Antagonistic Properties and n-Person Games

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Abstract In this note, we studied some classes of n-person games possessing properties of two person zero-sum games. We extend the deønition of a two-person almost strictly competitive game (Aumann 1961) to the n-person case. We show that the Nash equilibria of a n-person almost strictly competitive game induce the same payoœ; and we exhibit the connections between almost strictly competitive games and some classes of n-person games introduced by Kats and Thisse in 1992.

Introduction

In a two-person zero-sum game, the gain of one player is equal to the loss of his opponent. This class of games has some important features: when equilibria exist, they induce a unique payoœ, the set of Nash equilibria is convex, the equilibria are interchangeable[1]...
Some classes of two-person non zero-sum games having some of these nice properties have been introduced by diœerent authors. The deønitions of these classes are based on diœerent notions of antagonism. Indeed, zero-sum games correspond to the extreme case of competition between two players: what Player 1 wins is equal to what Player 2 loses. By weakening this notion of antagonism, we get some classes of non zero-sum games which satisfy

[1] Equilibria are interchangeable if for every equilibria (s_1, s_2) and (s'_1, s'_2), (s_1, s'_2) and (s'_1, s_2) are also equilibria (Nash 1951). Note that for the mixed extension of a ønite game, if the equilibria are interchangeable, then the set of Nash equilibria is convex. In fact, these two properties are equivalent for the mixed extension of every ønite two-person game but it is no longer true in the n-person case when $n > 2$ (Chin, Parthasarathy and Raghavan 1974).

some properties of zero-sum games.
The deønitions of some of these classes are also available for games with ønitely many players. The aim is the same as in the two-person case: to deøne classes of n-person games which possess some properties of two-person zero-sum games, as for example uniqueness of equilibrium payoœ. But the problematic is diœerent: we have to deøne the notion of antagonism between n players.
In section 1, we recall the deønition of n-person game of type A, B and C introduced by Kats and Thisse (1992). In section 2, we deøne the notions of saddle-point and value of a n-person game. With the help of these deønitions, we extend the deønitions of games of type I (introduced by Aumann (1961) under the name of almost strictly competitive games(ASC)), II and IV to the n-person case[2]. In section 3, we give some results concerning the connection between these diœerent classes. In section 4, we generalize Aumann's theorem concerning game of type I in extensive form (Aumann 1961) to the n-person case. At last some examples of games are given in section 5.

Notations

We denote by $G = (I, (S_i)_{i\in I}, (u_i)_{i\in I})$ a n-person game where

- $I = \{1, \dots, n\}$ is the set of players, $n \geq 2$.
- S_i is the set of strategies of Player i.
- u_i is the payoœ function of Player i; $u_i : S_1 \times \cdots \times S_n \to \mathbb{R}$ where $\mathbb{R}$ stands for the set of real numbers.

Let $S = \prod_{i\in I} S_i$. For each Player $i \in I$, $-i$ denotes the set $I \setminus \{i\}$ (i.e. $-i$ is the set of opponents of Player i). S_A terms the set $\prod_{i\in A} S_i$ $(A \subseteq I)$.
From now, we assume the following property:

Hypothesis 1 The sets S_i and the payoœ functions u_i are such that the game $G = (I, (S_i)_{i\in I}, (u_i)_{i\in I})$ has a Nash equilibrium.

For example, Hypothesis 1 is satisøed if each set of strategies is a convex compact subset of an Euclidian space and if the payoœ function of each player is continuous and quasi-concave in his own action (Glicksberg 1952).
We denote by $NE(G)$ the set of Nash equilibria of G and by $NEP(G)$ the set of its Nash equilibrium payoœs.

1 n-person game of type A, B and C

The antagonism for these three classes of (non zero-sum) games is deøned by comparing diœerent n-tuple of strategies according to several evaluation rules (see Figure 1):

[2] Games of type II and IV are generalizations of ASC games (Beaud 1999).

Type	Couples of strategies	Evaluation rule
A	Compare $s = (s_i, s_{-i})$ with $\tilde{s} = (\tilde{s}_i, \tilde{s}_{-i})$	$u_i(s) \geq u_i(\tilde{s}) \Leftrightarrow u_j(s) \leq u_j(\tilde{s}), \forall j \in I \setminus \{i\}$
B	Compare $s = (s_i, s_{-i})$ with $\tilde{s} = (\tilde{s}_i, s_{-i})$	$u_i(s) \geq u_i(\tilde{s}) \Leftrightarrow u_j(s) \leq u_j(\tilde{s}), \forall j \in I \setminus \{i\}$
C	Compare $s = (s_i, s_{-i})$ with $\tilde{s} = (\tilde{s}_i, s_{-i})$	$u_i(s) > u_i(\tilde{s}) \Rightarrow u_j(s) \leq u_j(\tilde{s})$ and $u_i(s) = u_i(\tilde{s}) \Rightarrow u_j(s) = u_j(\tilde{s}), \forall j \in I \setminus \{i\}$

Fig. 1 Deønitions of the classes.

This leads to the following deønitions:

Deønition 1 (Kats-Thisse, 1992) Let $G = (I, (S_i)_{i\in I}, (u_i)_{i\in I})$ be a n-person game.

- G is a game of type A if for all $i \in I$, $s, s' \in S$, $u_i(s) \geq u_i(s') \Leftrightarrow u_j(s) \leq u_i(s')$ $\forall j \in I \setminus \{i\}$.
- G is a game of type B if for each $i \in I$, for all $s_i, s_i' \in S_i$ and for all $s_{-i} \in S_{-i}$, we have

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \Leftrightarrow u_j(s_i, s_{-i}) \leq u_j(s_i', s_{-i}) \; \forall j \in I \setminus \{i\} \quad (1.1)$$

- G is a game of type C if for each $i \in I$, for all $s_i, s_i' \in S_i$ and all $s_{-i} \in S_{-i}$, we have

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \Rightarrow u_j(s_i, s_{-i}) \leq u_j(s_i', s_{-i}) \; \forall j \in I \setminus \{i\} \quad (1.2)$$

and

$$u_i(s_i, s_{-i}) = u_i(s_i', s_{-i}) \Rightarrow u_j(s_i, s_{-i}) = u_j(s_i', s_{-i}) \; \forall j \in I \setminus \{i\} \quad (1.3)$$

Remarks:

1. By deønition, every game of type A is of type B and every game of type B is of type C.
2. Two-person games of type A have been introduced under the name of strictly competitive games (Friedman 1983, Moulin 1976). Games of type B and C are also called unilaterally competitive games and weakly unilaterally competitive games (Kats and Thisse 1992).

Kats and Thisse (1992) have shown that every game of type C has a unique equilibrium payoœ, and that equilibria of a game of type B are interchangeable under some conditions on the sets of strategies and on the payoœ functions.

2 *n*-person game of type I, II and IV

The classes of games of type A, B and C are deøned directly be the preferences of each player without resorting to other concepts. This is no more the case for the classes we introduce now: we compare the strategic behavior of the players with the help of the notion of twisted equilibrium.

2.1 $n = 2$

Let $G = (S_1, S_2, u_1, u_2)$ be a two-person game. We associate to G the game $\bar{G} = (S_1, S_2, -u_2, -u_1)$. $\bar{G}$ is called the twisted game.
$s \in S$ is a twisted equilibrium of G if s is a Nash equilibrium of $\bar{G}$ (Aumann 1961).
$\mathbf{e} \in \mathbb{R}^2$ is a twisted equilibrium payoœ of G if there exists a twisted equilibrium s such that $u_i(s) = e_i$ for each $i = 1, 2$.
Aumann gives the following deønition of an almost strictly competitive game when $n = 2$ (Aumann 1961).

Deønition 2 G is an almost strictly competitive (ASC) game if

(i) there exists $s \in S$ which is a Nash and a twisted equilibrium;
(ii) the set of Nash equilibrium payoœs is equal to the set of twisted equilibrium payoœs.

Condition (i) of Deønition 2 may be deøned using the notion of a saddle-point of a two-person game (Beaud 1999):

Deønition 3 $\tilde{s} \in S$ is a saddle-point of the game G if for all $s \in S$, $i \in I$,

$$u_i(s_i, \tilde{s}_{-i}) \leq u_i(\tilde{s}) \leq u_i(\tilde{s}_i, s_{-i})$$

.

It is shown that the set of saddle-points of G, denoted by $S(G)$, is equal to the intersection of the sets of Nash and twisted equilibria of G. Hence, condition (i) of Deønition 2 is equivalent to: $S(G) \neq \emptyset$.
Aumann has shown that every almost strictly competitive game has a unique Nash equilibrium payoœ.

2.2 $n \geq 3$: saddle-point and value of a n-person game

The deønition of a twisted game does not extend when the number of players is greater than 2. In this latter case, how can we generalize the notion of a twisted equilibrium? Kats and Thisse suggest the following deønition of a twisted equilibrium (Kats and Thisse 1992):

Deønition 4 $\bar{s} \in S$ is a twisted equilibrium of a game G if $u_j(\bar{s}) \leq u_j(s_i, \bar{s}_{-i})$ for all $i \in I$, $s_i \in S_i$ and for all $j \in I \setminus \{i\}$.

By using this deønition of a twisted equilibrium, the deønition of an almost strictly competitive game can be extended to n-person games.
Unfortunately, Kats and Thisse's deønition is not satisfactory: we give now an example of a three-person almost strictly competitive game having two dioeerent Nash equilibrium payooes.
Example 1. $n = 3$, $S_i = \{A_i, B_i\}$ for each $i \in I$.
If $s_3 = A_3$:

$$\begin{array}{cc} & \begin{array}{cc} A_2 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ B_1 \end{array} & \begin{pmatrix} 1,3,5 & 1,3,5 \\ 1,4,5 & 1,4,5 \end{pmatrix} \end{array}$$

If $s_3 = B_3$:

$$\begin{array}{cc} & \begin{array}{cc} A_2 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ B_1 \end{array} & \begin{pmatrix} 1,4,5 & 1,4,5 \\ 1,4,5 & 1,4,5 \end{pmatrix} \end{array}$$

There are two Nash equilibria: (A_1, B_2, A_3) and (B_1, B_2, B_3). Hence $NEP = \{(1,3,5),$
$(1,4,5)\}$.
(A_1, B_2, A_3) and (B_1, B_2, B_3) are also the only twisted equilibria. Hence, $TEP = \{(1,3,5), (1,4,5)\}$.
So, there exists a prøle of strategies which is a twisted and a Nash equilibria, and the sets of Nash equilibrium payooes and twisted equilibrium payooes coincide: the game is almost strictly competitive and have two distinct Nash equilibrium payooes contrary to the two-person case.
Let us ørst generalize the notion of saddle-point to n-person games.

Deønition 5 $\tilde{s} \in S$ is a saddle-point of the game $G = (I, (S_i)_{i\in I}, (u_i)_{i\in I})$ if for every $i \in I$, for every $s \in S$,

$$u_i(s_i, \tilde{s}_{-i}) \leq u_i(\tilde{s}) \leq u_i(\tilde{s}_i, s_{-i}) \tag{2.4}$$

We denote by $S(G)$ the set of saddle-points of G.

Equation (2.4) means that for every $i \in I$, $\tilde{s}$ is a saddle-point of the function u_i with respect to maximizing in s_i and minimizing in s_{-i} (Rockafellar 1970).
This leads to the following deønition:

Deønition 6 $\bar{s} \in S$ is a strong twisted equilibrium of a n-person game G if:

$$\forall i \in I,\ \forall s_{-i} \in S_{-i},\ u_i(\bar{s}) \leq u_i(\bar{s}_i, s_{-i}) \tag{2.5}$$

We denote by $STE(G)$ (resp. $STEP(G)$) the set of strong twisted equilibria (resp. the set of the payooes induced by the strong twisted equilibria).

Remarks:

1. In deønition 4, any (unilateral) deviation of Player i induces a gain for all the other players whereas in deønition 6, any deviation of (part of) the other players induces a gain for Player i.
2. Deønition 6 is the same as the deønition of a twisted equilibrium when $n = 2$.
3. In the above example, $(1, 4, 5)$ is not a strong twisted equilibrium payoœ.

Deønition 7 $G = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$ is a game of:

- type I if
  a) there exists a proøle of strategies which is a Nash and a strong twisted equilibrium,
  b) the set of Nash equilibrium payoœs is equal to the set of strong twisted equilibrium payoœs;
- type II if
  a) there exists a proøle of strategies which is a Nash and a strong twisted equilibrium,
  b') the intersection between the set of Nash equilibrium payoœs and the set of strong twisted equilibrium payoœs is non empty;
- type IV if
  b') the intersection between the set of Nash equilibrium payoœs and the set of strong twisted equilibrium payoœs is non empty.

Example 2. Let $n = 3$, $S_i = \{A_i, B_i\}$ for each $i \in I$.
If $s_3 = A_3$:

$$\begin{array}{cc} & \begin{array}{cc} A_2 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ B_1 \end{array} & \begin{pmatrix} 1,0,0 & 0,1,0 \\ 0,0,1 & 0,0,0 \end{pmatrix} \end{array}$$

If $s_3 = B_3$:

$$\begin{array}{cc} & \begin{array}{cc} A_2 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ B_1 \end{array} & \begin{pmatrix} 0,0,0 & 0,0,1 \\ 0,1,0 & 1,0,0 \end{pmatrix} \end{array}$$

This game has two Nash equilibria, (B_1, B_2, A_3) and (A_1, A_2, B_3), which induce a payoœ equal to $(0, 0, 0)$. Indeed (B_1, B_2, A_3) and (A_1, A_2, B_3) are saddle-points.
By deønition of a Nash and of a strong twisted equilibrium, we get the following property:

Property 21 For every n-person game G, $S(G) = NE(G) \cap STE(G)$.

When $n = 2$, saddle-points are interchangeable (Beaud 1999). This is no more the case when $n > 2$. In the example above, (B_1, B_2, A_3) and (A_1, A_2, B_3) are saddle-points, but not (A_1, A_2, A_3).

2.2.1 Value of a n-person game

We can associate to each Player i two quantities:

1. The max-min of Player i: $\underline{\nu}_i = \max_{S_i} \min_{S_{-i}} u_i(\cdot,\cdot)$.
2. The min-max of Player i: $\bar{\nu}_i = \min_{S_{-i}} \max_{S_i} u_i(\cdot,\cdot)$.

Note that $\bar{\nu}_i \geq \underline{\nu}_i$ for all $i \in I$.

Deønition 8 The n-person game $G = (I, (S_i)_{i\in I}, (u_i)_{i\in I})$ has a vector value $\nu \in \mathbb{R}^n$ if $\bar{\nu}_i = \underline{\nu}_i = \nu_i$ for all $i \in I$.

For example, it is well known that every two-person zero-sum game has a value (recall Hypothesis 1).
De Wolf (1999) generalizes this result to n-person games of type C. In fact, we have this stronger result (see Section 3):

Property 22 Every n-person game of type IV has a value, and this value is the unique Nash equilibrium payoœ.

Proof:
Let $i \in I$ and $\mathbf{e} \in NEP(G) \cap STEP(G)$.
Consider $s^\star \in NE(G)$ and $\bar{s} \in STE(G)$ such that $u_i(\bar{s}) = u_i(s^\star) = e_i$. We have

$$e_i = u_i(s^\star) \geq \max_{S_i} u_i(s_i, s^\star_{-i}) \geq \min_{S_{-i}} \max_{S_i} u_i(s_i, s_{-i})$$

$$e_i = u_i(\bar{s}) \leq \min_{S_{-i}} u_i(\bar{s}_i, s_{-i}) \leq \max_{S_i} \min_{S_{-i}} u_i(s_i, s_{-i})$$

Hence $\bar{\nu}_i \leq e_i \leq \underline{\nu}_i$.
So $e_i = \underline{\nu}_i = \bar{\nu}_i = \nu_i$. ■

3 Connection between the diœerent classes

The deønitions of the diœerent classes of games imply that every game of type A (respectively B, I,II) is a game of type B (resp. C, II, IV).
When $n = 2$, it is known that every game of type C is of type II (Beaud 1999). When $n > 2$, this is still the case. De Wolf has proved that for a game of type C, for each player $i \in I$, if any players $-i$ deviate from their equilibrium strategy, then Player i's payoœ increases (De Wolf 1999). This implies that for every game of type C, $NE(G)$ is a subset of $STE(G)$. Hence:

Property 31 Every n-person game of type C is a game of type II.

Remark: Example 2 is an example of a game of type I but not of type C: when $s_2 = B_2$ and $s_3 = A_3$, Player 1 is indiœerent between A_1 and B_1, but not Player 2. There exists also game of type C but not of type I (Beaud 1999, Example 2.3) [3].

[3] A øgure showing the connections between the diœerent classes is placed at the end of this paper.

4 Extensive form game

4.1 n-person game of type I

The aim of this section is to generalize Theorem D of Aumann (1961) to the n-person case. We refer to Owen (1995) for the deønition of an extensive game and its properties.

Theorem 1 Let G be a n-person extensive game which decomposes at a move X and G^X be of type I. Let G^D be the diœerence game, where the payoœ to G^D at (the terminal note) X is the value of G^X. Assume that G^D is of type I. Then G is of type I, $NEP(G) = \nu(G^D)$, and the composition of saddle points in G^X and G^D yields a saddle-point in G.

Proof of the Theorem
Let s be a strategy proøle. We denote by s^X the couple of strategies obtained by restricting s to G^X. We deøne similarly s^D. We denote by u_i^{Γ} the payoœ of player i in the game Γ.
We need the following result

Theorem 2 Let G decomposes at X, and let s be a strategy such that (i) s^X is a strong twisted equilibrium of G^X, and (ii) $s^{G\setminus X}$ is a strong twisted equilibrium of $G \setminus X$ with payoœ $u(s^X)$ assigned to the terminal payoœ X. Then s is a strong twisted equilibrium of G.

Proof: Let s be an n-tuple of strategies which veriøes (i) and (ii), $i \in I$ and $s'_{-i} \in S_{-i}$. From (i), we have

$$u_i(s_i^X, s_{-i}^{'X}) \geq u_i(s^X) \tag{4.6}$$

We denote by G_s^D the demand game where the payoœ associated to the (terminal) node X is $u(s^X)$. From (ii), we have in the game G_s^D:

$$u_i(s_i^D, s_{-i}^{'D}) \geq u_i(s^D) \tag{4.7}$$

But the payoœ of player i induced by (s_i, s'_{-i}) is greater in $G^D_{s_i,s'_{-i}}$ than in G_s^D. Hence, $u_i(s_i, s'_{-i}) \geq u_i(s)$ (Owen 1995, Theorem I.4.3). ■

Lemma 1 Let ν be the unique equilibrium payoœ of G^D. Then every equilibrium payoœ in G is equal to ν.

Proof: The proof in the n-person case is similar as the proof in the 2-person case (Aumann 1961). ■

Let s be a strong twisted equilibrium in G. We denote by s^X the strategy obtained by restricting s to G^X. We denote by P_s the probability over nodes induced by s.

Lemma 2 Every strong twisted equilibrium payoœ of G is equal to ν.

Proof: First, we prove that: (A) if $P_s(X) > 0$, then s^X is an strong twisted equilibrium of G^X; and (B) s^D is an equilibrium of G^D.
(A): The proof of (A) is the same as for Nash equilibrium (Aumann 1961).
(B): If $P_s(X) > 0$, s^X is a strong twisted equilibrium of G^X, then $G_s^D = G^D$ and if s^D is not a twisted equilibrium of G^D, we can construct a strategy such that s is not a strong twisted equilibrium of G.
If $P_s(X) = 0$, the payoœ in G_s^D is the same as the one in G^D. Let s'_{-i} be such that (2.5) is not satisøed. Let $\tilde{s}^X_{-i}$ be a saddle-point of G^X. Then

$$\begin{aligned} u_i^G(s_i,(s'_{-i},\tilde{s}^X_{-i})) &\leq u_i^{G^D}(s_i,(s'_{-i},\tilde{s}^X_{-i})) \\ &= u_i^{G_s^D}(s_i,(s'_{-i},\tilde{s}^X_{-i})) \\ &< u_i^{G_s^D}(s^D) = u_i^G(s) \end{aligned}$$

which is impossible. So (B) is true.
Now, we apply Theorem I.4.3 in Owen (1995): for all $i \in I$, $u_i(s) = u_i^{G^D}(s^D) = \nu(G^D) = \nu$. ■

Lemmata 1 and 2 imply that condition b) is satisøed.

Lemma 3 The composition of a Nash (resp. strong twisted) equilibrium of G^X and of G^D yields a Nash (resp. strong twisted) equilibrium of G.

Proof
The proof for the Nash equilibria is the same as in (Aumann 1961). For the strong twisted equilibria, it is a consequence of theorem 2 because $G_s^D = G^D$.
■

Let s^X (resp. s^D) be a Nash and a strong twisted equilibrium of G^X (resp. G^D). By Lemma 3, the composition of s^X and s^D is a Nash and a strong twisted equilibrium of G. Hence, condition a) is satisøed and G is of type I. ■

5 Examples

5.1 Bertrand's model

n ørms produce the same item. The marginal cost is the same for each ørm and is equal to c. The ørms choose simultaneously their prices $p_1, \ldots, p_n \geq c$. The demand of the consumers is represented by a function $D(p)$ where $p = (p_1, \ldots, p_n)$ is the proøle of prices chosen by the ørms. (Kreps 1990). The proøt of ørm i is

$$\Pi_i(p_1, \ldots, p_n) = (p_i - c) D_i(p_1, \ldots, p_n)$$

where $D_i(p) = \frac{D(p_i)}{|\arg\min\{p_k\}_{k=1,\ldots,n}|}\mathbf{1}_{\{p_i \in \arg\min\{p_k\}_{k=1,\ldots,n}\}}$, $|L|$ denoting the cardinality of the ønite set L.
The aim is to show that Bertrand's model is a game of type I, but not of type C.

Lemma 4 $(c, \ldots, c)$ is a Nash equilibrium of this game.

Lemma 5 $(0, \ldots, 0)$ is the unique Nash equilibrium payoœ.

Proof:
Let $p^\star$ be an equilibrium and let us suppose that Player i's payoœ is positive for some $i \in I$. This implies that $p_i^\star > c$. But then player i has always incentive to deviate in playing $\min_{j\in I}\{p_j^\star\} - \varepsilon$ for some ε suŒciently small, $\varepsilon > 0$. ■

Lemma 6 $(c, \ldots, c)$ is a strong twisted equilibrium of this game, and each strong twisted equilibrium induces a payoœ of 0 to every player.

Proof:
Let $i \in I$. Then $\Pi_i(c, \ldots, c) = \Pi_i(c, p_{-i}) = 0$ for all p_{-i}. Moreover at every strong twisted equilibrium, at least one player plays c. ■

The Bertrand's model is a game of type I: by Lemmas 4 and 6, $NE \cap STE \neq \emptyset$ and by Lemma 5 we have that $NEP = STEP$. But it is not a game of type C: suppose $n = 3$, then $\Pi_1(c, 2c, 2c) = \Pi_1(3c, 2c, 2c) = 0$ and $\Pi_2(c, 2c, 2c) = 0 < \Pi_1(3c, 2c, 2c) = D(2c)/2$.

5.2 Auctions

A divisible item is sold by auction (see for example Wolfstetter (1996)). Player i's valuation of the item is v_i. We assume that everybody knows the valuation of the other players. The bid of Player i belongs to the set $S_i = \{1, \ldots, v_i - 1\}$ [4]. Player i bids $s_i \in S_i$. The player who has done the greatest bid wins the auction. If there is more than one winner, the item is divided. The payoœ function of Player 1 is equal to $u_i(s) = \frac{1}{\varphi(s)}(v_i - s_i)\mathbf{1}\{s_i = \max_{j\in I} s_j\}$ where $\varphi(s) = |\arg\max\{s_1, s_2, s_3\}|$.
The game $(I, (S_i)_{i\in I}, (u_i)_{i\in I})$ fulølls hypothesis 1: (98,97,97) is an equilibrium.

Lemma 7 This game is a game of type C.

Proof: Let $i \in I$, $s \in S$ and $s'_{-i} \in S_i$. We denote $W(s) = \{i \in \arg\max_{i\in I} s_i\}$.

1. Suppose that $u_i(s_i, s_{-i}) = u_i(s'i, s_{-i}) = \alpha$.

[4] Note that we restrict here the bids available to Player i.

(a) $\alpha > 0$. Then $i \in W(s)$, and $s_i = s'_i$ and $u_j(s) = u_j(s'_i, s_{-i})$ for all $j \neq i$.
(b) $\alpha = 0$. This implies that i does not belong to $W(s)$ and $W(s'_i, s_{-i})$. $W(s) = W(s'_i, s_{-i})$, hence $u_j(s) = u_j(s'_i, s_{-i})$ for all $j \neq i$.

2. Suppose that $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$. Then $i \in W(s)$ and $u_j(s_i, s_{-i}) \leq u_j(s'_i, s_{-i}) = \alpha$ for all j.

■

For other economical examples, see De Wolf (1999).

5.3 ıPerturbation" of two-person zero-sum games

Let $\Gamma = (S_1, S_2, u, -u)$ be a two-person zero-sum game and let $\delta_i : S_{-i} \to \mathbb{R}$ for $i = 1, 2$.
We associate to Γ the non-zero sum game $G = (S_1, S_2, \mathcal{U}_1, \mathcal{U}_2)$ where $\mathcal{U}_1(s_1, s_2) = u(s_1, s_2) - \delta_1(s_2)$ and $\mathcal{U}_2(s_1, s_2) = -u(s_1, s_2) - \delta_2(s_1)$. G may be considered as a perturbation of the zero-sum game Γ.
It is easy to check that G is a game of type B, and that G and Γ have the same set of Nash equilibria.

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Fig. 2 Connection between the diœerent classes

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