

# Unique Nash Implementation for a Class of Bargaining Solutions

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## **Abstract**

The paper presents a method of supporting certain solutions of two-person bargaining games by unique Nash equilibria of associated games in strategic form. Among the supported solutions is the Kalai-Smorodinsky solution.

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# 1 Introduction

There has been some discussion in the recent literature about the relation between the Nash program in game theory and the theory of mechanism design [cf. Dagan and Serrano (1998), Serrano (1997), Trockel (1999a), (1999b), and Naeve (1999)].

The Nash program aims to support axiomatic solutions of cooperative games in strategic or extensive form. In mechanism theory one is interested in the design of game forms implementing a prespecified cooperative solution considered as a (set of) desired social state(s). Implementation is achieved if any population in a certain domain by playing equilibrium strategies in the game defined by the game form and this population's profile of utility functions establishes (an element of) the solution. That the Nash program can be considered as a part of mechanism theory has been established by Trockel (1999a) [see also Dagan and Serrano (1998), Serrano (1997)]. There it is shown that any support result for a solution of cooperative NTU-games can be extended to an implementation result.

In this paper we present a specific class of two-person games in strategic form whose unique Nash equilibria support given bargaining solutions. In particular the Kalai-Smorodinsky solution is among those solutions.

An interesting feature of our games under the aspect of using reasonable and sensitive strategic models for supporting or implementation results is the fact that they enable the players to reach any of the utility allocations of bargaining games by suitable strategic choices.

Our games are based on the extensive form game that Trockel (1999b) has used for the subgame perfect equilibrium implementation of the Nash solution and which has been modified by Haake (1998) to implement the Kalai-Smorodinsky solution.

## 2 Background

We need a short sketch of the two-person extensive form game in Trockel (1999).

Let a two-person bargaining situation be given by some strictly convex, comprehensive (with respect to  $\mathbb{R}_+^2$ ) compact subset  $S$  of  $\mathbb{R}_+^2$  with non-empty interior. We assume for convenience that the bargaining game is normalized such that  $\text{proj}_1(S) = [0, 1] \setminus \{0\}$  and  $\text{proj}_2(S) = \{0\} \cup [0, 1]$ . We let  $\mathcal{S}_1 = \mathcal{S}_2 = [0, 1]$  be the strategy sets of the two players of a game in strategic form.

Let  $B$  denote the set of all two-person bargaining games normalized in the way described above. A solution is a map  $L : B \rightarrow \mathbb{R}_+^2 : S \mapsto L(S) \in S$ .

Now suppose that for a given efficient solution  $L$  there exists a continuous function  $z_1^L : \mathcal{S}_1 \rightarrow \mathcal{S}_1 : x_1 \mapsto z_1^L(x_1)$  which is strictly decreasing and strictly concave such that for all  $S \in B$

$$(MM) \quad L_1(S) = \arg \max_{x_1 \in \mathcal{S}_1} \min(x_1, z_1^L(x_1)).$$

Notice that by continuity of the mappings  $z_1$  and  $\text{id}_{\mathcal{S}_1}$  we have  $L_1(S) = z_1(L_1(S))$ .

In this case the solution  $L$  can be supported by the unique subgame perfect equilibrium of the following two-step game in extensive form. We shall denote the efficient boundary of  $S$  by  $\partial S$ .

On stage 1 player 1 chooses  $x_1 \in [0, 1]$ . Then on stage 2 player 2 chooses one element from the set  $\{X, Z\}$ . The resulting payoff vector for the strategy choices  $(x_1, Y) \in [0, 1] \times \{X, Z\}$  is the unique payoff vector  $(x_1, x_2) \in \partial S$  or  $(z_1(x_1), z_2) \in \partial(S)$ , if  $Y = X$  or  $Y = Z$ , respectively. Note that player 1 prefers one of the points  $(x_1, x_2), (z_1(x_1), z_2)$  if and only if the other player prefers the other point. Therefore backward induction leads to  $(L_1(S), L_2(S)) = L(S)$  as the unique equilibrium outcome for the two subgame perfect equilibria  $(L_1(S), X)$  and  $(L_1(S), Z)$ . This non-uniqueness is harmless because the uniqueness of the first player's equilibrium strategy together with the uniqueness of the equilibrium payoff vector prevents any need for coordination, and is, therefore, for predictive purposes as good as uniqueness. The above reasoning has been used by Trockel (1999b) and subsequently by Haake (1998) for the special cases of the Nash solution ( $N$ ) and the Kalai-Smorodinsky solution ( $KS$ ). Both belong to the class of solutions allowing an argmax min-characterization as described above. But only the Kalai-Smorodinsky solution is in the class covered by our proposition.

We shall first derive a unique Nash equilibrium support result for any solution  $L$ , satisfying condition (MM). Then we shall apply this result to the specific function  $z_1^{KS}$  to support the Kalai-Smorodinsky solution.

### 3 The support result

The first attempt to achieve a Nash equilibrium support result for a solution  $L$  satisfying condition (MM) is to mimic the players' strategic behavior in the two-stage game described above but to let them make their choices simultaneously. The resulting game  $S^L$  for a given bargaining game would be

$$S^L = ([0, 1], \{X, Z\}; \pi_1, \pi_2)$$

with

$$\pi_1(x_1, X) = x_1, \quad \pi_2(x_1, X) = x_2, \quad (x_1, x_2) \in \partial S$$

$$\pi_1(x_1, Z) = z_1^L(x_1), \quad \pi_2(x_1, Z) = z_2, \quad (z_1^L(x_1), z_2) \in \partial S$$

Unfortunately, this game has no Nash equilibrium (in pure strategies). This can be seen as follows:

case 1:  $(x_1, X), x_1 < L_1(S)$

Player 1 can improve by choosing a larger  $x'_1$  and getting  $x'_1$  rather than  $x_1$ .

case 2:  $(x_1, X), x_1 > L_1(S)$

Player 2 can improve by choosing  $Z$  and getting  $z_2$  rather than  $x_2$ .  
As  $z_1^L(x_1) < L_1(S) < x_1$  we must have  $z_2 > x_2$ .

case 3:  $(x_1, Z), x_1 < L_1(S)$

Player 2 can improve by choosing  $X$  and getting  $x_2$  rather than  $z_2$ .  
As  $x_1 < L_1(S) < z_1^L(x_1)$ , we must have  $x_2 > z_2$ .

case 4:  $(x_1, Z), x_1 > L_1(S)$

Player 1 can improve by choosing a smaller  $x'_1$ . He gets then  $z_1^L(x'_1)$  rather than  $z_1^L(x_1)$ . As  $x'_1 < x_1$ , we have  $z_1^L(x'_1) > z_1^L(x_1)$ .

Therefore, we consider instead the partially mixed extension  $S^L$  of  $S^L$  which we define as follows:

$$S_L := ([0, 1], [0, 1]; \alpha_1, \alpha_2)$$

with

$$u_1(x_1, \alpha) := \alpha u_1(x_1, X) + (1 - \alpha) u_1(x_1, Z)$$

$$u_2(x_1, \alpha) := \alpha u_2(x_1, X) + (1 - \alpha) u_2(x_1, Z).$$

Notice that the strategy spaces of the two players, though formally identical, have very different meanings, as can be seen from the payoff functions. Player 1 still chooses utility claims for himself between 0 and 1, while player 2 chooses lotteries over the set  $\{X, Z\}$ . The resulting payoffs are therefore in terms of expected utilities.

We state now the result of the present paper.

Proposition:

Let  $L$  be a bargaining solution satisfying condition  $(MM)$ . Then for any  $S \in B$  the game  $S_L$  has a unique Nash equilibrium (in pure strategies)  $(L_1(S), \alpha_L(S))$  whose equilibrium payoff vector is  $L(S)$ .

Proof:

In the first step we show that no point  $(x_1, \alpha) \in [0, 1]^2$  with  $x_1 \neq L_1(S)$  can be an equilibrium.

Consider such a point  $(x_1, \alpha)$ . The cases  $\alpha \in \{0, 1\}$  can be excluded because of our above reasoning. Consider, therefore,  $\alpha \in ]0, 1[$ . If  $x_1$  is smaller (resp. larger) than  $z_1^L(x_1)$ ,  $x_2$  is larger (resp. smaller) than  $z_2$ . So player 2 can increase his payoff by increasing (resp. decreasing)  $\alpha$ .

Hence, the only candidates for equilibria are the points  $(L_1(S), \alpha)$ ,  $\alpha \in ]0, 1[$ . All of these result in the same payoff for player 2, as  $L_2(S) = z_2^L(L_1(S))$ . Accordingly, each  $\alpha$  is a best reply of player 2 to player 1's strategy  $L_1(S)$ . Obviously,  $L_1(S)$  is not the best reply for player 1 to each  $\alpha \in ]0, 1[$ . For a very small  $\alpha$  player 1 could improve by a smaller  $x_1$ , for an  $\alpha$  close to 1 he could improve by an  $x_1$ , close to 1. So we are looking for some  $\alpha$  to which  $L(S)$  is player 1's best response.

So for any  $\alpha$  we can look at the solutions of the optimization problem  $\arg \max_{x_1 \in [0, 1]} f(x_1, \alpha)$ , where  $f(x_1, \alpha) := \alpha u_1(x_1, X) + (1 - \alpha) u_1(x_1, Z)$ .

As only  $x_1 = L_1(S)$  is consistent with the Nash equilibrium requirement for  $(x_1, \cdot)$ , the question remains for which  $\alpha \in ]0, 1[$  we have  $L_1(S) \in \arg \max_{x_1 \in [0,1]} f(x_1)$ .

Consider the function

$$f : [0, 1] \times [0, 1] \rightarrow \mathbb{R} : (x_1, \alpha) \mapsto f(x_1).$$

As  $z_1^L$  is strictly concave and  $id_{[0,1]}$  is concave the function  $f = f(\cdot, \alpha)$  is strictly concave for any  $\alpha \in ]0, 1[$ . Let for  $\alpha \in [0, 1]$

$$f(\alpha) := \max_{x \in [0,1]} f(x).$$

and

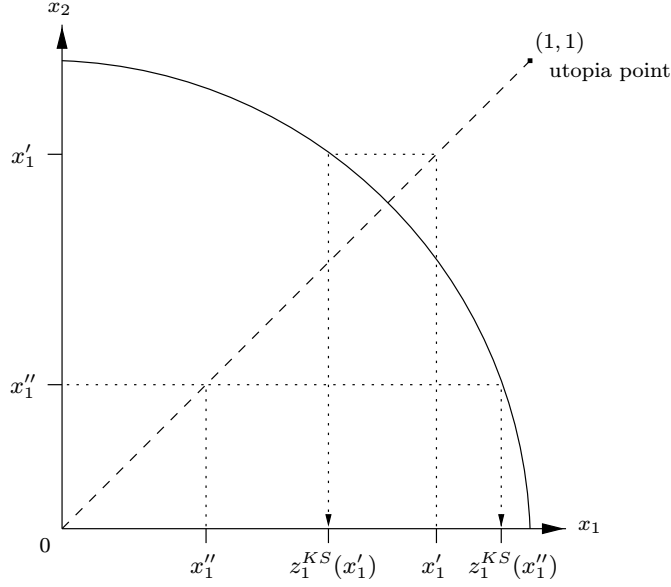
$$\mathfrak{D}(\alpha) := \arg \max_{x \in [0,1]} f(x).$$

We define the maximizer function  $x_1 : [0, 1] \rightarrow [0, 1]$  by  $\{x_1(\alpha)\} := \mathfrak{D}(\alpha)$  for  $\alpha \in [0, 1]$ . By the Maximum Theorem under Convexity [cf. Sundaram (1996, Theorem 9.17)] our assumptions imply that  $f$  is a continuous strictly concave function and that  $\mathfrak{D}$  is an upper hemicontinuous single-valued correspondence. The latter fact says that  $x_1$  is a continuous function.

Now observe that  $x_1(0)$  maximizes  $z_1^L$  on  $[0, 1]$ . As  $z_1^L$  is strictly decreasing this implies  $x_1(0) = 0$  and  $f(0) = z_1^L(0) > 0$ . Similarly  $x_1(1)$  maximizes  $id_{[0,1]}$  on  $[0, 1]$ . Hence,  $x_1(1) = 1$  and  $f(1) = 1$ . As  $x_1$  is continuous there exists some  $\alpha \in [0, 1]$  such that  $x_1(\alpha) = L_1(S) \in [0, 1]$  with  $f(\alpha) = L_1(S)$ . Therefore  $(L_1(S), \alpha)$  is a Nash equilibrium of  $\frac{S}{L}$ .

To establish the uniqueness of the equilibrium we employ the strict concavity of the function  $f$ . We have  $f'(\alpha) = x(\alpha) - z_1^L(x(\alpha))$ , for all  $\alpha \in [0, 1]$ . In particular, we have  $f'(0) = z_1^L(0) < 0, f'(1) = 1 - z_1^L(0) > 0$  and  $f'(\alpha) = x(\alpha) - z_1^L(x(\alpha)) = 0$ . The concavity excludes  $f'(\hat{\alpha}) = 0$  for any  $\hat{\alpha} \neq \alpha$ , which excludes  $(L(S), \hat{\alpha})$  as an equilibrium.

## 4 Application to the Kalai-Smorodinsky solution



Let  $S$  be again a normalized bargaining game in the class  $B$ .

In the case that the solution  $L$  is the Kalai-Smorodinsky solution  $KS$  we define the function  $z_1^{KS} : [0, 1] \rightarrow [0, 1]$  as follows:

Any point  $x_1 \in [0, 1]$  defines a unique point  $(x_1, x_1)$  on the diagonal of  $\mathbb{R}_+^2$ . Projecting this point along the two axes to the efficient boundary generates two different points  $(x_1, x_2)$  and  $(z_1^{KS}(x_1), x_1)$ . They are identical only if  $(x_1, x_1) = KS(S)$ . The function  $z_1^{KS}$  defined in this way satisfies the assumptions of our proposition. In particular, we get:

$$z_1^{KS}(0) = 1, z_1^{KS}(KS(S)) = KS_1(S), z_1^{KS}(1) = 0.$$

Accordingly, the proposition asserts the support of the Kalai-Smorodinsky solution by the unique Nash equilibrium of the game  $\frac{S}{KS}$ .

## 5 Concluding Remarks

The present paper presents a unique Nash support result for a class of solutions of two-person bargaining games containing the Kalai-Smorodinsky solution. Application of the imbedding principle in Trockel (1999a) immediately transforms this contribution to the Nash program into a result on unique Nash implementation in the sense of mechanism theory. The class of solutions satisfying the assumptions of our proposition is characterized by the fact that each of the solutions can be implemented in subgame perfect equilibrium in some divide-and-choose-mechanism. This yields an interesting alternative to implementations in the literature based on auction mechanisms [cf. for instance Moulin (1984)].



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