The Endogenous Formation of Cartels

Joachim Rosenmüller

Institute of Mathematical Economics
IMW
University of Bielefeld
D-33615 Bielefeld
Germany

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Abstract

We discuss large but finite linear production games or market games. These games represent markets such that the agents decompose into finitely many disjoint groups each of which holds a corner of the market. In such a market most solution concepts like the core, the Shapley value, or the Walrasian equilibrium tend to favor the short side of the market excessively. That is, in the replicated limit or in the continuum version, the short side is awarded all the possible profits even though cooperation within the grand coalition is required. This kind of behavior is also observed with the Walrasian equilibrium. We show that vNM–Stable Sets differ markedly. For large but finite player sets we demonstrate that this concept is capable of assigning considerable profits towards the long side of the market. Also, it turns out that the shape of the generic vNM–Stable Set suggests cartelization of the market. Thus, it turns out that the long side first agrees to form a cartel and then forces the short side to make concessions. This way the long side profits from the game as well.

1 Introduction

Within this paper we attempt to explain the *endogenous* formation of cartels in large but finite markets or economies. We assume that contracts generating cartels are legally permissible and can be enforced. This is not all together unusual. While anti trust laws prevail in many western states, it is also observed that cartels are legal (e.g., in Switzerland in previous decades the formation of cartels was legally permitted). Also, we may consider a union representing a group of workers with (approximately) equal characteristics to be a cartel in the technical sense. Then obviously at least parts of the society are legally cartelized in most western countries.

General equilibrium theory or related approaches via coalition formation in exchange economies are apparently unable to *predict* the endogenous formation of cartels, even in simple situations which seem to call for such kind of organization. A simple version of such a situation is what is called the glove market or glove game.

In such an economy the traders may be seen to command *corners* of different (nonoverlapping) sets of commodities. When contracts are feasible – and can be legally enforced – there seems to be a strong incentive for agents to form (at least intermediately) *cartels* ot *syndicates* by joining forces in subcoalitions. These cartels may then act as players or agents themselves, so that the responsibility for bargaining is delegated to them and the result is implemented to the grand market accordingly.

This procedure points to a different game in which few players act to the benefit of those they represent. If we consider the game in which the various cartels act as players, then their bargaining power may be quite different from the one in the original market and they may force (members of) opposing cartels to accept a distribution of profits (allocations, imputations) that is much more favorable compared to the result obtained without cartelization. The latter situation might be represented by the core. Results from bargaining via cartels cannot be seen as consistent with the concept of the core.

In view of equivalence theorems for large markets, the result of cooperation within cartels also differs from what it turns out to be if agents show price taking behavior. The same is true, to mention a further concept of Game Theory, for the Shapley value. This concept measures the marginal contribution of traders on average. For large games, as all coalitions look almost like the grand coalition, the Shapley value represents eventually the marginal contribution of traders to the grand coalition – which is zero for agents living in an excess supply corner.

In a paper "Formation of Cartels in Large Markets" [4] Sergiu Hart discusses this situation from the viewpoint of a different solution concept, the vNM–Stable Set. He argues that in markets with disjoint corners, the formation of cartels has to be a *result* of the solution concept employed, it should be an *endogenous* concept. And he points to the vNM–Stable Set which (for the non-atomic case and other than the core etc.) *does* indicate this kind of behavior.

Hart's argument essentially is that there are vNM–Stable Sets which are obtained from finite vNM–Stable Sets in a symmetric way (treating all players of the same type alike). This, he goes on, shows that coalition *of types* have been formed, acted as players (in the finite game) and distributed the profits obtained this way symmetrically among their members. As all the solutions in the continuous case are of this shape (his main result) he goes even further in holding that society *has* to organize itself this way.

Hart's cartels consist essentially of types. His arguments reflects the fact that continuous solutions can be obtained from finite ones by an embedding procedure. This he interprets as a representation of the continuous mass of traders of a cartels by their representative, a player in the finite game.

Hart does not explicetely construct vNM–Stable Sets. Rather he exploits the similarities (symmetries) between the continuous non-atomic version and the finite one.

Also, there seems to be a tendency in his arguing that the actual shape of a vNM–Stable Set essentially does not matter. Rather he points out that the *coalitions* predicted by this solution concept do indeed reflect cooperation within cartels – which the core does not.

Thus, the claim that society organizes itself in cartels again fails in pointing out the result of this organization. But, after all, it is the distribution of wealth that matters in the public discussion. The formation of cartels as such is only the first step. Certainly, one would like to see the the actual nature of the resulting distribution of wealth as well.

In a more recent paper [10], Rosenmüller and Shitovitz classify all vNM–Stable Sets in the non-atomic context. There appears a further version of cartelization of the market: it is seen that in each corner of the market a distribution of wealth is agreed upon which, to some extent, reflects the initial holdings of the agents within this corner. Indeed, cartels distribute the result of their “internal game” symmetrically among their members in a most plausible fashion: on the short side each trader gets a share exactly proportional to his holdings, and on the long side a share is not exceeding the holdings.

Thereafter the various cartels agree about some mixture of these distributions. This may be seen as the result of cooperation within the representatives of the cartels.

The result in [10] bears a consequence to the finite case only if we assume uniform distribution of commodities in each corner. So it does imply the construction of finite games representing the cartels in Hart’s sense – but only for a very limited (fully symmetric) class of glove games.

In this paper we construct much more general vNM–Stable Sets for games that are large but finite. Thus, we actually finish the discussion initiated by Hart by pointing to the finite case and indicating solutions. This is achieved for a reasonable class of *large* games: necessarily, there has to be a sufficient number of small players located in each corner of the market. But players with large chunks of initial holdings are allowed and may also be arbitrarily many – a result which also permits an interpretation which does not exclude “atoms” in the market.

The construction we are dealing with offers some further insight into the formation of stable situations or cartels. We point to some type of “bandwagon process” which is used to block off assignments that are outside the solution.

We would like to argue that the situation has greatly improved since Hart’s paper. There are now existence results concerning vNM–Stable Sets which support his view. But much more has been achieved: the endogenous formation of cartels representing each corner is explained as well. We can see that finite games allow for the actual construction of vNM–Stable Sets which reflect the same situation as in the continuum. Thus, agents on the short side of the market benefit exactly in accordance with their initial holdings and the long side has a certain freedom to distribute payoffs in accordance with boundedness by the holdings.

2 Definitions, Simple Properties

We start out with a finite set $I = \{1, \ldots n\}$, the set of ***agents*** or ***players***. The power set $\underline{\underline{\mathbf{P}}} = \mathcal{P}(I)$ describes the system of ***coalitions***. If $\boldsymbol{v} : \underline{\underline{\mathbf{P}}} \to \mathbb{R}_+$ is a real valued function satisfying $\boldsymbol{v}(\emptyset) = 0$, then the triple $(I, \underline{\underline{\mathbf{P}}}, \boldsymbol{v})$ constitutes a ***game***; however, we shall also apply the term to the function $\boldsymbol{v}$ as such. $\boldsymbol{v}$ is otherwise also called the ***coalitional function***. The quantity $\boldsymbol{v}(S)$ is the (monetarian) worth a coalition $S \in \underline{\underline{\mathbf{P}}}$ may achieve by cooperation (and distribute among its members according to agreement).

An ***additive*** (and nonnegative) set function or ***measure*** $\boldsymbol{\mu}$ on $\underline{\underline{\mathbf{P}}}$ is tantamount to a vector $\mu = (\mu_1, \ldots, \mu_n)$ via the convention

$$\boldsymbol{\mu}(S) \; = \; \sum_{i \in S} \mu_i \quad (S \in \underline{\underline{\mathbf{P}}}) \; .$$

We regard additive functions as distributions of wealth coalitions may agree upon rather than games. The set of measures is denoted by $\boldsymbol{A}_+$. The ***carrier*** of $\boldsymbol{\mu} \in \boldsymbol{A}_+$ is the set $C(\boldsymbol{\mu}) \; = \; \{i \in I \mid \mu_i > 0\}$.

The ***core*** of a game $(I, \underline{\underline{\mathbf{P}}}, \boldsymbol{v})$ (or of $\boldsymbol{v}$) is the set of all distributions of the wealth of the grand coalition that cannot be improved upon by any smaller coalition, formally:

$$\mathcal{C}(\boldsymbol{v}) \; := \; \left\{ \boldsymbol{\mu} \in \boldsymbol{A}_+ \,\middle|\, \boldsymbol{\mu}(I) = \boldsymbol{v}(I), \; \boldsymbol{\mu}(S) \geq \boldsymbol{v}(S) \quad (S \in \underline{\underline{\mathbf{P}}}) \right\}$$

The class of ***market games*** results from exchange economies and according to Shapley – Shubik [14] is equivalent to the class of ***totally balanced*** games. This class can be described as to be the set of all games such that the restriction to every (nonempty) coalition has a nonempty core (the Bondareva – Shapley Theorem, [2], [13]). A further observation (Kalai – Zemel, [5]) establishes the fact that this class can as well be described as the class of ***linear production***– or ***LP.–games***. As this is the version we shall be concerned with, we provide the following description.

LP.–games are games which can be represented as minima of finitely many nonnegative additive set functions (measures) $\boldsymbol{\lambda}^1, \cdots, \boldsymbol{\lambda}^r \in \mathbf{A}_+$ via

(1) $$\boldsymbol{v}(S) \; = \; \min \{ \boldsymbol{\lambda}^\rho(S) \mid \rho = 1, \ldots, r \} \, .$$

We write this

(2) $$\boldsymbol{v} = \bigwedge \{ \boldsymbol{\lambda}^1, \cdots, \boldsymbol{\lambda}^r \} \; ,$$

thus indicating a min-operation within games.

Regardless of the representation we choose for the coalitional function, our view is that we are dealing with market games. The most notorious example is the one of the glove game. There are two *orthogonal* measures $\boldsymbol{\lambda}^1$, $\boldsymbol{\lambda}^2$ involved, each of them representing "uniform distribution", that is, as a vector each of them equals $(1, \ldots, 1)$. The well known interpretation is that owners of right hand gloves and left hand gloves occupy different "corners" of the market. By cooperation it is possible to sell pairs of gloves profitably, but each of the corners on its own is unprofitable.

We deal with a generalization of this version and, by various reasons restrict our discussion to the ***orthogonal*** case. This means that with respect to a game represented as in (2), we assume that the measures $\boldsymbol{\lambda}^1, \cdots, \boldsymbol{\lambda}^r \in \mathbf{A}_+$ are orthogonal. In this case, the game or the coalitional function will also be called orthogonal. Note that the representation of $\boldsymbol{v}$ by means of orthogonal measures is actually unique (see [9], Chapter 5 for a discussion of this topic)

The ***carrier*** of $\boldsymbol{\lambda}^\rho$ is denoted by $C(\boldsymbol{\lambda}^\rho) = C^\rho$, $(\rho = 1, \ldots, r)$, orthogonality means that $I = \sum_{\rho=1}^{r} C^\rho$ describes a partition of I (disjoint unions are written as sums).

Also, we shall generally assume that each $\boldsymbol{\lambda}^\rho$ is integer-valued. By reasons of continuity, this assumption is not too strong; rational–valued measures are dense in $\boldsymbol{A}_+$ anyway. We identify the carriers with the various ***corners*** of the market. Each commodity in one of the corners is indispensable for profitable exchange (or efficient production, when seen as a production game). For, if one corner of the market is missing, then a coalition obtains zero worth. Thus, the orthogonal game is a game with distinct corners of the market.

While we focus on orthogonal games, we do not intend to restrict our discussion to glove markets. The agents in the various corners may own different quantities of the commodity available. Thus we introduce the notion of *types* as follows:

Each corner (carrier) C^ρ is decomposed into finitely many sets, the sets of ***types*** via

$$C^\rho = \sum_{\rho=1}^{T^\rho} K_\tau^\rho \quad (\tau = 1, \ldots, T^\rho) \tag{3}$$

(recall that the sum indicates the union of disjoint coalitions). Within each type (each set K_τ^ρ representing a type), all players have the same ***weight***, i.e.,

$$\lambda_i^\rho = w_\tau^\rho \quad (i \in K_\tau^\rho). \tag{4}$$

Thus, in corner $^\rho$ player i is of type $_\tau$ whenever he owns a quantity of $\lambda_i^\rho = w_\tau^\rho$ units of commodity $^\rho$.

Equivalently, this can be written

$$\boldsymbol{\lambda}^\rho(\bullet) = \sum_{\tau=1}^{T^\rho} |\bullet \cap K_\tau^\rho| w_\tau^\rho, \quad (\rho = 1, \ldots, r). \tag{5}$$

Within each corner we assume the weights to be ordered increasingly and the smallest weight to be 1,

i.e.,

$$1 = w_1^\rho \leq \ldots \leq w_{T^\rho}^\rho \quad (\rho = 1, \ldots, r). \tag{6}$$

The size of type τ in corner ρ is denoted by $k_\tau^\rho = |K_\tau^\rho|$, thus the vector $k^\rho = (k_1^\rho, \ldots, k_{T^\rho}^\rho)$ reflects the size of the type and $k = (k^1, \ldots, k^r)$ can be seen as to indicate the distribution of the players over the types.

Every coalition $S \in \underline{\underline{\mathbf{P}}}$ decomposes naturally into

$$S = \sum_{\rho=1}^{r} S^\rho \text{ with } S^\rho = S \cap C^\rho \ (\rho = 1, \ldots, r), \tag{7}$$

we call the-members of S^ρ the ***partners*** of coalition S in corner $^\rho$.

Clearly we have $\boldsymbol{\lambda}^\rho(S) = \boldsymbol{\lambda}^\rho(S^\rho) = w^\rho s^\rho \quad (S \in \underline{\underline{\mathbf{P}}}, \ \rho = 1, \ldots, r)$. We use the abbreviation M^ρ in order to indicate the total amount of commodity or ***mass*** available in corner $^\rho$, that is,

$$M^\rho \ := \ \boldsymbol{\lambda}^\rho(I) \ = \ \boldsymbol{\lambda}^\rho(C^\rho) \ = \ \sum_{i \in C^\rho} \boldsymbol{\lambda}_i^\rho \ = \ \sum_{\tau=1}^{T} k_\tau^\rho w_\tau^\rho \ = \ k^\rho w^\rho. \tag{8}$$

We shall also assume that the groups are ordered according to total mass, i.e., $M^1 \leq \ldots \leq M^r$.

An important system of coalitions is provided by the ***diagonal*** which is

$$\underline{\underline{\Delta}} \ := \ \{S \in \underline{\underline{\mathbf{P}}} | \boldsymbol{\lambda}^\rho(S) = \boldsymbol{v}(S) \ (\rho = 1, \ldots, r)\}. \tag{9}$$

We call the elements of $\underline{\underline{\Delta}}$ ***diagonal*** sets as the vector–valued measure provided by the representation (2), i.e., $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^1, \cdots, \boldsymbol{\lambda}^r)$ maps such sets into diagonal vectors in $\mathbb{R}^r$. On diagonal sets, $\boldsymbol{v}$ behaves *additively*, that is, for $S, T \in \underline{\underline{\Delta}}$ it follows that

$$\boldsymbol{v}(S) + \boldsymbol{v}(T) \;=\; \boldsymbol{v}(S+T)$$

holds true. As a consequence, it is observed that the behavior of $\boldsymbol{v}$ on the diagonal has a strong influence on the core. Indeed, suppose the carrier C^1 can be matched by partners in each corner, i.e., there is, for every $(\rho = 2, \ldots, r)$, a coalition D^ρ satisfying $\boldsymbol{\lambda}^\rho(D^\rho) = \boldsymbol{\lambda}^1(C^1)$. Then it follows immediately, that $\boldsymbol{v}$ equals any core element on $\underline{\underline{\Delta}}$, formally:

$$\boldsymbol{v}(S) \;=\; \boldsymbol{\mu}(S) \quad (S \in \underline{\underline{\Delta}},\ \boldsymbol{\mu} \in \mathcal{C}(\boldsymbol{v})\ ) \ . \tag{10}$$

Diagonal coalitions can be seen as efficient: no input is wasted and the exact quantity of commodity $^\rho$ necessary to achieve $\boldsymbol{v}(S)$ is available from every corner $^\rho$ of the market. Also (this is the meaning of additivity), the result of cooperation in the grand coalition or within a system of smaller but diagonal coalitions is the same. The worth achieved in diagonal coalitions decomposing the grand coalition is just added up.

Therefore, diagonal coalitions play an important role. With increasing size of the market or game, the core is eventually determined by the diagonal coalitions: if there are "sufficiently many" diagonal coalitions the equations suggested by (10) will precisely characterize the core elements. This can as well be formulated as an "equivalence theorem" between the core and the set of shadow price–generated imputations, see [7], [8], [9] for the details.

In our present context, the shadow price–generated imputations are given by the convex hull of those measures in the representation (2) that have minimal mass (the *minimizing measures*). For simplicity, we shall always assume that the extreme points of the core are among the measures used in (2), that is, that the equivalence theorem is satisfied. We refer to this fact (sometimes sloppily) by saying that we deal with a ***large game***.

This property could be achieved by just adding the extremals of the core to the representation. But this might disturb orthogonality. Thus, we have to assume that "sufficiently many players" are present in order to achieve an equivalence theorem. It is not hard to see that an equivalence theorem holds true if there are sufficiently many players of the smallest weight $(w_1^\rho = 1)$ available in each corner $^\rho$. More precisely, given the weights of the larger players, there is an integer $\boldsymbol{N}_0$ (depending on these weights only) such that, whenever $k_1^\rho \geq \boldsymbol{N}_0(\rho = 1, \ldots, r)$ is satisfied, then the core is the convex hull of the minimizing measures. This can be verified by a small change of an equivalence theorem as presented in Theorem 3.23 in CHAPTER 5 of [9].

Eventually we shall argue that the core is a significant solution concept for orthogonal games only for exact games. A game $\boldsymbol{v}$ is called ***exact*** if, for every coalition $S \in \underline{\underline{\mathbf{P}}}$, there exists a measure $\boldsymbol{\mu} \in \mathcal{C}(\boldsymbol{v})$ such that

$$\boldsymbol{\mu}(S) \;=\; \boldsymbol{v}(S)$$

holds true. An exact game is totally balanced and it is not hard to see that a totally balanced game is exact if and only if there exists a representation (2) of $\boldsymbol{v}$ by means of finitely many measures such that each of them has the same total mass, i.e.,

$$\boldsymbol{\lambda}^1(I) \;=\; \ldots \;=\; \boldsymbol{\lambda}^r(I) \;=\; \boldsymbol{v}(I)$$

holds true. In other words, for an exact game, the grand coalition is diagonal.

In the continuum, the core of the exact game is exactly the convex hull of the measures establishing the representation (Billera and Raanan [1], a continuous game is large in the sense explained above). In this case the core seems to derive an additional strength from the fact that it has a tendency to be a ***vNM-Stable Set***. This means that the core dominates all imputations outside of itself. For a continuum of players, this has been established by Einy et. al. [3].

Again it turns out that diagonal coalitions play an important role in order to establish this property for certain large but finite games. We describe this result in SECTION 3.

The concept of a ***vNM–Stable Set*** is due to von Neumann-Morgenstern([15]). This is a set $\mathcal{S}$ of imputations such that no internal domination occurs while any feasible payoff measure outside of $\mathcal{S}$ can be dominated from inside. For completeness, we offer a definition of the concept as follows.

Definition 2.1. *1. Let* $(I, \underline{\underline{\mathbf{P}}}, \boldsymbol{v})$ *be a game. An* ***imputation*** *is a measure* $\boldsymbol{\xi}$ *such that* $\boldsymbol{\xi}(I) = \boldsymbol{v}(I)$ *and* $\xi_i \geq \boldsymbol{v}(\{i\}), \quad (i \in I)$ *holds true.*

2. An imputation $\boldsymbol{\xi}$ ***dominates*** *an imputation* $\boldsymbol{\eta}$ *w.r.t a coalition* $S \in \underline{\underline{\mathbf{P}}}$ *if* $\boldsymbol{\xi}$ *is* ***effective*** *for* S, *i.e.,*

$$\boldsymbol{\xi}(S) \leq \boldsymbol{v}(S) \tag{11}$$

and if

$$\boldsymbol{\xi}(T) > \boldsymbol{\eta}(T) \quad (T \in \underline{\underline{\mathbf{P}}},\ T \subseteq S) \tag{12}$$

holds true, that is, every sub-coalition of S *(every player in* S*) strictly improves its payoff at* $\boldsymbol{\xi}$ *versus* $\boldsymbol{\eta}$*. We write* $\boldsymbol{\xi}\ \mathrm{dom}_S\ \boldsymbol{\eta}$ *to indicate domination.*

3. *An imputation* $\boldsymbol{\xi}$ ***dominates*** *an imputation* $\boldsymbol{\eta}$ *(written* $\boldsymbol{\xi}$ dom $\boldsymbol{\eta}$*) if there is a coalition* $S \in \underline{\underline{\mathbf{P}}}$ *such that* $\boldsymbol{\xi}$ *dominates* $\boldsymbol{\eta}$ *with respect to* S.

4. *Let* $\boldsymbol{v}$ *be a game. A set* $\mathcal{S}$ *of imputations is called a* ***vNM–Stable Set*** *if*

   - *there is no pair* $\boldsymbol{\xi}, \boldsymbol{\mu} \in \mathcal{S}$ *such that* $\boldsymbol{\xi}$ dom $\boldsymbol{\mu}$ *takes place,*
   - *for every imputation* $\boldsymbol{\eta} \notin \mathcal{S}$ *there exists* $\boldsymbol{\xi} \in \mathcal{S}$ *such that* $\boldsymbol{\xi}$ dom $\boldsymbol{\eta}$ *holds true.*

The interpretation of vNM–Stable Sets generally is quite involved. A well known quotation from [15] states that it is a “standard of behavior.” In our context, Hart emphasizes that vNM–Stable Sets predict the coalitions that will actually form.

In Section 2 we show that for large but finite exact games the core is a (the unique) vNM–Stable Set. This is the finite version of the result of Einy et al. [3].

Eventually we want to deal with non-exact games. Here, the result available for a continuum of players states that all (convex, polyhedral) vNM–Stable Sets can be characterized and (other than the core) indicate the bargaining power of the long side of the market via cartelization (see Rosenmüller and Shitovitz [10]). We obtain this result (partially) for the case of a finite but large game (Section 4). In this situation, the difficulties of interpreting vNM–Stable Sets are nicely overcome. For, the convex polyhedral type we exhibit obviously admits of the interpretation Hart had in mind: the formation of cartels. We see clearly, that all players in a corner of the market derive a certain strength from cartelization: on one hand (first off all ?) they agree about a distribution of worth inside their corner, this way establishing the cartel. On the hand (thereafter ?), the cartels bargain (via representatives ?) on an equal footing since all of them are absolutely necessary for generating wealth at all.

Thus, the concept of vNM–Stable Sets all of a sudden appears to be most successful: other that the core (or the Shapley value or the Walrasian equilibrium) they provide an (endogenously justified) successful participation of the long side of the market in bargaining process.

3 Exact Games and vNM–Stability of the Core

Within this section we focus on large exact (totally balanced) games. The solution concept we are dealing with is the core - but only tentatively so: eventually we want to deal with vNM–Stable Sets. Our present aim is to obtain the finite version of a theorem which is provided by Einy et al. ([3]) within the continuous context. It states that the core is vNM–stable.

We cannot expect this statement to hold true quite generally in the finite case (it is not true). Rather, we are going to show that, in some well defined case, the theorem is true for *large* games. Thus, eventually we shall exhibit a class of exact games generated by integer (orthogonal) measures and having many members of the smallest type in each corner: this class allows for a core that is a vNM–Stable Set.

To begin with, consider an *exact* game, i.e., a totally balanced game represented as

$$\boldsymbol{v} = \bigwedge \{\boldsymbol{\lambda}^1, \cdots, \boldsymbol{\lambda}^r\}$$

such that the total mass of each of the measures $\boldsymbol{\lambda}^\rho$ is the same,

$$(1) \qquad M^1 \;=\; \boldsymbol{\lambda}^1(I) \;=\; M^\rho \;=\; \boldsymbol{\lambda}^\rho(I) \;\; (\rho = 1, \ldots, r).$$

As has been emphasized, we assume the game to be large, so the core is the convex hull of the measures used for the representation. Within the framework of exact games, the *diagonal* again plays an important role. Our first task is to describe a subsystem of diagonal sets which partitions the player set I in a most delicate way. We may think of this system as of a *coalition formation process.* Coalitions form successively by the introduction of new players who organize themselves in partnerships within the various corners of the market.

More specifically, consider a system $\underline{\underline{\mathbf{S}}} \subseteq \underline{\underline{\boldsymbol{\Delta}}}$ of diagonal sets which constitutes a ***partition*** of I. Assume that there is an ordering $\preceq$ defined on $\underline{\underline{\mathbf{S}}}$. For every corner ρ and any $S \in \underline{\underline{\mathbf{S}}}$ consider the partners of S in C^ρ given by $S^\rho \;=\; S \cap C^\rho$. Clearly, the ordering of $\underline{\underline{\mathbf{S}}}$ induces a consistent ordering on each system $\underline{\underline{\mathbf{S}}}^\rho \;=\; \{S^\rho\}_{S \in \underline{\underline{\mathbf{S}}}}$. This means that, whenever the partners S^ρ of S precede the partners T^ρ of T within one corner ρ, then the same holds true for any other corner $^{\rho'}$. This way we obtain as well an ordering within each corner. Clearly we write $S^\rho \preceq T^\rho$ for the induced ordering; the same symbol $\preceq$ is used for the ordering on any $\underline{\underline{\mathbf{S}}}^\rho$ as well as the one on $\underline{\underline{\mathbf{S}}}$. There is little danger that this will cause confusion.

Formally we supply the following definition:

Definition 3.1. *A system of diagonal coalitions* $\underline{\underline{\mathbf{S}}} \subseteq \underline{\underline{\Delta}}$ *is* ***consistently ordered*** *if*

1. $\underline{\underline{\mathbf{S}}}$ *constitutes a partition of* I*, i.e.,* $\sum_{S \in \underline{\underline{\mathbf{S}}}} S = I$*, and*
2. *there is a total ordering* $\preceq$ *defined on* $\underline{\underline{\mathbf{S}}}$ *which consistently induces a total ordering on every* $\underline{\underline{\mathbf{S}}}^\rho$ $(\rho = 1, \ldots, r)$*, that is,*

$$S \preceq T \iff S^\rho \preceq T^\rho \quad (\rho = 1, \ldots, r)$$

holds true.

Intuitively, if we adopt the idea that diagonal coalitions produce or trade most efficiently, then the above definition means that we have to arrange the players in each corner to form coalitions with partners in every other corner such that coalitions with equal weight join across the corners to reach maximal efficiency. This kind of arrangement will proceed in a certain ordering which, eventually, reflects the fact that smaller players "support the introduction of larger players into the system'. The introduction of larger players is more difficult because these players, owing a large chun of commodity in some corner, are more problematically to arrange into efficiently producing coalitions. On the other hand they are "stronger" in some obvious sense. The problem is to arrange for an "adjusted" worth of these large players with respect to a core distribution. This reflects a "coalition formation process".

Consequently, we want this partition to satisfy some additional conditions. Let us describe how large players have to be ***introduced*** by smaller ones into the groups of efficiently producing coalitions. To this end we use the ordering $\preceq$ in a slightly extended way: if $\bar{S}^\rho = \bar{S} \cap C^\rho$ represents the $\rho-$partners of some $\bar{S} \in \underline{\underline{\mathbf{S}}}$ and $T^\rho \subseteq C^\rho$ is an *arbitrary* coalition in corner ρ, then we write $T^\rho \preceq \bar{S}^\rho$ if all members of T^ρ belong to coalitions S^ρ that precede $\bar{S}^\rho$, formally

$$T^\rho \preceq \bar{S}^\rho \;\text{ iff }\; T^\rho \subseteq \bigcup_{S^\rho \in \underline{\underline{\mathbf{S}}}^\rho,\; S^\rho \preceq \bar{S}^\rho} S^\rho \;\text{ holds true .}$$

Definition 3.2. *Let* $\underline{\underline{\mathbf{S}}} \subseteq \underline{\underline{\Delta}}$ *be a system of diagonal coalitions which is consistently ordered (cf. Definition 3.1). Let* $S \in \underline{\underline{\mathbf{S}}}$ *and let* $S^\rho = S \cap C^\rho$ *be the partners of* S *in corner* $^\rho$*. We shall say that* $T^\rho \subseteq C^\rho$ ***supports the introduction*** *of* $i \in S^\rho$*, if the following conditions are satisfied:*

1. *$T^\rho \preceq S^\rho$.*

2. *All players in T^ρ are of the same type as i.*

3. *T^ρ and S^ρ have the same weight, i.e., $\boldsymbol{\lambda}^\rho(T^\rho) = \boldsymbol{\lambda}^\rho(S^\rho)$.*

For short we shall say that the introduction of S^ρ is "supported by predecessors" if, for every $i \in S^\rho$, a suitable T^ρ supports its introduction. Sloppily, we shall also say that i "is introduced by predecessors".

Note that the coalition T^ρ that appears in the above definition is not necessarily the partnership of some $T \in \underline{\underline{\mathbf{S}}}$. All we require is that T^ρ precedes S^ρ in the above mentioned sense. Also, the third requirement essentially states that, whenever S^ρ consists of a singleton, then the type of this player must have appeared "earlier" - again in the above mentioned sense.

A first vague idea concerning this definition is described as follows. Suppose that with respect to the formation of $S \in \underline{\underline{\mathbf{S}}}$, the partners in corner $^\rho$ consider how to justify their claims (vs. the other corners and inside their coalition). Now, by some (inductive) procedure, player $i \in S^\rho$ can point to a coalition T^ρ of players that are already "in the system" and which are all of his type. These players may belong to various coalitions which appeared during the process of forming the system so far. But as the game along diagonal coalitions is additive, this may not matter to much: they could now form a coalition which has the same power (initial assignment) as S^ρ and in which their share already has been established. Then they could joint the large coalition of all players players "preceding" $\underline{\underline{\mathbf{S}}}$ - which is diagonal and produces efficiently. This way the share of player i is considered to be established by the previous bargaining successes of all players of his type in his corner.

Now the successive introduction of players is formally described as follows:

Definition 3.3. *Let $\underline{\underline{\mathbf{S}}} \subseteq \underline{\underline{\boldsymbol{\Delta}}}$ be a partition of I. We shall say that $\underline{\underline{\mathbf{S}}}$ is* ***universally ordered*** *by $\preceq$, if $\underline{\underline{\mathbf{S}}}$ is consistently ordered (c.f. Definition 3.1) and, in addition, the following conditions are satisfied.*

1. *Let R be the $\preceq$-first coalition. Then, for every corner $^\rho$, the partnership $R^\rho = R \cap C^\rho$ consists of just one player of the smallest type.*

2. *Let $S \in \underline{\underline{\mathbf{S}}}$, $S \neq R$. Then, for any S^ρ which is not a singleton, introduction is supported by predecessors. (cf. Definition 3.2).*

3. *For every* $S \in \underline{\underline{\mathbf{S}}}$, $S \neq R$ *there exists* $\rho \in \{1, \cdots, r\}$ *such that introduction of* S^{ρ} *is supported by predecessors (cf. Definition 3.2).*

This definition is interpreted as to incorporate the idea of successive introduction as follows: the $\preceq$ −first coalition consists of just one of the smallest players in each corner. They are starting the coalition formation process represented by $\underline{\underline{\mathbf{S}}}$ and their "introduction" is immediate.

For every coalition following later, consider the partners in corner $^{\rho}$: if this is not a singleton, their introduction is necessarily supported by predecessors. If it is a singleton, then this requirement is not necessary. However, it cannot happen that all partner coalitions are singletons that have not been supported by predecessors. Rather, in at least one corner, there is a partner set the introduction of which by predecessors is guaranteed. This way we have explained the way that singletons are ***introduced by partners***: they are not directly linked to the predecessors in the process, but they are partners to at least one coalition that is linked to players already in the system.

Example 3.4. In the following figure we represent each $S \in \underline{\underline{\mathbf{S}}}$ by a block. There are three corners which correspond to the rows in each block. The first coalition contains a player of weight 1 from each corner, hence for the next two blocks introduction is trivially feasible.

In order to introduce a player with weight 3 in the middle corner of the third block, three players of weight 1 in each of the other corners are available, introduction for each of these coalitions is feasible, since the 1 appears three times previously on both levels.

1	1	1	111	111	3	3	6	6	9	66	66	6666
1	1	1	3	3	3	3	33	33	333	12	12	12 3333
1	1	1	111	111	3	3	111111	33	333	3333	3333	24

Consider the last block: in order to introduce the player with weight 24 in the lowest corner, we need four times a preceding 6 in the uppermost corner so as to generate a total of 24. This way, the introduction of the coalition in the uppermost corner is supported by predecessors. In the middle corner we need twice the 12 and eight times the 3 in the preceding coalitions; hence again the introduction is supported by predecessors. Eventually we see that the introduction of the player with weight 24 is supported by his partners in the other corners because they are all linked to previously introduced players / coalitions.

We shall now make a further effort to interpret this definition, viewing the solution concept we have in mind.

As $\boldsymbol{v}$ is additive on the diagonal, it is clear that efficient production can as well be organized in the grand coalition I of the exact game as in any decomposition of I into diagonal sets. The above definition is meant to stabilize the core in the sense of vNM–stability (see the Theorem below). In order to achieve this, players must be successively "introduced" into the diagonal system. This process is now seen with a particular regard towards blocking an imputation which is outside the solution concept, hence considered to be "illegal" in the sense of vNM–Stability.

To this end, first of all the smallest players in each corner organize themselves into an efficiently producing (diagonal) coalition by joining in the $\prec$ −first coalition R which contains one player of this smallest type in each corner.

Now suppose a certain number of diagonal coalitions has already been formed. Then typically a player with considerable weight which so far was not in the system is "introduced". The player (as a singleton) is matched by smaller players in the various corners other than his own so that they form a efficiently producing (i.e., diagonal) coalition. Each of these partners has a smaller weight. The introduction of players of this weight has been supported earlier – so now his introduction can easily be supported.

This procedure now permits to compute the suitable share (in the core and with the aim to block an imputation outside the core) of the partners by "previous" agreements and to compare it with the suitable share of the new player to be introduced. This way it must be possible to introduce successively all players via efficiently trading / producing coalitions and to determine their proper share when an element outside the core (an illegal proposition in the sense of vNM) has to be dominated. The result of the process is nevertheless an imputation, i.e., a distribution of profits that suggests the grand coalition. However, producing / trading in the grand coalition and in a system of diagonal ones has the same effect. The final result will be that the "legal" propositions (in the sense of vNM) obtained essentially by this procedure point (in the non-exact case) to the endogenous formation of cartels.

Now it turns out that the possibility to arrange cooperation simultaneously in small but efficient coalitions as described above renders the core to be vNM-stable. We borrow this theorem from [9].

Theorem 3.5. *Let* $\boldsymbol{v} = \bigwedge \left\{\boldsymbol{\lambda}^1, \cdots, \boldsymbol{\lambda}^r\right\}$ *be a large exact game. If* $\boldsymbol{\lambda}$ *admits of a universally ordered system* $\underline{\underline{\mathbf{S}}} \in \underline{\underline{\mathbf{D}}}$*, then the core of* $\boldsymbol{v}$ *is a vNM-Stable Set.*

We do not want to enter into the details of the proof of the above theorem. The point is, that, for any imputation outside the core, there exists a rearrangement of the coalitions of the universally ordered system which uniquely determines a core element. This core element dominates the outside imputation; hence the core is vNM–stable.

The main result of this section is that we can construct universally ordered systems if there are sufficiently many small players present in every corner. More precisely, given an exact game $\boldsymbol{v} = \bigwedge \{\boldsymbol{\lambda}^1, \cdots, \boldsymbol{\lambda}^r\}$, we can point out certain bounds $\boldsymbol{N}^\rho \quad (\rho = 1, \ldots, r)$, depending on the large weights w^ρ_τ only, with the following property. Whenever the number of players of weight 1 in corner ρ is at least $\boldsymbol{N}^\rho$, then there exists a universally ordered system partitioning I, hence $\mathcal{C}(\boldsymbol{v})$ is stable.

Necessarily we have to introduce some notation. We denote tentatively the set of *types* of corner ρ *without the smallest one* by $\mathcal{T}_\rho = \{2, \ldots, T_\rho\}$. Following tradition, $\tau_{-\rho}$ indicates the coordinates of a vector $(\tau_1, \ldots, \tau_r)$ *without* coordinate ρ. The analogous notation for taking away *two* indices is $\tau_{\{\rho\sigma\}}$. The same is used for index sets and their Cartesian product, say $\mathcal{T}_1 \times \ldots \times \mathcal{T}_r$.

Also, we introduce the ***products*** of weights

$$W_{\tau_1,\ldots,\tau_r} := w^1_{\tau_1} \cdot \ldots \cdot w^r_{\tau_r} \tag{2}$$

and the ***greatest common divisor*** of weights

$$d_{\tau_1,\ldots,\tau_r} := g.c.d.\{w^1_{\tau_1}, \ldots, w^r_{\tau_r}\}. \tag{3}$$

These quantities are now used in order to define our lower bounds for the number of small players. For $\sigma = 1, \ldots, r$ and $\tau_\sigma \in \mathcal{T}_\sigma$ we define

$$\boldsymbol{H}^\sigma_{\tau_\sigma} := \max_{\rho \neq \sigma} \sum_{\tau_\rho = 2}^{T_\rho} \max_{\tau_{-\{\rho,\sigma\}} \in \mathcal{T}_{-\{\rho,\sigma\}}} \frac{W_{\tau_1\tau_2\ldots\tau_r}}{d_{\tau_1\tau_2\ldots\tau_r} w^\sigma_{\tau_\sigma}} \,. \tag{4}$$

Then our bounds are given by

$$\boldsymbol{N}^\rho := \max\left\{w^\sigma_{T_\sigma} \,\middle|\, \sigma = 1, \ldots, r\right\} + \max\left\{\min\left\{k^\rho_{\tau_\rho}, \boldsymbol{H}^\rho_{\tau_\rho}\right\} \,\middle|\, \tau_\rho \in \mathcal{T}_\rho\right\}. \tag{5}$$

Let us shortly consider this expression. The essential clue is that these bounds depend on the weights w^ρ_τ only. Indeed, for the first term this is obvious. As to the second term, the products of the weights are involved (the reader who wishes to skip the details may as well exchange the term for the maximum product of all weights). The minimum to be taken between the number of players $k^\rho_\bullet$ and the quantities $\boldsymbol{H}^\rho_{\tau_\rho}$ as well as the combinatorial niceties of involving the g.c.d.'s are technical and not too important for the interpretation. The point is that, given the weights of the large types, the number of players of the smallest type which is necessary for the core to be a vNM-Stable Set, can be determined. In this sense, the core is vNM-stable for large games. The precise version is as follows:

Theorem 3.6. *Let* $\boldsymbol{v} = \bigwedge \{\boldsymbol{\lambda}^1, \cdots, \boldsymbol{\lambda}^r\}$ *be a large exact game. Assume that, for every corner* ρ *the number of players of weight* 1, *i.e.* k^ρ_1, *exceeds* $\boldsymbol{N}^\rho$. *Then the core of the game* $\mathcal{C}(\boldsymbol{v})$ *is a vNM-Stable Set.*

Proof:

The proof rests on Theorem 3.5. We are going to construct a universally ordered system $\underline{\underline{\mathbf{S}}} \subseteq \underline{\underline{\boldsymbol{\Delta}}}$ of diagonal coalitions. To this end we arrange the players in "blocks', each "block" representing a diagonal coalition S such that the members of the various corners are represented by their weights. The first set of blocks is given as follows:

$$\text{(6)} \qquad \boxed{\begin{matrix}1\\ \vdots\\ 1\end{matrix}}\ \boxed{\begin{matrix}1\\ \vdots\\ 1\end{matrix}}\ \cdots\cdots\ \boxed{\begin{matrix}1\\ \vdots\\ 1\end{matrix}}$$

Thus we see, that from each corner a player of weight 1 joins a block (diagonal coalition) S, these weights are listed horizontally each row indicating a corner. The (vector-valued) measure of such a block is of course

$$\boldsymbol{\lambda}(S) = (1, \ldots, 1).$$

As for the number of these blocks we take

$$\text{(7)} \qquad \max_{\rho=1,\ldots,r} \max_{\tau=1,\ldots,T} w^\rho_\tau = \max_{\rho=1,\ldots,r} w^\rho_{T_\rho}$$

(the weights are ordered increasingly). There are enough players of weight 1 in each corner available for this construction, this is just ensured by the first term that appears in the definition (5) of $\boldsymbol{N}^\rho$.

Note also that the blocks introduced so far satisfy the conditions to be imposed on a universally ordered system. The first block starts the procedure of introduction and the next blocks obviously consist of partner sets the introduction of which is supported by predecessors.

The next type of blocks is obtained as follows: from some corner ρ we take a player of weight w^ρ_τ and from each other corner we take just w^ρ_τ players of weight 1. One block of this type is represented by

$$(8) \qquad \boxed{\begin{array}{ccc} 1 & \dots & 1 \\ \vdots & \vdots & \vdots \\ 1 & \dots & 1 \\ & w^\rho_\tau & \\ 1 & \dots & 1 \\ \vdots & \vdots & \vdots \\ 1 & \dots & 1 \end{array}} \; .$$

This way we introduce successively the players with weights w^ρ_τ into our system, again this is compatible with the requirement of a universally ordered system. For, the partner sets that are non-singletons are introduced by predecessors. Hence the singleton, consisting of one player of weight w^ρ_τ, is introduced by its partners.

As for the number of such blocks, we require that

$$(9) \qquad \text{the number of blocks (8) is } \boldsymbol{H}^\rho_\tau.$$

This is feasible by our assumption regarding the numbers of small players available: indeed, whenever it occurs that the number of players with the large weight w^ρ_τ is exhausted, then all players of this weight appear already in our system and can be disregarded henceforth. In this case we save a few players of weight 1 as well, as only k^ρ_τ of them are necessary. On the other hand, if there are more players of the large weight w^ρ_τ, the fact that enough small players are available in each corner is exactly ensured by the appearance of the term $\boldsymbol{H}^\rho_\tau$ in the second part of the definition of $\boldsymbol{N}^\rho_\tau$ as spelled out in (5).

Thus there is now a huge number of blocks of shape (8) in our system, all of them are diagonal (with measure $\boldsymbol{\lambda}(S) = (w^\rho_\tau, \dots, w^\rho_\tau)$) and so is their union. We may represent the part of the system $\underline{\underline{\mathbf{S}}}$ constructed thus far by

$$(10)\qquad \boxed{\begin{array}{c}1\\ \vdots\\ \vdots\\ .\\ \vdots\\ \vdots\\ 1\end{array}}\;\boxed{\begin{array}{c}1\\ \vdots\\ \vdots\\ .\\ \vdots\\ \vdots\\ 1\end{array}}\;\cdots\cdots\;\boxed{\begin{array}{c}1\\ \vdots\\ \vdots\\ .\\ \vdots\\ \vdots\\ 1\end{array}}\boxed{\begin{array}{ccc}&w_1^1&\\ 1&\dots&1\\ \vdots&\vdots&\vdots\\ 1&\dots&1\\ 1&\dots&1\\ \vdots&\vdots&\vdots\\ 1&\dots&1\end{array}}\;\cdots\;\boxed{\begin{array}{ccc}1&\dots&1\\ \vdots&\vdots&\vdots\\ 1&\dots&1\\ &w_\tau^\rho&\\ 1&\dots&1\\ \vdots&\vdots&\vdots\\ 1&\dots&1\end{array}}\;\cdots\;\boxed{\begin{array}{ccc}1&\dots&1\\ \vdots&\vdots&\vdots\\ 1&\dots&1\\ 1&\dots&1\\ \vdots&\vdots&\vdots\\ 1&\dots&1\\ &w_{T_r}^r&\end{array}}$$

At this stage we observe, that players of every weight have been introduced by predecessors in sufficient numbers.

3rdSTEP : Therefore, further players with weight 1 are not necessary, it is now possible to introduce all other players still remaining into the the system. To explain this procedure we proceed as follows.

The sum (disjoint union) of all blocks collected so far and indicated in (10) is a diagonal coalition. As we are presently dealing with an exact game, the complement or *remainder coalition* is a diagonal coalition as well. This remainder at this stage is a coalition containing players of all possible weights, let us represent it by a block

$$(11)\qquad \boxed{\begin{array}{ccccccccc}1&\dots&1&w_2^1&\dots&w_2^1&\dots&w_{T_1}^1&\dots&w_{T_1}^1\\ \vdots&&\vdots&\vdots&\vdots&\vdots&&\vdots&\vdots&\vdots\\ 1&\dots&1&w_2^\rho&\dots&w_2^\rho&\dots&w_{T_\rho}^\rho&\dots&w_{T_\rho}^\rho\\ \vdots&&\vdots&\vdots&\vdots&\vdots&&\vdots&\vdots&\vdots\\ 1&\dots&1&w_2^r&\dots&w_2^r&\dots&w_{T_r}^r&\dots&w_{T_r}^r\end{array}}\;.$$

The number of weights 1 does not have to be equal in each row. In some rows these weights may be missing. Also, in some rows certain other weights greater that 1 may be missing. All of this does not change the argument we are now presenting. This argument is provided for the case that there are no weights 1 in the above list and that all the other weights are present. This induces a simplified notation and has no serious consequences. Therefore, in what follows, the weights in each corner start out with an index 2 for each type.

Let us assume that the number of weights w_τ^ρ that appears in the above list (11) is a_τ^ρ. Since the coalition represented by the sum of all blocks in (11) is a diagonal one, the mass accumulated in every corner is the same, say Λ. This means that we have a set of equations

$$\sum_{\tau=1}^{T_\rho} a^\rho_\tau w^\rho_\tau \;=\; \Lambda \quad (\rho = 1, \dots, r). \tag{12}$$

Now we want to show that we can introduce all members of this remainder coalition into our system $\underline{\underline{\mathbf{S}}}$ without the help of further elements of weight 1 just by using the weights w^ρ_τ that are already present.

We fix an index in $\{1, \dots, r\}$, say $\rho = 1$ (any other index would be treated analogously, this is just for notational convenience).

Suppose that for some $\bar{\tau}_1$ we find that

$$a^1_{\bar{\tau}_1} \geq \max_{\rho=2,\dots,r} \sum_{\tau_\rho=2}^{T_\rho} \max_{\tau_{-\{1,\rho\}} \in \mathcal{T}_{-\{1,\rho\}}} \frac{W_{\bar{\tau}_1\tau_2\dots\tau_r}}{d_{\bar{\tau}_1\tau_2\dots\tau_r} w^1_{\bar{\tau}_1}} \tag{13}$$

is the case. Then, for every σ not all indices τ_σ can satisfy

$$a^\sigma_{\tau_\sigma} < \max_{\rho\neq\sigma} \sum_{\tau_\rho=2}^{T_\rho} \max_{\tau_{-\{1,\sigma\}} \in \mathcal{T}_{-\{1,\sigma\}}} \frac{W_{\bar{\tau}_1\tau_2\dots\tau_r}}{d_{\bar{\tau}_1\tau_2\dots\tau_r} w^\sigma_{\tau_\sigma}} \quad . \tag{14}$$

For, in this case (14) would imply

$$\begin{aligned}
& \sum_{\tau_\sigma \in \mathcal{T}_\sigma} a^\sigma_{\tau_\sigma} w^\sigma_{\tau_\sigma} \\
< \; & \sum_{\tau_\sigma \in \mathcal{T}_\sigma} \max_{\tau_{-\{1,\sigma\}} \in \mathcal{T}_{-\{1,\sigma\}}} \frac{W_{\bar{\tau}_1\tau_2\dots\tau_r}}{d_{\bar{\tau}_1\tau_2\dots\tau_r}} \\
\leq \; & \max_{\rho=2,\dots,r} \sum_{\tau_\rho=2}^{T_\rho} \max_{\tau_{-\{1,\rho\}} \in \mathcal{T}_{-\{1,\rho\}}} \frac{W_{\bar{\tau}_1\tau_2\dots\tau_r}}{d_{\bar{\tau}_1\tau_2\dots\tau_r}} \\
\leq \; & a^1_{\bar{\tau}_1} w^1_{\bar{\tau}_1} \quad \text{(in view of (13))} \\
\leq \; & \Lambda \quad \text{in view of (12)} \quad .
\end{aligned} \tag{15}$$

This is clearly a contradiction to (12) and hence, for every σ there is at least one $\bar{\tau}_\sigma \in \mathcal{T}_\sigma$ such that

$$a^\sigma_{\bar{\tau}_\sigma} \geq \max_{\tau_{-\{1,\sigma\}} \in \mathcal{T}_{-\{1,\sigma\}}} \frac{W_{\bar{\tau}_1\dots\bar{\tau}_\sigma\dots\tau_r}}{d_{\bar{\tau}_1\dots\bar{\tau}_\sigma\dots\tau_r} w^\sigma_{\bar{\tau}_\sigma}} \tag{16}$$

holds true. Hence we have obtained from every corner σ a type $\bar{\tau}_\sigma$ such that

$$a^\sigma_{\bar{\tau}_\sigma} \geq \frac{W_{\bar{\tau}_1 \ldots \bar{\tau}_\sigma \ldots \bar{\tau}_r}}{d_{\bar{\tau}_1 \ldots \bar{\tau}_\sigma \ldots \bar{\tau}_r} w^\sigma_{\bar{\tau}_\sigma}} \tag{17}$$

is satisfied. As a consequence, we can take from every corner of the remainder coalition as indicated by (11) a number of

$$\frac{W_{\bar{\tau}_1 \ldots \bar{\tau}_\sigma \ldots \bar{\tau}_r}}{d_{\bar{\tau}_1 \ldots \bar{\tau}_\sigma \ldots \bar{\tau}_r} w^\sigma_{\bar{\tau}_\sigma}} \tag{18}$$

players of weight $w^\sigma_{\bar{\tau}_\sigma}$, the total weight being in each case

$$\frac{W_{\bar{\tau}_1 \ldots \bar{\tau}_\sigma \ldots \bar{\tau}_r}}{d_{\bar{\tau}_1 \ldots \bar{\tau}_\sigma \ldots \bar{\tau}_r}} \quad . \tag{19}$$

This defines indeed a sub-coalition of the remainder coalition (11). We may indicate this sub-coalition by a block

$$\boxed{\begin{matrix} w^1_{\bar{\tau}_1} & \ldots & w^1_{\bar{\tau}_1} \\ \vdots & \vdots & \vdots \\ w^r_{\bar{\tau}_r} & \ldots & w^r_{\bar{\tau}_r} \end{matrix}} \tag{20}$$

each row σ of which (representing a corner of the corresponding coalition) has the number of elements indicated in (18) and hence represents a weight as given in (19).

By taking out this coalition (i.e., block (20)) from the remainder coalition (i.e., from (11)) and adding it to system $\underline{\underline{\mathbf{S}}}$ that has been constructed, we can effectively *decrease* the coefficient $a^1_{\bar{\tau}_1}$ that we have focused upon by considering (13) (the same can be done, of course if the index specified is not $\rho = 1$). And adding (20) to the system constructed so far is indeed permitted because the blocks (10) describing the "preceding" coalitions show enough players in order to support the introduction of every player of (20); this is guaranteed by the requirement (9).

After repeating this procedure as often as necessary we may eventually assume that no inequality like (13) prevails. That is, we may eventually assume that

$$a^\sigma_{\tau_\sigma} \leq \max_{\rho \neq \sigma} \sum_{\tau_\rho = 2}^{T_\rho} \max_{\tau_{-\{\sigma,\rho\}} \in \mathfrak{T}_{-\{\sigma,\rho\}}} \frac{W_{\tau_1 \ldots \tau_r}}{d_{\tau_1 \ldots \tau_r} w^1_{\tau_1}} \tag{21}$$

holds true.

But in view of (12), this means exactly that the remainder coalition (11) can be introduced by predecessors, i.e., by the members of the previously constructed system (10).

q.e.d.

This completes our treatment of the exact case. If there are sufficiently many players of the smallest type available, then the procedure of "successive introduction of players" as described by the notion of a universally ordered partition of I into diagonal sets ensures the core to be stable in the vNM sense. Therefore, proposals from outside the core can be dominated from inside.

Now we leave the territory of exact games. The cores loses its property of external stability. However, we believe that external domination is quite important in the general context and that it helps to understand the formation of cartels as will be argued in the following section.

4 The Formation of Cartels in Large Games

Within this section we turn to the general totally balanced and orthogonal game; explicitly we want to deal with non-exact games. Thus we assume that we are given a representation by means of finitely many measures $\boldsymbol{\lambda}^1, \cdots, \boldsymbol{\lambda}^r \in \mathbf{A}_+$ via

$$\boldsymbol{v} \; = \; \bigwedge \{\boldsymbol{\lambda}^1, \cdots, \boldsymbol{\lambda}^r\} \; . \tag{1}$$

Again we assume that the measures are *integer valued* and ordered according to mass, i.e.,

$$\boldsymbol{v}(I) \; = \; M^1 \; = \; \boldsymbol{\lambda}^1(C^1) \; \leq \; \ldots \; \leq \; M^r \; = \; \boldsymbol{\lambda}^r(C^r); \tag{2}$$

at least one of the inequalities in (2) is a strict one.

In this case, and for a sufficiently large game, the core consists of the convex hull of all the measures with minimum mass, i.e., of the measures

$$\{\boldsymbol{\lambda}^\rho \mid \boldsymbol{\lambda}^\rho(I) \; = \; \boldsymbol{\lambda}^1(I) \; = \; \boldsymbol{v}(I)\} \; . \tag{3}$$

Thus, the payoffs in the core favors the short side of the market excessively and the bargaining power of the long side is regarded to be neglectable. The reason behind this extreme behavior is the ability of the coalition of short-siders to pick a suitable proper sub-coalition of the long side for cooperation and to thereby exploit the members of the long side arbitrarily.

More precisely, assume for the moment that we argue for the case of just two corners in the market, the second (C^2) reflecting the long side. Given a core element $\boldsymbol{x} \in \mathcal{C}(\boldsymbol{v})$, every sufficiently small coalition $T \subseteq C^2$ has the bad luck that its complement contains a suitable subset, say I^2 such that the pair C^1, I^2 constitutes a diagonal coalition (i.e., $\boldsymbol{\lambda}^1(C^1) \; = \; \boldsymbol{\lambda}^2(I^2)$) the worth of which in terms of the game is $\boldsymbol{v}(I)$.

This means that $\boldsymbol{x}$ yields the total payoff $\boldsymbol{x}(C^1 + I^2) \; = \; \boldsymbol{v}(I)$ which equals the total payoff $\boldsymbol{x}(I)$. Consequently, every sufficiently small coalition receives zero-payment because the complement can cooperative efficiently with C^1.

This way we observe that, at any core element, small coalitions on the long side of the market receive zero because their complements cooperative efficiently with the short side. However, as everyone on the long side is a member of a small coalition (assuming suitable conditions for to speak of a 'large game'), everyone gets zero payment. This way the short side of the market plays all the small coalitions on the long side off against each other and this is why no one on the long side benefits from a core element.

The essential point is that the core derives its strength from the ability of coalitions to *achieve* something (consider coalition I^2 above). However, it is also important to consider the ability of a coalition to *prevent* opposing coalitions from achievements. The term 'blocking', while it is generally not considered to be justified in the context of the core, may be justified when used for the preventive power of a coalition.

How can the preventive power of the long side of the market be exercised? The bargaining procedure is obvious: Let the long side form a tentative coalition which agrees that no smaller coalition will join the short side unless there is a general agreement. They may also agree on the distribution of the resulting benefits among their members respecting in some sense or other the initial assignments.

Then the long side bargains (say by some representatives) with the short side regarding a distribution of the benefits between both sides. Thereafter these benefits are distributed inside the rank and file.

This is what is meant by the formation of cartels. Can we recognize this kind of bargaining behavior by the predicting power of a solution concept? This would be the case if a solution concept typically proposes a distribution of wealth for each corner of the market which yields the same total wealth for both sides. More precisely, there should be a set of mutually orthogonal measures each of them absolutely continuous and with bounded density with respect to the distribution of the initial assignments.

If these measures are specified in order to represent the the distribution of wealth *inside* each corner, then convex combinations to be taken between these wealth distributions would reflect the bargaining procedure within the cartels.

It turns out that vNM-Stable Sets reflect just this procedure. It is a strength of the vNM-Stable Set that it requires not only internal stability (which the core achieves as well) but also external stability (which the core cannot guarantee when there is a long side on the market as it is not vNM stable in this case). That is, the vNM-Stable Sets seem to reflect some prevention or blocking power by the very definition of external stability.

It is therefore the most interesting fact that vNM-Stable Sets reflect cartelization of the market in the above mentioned sense.

This is now demonstrated by the construction of such kind of solutions as follows.

Theorem 4.1. *Let* $\boldsymbol{v} = \bigwedge \{\boldsymbol{\lambda}^1, \cdots, \boldsymbol{\lambda}^r\}$ *be a totally balanced and orthogonal game and let* $\boldsymbol{\mu}^1, \dots, \boldsymbol{\mu}^r$ *be normalized measures, i.e., satisfy* $\boldsymbol{\mu}^\rho(I) = \boldsymbol{v}(I) \quad (\rho = 1, \dots, r)$. *Assume that*

$$\boldsymbol{\mu}^\rho \leq \boldsymbol{\lambda}^\rho \quad (\rho = 1, \dots, r) \tag{4}$$

is satisfied. Then there exist numbers $\hat{\boldsymbol{N}}^\rho \quad (\rho = 1, \dots, r)$, *depending on the weights* $w^\rho_\tau \quad (\rho = 1, \dots, r; \quad \tau = 2, \dots T^\rho)$ *of* $\boldsymbol{\mu} = (\boldsymbol{\mu}^1, \dots, \boldsymbol{\mu}^r)$ *only, with the following property: whenever the number of players of weight* 1 *in corner* ρ *exceeds* $\hat{\boldsymbol{N}}^\rho$, *then the convex hull* $\mathcal{S} = \boldsymbol{CnvH}\{\boldsymbol{\mu}^1, \dots, \boldsymbol{\mu}^r\}$ *is an externally vNM–Stable Set for* $\boldsymbol{v}$.

Proof:

The proof rests on Theorem 3.6 and hence uses the result in [9] concerning the vNM-stability of the core for large *exact* games.

Given the measures $\boldsymbol{\mu}^\rho \quad (\rho = 1, \dots, r)$, we consider the new game $\boldsymbol{u}$ given by

$$\boldsymbol{u} = \bigwedge \{\boldsymbol{\mu}^1, \dots \boldsymbol{\mu}^r\}. \tag{5}$$

It follows from $\boldsymbol{\mu}^\rho \leq \boldsymbol{\lambda}^\rho$ and from $\boldsymbol{\lambda}^1(I) = \boldsymbol{v}(I) = \boldsymbol{\mu}^1(I) = \dots = \boldsymbol{\mu}^r(I)$ that

$$\boldsymbol{u} \leq \boldsymbol{v}, \quad \boldsymbol{u}(I) = \boldsymbol{v}(I). \tag{6}$$

holds true. Clearly, the game $\boldsymbol{u}$ is exact and orthogonal. In addition, we can assure that $\boldsymbol{u}$ is large. To this end, we have to take sufficiently many players of the smallest type in each corner. The bound $\boldsymbol{N}_0$ which has to be exceeded depends on the larger weights of $\boldsymbol{\mu}$ only (see SECTION 2). Therefore, we take the number $\hat{\boldsymbol{N}}^\rho$ to be the maximum of $\boldsymbol{N}_0$ and the number $\boldsymbol{N}^\rho$ which is determined by Theorem 3.6 with regard to the game $\boldsymbol{u}$.

Obviously the imputations for both games, $\boldsymbol{u}$ and $\boldsymbol{v}$, are the same. Moreover, for two imputations $\boldsymbol{x}$ and $\boldsymbol{\xi'}$, we can infer that

$$\boldsymbol{x} \,\mathrm{dom}^{\boldsymbol{u}}\, \boldsymbol{\xi'} \quad \text{implies} \quad \boldsymbol{x} \,\mathrm{dom}^{\boldsymbol{v}}\, \boldsymbol{\xi'}, \tag{7}$$

this is a rather immediate consequence of (6).

Now, according to Theorem 3.6, $\mathcal{S} = \boldsymbol{CnvH}\{\boldsymbol{\mu}^1, \dots, \boldsymbol{\mu}^r\}$ is the unique vNM–Stable Set of $\boldsymbol{u}$. Therefore, the imputations in $\mathcal{S}$ dominate all imputations outside of $\mathcal{S}$ with respect to $\boldsymbol{u}$. In view of the above observation it follows that the imputations in $\mathcal{S}$ *a fortiori* dominate everything outside of $\mathcal{S}$ with respect to $\boldsymbol{v}$. **q.e.d.**

Corollary 4.2. *Let* $\boldsymbol{v} = \bigwedge \left\{\boldsymbol{\lambda}^1, \cdots, \boldsymbol{\lambda}^r\right\}$ *be a totally balanced and orthogonal game and let* $\boldsymbol{\mu}^1, \dots, \boldsymbol{\mu}^r$ *be normalized measures. Let* $\boldsymbol{\mu}^\rho$ *have the same weights as* $\boldsymbol{\lambda}^\rho$ $(\rho = 1, \dots, r)$ *and let*

$$\boldsymbol{\mu}^\rho \leq \boldsymbol{\lambda}^\rho \quad (\rho = 1, \dots, r) \tag{8}$$

be satisfied. Then there exist numbers $\tilde{\boldsymbol{N}}^\rho$ $(\rho = 1, \dots, r)$*, depending on the weights* w^ρ_τ $(\rho = 1, \dots, r;\ \ \tau = 2, \dots T^\rho)$ *of* $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^1, \cdots, \boldsymbol{\lambda}^r)$ *only, with the following property: whenever the number of players of weight* 1 *with respect to* $\boldsymbol{\mu}^\rho$ *exceeds* $\tilde{\boldsymbol{N}}^\rho$*, then the convex hull* $\mathcal{S} = \boldsymbol{CnvH}\{\boldsymbol{\mu}^1, \dots, \boldsymbol{\mu}^r\}$ *is an externally vNM–Stable Set for* $\boldsymbol{v}$*.*

The proof follows immediately from the previous theorem. The advantage is, that the bounds are "universal". They depend, in this case, on the representation given by $(\boldsymbol{\lambda}^1, \cdots, \boldsymbol{\lambda}^r)$ only. Hence, an externally vNM–Stable Set is constructed by taking the weights available for the large players and sufficiently many players from the smallest type in each corner. Note that each $\boldsymbol{\mu}^\rho$ equals the corresponding $\boldsymbol{\lambda}^\rho$ whenever the latter one in minimizing, i.e., yields $\boldsymbol{\lambda}^\rho(I) = \boldsymbol{v}(I)$.

We are now going to show that the type of solution introduced by the previous theorem is indeed a vNM–Stable Set. More precisely, we obtain the following theorem:

Theorem 4.3. *Let* $v = \bigwedge \left\{\boldsymbol{\lambda}^1, \cdots, \boldsymbol{\lambda}^r\right\}$ *be totally balanced and orthogonal and let* $\boldsymbol{\mu}^1, \dots, \boldsymbol{\mu}^r$ *be normalized measures such that* $\boldsymbol{\mu}^\rho \leq \boldsymbol{\lambda}^\rho$, $(\rho = 1, \dots, r)$ *holds true. Then* $\mathcal{S} = \boldsymbol{CnvH}\{\boldsymbol{\mu}^1, \dots, \boldsymbol{\mu}^r\}$ *is internally stable. Hence, if there are sufficiently many small players in each corner of the market, then* $\mathcal{S}$ *is a vNM–Stable Set.*

Proof: We have to prove internal stability only, as external stability follows from Theorem 4.1 or Corollary 4.2.

Let $\boldsymbol{\mu} = \sum_{\rho=1}^r c_\rho \boldsymbol{\mu}^\rho$ and Let $\boldsymbol{\nu} = \sum_{\rho=1}^r d_\rho \boldsymbol{\mu}^\rho$ be imputations in $\mathcal{S}$ and assume that we have

$$\boldsymbol{\mu} \ \text{dom}_S \ \boldsymbol{\nu}$$

with a suitable coalition S. Because of

$$\boldsymbol{v}(S) \geq \boldsymbol{\mu}(S) > \boldsymbol{\nu}(S)$$

we know that $\boldsymbol{\lambda}^\rho(S) > 0 \quad (\rho = 1, \ldots, r)$ holds true, this follows from the representation of $\boldsymbol{v}$ in view of

$$\boldsymbol{v}(S) \;=\; \min\{\boldsymbol{\lambda}^\rho(S) \mid \rho = 1, \ldots, r\}.$$

Consequently we obtain

$$S \cap C^\rho \;\neq\; \emptyset \quad (\rho = 1, \ldots, r).$$

For every ρ we choose $i \in S \cap C^\rho$. Then, using the fact that our measures are orthogonal, we obtain

$$\begin{aligned} c_\rho \boldsymbol{\mu}_i^\rho \;&=\; \sum_{\sigma=1}^{s} c_\sigma \boldsymbol{\mu}_i^\sigma \;=\; \boldsymbol{\mu}_i \\ > \boldsymbol{\nu}_i \;&=\; \sum_{\sigma=1}^{s} d_\sigma \boldsymbol{\mu}_i^\sigma \;=\; d_\rho \boldsymbol{\mu}_i^\rho \end{aligned}$$

and hence

$$c_\rho > d_\rho \quad (\rho = 1, \ldots, r).$$

This obviously contradicts

$$1 \;=\; \sum_{\rho=1}^{r} c_\rho \;=\; \sum_{\rho=1}^{r} d_\rho.$$

q.e.d.

The reader may wish to compare the corresponding proofs in the continuous case, these are provided in [10]. Of course, the continuous result so far is much stronger: there we obtain a complete characterization of convex (polyhedral) vNM–Stable Sets. Presently we claim that the situation is not so different in the large but finite case. Nevertheless, a complete characterization seems to be out of the question: Ljapunovs Theorem is crucial for the proof of Theorem 4.6. of [10] (and hence for the Characterization Theorem). On the other hand, the continuous result allows for conclusions with respect to the discrete version only if all measures $\boldsymbol{\lambda}^\rho$ represent uniform distribution – this we have surpassed in our present treatment.

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