be used to show that a plan c is optimal if it satisfies (10), that is for all stopping times s and some constant L

$$\mathbb{E}\left[\int_s^\infty e^{-\delta t}u'(c_t)dt|\mathcal{F}_s\right] \leq L\psi_s$$

with equality if $dc_s > 0$. Of course, it is in general difficult to obtain c from this inequality. However, the methods I developed with Peter Bank in (Bank and Riedel 2001) suggest that c can still be identified as the running maximum of some process L. One might conjecture that L will be related to yet no longer be simply equal to the consumption plan a time–additive investor would choose.

References

BACK, K. (1991): "Asset Pricing for General Processes," *Journal of Matthematical Economics*, 20, 371–396.

BANK, P., AND F. RIEDEL (2001): "Existence and Structure of Stochastic Equilibria with Intertemporal Substitution," *Finance and Stochastics*, 5, 487–509.

BERTOIN, J. (1996): *Lévy Processes*. Cambridge University Press.

CONSTANTINIDES, G. (1990): "Habit Formation: A Resolution of the Equity Premium Puzzle," *Journal of Political Economy*, 98, 519–543.

COX, J. C., AND C.-F. HUANG (1989): "Optimal Consumption and Portfolio Policies When Asset Prices Follow a Diffusion Process," *Journal of Economic Theory*, 49, 33–83.

DUFFIE, D. (1992): *Dynamic Asset Pricing Theory*. Princeton University Press, Princeton.

DYBVIG, P. H. (1995): "Duesenberry's Ratcheting of Consumption: Optimal Dynamic Consumption and Investment Given Intolerance for any Decline in Standard of Living," *Review of Economic Studies*, 62, 287–313.

JACOD, J., AND A. SHIRYAEV (1987): *Limit Theorems for Stochastic Processes*. Springer.

KARATZAS, I., J. P. LEHOCZKY, AND S. E. SHREVE (1987): "Optimal Portfolio and Consumption Decisions for a 'Small Investor' on a Finite Horizon," *SIAM Journal of Control and Optimization*, 25, 1557–1586.

MERTON, R. C. (1990): *Continuous-Time Finance*. Blackwell Publishers.

SUNDARESAN, S. (1989): "Intertemporally Dependent Preferences and the Volatility of Consumption and Wealth," *Review of Financial Economics*, 2, 73–89.