The Electricity Market

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Abstract

We describe a liberalized competitive market for flows on a network. The model can serve to understand the strategic interaction of firms producing electricity, gas, or telephone service and satisfying the demand of consumers via a network. The network fees are fixed externally. The consumers satisfy their demand according to the prices set by the entrepeneurs and the network costs via some "Bertrand" demand function. The strategic behavior of the firms is analyzed and seen to involve price setting as well as network specifications. As a consequence, we obtain a generalized (oligopolistic, "Bertrand") n-person game of the power companies. The equilibria of this game may serve to support market decisions of power companies, telefone service providers and others.

1 Electricity Networks

Network–flow problems and minimal cost problems on networks are a standard topic in OR. Accordingly, great parts of the model we introduce are discussed in textbooks and monographs, see e.g. SCHRIJVER[5], Chapter 12.

We consider networks that are capable of transporting a certain flow within well specified limitations. There are various examples in economical context. The main application we have in mind is the case of electricity transportation. Here we have the typical network with capacity constraints. The constraints limit the transportation of electricity through the lines as well as through certain nodes. The nodes may vary in their character: First of all there are the power plants which essentially generate electricity. We have also consumers which use electricity as an input. A further type of node is given by some transformer station. Here the output and input may be considered to be the same but again, a transformer has a limited capacity.

A second feature of the electricity network is given by cost generating quantities. We assume that the power companies have an incentive to maximize revenue. They may be able to set prices and this way influence their net output of electricity. This output, however, will depend on the reaction of the consumers. A further cost generating feature is given by the network fees. In a partially liberalized market we assume that the network fees are specified externally. They are the result of some induced price setting procedure which may be implemented by a government agency. On a higher scale we could think of a *cooperative game* that takes place *ex ante* and results in a specification of network fees.

We regard the power companies as active entrepeneurs who compete on an oligopolistic market. The actual nature of their strategic behavior has to be determined and will result in an n-person non-cooperative game. The (Nash) equilibria of this game reflect the result of the balance of the market forces. They may be useful to explain competitive behavior of power companies in a liberalized electricity market.

The model is based on the one developed by H.–M. WALLMEIER [6]. This author introduces strategic considerations into a liberalized electricity market. However, the strategic variables are just prices. Oligopolistic competition is discussed, but the market mechanism is assumed to result from some planning agency that, given the prices of all firms, minimizes aggregated total cost of providing electricity to the consumer(s). The author suggests that equilibria in mixed strategies can be found.

By contrast, we shall attempt to provide a model in which there is a suitably modified version of Bertrand equilibrium given that firms are "similar". (WALLMEIER speaks of firms of "approximately the same size", so this assumption may be acceptable for the practitioners).

It should be mentioned that, for the electricity market, there are by now various approaches. See e.g. BALASKO[1], who explicitely states that "This approach ignores the important spatial features of the electricity power industry like transmission and distribution networks". This author discusses a problem of general equilibrium with production in which one firm (the power company) faces a finite number of consumers. One could argue that this author discusses Walrasian equilibrium while the present paper deals with Bertrand equilibrium – all setups adapted to the specific needs of the special type of economic consumption good called "electricity".

This first section provides the modelling definitions. We will first speak about the basic network which supplies the natural background for the market. We then turn to the cost structure and the utility functions of the entrepeneurs. Finally we discuss some operations that take place on networks: they can be united, summed up, or be disintegrated and we would like to study the consequences of such operations.

We start out with the definition of a network.

Definition 1.1. *A* ***network*** *(with capacity constraints) is a pair of data*

$$\mathcal{N} := (\boldsymbol{N}, \boldsymbol{C}) \tag{1.1}$$

with the following specifications and interpretations.

1. $\boldsymbol{N}$ *is a finite set, the set of* ***nodes****. The elements of* $\mathcal{K} := \boldsymbol{N} \times \boldsymbol{N}$ *are called the* ***edges****; we imagine that* $(\boldsymbol{N}, \mathcal{K})$ *represents a finite graph such that* $(h,k) \in \mathcal{K}$ *is the* link *between* h *and* k.

2. $\boldsymbol{C} := \big((c_{hk})_{(h,k)\in\mathcal{K}},\ \ (L_k)_{k\in\boldsymbol{N}}\big)$ *is a set of data as follows:*

   (a) *The first matrix describes* ***capacities*** *of the edges. Thus,*

   $$c_{hk} \geq 0 \tag{1.2}$$

   is the maximal amount of flow to be sent along $h \to k$ *An edge with capacity* 0 *can as well considered to be nonexistent. Flows may be directed in both directions,* $h \to k$ *and* $k \to h$.

(b) *The second set of data descibes the characteristics of the nodes to either produce flow (sources), to consume flow (sinks) or to just transfer ('transform') it. Intuitively, L_k is the **maximal net output** of node k; negative outputs are consumption.*

3. *There are restrictions of the node characteristics that declare the nodes to be of a certain **type**.*

   *If $L_k = 0$ holds true, then k is a **transformer**, we indicate this by writing $k \in \boldsymbol{T}$, hence $\boldsymbol{T} \subseteq \boldsymbol{N}$ is the set of transformers.*
   *If $L_j =: -d_j < 0$ holds true, then j is a **consumer**, we write $j \in \boldsymbol{J}$,*
   *If $L_i > 0$ holds true, then i is a (flow or electricity producing) **power company**, we write $i \in \boldsymbol{I}$. Later on we will assign an active role to the power companies as **players** of a (Bertrand oligopoly) game.*

   This way $\boldsymbol{N} = \boldsymbol{I} + \boldsymbol{J} + \boldsymbol{T}$ is a decomposition of the set of nodes. (Our convention is to use $+$ for disjoint unions and to write $i \in \boldsymbol{I},\ j \in \boldsymbol{J}$, etc. and, if possible, L^i, d_j, etc.)

4. *$\underline{\overline{\mathbf{N}}}$ denotes the set of networks living on a finite set of nodes $\boldsymbol{N}$ taken from a fixed* universe *which is thought of to be a countable set. Given some fixed $\boldsymbol{N}$, the data specifying the capacities are real numbers, hence the system $\underline{\overline{\mathbf{N}}}_{\boldsymbol{N}}$ of networks living on $\boldsymbol{N}$ is essentially a Euclidian space, we have*

$$\underline{\overline{\mathbf{N}}}_{\boldsymbol{N}} \;\cong\; \mathbb{R}_+^{\mathcal{K}} \times \mathbb{R}_+^{\boldsymbol{N}}. \tag{1.3}$$

We now assign feasible flows to our network. A flow is a certain state of nature in which a fixed quantity of electricity passes through an edge of the network. Accordingly, at each node there occurs an input and an output of electricity the net difference of which has to be limited by the capacity constraints.

Definition 1.2. *1. A **flow** is an assignment of flow quantities to the edges. We write this*

$$X = (x_{hk})_{(h,k)\in\mathcal{K}}\ . \tag{1.4}$$

The (net) ***output*** *at* $k \in \boldsymbol{N}$ *at* X *is given by*

$$Y^k(X) \; := \; \sum_{h \in \boldsymbol{N}} x_{kh} - \sum_{r \in \boldsymbol{N}} x_{rk} \; . \tag{1.5}$$

An analogous definition explains the ***consumption*** Y_j *of* j *at a flow* X *which is given by*

$$Y_j(X) = \sum_l x_{lj} - \sum_r x_{jr} \; . \tag{1.6}$$

2. *A flow* $\mathcal{X}$ *is* ***feasible*** *if it respects the capacity restrictions of the whole network. That is, the capacitiy conditions at the edges*

$$0 \leq x_{hk} \leq c_{hk} \quad ((h,k) \in \mathcal{K}), \tag{1.7}$$

are satisfied. Also, the the nature of the nodes are reflected, we have:

$$0 \leq Y^i(X) \leq L^i \quad (i \in \boldsymbol{I}), \tag{1.8}$$

$$Y_j(X) \; = \; d_j \quad (j \in \boldsymbol{J}), \tag{1.9}$$

and

$$Y^k(X) = Y_k(X) \; = \; 0 \quad (k \in \boldsymbol{T}). \tag{1.10}$$

3. *The set of feasible flows for* $\mathcal{N}$ *is denoted by* $\mathcal{X}$ *or* $\mathcal{X}^{\mathcal{N}}$*.*

We want to consider some natural structures on $\underline{\overline{\mathbf{N}}}$ which will be usefull with respect to capacity considerations later on. For instance we would like to add networks which essentialy means that we add their capacities. In this context it is not always necessary to regard the specification of capacities as defining a real network. It could as well mean a requirement issued by a company or a group of companies specifying certain capacity demands. Roughly speaking the various power companies could be forced to specify subnetworks which reflect their demands for capacities on certain parts of the market. To be able to operate with such capacity requirements is essential in our context.

A particular network is given by $\mathcal{O} \in \underline{\overline{\mathbf{N}}}$, the zero capacity network or ***zero network*** which is specified by all data being 0. We don't have to specify the set of nodes $\mathcal{O}$ is living on. Actually the detailed specification of the nodes a network is living on is less important. A network can always be

extended to some larger set of nodes by specifying all additional nodes to be transformers with capacity 0. That is, the extension is supposed to look like the zero network on the new nodes. The operation of ***restriction*** can also be defined in a quite obvious manner.

There is a partial ordering $\preceq$ on networks defined via

$$\widetilde{\mathcal{N}} \preceq \mathcal{N}$$

$$\Longleftrightarrow \tag{1.11}$$

$$\widetilde{c}_{hk} \leq c_{hk} \quad ((h,k) \in \mathcal{K}), \qquad \widetilde{L}_i \leq L_i \quad (i \in \boldsymbol{I}),$$
$$\widetilde{d}_j \leq d_j \quad (j \in \boldsymbol{J}).$$

Clearly, the 'smaller network' in the sense of $\preceq$ has less capacity with respect to every node. Therefore we may conclude that

$$\mathcal{X}^{\widetilde{\mathcal{N}}} \subseteq \mathcal{X}^{\mathcal{N}}$$

holds true whenever $\widetilde{\mathcal{N}} \preceq \mathcal{N}$ happens to be the case.

It is not hard to see that, with respect to $\preceq$, the set of networks constitutes a ***lattice***, that is, the operations $\wedge$ (minimum) and $\vee$ (maximum) are well defined. The minimum net allows for every flow that may occur in both of the nets involved (the minimum flow, that is) and the maximum net allows for every flow that is the maximum flow of two flows occurring in the networks involved. (The operations discussed may as well be defined on flows).

Finally, we can speak about the ***sum*** of networks. This means that under certain circumstances we may add capacities (and flows) for various networks. Thus, we say that a network $\mathcal{N}$ is the sum of a family of networks

$$(\mathcal{N}^{\rho}), \quad (\rho = 1, \ldots, r)$$

if

$$\begin{aligned} c_{hk} &= \sum_{\rho=1}^{r} c^{\rho}_{hk}, \quad ((h,k) \in \mathcal{K}) \\ L^{i} &= \sum_{\rho=1}^{r} L^{i\rho}, \quad (i \in \boldsymbol{I}) \\ d_j &= \sum_{\rho=1}^{r} d^{\rho}_{j}, \quad (j \in \boldsymbol{J}) \end{aligned} \tag{1.12}$$

holds true.

Having these operations at our disposal we now define a further structure on networks: the fees requested for transportation of flow through an edge.

Definition 1.3. *An **Electricity Network** is a tripel*

$$\mathcal{E} := (\boldsymbol{N}, \boldsymbol{C}, \boldsymbol{\beta}) \tag{1.13}$$

such that $\mathcal{N} = (\boldsymbol{N}, \boldsymbol{C})$ *is a network as specified above (Definition 1.1) and the quantity*

$$\boldsymbol{\beta} = (\beta_{hk})_{(h,k)\in\mathcal{K}} \geq 0 \tag{1.14}$$

is a matrix attached to the edges of $\mathcal{N}$. $\boldsymbol{\beta}$ *is interpreted as a fixed price scheme or **service fee** schedule. The meaning is that a unit flow (an MWh of electricity) to be send from* h *to* k *generates a fee of* β_{hk} *units of money.*

Sometimes we wish to refer to the data separately. E.g., if a network $\mathcal{N}$ *is given, then a fee schedule* $\boldsymbol{\beta}$ *will be called suitable for* $\mathcal{N}$*, if the matrix has the right dimensions, i.e., constitutes a function on* $\mathcal{K}$*.*

The next sections will serve to gradually built up the electricity network game. To this end we proceed from the most primitive one company/one consumer network to the most involved version in which we have overlapping demand and supply on a free market.

In SECTION 2 we discuss the primitive version. The main purpose is to discuss the cost functions that result from varying demand of a single consumer. This cost function appears in later sections and is part of both, the entrepeneurs and the consumers strategic problem.

In SECTION 3 we enlarge the network so that various companies serve one single consumer. We assume that this can be done by completely disjoined networks owned by the companies. Yet, at this instant our view is slightly changed: we conceive the management of the power plant as an oligopolist in a competitive market and, as a consequence, concepts of *Game Theory* enter the scene. The set $\boldsymbol{I}$ of flow producing entities or power plants receives now additional weight via the interpretation as the *set of players*.

In this context the game is rather easy to describe and the strategic behavior of the power companies is just explained by their price setting policies. Because the network constraints can just be considered separately, there appears a cost function for each company. The consumer is seen as a price taker. He has no strategic considerations in mind but, given the companies prices, just minimizes his expenses. Therefore, eventually a version of Bertrand Oligopoly emerges.

In Section 4 we combine the features discussed previously. Now we have overlapping networks and electricity may freely be exchanged though network costs occur and prices may change according to the strategic behavior of the power companies. It turns out that the strategic behavior of the companies involves capacity planning. They have to set aside a certain capacity specification for each consumer they want to serve. Within these constrains they try to route electricity towards the consumer at minimal prices. Since the strategy understood this way is combined by price settings and networks specifications, the results of Section 3 can be applied in an intricate manner.

As competitive specification of network requirements may result in excess capacity demands we introduce penalty functions, which generate further costs (towards the entrepeneurs) for capacity requirements exceeding the global networks capability. In equilibrium, however, there will be no excess capacity demands as a single entrepeneur would always improve the situation by reducing his capacity demand.

2 Monopoly: The Cost Function

Within this section, we restrict ourselves to networks $\mathcal{N}$ with capacity constraints such that there is a *single power company* i which supplies electricity to a *a single consumer* j. Formally, we assume the existence of at least on 'intermediate'transformer. Consider the demand $d = d_j$ of the consumer and suppose that, for the moment, we regard this as a variable. The resulting network is then called $\mathcal{N}^d$. The system of feasible flows is $\mathcal{X}^d$. Note that, within this simple framework, a feasible flow clearly satisfies

$$Y^i(X) = Y_j(X) = d , \tag{2.1}$$

which reflects the fact that the full demand of consumer j is just satisfied by the production of the only power company i.

Given a suitable fee schedule $\boldsymbol{\beta}$, the network fees constitute a cost function as follows. First of all, the capacity constraints yield an upper bound for the supply of electricity, say $\bar{d}$.

Now the power plant schedules the total flow to the consumer such that network fees are minimized. Then the resulting cost function is described by

$$\begin{aligned} C &= C_{\boldsymbol{\beta}}^{\mathcal{N}^\bullet} = C^{\mathcal{N}}(\bullet) \ : \ [0, \bar{d}\,] \to \mathbb{R}_+ \\ C(d) &= C_{\boldsymbol{\beta}}^{\mathcal{N}^d} := \min \left\{ \sum \beta_{lk} x_{lk} \,\middle|\, X \in \mathcal{X}^d, \right\} \quad (d \in [0, \bar{d}\,]). \end{aligned} \tag{2.2}$$

Clearly we have

Theorem 2.1. *Given some network $\mathcal{N}$ and a suitable fee schedule $\boldsymbol{\beta}$, the cost function $C_{\boldsymbol{\beta}}^{\mathcal{N}^d}$ is monotone, continuous, convex, and piecewise linear.*

We will only consider costfunctions of this type. Let us agree on suitable description.

A strictly increasing, convex, and piecewise linear function C is written

$$\begin{aligned} C &: \quad [0, \bar{d}] \to \mathbb{R}_+, \\ C(t) &: \quad \max \{ A_k t - B_k \mid k = 0, \ldots K \} \quad (t \in [0, d_0]). \end{aligned} \tag{2.3}$$

Here $(A_k)_{k=0,\ldots K}$ and $(B_k)_{k=0,\ldots K}$ are real numbers *strictly increasing* in k and satisfying $A_0 \geq 0, \ B_0 = 0$. We put

$$\Delta_0 := 0, \ \Delta_k := \frac{B_k - B_{k-1}}{A_k - A_{k-1}}. \tag{2.4}$$

and assume that Δ_k is as well *strictly increasing* in k.

The numbers Δ_k describe the arguments at which the function shows kinks: it is seen that

$$C(t) \;=\; A_k t - B_k \quad (t \in [\Delta_k, \Delta_{k+1}]) \tag{2.5}$$

holds true (cf. Figure 2.1).

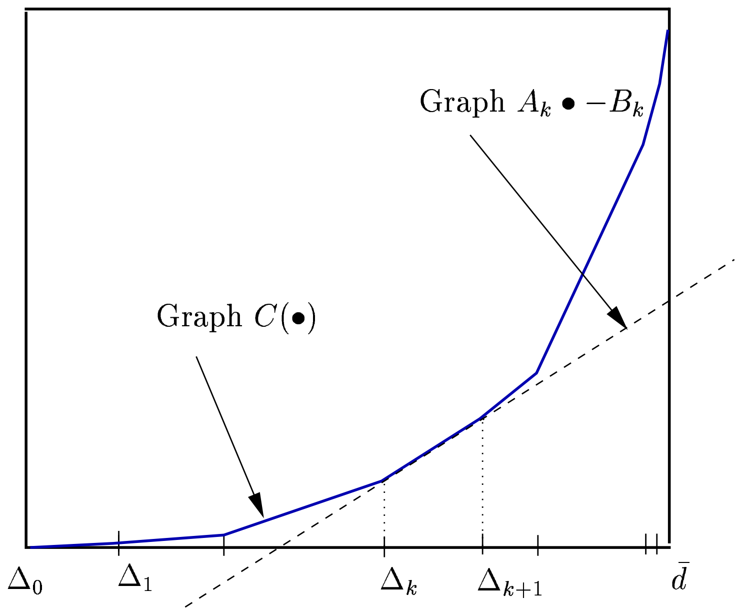

Figure 2.1: A Cost Function

Now we exhibit the connection between costfunctions and networks.

Definition 2.2. *A network $\mathcal{N}$ is called a* ***chain*** *if it consists of one power plant i, one consumer j and a set of transformers (at least one) connecting both along a unique path (cf. Fig. 2.2). A chain is* ***uniform*** *if all capacities along the path are equal. A chain is* ***minimal*** *if it is uniform and contains just one transformer. Finally, a network is called* ***minimal*** *if it is the sum of finitely many minimal chains each of which having the same power plant and the same consumer while the intermediate nodes as well as the capacities are all different.*

Clearly, a chain $\mathcal{C}$ generates a linear cost function on an interval. More generally, we can state:

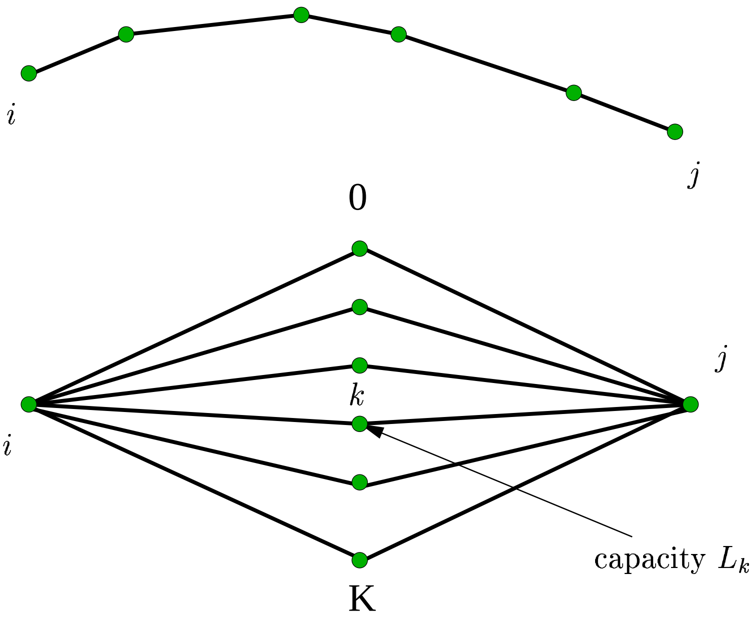

Figure 2.2: Chain and Minimal Network

Theorem 2.3. *Let C be a cost function. Then there is a minimal network $\mathcal{N}$ and a fee schedule $\boldsymbol{\beta}$ such that*

$$C \;=\; C^{\mathcal{N}}_{\boldsymbol{\beta}} \tag{2.6}$$

holds true. The pair $(\mathcal{N}, \boldsymbol{\beta})$ is essentially unique (i.e., up to choosing the nodes in the universe).

Proof: This is obvious: if C is given via

$$C(t) \;:=\; \max\{A_k t - B_k \mid k = 0, \ldots K\} \quad (t \in [0, \bar{d}]),$$

then the generating network consists of a power plant i, a consumer j, and $K+1$ intermediate nodes $0, \ldots, K$. The transformer capacities and the cost schedule are given by

$$\begin{array}{rclcrcl}
\beta_0 & = & A_0 & & c_{i0} = L_0 = c_{Kj} & = & \Delta_1, \\
\beta_1 & = & A_1 & & c_{i1} = L_1 = c_{Kj} & = & \Delta_2 - \Delta_1, \\
. & . & . & . & . & . & . \\
\beta_k & = & A_k & & c_{ik} = L_k = c_{Kj} & = & \Delta_{k+1} - \Delta_k, \\
. & . & . & . & . & . & . \\
\beta_K & = & A_K & & c_{iK} = L_K = c_{Kj} & = & \bar{d} - \Delta_K,
\end{array} \tag{2.7}$$

the remaining quantities can obviously chosen to be large enough. (The construction can also be changed by chosing the node capacities suitable and the edge capacities large).

q.e.d.

Thus, up to the capacity of the least expensive chain, this chain is solely used for transportation of electricity. Thereafter, the second cheapest chain is used etc. This generates exactly the cost function prescribed.

Remark 2.4. *Clearly, each chain of the minimal network constructed corresponds to exactly one linear function $A_k \bullet - B_k$ generating C. More generally, if a network $\mathcal{N}$ is the sum of finitely many uniform chains which share only the one and only one power plant and consumer, then each chain can be associated to one supporting linear function of the cost function. The mapping constructed is unique if the total costs of transporting a unit along a chain are all different.*

Theorem 2.5. *A Network $\mathcal{N}$ can be decomposed into a sum of networks*

$$\mathcal{N} = \mathcal{N}^\star + \mathcal{C}^0 + \mathcal{C}^1 + \ldots + \mathcal{C}^K \tag{2.8}$$

such that the $\mathcal{C}^k$ $(k = 0 \ldots K)$ are uniform chains while $\mathcal{N}^*$ has zero capacity.

Proof: This is a standard result from *Network-Flow* Theory (FORD–FULKERSON [2][3]). Consider the case of of one power plant and one consumer only. Pick a feasible flow with maximal output and, at each node beginning at the power company, mark a direction of maximal output. One reaches either a consumer or generates a loop. This defines a chain or a loop. The minimum of all flow quantities along this chain/loop defines a uniform set of capacities which can be deducted. After removing all loops successively, the remaining chains are assigned this uniform capacities and deducted successively as well. Each one actually exhaustes one arc completely. The final remainder is a network of zero capacity.

q.e.d.

Corollary 2.6. *Let $\mathcal{N}$ be a network and let $\boldsymbol{\beta}$ be a fee schedule. Then $\mathcal{N}$ can be decomposed into a sum of networks*

$$\mathcal{N} = \mathcal{N}^\star + \mathcal{C}^0 + \mathcal{C}^1 + \ldots + \mathcal{C}^K \tag{2.9}$$

as follows: the networks $\mathcal{C}^k$ $(k = 0 \ldots K)$ are uniform chains while $\mathcal{N}^*$ has zero capacity. Each $\mathcal{C}^k$ can be assigned to a linear support function of the cost function $C_{\boldsymbol{\beta}}^{\mathcal{N}}$ in the sense of Corollary 2.4. This cost function is the same as the one generated by $\mathcal{C}^0 + \mathcal{C}^1 + \ldots + \mathcal{C}^K$.

Proof: Let Δ_1 be the last point in the domain of definition of $C_{\boldsymbol{\beta}}^{\mathcal{N}}$ at which the first linear function $A_0 \bullet -B_0 = A_0\bullet$ supports $C_{\boldsymbol{\beta}}^{\mathcal{N}}$. Take a flow with output $d = \Delta_1$ which is cost minimizing, hence costs $C_{\boldsymbol{\beta}}^{\mathcal{N}}(\Delta_1)$. We may assume that there are no loops. Then decompose this flow according to the FORD–FULKERSON method indicated by Theorem 2.5. This generates a set of chain flows wich in turn can be used to define uniform chain networks. Along this networks the marginal cost has necessarily to be the minimal one possible, i.e., A_0. Necessarily, this is the sum of all costs occurring along the chain. Therefore, the fee schedule for this chain is defined in a canonical way, i.e., the same as for the original network.

The rest is done by induction.

q.e.d.

Definition 2.7. *We call two pairs $(\mathcal{N}, \boldsymbol{\beta})$ and $(\mathcal{N}', \boldsymbol{\beta}')$* ***equivalent*** *if they generate the same cost function C. The minimal network generating the cost function according to Theorem 2.3 (which is an element of the equivalence class "of C") is called the* ***minimal representative****.*

Remark 2.8. *Given $(\mathcal{N}, \boldsymbol{\beta})$ the minimal representative of the equivalence class can be constructed at once. Decompose the network according to Corollary 2.6. Dispose of the zero capacity network. Now replace every chain (which is uniform) by a minimal chain with the same capacity generating the same costs. This way we obtain the minimal network generating the same cost function.*

Remark 2.9. *Similarly to the situation with cost functions, we may introduce a metric on pairs of equivalence classes $(\mathcal{N}, \boldsymbol{\beta})$. To this end, we take the minimal representatives, assume w.l.o.g that they are defined on the same $\boldsymbol{N}$, and apply a Euclidean metric w.r.t. the (pairs of) data $(\boldsymbol{C}, \boldsymbol{\beta})$. Clearly, equivalence classes are close if and only if the corresponding cost functions are close. We will (sloppily) say that networks and cost schedules are close if this is the case for the equivalnece class, that is, for the minimal representatives.*

The following theorem discusses the relation between comparable El–networks and their cost functions.

Theorem 2.10. *Let $\mathcal{E}, \mathcal{E}'$ be two El-networks such that $\mathcal{N} \preceq \mathcal{N}'$ and $\boldsymbol{\beta} \geq \boldsymbol{\beta}'$ holds true. Then, on the domain of $C^{\mathcal{N}}$, the corresponding cost functions satisfy*

$$C^{\mathcal{N}} \geq C^{\mathcal{N}'}. \tag{2.10}$$

If $\boldsymbol{\beta} = \boldsymbol{\beta}'$, then $C^{\mathcal{N}}$ has at most the slopes that appear in $C^{\mathcal{N}'}$.

Proof: We write $C := C^{\mathcal{N}}$ and $C' := C^{\mathcal{N}'}$.

It is rather clear that C has the smaller domain on which it dominates C'. For, whenever we consider a chain of $\mathcal{N}$, then this will constitute a chain of $\mathcal{N}'$, but the expenses are lower at the larger network.

In particular, if both cost schedules are equal, consider the decomposition $\mathcal{N}$ of as indicated by Corollary 2.6 into chains correponding to the slopes of the cost function. Then, if we take the chain corresponding to some slope A_k, we will be able to send this flow through this chain with respect to the net $\mathcal{N}'$ as well. For small demand, this flow will possibly not be cost minimizing with respect to $\mathcal{N}'$. But with increasing demand one is eventually forced to use all the edges and nodes within this chain in order to transport further flow in $\mathcal{N}$.

Thus, the slope A_k will appear in C'. Proceding this way, we observe that C is obtained by "shrinking" $\mathcal{N}'$, i.e., the same slopes appear successively, but on possibly shorter (or empty) intervals. **q.e.d.**

We are now in the position to formulate a standard monopoly model. We introduce a demand function for the consumer and a payoff function for the power plant.

The demand of the consumer is modelled by a decreasing continuous and convex function

$$D : \mathbb{R}_+ \to \mathbb{R}_+ \tag{2.11}$$

which, given a price $p \in \mathbb{R}_+$, yields the demand $D(p)$ of the consumer. For technical reasons we shall later on (in the context of oligopoly) assume that the demand function generates *increasing marginal expenditure*. Formally we require that it is *slowly decreasing*, i.e., satisfies $\frac{D(t)}{t} \geq -D'(t)(t \in \mathbb{R}_+)$ (Cf. [4]).

Now, if the power plant fixes a price $p \in \mathbb{R}_+$, then electricity is sold in a quantity resulting from demand and the payoff to the power plant is obtained by a function which combines revenue and costs. This yields a payoff or utility for the power plant

for $p \in \mathbb{R}_+$

$$U(p) = pD_j(p) - C(D(p)) \quad (p \in \mathbb{R}_+). \tag{2.12}$$

Here we assume that maximum demand at zero price can be transported throug the network, hence is located in the domain of C. If fixed costs and production costs are to be incorporated, then the appearance of the payoff function changes into

$$U(p) \;=\; pD(p) - \alpha - C^0(D(p)) - C(D(p)). \tag{2.13}$$

Formally, this function is of the same type if we assume that C^0 is piecewise linear and convex as well. So the additional terms can just be thought of as incorporated in the cost function C. Therefore, we will mainly discuss the version (2.12).

Now the power company will maximize its payoff resulting from the price setting procedure and the network fees. It is the task of the power companies management to find

$$\max\{U(p) \mid p \in \mathbb{R}_+\}. \tag{2.14}$$

This kind of monopoly is a textbook problem. Usually one assumes that expenditure is concave and that a price can be found at which marginal expenditure and marginal cost coincide. Another version is as follows:

Theorem 2.11. *Let p_0 be the maximal price in the domain of the cost function C and let $p \leq p_0$ be such that $U(p) \geq 0$, $U(p_0) = 0$ holds true.*

Then the quantity (2.14) *exists and a maximizer correponding to the problem* (2.14) *establishes an optimal price for the power plant.*

This follows from the fact that $U(0) = 0$ is assumed (implied by zero cost at zero production).

Next we will consider ologopolistic competition in the market.

3 Oligopoly: Disjoint Networks

Within this section we consider an Electricity Network $\mathcal{E}$ with just *one consumer* j and a finite number of power plants. Also, we will assume that the power companies act on almost *disjoint networks*, each of them supplying electricity to the consumer via its own separated network. Formally, this is expressed by the existence of finitely many networks $\mathcal{N}^i$ $(i \in \boldsymbol{I})$ each of them containing just one single power company i. All of them are dominated by the global network ($\mathcal{N}^i \preceq \mathcal{N}$ $(i \in \boldsymbol{I})$) such that (with some abuse of notation)

$$\mathcal{N} \;=\; \bigvee_{i \in E} \mathcal{N}^i, \quad \mathcal{N}^i \wedge \mathcal{N}^{i'} \;:=\; \{v\} \quad (i, i' \in \boldsymbol{I}) \tag{3.1}$$

holds true.

Within each of these disjoint networks we can speak of feasible flows, we write

$$\mathcal{X}^i \;:=\; \left\{ X \,\middle|\, X \;=\; (x_{hk})_{(h,k)\in\mathcal{K}^i}, \quad X \text{ feasible for } \mathcal{N}^i \right\} \tag{3.2}$$

If a fee schedule $\boldsymbol{\beta}$ is suitably defined for $\mathcal{N}$, then obviously the restriction to any of the subnetworks defines a fee schedule suitable for this subnetwork. It is not necessary to carry the index ^{i} in this case. Thus, the above mentioned decomposition naturally induces a decomposition of the global structure $\mathcal{E}$ into local (power company owned) structures $\mathcal{E}^i$.

As a consequence, we can assign cost functions to the various local networks. Therefore, the framework of SECTION 2 which refers to a monopolistic network can be applied to each of the structures $\mathcal{N}^i$ and $\mathcal{E}^i$. This way we obtain, for any $i \in \boldsymbol{I}$, a ***cost function***

$$\begin{aligned} C^i \;&:\; \quad [0, \bar{d}^i] \to \mathbb{R}^+ \\ C^i(d) \;&=\; \quad \min \left\{ \sum \beta_{hk} x_{hk} \,\middle|\, X \in \mathcal{X}^{id}, \; \right\} \;. \end{aligned} \tag{3.3}$$

Here, a flow $\mathcal{X}^{id}$ referes to the network $\mathcal{N}^{id}$ which is obtained from $\mathcal{N}^i$ by altering the input of the single consumer j from d_j to d. The detailes are treated exactly as in SECTION 2. Not that the cost functions are all defined on a possibly different domain, the interval $[0, \bar{d}^i]$. If need be, one can choose $d_0 \;:=\; \min \left\{ \bar{d}^i \,\middle|\, i \in \boldsymbol{I} \right\}$ for the joint domain of definition.

We assume that the consumer is characterized by a ***demand function*** $D = D^i$ in the sense of [4], i.e., a convex and slowly decreasing function which is continuous at 0. In order to obtain a compact range, one can choose p_0 sufficiently large as to satisfy all the conditions that are specified for a piecewise linear Bertrand oligopoly (PLBO) in [4]. Now we recall the Bertrand demand function derived in this context.

The behavior of this consumer is described by the assumption that he distributes his demand equally among those competitors who offer the minimal price.

Definition 3.1. *Let $p = (p^i)_{i\in \boldsymbol{I}} \in \mathbb{R}_+^{\boldsymbol{I}}$ be a price vector. We put*

$$M_{\boldsymbol{I}}^p := \left\{ i \in \boldsymbol{I} \,\middle|\, p^i = \min_{k\in\boldsymbol{I}} p^i \right\} = arg\ min_{\boldsymbol{I}}\ p. \tag{3.4}$$

Also, let $D_j : [0, p_0] \to [0, d_0]$ denote the demand function of consumer j. We put for $p \in [0, p_0]_+^{\boldsymbol{I}}$

$$D_j^i(p) := \begin{cases} 0 & i \notin M_{\boldsymbol{I}}^p \\ \frac{D_j(p^i)}{|M_{\boldsymbol{I}}^p|} & i \in M_{\boldsymbol{I}}^p \end{cases} \tag{3.5}$$

The functions D_j^i describe the demand resulting from the price settings of the power companies. This, in terms of oligopoly, is the market reaction function.

Now we are in the position to describe oligopolistic competition. Suppose, that all power companies fix a certain price p^i. Then, for $i \in \boldsymbol{I}$, payoff to power company i is given by

$$\begin{aligned} U^i(p) &= U^i((p^k)_{k\in\boldsymbol{I}}) \\ &= p^i D_j^i(p) - C^i(D_j^i(p)). \end{aligned} \tag{3.6}$$

Finally we have the $n-$person game generated by the electricity network. This game is now well defined as we have described the strategic behavior of the power companies and defined their payoffs accordingly.

Definition 3.2. *The* ***El-Network game*** *generated by the Electricity Network $\mathcal{E}$ is the noncooperative $|\boldsymbol{I}|-$person game*

$$\Gamma = \Gamma_{\mathcal{E}} := (\mathbb{R}^{\boldsymbol{I}}, (U^i)_{i\in\boldsymbol{I}}) . \tag{3.7}$$

The El-Network game is a Bertrand Oligopoly. The power companies appear in the role of the players, the consumer just provides the price dependent demand function, his behaviour is not strategic but price taking. We can apply standard results from Oligopoly Theory, like e.g. Theorem 3.4. of [4] and Remark 2.9 in order to come to the following conclusion:

Theorem 3.3. *Let $\mathcal{E}$ be an El–Network with disjoint subnetworks $\mathcal{N}^i$ of the powerplants. Suppose that these networks are close in the sense of Remark 2.9 and hence the cost functions define a PLBO with similar firms (Definition 2.9. of [4]). Also, assume that demand and supply are interceding (Definition 3.2 of [4]). Then there exists a Bertrand equilibrium. Within a certain neighborhood, the Bertrand equilibrium correspondence is* uhc *in the data of $\mathcal{N}$.*

4 The El-Network Game

Our next aim is the construction of the El–network game and of equilibrium in the general case of overlapping networks. It seems that the approach which was successful in SECTION 3 is not sufficient in the general environment. The problem is that the power companies will not only have to balance their prices in equilibrium. Also they will have to take care that electricity is transported with minimal networking fees towards the consumers.

In this context minimizing the networking fees can only mean that, in equilibrium, the single power company optimizes against its opponents. Given this viewpoint the special capacity demand of the power companies or, more precisely, a network specification, will be part of the equilibrium. Clearly, this raises the question of the meaning of strategic behavior in our context.

The first idea is that, next to the price setting mechanism, the flows of electricity as planned by the power companies are part of the strategic behavior. Yet, because of non-uniqueness of the (minimizing) flows we are confronted with a selection problem. In addition, the concept of maximizing the payoff depending on the price against given network specifications of the opponents appears inherently in SECTION 3. One cannot recognize any concept of maximizing against the opponents chosen transportation of flow.

We shall, therefore, adopt the idea that a strategy of a player (a power company) is a price plus a network specification. Among the advantages we note the one that a nework specification implies a cost function.

On the other hand, there appears a further problem which, however, is natural in this context. Whenever a choice of prices and network specifications of the companies is specified, then possibly the the capacity constraints of the global network cannot be met. The equilibrium calculus will not *a priori* lead to capacity constraints. Our solution is obtained by introducing penalties attached to an overflow at some node or edge of the network. Again this supports the idea of introducing network specifications: to impose penalties for the concrete planning of flows does not seem to be sensible. To introduce penalties as a punishment for exceeding capacity demands appears to be most natural.

Penalties will be a natural but only intermediary tool. Given sufficiently large penalty payments, we expect the network in equilibrium to be organized in a way such that penalties are avoided and the capacity constraints are met.

We first describe a simple version of penalties. Within this context we use

α^+ in order to denote $\max\{\alpha, 0\}$ for real numbers α.

Definition 4.1. *A **penalty code** (given a network $\mathcal{N}$) is a family $\boldsymbol{Q} := (Q, R)$ of positive quantities such that*

$$Q := (Q_{hk})_{(h,k)\in\mathcal{K}} > 0 \tag{4.1}$$

is a matrix and

$$R := (R^i)_{i\in\boldsymbol{I}} > 0 \tag{4.2}$$

is a vector.

We suggest the following interpretation of the penalty code. As previously, we can disregard the particular set of nodes and extend $\boldsymbol{Q}$ canonically to all networks in the universe (the values are zero outside the original finite $\boldsymbol{N}$).

Now we introduce the the ***penalty function***. This function is defined on *pairs* of networks. We use the letter $\boldsymbol{Q}$ for this version as well, no confusion will arise. We define

$$\boldsymbol{Q} = \boldsymbol{Q}_{\mathcal{N}} : \underline{\overline{\mathbf{N}}} \times \underline{\overline{\mathbf{N}}} \to \mathbb{R}_+ \tag{4.3}$$

by

$$\begin{aligned} \boldsymbol{Q}(\widetilde{\mathcal{N}}, \widehat{\mathcal{N}}) := & \sum_{(h,k)\in\mathcal{K}} Q_{hk}\widetilde{c}_{hk}(\widehat{c}_{hk} - c_{hk})^+ \\ & + \sum_{i\in\boldsymbol{I}} R^i \widetilde{L}^i (\widehat{L}^i - L^i)^+ \end{aligned} \tag{4.4}$$

The penalty function is interpreted to be arranged for the punishment of overcapacities required from the network $\mathcal{N}$ by the power companies. We interprete the last formula as follows. There is the underlying net $\mathcal{N}$ which is available to all players. Based on the strategic decision of *all* players, there is a total or "public" *demand for capacity* represented by the network $\widehat{\mathcal{N}}$. This network is hypothetical. There is also a *privat demand* of a player represented by $\widetilde{\mathcal{N}}$ which is hypothetical as well. The excess demand for capacity that may occur in the public network is compared to the capacity available at $\mathcal{N}$. Now, in order to punish overcapacity demands, some penalty charge is raised. The private network is charged in accordance with its own capacity demands only if public excess demand occurs at all.

Having cleared this point, we can now turn to the formulation of strategic behavior in the present context.

Definition 4.2. *Let $\mathcal{N}$ be a network and let $i \in \boldsymbol{I}$ be player (a power company). A **strategy** of i is a price/network specification*

$$s^i := (p^i_\bullet, \mathcal{N}^i_\bullet) \tag{4.5}$$

with the following properties:

1. *The first data*

   $$p^i_\bullet = (p^i_j)_{j \in \boldsymbol{J}} \geq 0 \tag{4.6}$$

   *is a nonnegative vector, called a **price vector**.*

2. *The second data*

   $$\mathcal{N}^i_\bullet = (\mathcal{N}^i_j)_{j \in \boldsymbol{J}} \tag{4.7}$$

   is a family of networks such that

   $$\sum_{j \in \boldsymbol{J}} \mathcal{N}^i_j \preceq \mathcal{N} \tag{4.8}$$

   *is satisfied. This family is called the **network specifications** or **capacity requirements**.*

*The **strategy spaces** are denoted by*

$$\mathcal{S} = \mathcal{S}^1 \times \ldots \times \mathcal{S}^n.$$

Thus, the company plans to sell electricity at prices varying with the consumers. Also, it plans to set aside a certain set of capacity requirements (represented by a set of networks) for each customer such that the total capacity does not exceed the capacity of the network available.

Of course, the independent planning may lead to inconsistencies – it is just the task of the forces establishing equilibrium that should eventually take care of a consistent global planning.

Given the subnetworks resulting from strategic choice, we may now proceed as in SECTION 3 in order to determine market demand. That is we consider the 'local' capacity constraints and define the cost function and the cost schedule accordingly. This means that, given the local capacity constraints and a feasible demand of a consumer, the power plants compute the minimal costs and add it to their fixed costs if any.

Definition 4.3. *Let $\mathcal{N}$ be a network and let $i \in \boldsymbol{I}$ be a player (power company) and $j \in \boldsymbol{J}$ be a consumer.*

1. *Let $s^i := (p^i_\bullet, \mathcal{N}^i_\bullet)$ be a strategy of i. Then $\mathcal{X}^i_j$ denotes the set of feasible flows within $\mathcal{N}^i_j$.*
2. *Let $\boldsymbol{\beta}$ be a fee schedule suitable for $\mathcal{N}$. Then $\boldsymbol{\beta}$ constitutes also a fee schedule for $\mathcal{N}^i_j$. Hence there is a well defined cost function*

$$
\begin{aligned}
C^i_j(s^i, \bullet) \ &: \quad [0, \bar{d}^i_j] \to \mathbb{R}^+ \\
C^i_j(s^i, d) \ &:= \quad C_{\boldsymbol{\beta}}^{\mathcal{N}^{id}_j} \\
&= \quad \min \left\{ \sum \beta_{hk} x_{hk} \,\middle|\, X \in \mathcal{X}^{id}_j, \ \right\}
\end{aligned}
\tag{4.9}
$$

   defined on the interval of possible delivery of i towards j. This cost function is specified exactly as in SECTION *2 and* SECTION *3. That is, $\mathcal{N}^{id}_j$ refers to the network $\mathcal{N}^i_j$ (part of the strategy!) in which the demand of consumer j has been changed to be d.*

3. *Accordingly, given the demand function of consumer j, there is the specific (Bertrand) demand function of consumer j towards player i, i.e., the function $D^i_j : \mathbb{R}_+ \to \mathbb{R}_+$, which is exactly defined as in formula* (3.5) *of* SECTION *3.*
4. *Let $D^i(\bullet) := \sum_{j \in \boldsymbol{J}} D^i_j(\bullet)$. The* ***oligopolistic demand*** *is the function*

$$
D : \mathcal{S} \to \mathbb{R}^{\boldsymbol{I}}; \quad s \to (D^i(s))_{i \in \boldsymbol{I}} \quad . \tag{4.10}
$$

Note that the capacity constraints of a single power company are reflected in its strategic choice. Not so the total capacities of the global electricity network. The oligopolistic demand curve describes the demand pattern of the market. In a sense we have justified the demand curve by a microeconomic consideration. Now we describe the payoff functions of the players employing the penalty functions for overdemand of capacities.

Definition 4.4. *Let $\mathcal{E}$ be an electricity network.*

1. *Let $s^i := (p^i_\bullet, \mathcal{N}^i_\bullet)$ be a strategy of $i \in \boldsymbol{E}$. Then*

$$
\mathcal{N}^i(s^i) := \sum_{j \in \boldsymbol{J}} \mathcal{N}^i_j \tag{4.11}
$$

   reflects the ***capacity demand*** *of i at s^i.*

2. *Moreover, if s is a strategy $|\boldsymbol{I}|$−tuple, then*

$$\widehat{\mathcal{N}}(s) := \sum_{i\in\boldsymbol{I}} \mathcal{N}^i(s^i) \tag{4.12}$$

*reflects the **total capacity demand** at s.*

3. *The **payoff function** of player $i \in \boldsymbol{I}$ is the mapping*

$$\begin{aligned} U^i :\quad & \mathcal{S} \to \mathbb{R}; \\ U^i(s) := \quad & p^i D^i(p) - \sum_{j\in\boldsymbol{J}} C^i_j(D^i_j(p)) - \boldsymbol{Q}(\mathcal{N}^i(s^i), \widehat{\mathcal{N}}(s)) \\ := \quad & \sum_{j\in\boldsymbol{J}} \left[p^i_j D^i_j(p) - C^i_j(D^i_j(p))\right] - \boldsymbol{Q}(\mathcal{N}^i(s^i), \widehat{\mathcal{N}}(s)) \\ & (s \in \mathcal{S}). \end{aligned} \tag{4.13}$$

4. *The **El–Network Game** (a noncooperative n-person game) of the power companies is given by*

$$\Gamma := (\mathcal{S}, (U^i)_{i\in\boldsymbol{I}}) \quad . \tag{4.14}$$

Penalties are generated only when the total capacity is exceeded. In this case each company pays proportionally to its demands. It makes sense to assume that penalties are large compared to fees. Sofar as the penalty function will be used to enforce stability of equilibrium there should be no incentive to accept penalties in exchange for more capacities alloted.

In what follows, we assume that there is just one consumer in the market, hence the label j is fixed. We are going to construct an equilibrium within this context. The generalizations are obvious.

First of all we construct, for each player i, the network $\overset{\circ}{\mathcal{N}}{}^i$ which results from player i's exclusive use of the network. To this end, given a player $i \in I$, we render all nodes (of players) $i' \neq i$ to become transformers. That is, we change $L^{i'} > 0$ to $\overset{\circ}{L}{}^{i'} = 0$.

Next, we decompose the remaining network (which includes one player and one consumer only), according to the procedure indicated by Corollary 2.6. This generates a set of chains $\mathcal{C}^{ik}$ and some network $\mathcal{N}^\star$, the first group representing the slopes of the corresponding cost function c^i and the last one having no overall capacity. We put

(4.15)
$$\overset{\circ}{\mathcal{N}}{}^{i} := \sum_{i=1}^{K} \mathcal{N}^{ik}.$$

Definition 4.5. *The network* $\overset{\circ}{\mathcal{N}}{}^{i}$ *together with the original cost schedule* $\boldsymbol{\beta}$ *and player* i*'s utility function* U^i *define player* i*'s* ***monopolistic*** *El–network* $\overset{\circ}{\mathcal{E}}{}^{i}$

Next we begin with the construction of the (hypothetical) equilibrium networks $\overline{\mathcal{N}}^{i} \quad (i \in I)$. To this end, consider the system

(4.16)
$$\boldsymbol{\mathcal{N}} := \left\{ \left(\mathcal{N}^1, \ldots, \mathcal{N}^n\right) \middle| \; \mathcal{N}^i \prec \overset{\circ}{\mathcal{N}}{}^{i} (i \in I), \quad \sum_{i \in \boldsymbol{I}} \mathcal{N}^i \preceq \mathcal{N}, \quad \mathcal{N} - \sum_{i \in \boldsymbol{I}} \mathcal{N}^i \quad \text{has zero capacity} \right\}$$

Definition 4.6. *Let*

(4.17)
$$M := \min_{i, i' \in \boldsymbol{I}, \; i \neq i'} \left\{ \left| C^{\mathcal{N}^i} - C^{\mathcal{N}^{i'}} \right| \; \middle| \; \left(\mathcal{N}^1, \ldots, \mathcal{N}^n\right) \in \boldsymbol{\mathcal{N}} \right\}$$

and let

$$\overline{\boldsymbol{\mathcal{N}}} := \left(\overline{\mathcal{N}}^1, \ldots, \overline{\mathcal{N}}^n\right)$$

be a minimizer of the above problem. Then $\overline{\boldsymbol{\mathcal{N}}}$ *is a* ***share network system*** *and the component* $\overline{\mathcal{N}}^i$ *is player* i*'s* ***share network****.*

Lemma 4.7. *The quantity* M *defined in* (4.17) *is well defined and share networks systems exist.*

Proof: First of all, have to show that the set on the right hand side of (4.17) is nonempty. To this end, we construct an element $\widehat{\boldsymbol{\mathcal{N}}}$ of this set.

For any i, The nodes of $\widehat{\mathcal{N}}^{i}$ are the ones of the monopolistic network $\overset{\circ}{\mathcal{N}}{}^{i}$. Moreover, consider any edge (h, k) of this network. Then, if (h, k) appears in the monopolistic network of exaktly K players, we assign $\frac{1}{K} c_{hk}$ to be the capacity of $\overline{\mathcal{N}}^i$ for each of these players. That is, those players who may need this edge will have to equally share its original capacity. Of course, this means that a player who solely uses a certain edge may use its full capacity. Clearly, this construction leads to a suitable system $\widehat{\mathbf{N}}$.

Next, the costfunctions are specified similarly to formula (2.3) of SECTION 2. Obviously, the data specifying such a costfunction are finitely many, hence the metric indicated in (4.17) is the one of a Euclidean metric space. Clearly, the data of the cost functions depend continuously on the data of the underlying neworks. Consequently, the function to be minimized in (4.17) is continuous with respect to the underlying data, that is, with respect to the argument $\widehat{\mathbf{N}}$. It remains to observe that the set under consideration in (4.17) is indeed compact (and convex), as it is defined by finitely many inequalities and equations invoving the underlying data of $n-$tuples of networks,

q.e.d.

Now we have

Theorem 4.8. *Suppose the shared El–networks $\overline{\mathcal{E}}^{\,i} = (\overline{\mathcal{N}}^i, \boldsymbol{B})$ of all players $i \in I$ are similar in the sense of Remark 2.9 and hence, with suitable p_0, the cost functions define a PLOB $\mathcal{O} = (p_0, (\bar{d}_i)_{i\in I}, D^j, (\boldsymbol{C}^i)_{i\in I})$ with similar firms (Definition 2.9. of [4]). Also, assume that demand and supply are interceding (Definition 3.2 of [4]). Let $\overline{C}^i = C_{\boldsymbol{\beta}}^{\overline{\mathcal{N}}^i}$ denote the cost function of player i resulting from his shared El–network and let $\overline{p}_i$ be the Bertrand equilibrium price of $\mathcal{O}$. Then, $(\overline{\mathcal{N}}^i, \overline{p}_i)$ is a Nash equilibrium in the El–Network game.*

Proof:

1st STEP : We start out with a discussion of the cost function that results from the construction of the sub–El–networks as indicated above. Note that the cost schedule has not been changed but the capacities have been decreased on certain edges. Therefore, it follows from Theorem 2.10 that $\overline{C}^{\,i}$ (on its domain) dominates the costfunction $C_{\boldsymbol{\beta}}^{\mathring{\mathcal{N}}^i} =: \mathring{C}^{\,i}$ that corresponds to the monopolistic network.

2nd STEP : Let us now assume that player i_0 deviates from the hypothetical equilibrium by employing a pair $(\widehat{\mathcal{N}}^{i_0}, \widehat{p}_{i_0})$. Denote his cost function resulting by $\widehat{C}^{i_0} := C_{\boldsymbol{\beta}}^{\widehat{\mathcal{N}}^{i_0}}$. We have to show that this player does not improve his payoff. Now, if his price exceeds the equilibrium price, the he will receive zero profit and we know by the nature of the oligopoly equilibrium that this does not improve his payoff. Hence we discuss the case $\widehat{p}_{i_0} \leq \overline{p}_{i_0}$ only.

3rd STEP : First off all, if $\widehat{\mathcal{N}}^{i_0} \preceq \overline{\mathcal{N}}^{i_0}$ is the case, then player i_0 has an increased cost function in view of Theorem 2.10 . Hence, as he was not able to improve by cutting down his price in the original oligopoly, he will all the more not be able to do so with an increased cost function.

4$^{\text{th}}$STEP :

A similar argument holds true if player i_0 tries to use some capacity that has not been used up in $\sum_{i\in I}\overline{\mathcal{N}}^i$ but is available in $\mathcal{N}$. This capacity must be located in his network $\mathcal{N}^{i\star}$ mentioned above, as appears in this players decomposition generating his monopolistic network. So, when collecting the cheapest chains generating the slope of the cost function $\overset{\circ}{C}{}^i$, none of the routes available to player i along this area appeared. Now, given the restrictions put on player i's choices, it may be feasible to use these routs in order to admit higher capacity. But the expenses will be higher. Thus the costfunction again dominates the one of player i in the equilibrium as well as the original monopolistic one.

Hence, player i cannot improve his payoff by the same argument as above: the cost function of the deviating network exceeds the one in the original network.

5$^{\text{th}}$STEP :

Now we have to consider the case that player i deviates by specifying a net that exceeds the capacity available. That is, we assume that

$$\widehat{\mathcal{N}}^i + \sum_{k\in I,k\neq i}\overline{\mathcal{N}}^k \preceq \mathcal{N}$$

is *not* true. In this case he has to pay penalty costs. As these exceed any cost induced from the fee schedule, the player will not improve his payoff.

q.e.d.

We can singel out certain networks such that Bertrand equilibria exist under conditions verifiable in advance. E.g., if the power plants can reroute feasible flows through their neighbors domain at low costs, then the cost functions will be similar and hence Bertrand Oligopoly for all players may exist.

Theorem 4.9. *Let $\mathcal{N} = (\mathcal{E}, \boldsymbol{\beta})$ be an El-network and let there be a power plant i_0 and the corresponding monpolistic cost function $\overset{\circ}{C}{}^i$ such for some $\eta > 0$ all cost functions in an $\eta-$vicinity are similar in the sense of Definition 2.9. of [4]. Assume that, for any two power plants i, i' there is a connecting chain*

$$i = i_1, i_2, \ldots, i_R = i'$$

such that

$$\sum_{l=1}^{R-1} \beta_{i_l, i_l+1} < \frac{\eta}{N} \tag{4.18}$$

and

$$c_{i_l i_{l+1}} \geq \mathring{\Delta}_K^i \quad (l = 1, \dots R-1) \tag{4.19}$$

holds true. Then there exists a n equilibrium of the El–Network–Game.

Proof: Obviously our condition ensures that in the monopolistic oligopoly all firms are similar in the sense of Definition 2.9 of [4]. For each player can route flows similar to every other player by adding costs of at most η to his expenses.

Now, the shared networks (cf. the proof of Theorem 4.8) appear to be the monopolistic networks with capacities divided by N. the corresponding cost-functions are shrinked versions (cf. Theorem 2.10) of the “monopolistic” cost functions, so they represent similar firms as well. Therefore, the oligopoly constructed with respect to these cost functions has a Bertrand equilibrium. Now the remainder of the proof of Theorem 4.8 can be copied.

q.e.d.

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