The Dummy Paradox of the Bargaining Set*

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Abstract

By means of an example of a superadditive 0-normalized game, we show that the maximum payoff to a dummy in the bargaining set may decrease when the marginal contribution of the dummy to the grand coalition becomes positive.

We consider the weighted majority game (N, v_0) which has the tuple $(3; 1, 1, 1, 1, 1, 0)$ as a representation (see (3)). The maximum payoff to the dummy (the last player) in the bargaining set of (N, v_0) is shown to be $2/7$ (see Remark 2). If we now increase $v_0(N)$ by δ, $0 < \delta < 2/3$, then the maximum payoff to the last player in the new game, in which this player is no longer a dummy and contributes δ to N, is smaller than $2/7$ and strictly decreasing in δ (see Lemma 3).

We recall some definitions and introduce relevant notations. A (cooperative TU) *game* is a pair (N, v) such that $\emptyset \neq N$ is finite and $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. For any game (N, v) let

$$I(N, v) = \{x \in \mathbb{R}^N \mid x(N) = v(N) \text{ and } x^i \geq v(\{i\}) \text{ for all } i \in N\}$$

denote the set of *imputations*. (We use $x(S) = \sum_{i \in S} x^i$ for every $S \subseteq N$.) Let (N, v) be a game, $x \in I(N, v)$, and $k, l \in N$, $k \neq l$. Let

$$\mathcal{T}_{kl} = \{S \subseteq N \setminus \{l\} \mid k \in S\}.$$

An *objection* of k against l at x is a pair (P, y) satisfying

$$P \in \mathcal{T}_{kl},\ y(P) = v(P), \text{ and } y^i > x^i \text{ for all } i \in P. \tag{1}$$

We say that k can *object* against l *via* P, if there exists y such that (P, y) is an objection of k against l. Hence k can object against l via P, if and only if $P \in \mathcal{T}_{kl}$ and $e(P, x, v) > 0$,

*The second author was partially supported by the Edmund Landau Center for Research in Mathematical Analysis and Related Areas, sponsored by the Minerva Foundation (Germany).

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where $e(S,x,v) = v(S) - x(S)$ is the excess of S at x for $S \subseteq N$. A *counter objection* to an objection (P,y) of k against l is a pair (Q,z) satisfying

$$Q \in \mathcal{T}_{lk},\ z(Q) = v(Q),\ z^i \geq y^i \text{ for all } i \in Q \cap P \text{ and } z^j \geq x^j \text{ for all } j \in Q \setminus P. \tag{2}$$

Aumann and Maschler (1964) introduced the concepts of objections and counter objections.

An imputation $x \in I(N,v)$ is *stable* if for every objection at x there exists a counter objection. The *bargaining set* $\mathcal{M}(N,v)$ is defined by $\mathcal{M}(N,v) = \{x \in I(N,v) \mid x \text{ is stable}\}$. The bargaining set was introduced by Davis and Maschler (1967).

Player $i \in N$ is a *dummy* of (N,v) if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \setminus \{i\}$. The game (N,v) is *superadditive* if $v(S) + v(T) \leq v(S \cup T)$ for all $S \subseteq N$ and $T \subseteq N \setminus S$.

Remark 1. *Let (N,v) be a game. We recall that the core of (N,v) is the set $\mathcal{C}(N,v) = \{x \in\ I(N,v) \mid\ e(S,x,v) \leq 0$ for all $S \subseteq N\}$. Also we remark (see [2]) that $\mathcal{C}(N,v) \subseteq \mathcal{M}(N,v)$.*

In the sequel let $N = \{1,\ldots,6\}$ and (N,v_0) be the weighted majority game mentioned above. That is, $v_0(S),\ S \subseteq N$, satisfies the following equation:

$$v_0(S) = \begin{cases} 0 & , \text{ if } |S \setminus \{6\}| \leq 2 \\ 1 & , \text{ if } |S \setminus \{6\}| \geq 3 \end{cases} \tag{3}$$

Then (N,v_0) is a superadditive game and player 6 is a dummy. Also, for every $\delta \in \mathbb{R}, \delta > 0$, let (N,v_δ) be the game which differs from (N,v_0) only inasmuch as $v_\delta(N) = 1 + \delta$.

If $0 \leq \delta \leq 2/3$, then define $x_\delta \in \mathbb{R}^N$ by

$$x_\delta^j = \frac{1}{7} + \frac{2}{7}\delta \text{ for } j \in N \setminus \{6\} \text{ and } x_\delta^6 = \frac{2}{7} - \frac{3}{7}\delta. \tag{4}$$

If $\delta \geq 2/3$, then define $x_\delta \in \mathbb{R}^N$ by

$$x_\delta^j = \frac{1}{3} \text{ for } j \in N \setminus \{6\} \text{ and } x_\delta^6 = \delta - \frac{2}{3}. \tag{5}$$

Remark 2. *For every $\delta \geq 0$, $x_\delta \in \mathcal{M}(N,v_\delta)$.*

Proof: Clearly $x_\delta \in I(N,v_\delta)$. If $\delta \geq 2/3$, then $x_\delta \in \mathcal{C}(N,v_\delta)$, thus $x_\delta \in \mathcal{M}(N,v_\delta)$ by Remark 1. Now we assume $0 \leq \delta < 2/3$. Then $\mathcal{C}(N,v_\delta) = \emptyset$. Let $k,l \in N,\ k \neq l$, and let (P,y) be an objection of k against l at x_δ. By (1), $|P \setminus \{6\}| \geq 3$. If $l \neq 6$ and $k \neq 6$, then let $Q = (P \setminus \{k\}) \cup \{l\}$. If $k = 6$, then there exists $i \in P$ and let $Q = (P \setminus \{k,i\}) \cup \{l\}$. If $l = 6$, then select $i \in P \setminus \{k\}$ satisfying $y^i \geq y^j$ for all $j \in P \setminus \{k\}$ and let $Q = N \setminus \{k,i\}$. Also, let $z \in \mathbb{R}^Q$ be given by

$$z^j = \begin{cases} y^j & , \text{ if } j \in Q \cap P \\ v(Q) - y(P \cap Q) - x_\delta(Q \setminus (P \cup \{l\})) & , \text{ if } j = l \\ x_\delta^j & , \text{ if } j \in Q \setminus (P \cup \{l\}). \end{cases} \tag{6}$$

Then (Q,z) is a counter objection to (P,y). **q.e.d.**

Lemma 3. *Let $\delta \in \mathbb{R}_+$. If $x \in \mathcal{M}(N, v_\delta)$, then $x^6 \leq x^6_\delta$.*

Proof: Let $x \in I(N, v_\delta)$ satisfy $x^6 > x^6_\delta$. It remains to show that $x \notin \mathcal{M}(N, v_\delta)$. Without loss of generality we may assume

$$x^1 \leq \cdots \leq x^5. \tag{7}$$

In what follows we shall construct a justified objection of 1 against 6 via the coalition $P = \{1, 2, 3\}$. We distinguish two cases:

(1) $\delta \geq 2/3$: Then 1 can object against 6 via P by (7) and the assumption that $x^6 > x^6_\delta$. Also, $\{2, 3, 4, 6\}$, $\{2, 3, 5, 6\}$, and $\{2, 4, 5, 6\}$ are the only coalitions in $\mathcal{T}_{61}$ which **might** have a nonnegative excess at x. Now, player 2 is a member of all of them and $e(P, x, v_\delta) > e(Q, x, v_\delta)$ for all $Q \in \mathcal{T}_{61}$, thus there exists $y \in \mathbb{R}^P$ such that $y(P) = v(P)$, $y^i > x^i$ for all $i \in P$, and $y^2 - x^2 > e(Q, x, v_\delta)$ for all $Q \in \mathcal{T}_{61}$. We conclude that (P, y) is a justified objection of 1 against 6 at x_δ.

(2) $0 \leq \delta < 2/3$: Again, 1 can object against 6 via P, because $x^6 > x^6_\delta$. Let $Q_{\{i\}}, i = 2, 3$, and $Q_{\{2,3\}}$ be the members of $\mathcal{T}_{61}$ defined by

$$Q_{\{i\}} = \{i, 4, 5, 6\},\ i = 2, 3, \text{ and } Q_{\{2,3\}} = \{2, 3, 4, 6\}.$$

Then

$$Q \in \mathcal{T}_{61},\ e(Q, x, v_\delta) \geq 0 \Rightarrow u(Q) = 1, \tag{8}$$

because $x \geq 0$ and $x^6 > 0$. Also, we have

$$Q \in \mathcal{T}_{61},\ u(Q) = 1 \Rightarrow e(Q, x, v_\delta) \leq e(Q_{Q \cap \{2,3\}}, x, v_\delta). \tag{9}$$

Indeed, every $Q \in \mathcal{T}_{61}$ satisfying $u(Q) = 1$, intersects $\{2, 3\}$, hence $Q_{Q\cap\{2,3\}}$ is defined. The inequality follows from (7). Also, $x \geq 0$, $x^6 > 0$, (7) - (9) imply that

$$e(P, x, v_\delta) > (e(Q, x, v_\delta))_+ \text{ for all } Q \in \mathcal{T}_{61}. \tag{10}$$

We claim that

$$e(P, x, v_\delta) > (e(Q_{\{2\}}, x, v_\delta))_+ + (e(Q_{\{3\}}, x, v_\delta))_+. \tag{11}$$

By (10) it suffices to show that

$$e(P, x, v_\delta) > e(Q_{\{2\}}, x, v_\delta) + e(Q_{\{3\}}, x, v_\delta), \tag{12}$$

which is equivalent to

$$1 - x(P) > 1 - x(Q_{\{2\}}) + 1 - x(Q_{\{3\}})$$

and, thus, to $-1 - x^1 + 2x(\{4, 5, 6\}) > 0$. By the observation that

$$-1 - x^1 + 2x(\{4, 5, 6\}) = -1 + x(N) - 2x^1 - x(\{2, 3\}) + x(\{4, 5, 6\}) \geq \delta + x^6 - 2x^1$$

it suffices to show that $\delta + x^6 - 2x^1 > 0$. By (7), $5x^1 + x^6 \leq 1 + \delta$, thus

$$\delta + x^6 - 2x^1 \geq \frac{3\delta + 7x^6 - 2}{5} > 0.$$

The last inequality is implied by the assumption that $x^6 > x^6_\delta = 2/7 - (3/7)\delta$.

Now the proof can be finished. By (10) and (11) there exists $t \in \mathbb{R}^P$ satisfying

$$\begin{aligned} &t(P) = e(P, x, v_\delta),\ t(\{2,3\}) > e(Q_{\{2,3\}}, x, v_\delta), \\ &t^i > (e(Q_{\{i\}}, x, v_\delta))_+, i \in \{2,3\},\ \text{and } t^1 > 0. \end{aligned} \tag{13}$$

Let $y = x^P + t$. By (13), (P, y) is a justified objection of 1 against 6 at x. **q.e.d.**

Remark 4. *The* **reactive** *bargaining set and the* **semi-reactive** *bargaining set, two variants of the bargaining set recently introduced by Granot and Maschler (1997) and Sudhölter and Potters (2001), do not show the dummy paradox. Indeed, in [4] it is shown, that both solutions, when restricted to superadditive games, satisfy the dummy property (that is, each member of the solution assigns $v(\{i\})$ to a dummy i).*

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