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Evan Shellshear

IMW · Bielefeld University
Postfach 100131
33501 Bielefeld · Germany

email: imw@wiwi.uni-bielefeld.de
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A Note on Apportionment Methods *

Evan Shellshear
Institute of Mathematical Economics, Bielefeld University
P.O. Box 100131, 33501 Bielefeld, Germany
Email: evan.shellshear@wiwi.uni-bielefeld.de

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Abstract

This paper investigates the suitability of apportionment methods based on the idea of preserving the coalition function of the simple game represented by the populations of the states. The results show that an apportionment method which satisfies desirable properties such as population monotonicity, house monotonicity, etc., does not exist. A classification of simple voting games via winning coalitions is also given.
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1 Introduction

The apportionment problem is a well known problem in political science concerned with determining how to divide a whole number of representatives or delegates among given states, territories, groups, etc., according to their respective sizes. The main goal of the theory is to assign a "fair" number of seats (representatives, etc.) to each state, territory, etc., according to their relative population (relative to the total population). Problems arise as the apportionment must provide integer values for each state, territory, etc., and the sum of the distributed seats must sum to some fixed number representing the number of seats in parliament, in the committee, etc. Apportionment

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methods have been studied in detailed by a number of researchers, in particular by M.L. Balinksi and H.P. Young. The aforementioned authors developed a number of results which culminated in the publishing of their book "Fair Representation", ([1]). This work summarises the most significant results based on their research up to 1982. This note is based on the ideas presented in their book. In this note a new type of apportionment method is investigated, as defined in section three. The idea behind the apportionment is that the populations of the states define a simple majority game and the new apportionment method is the method which preserves this game, when possible, or is the "nearest" of all possible simple majority games with the total weight of the states equalling the number of seats. The results concerning this new apportionment method demonstrate the unsuitability of the method more than anything. It is shown that the method does not satisfy many desirable properties of apportionment methods (eg. House monotonicity, etc.). The use of games to analyse apportionment methods, however, is not new, see for example, ([2]). Before the new apportionment method is investigated a short introduction and results relating to simple games and winning coalitions is provided. This first section develops a theorem about the number of winning coalitions in voting (simple) games.

2 Simple Games

Before the section on apportionment methods is introduced and the pertinent definitions are expounded, results relating to winning coalitions and simple games will be elucidated in this section. Simple games form the basis upon which the ideas in the next section are based. To begin the analysis a few definitions are requisite. First of all that of a n-person cooperative TU game. An n-person cooperative TU game is a pair (N, v) where N is a finite subset of the natural numbers (representing players) and v is a function, $v : 2^N \to \mathbb{R}$, satisfying $v(\emptyset) = 0$. Within this setting a simple game is defined as follows.

Definition 2.1. *A game (N, v) is a simple game if the coalition function v only takes on the values 0 or 1.*

Simple games have been used for numerous applications apart from purely mathematical research. One of the felicitous areas of research with numerous results is that of weighted majority games. In this section simple games with a focus on weighted majority games will be analysed. To begin this analysis, some notation and definitions are required to be able to present a particularly informative representation of simple games. In particular, the following definition will turn out to be quite befitting for future results.

Definition 2.2 (Winning Coalitions)**.** *A* ***winning coalition*** *is a coalition $S \subseteq N$ such that $v(S) = 1$. The set of winning coalitions will be denoted by W. A* ***minimal winning coalition*** *is a coalition $S \subseteq N$ such that*

$$v(S) = 1 \text{ but } v(T) = 0 \;\forall\; T \subsetneqq S$$

The set of minimal winning coalitions will be denoted by W^{min}.

Using this notation a simple game (N, v) can also be written as (N, W). With this notation, it is also normal to impose the following condition on simple games.

$$\text{If } S \in W \text{ and } S \subseteq T \;\Rightarrow\; T \in W \tag{2.1}$$

Equation (2.1) is normally referred to the **monotonicity condition** and is imposed so that the winning coalitions correspond to what one would usually expect of winning coalitions. If this assumption is used later on then it will be explicitly stated and referred to as the **monotonicity condition**. By imposing this condition one has the following result.

Lemma 2.3. *Let* $G_1 = (N, W_1)$ *and* $G_2 = (N, W_2)$ *be two simple games satisfying the monotonicity condition. Then* $W_1^{min} = W_2^{min}$ *if and only if* $W_1 = W_2$.

Proof: For the if direction let $W_1^{min} = W_2^{min}$. Then for $i = \{1, 2\}$ every winning coalition T is such that there exists a $S \in W_i^{min}$ such that $S \subseteq T$. As the game satisfies the monotonicity condition, $T \in W_1$ and $T \in W_2$. Now, for the only if part of the lemma, assume that $W_1 = W_2$. It then follows from the definition of W that $W_1^{min} = W_2^{min}$. **q.e.d.**

With regard to the structure and relationships between winning coalitions the following three definitions are quite germane.

Definition 2.4. *A simple game* (N, W) *is said to be proper if for all*

$$S \in W \;\Rightarrow\; N \backslash S \notin W.$$

It is called strong if

$$S \notin W \;\Rightarrow\; N \backslash S \in W.$$

Finally it is called weak if

$$V := \bigcap \{S \mid S \in W\} \neq \emptyset. \tag{2.2}$$

The set V is called the set of veto players. A game is called dictatorial if there exists $j \in N$ such that

$$S \in W \Leftrightarrow j \in S.$$

That is $|V| = 1$ and $j \in V$. With these concepts and ideas the definition of a weighted majority game can be introduced. The following definition follows that of, eg. B. Peleg ([4]).

Definition 2.5 (Weighted majority game)**.** *Let* $G = (N, W)$ *be a simple game.* G *is a* ***weighted majority game*** *if there exists a quota* $\mu > 0$ *and weights* $w_1 \geq 0, \ldots, w_n \geq 0$, $w_j \in \mathbb{R}$ *for all* $j \in \{1, \ldots, n\}$, *such that*

$$S \in W \Leftrightarrow w(S) \geq \mu.$$

The $(n+1)$*-tuple,* $[\,\mu\,; w_1, \ldots, w_n]$, *is called a representation of* G *and one writes* $G = [\,\mu\,; w_1, \ldots, w_n]$.

It will be assumed later on that all the w_j actually satisfy $w_j \in \mathbb{N}$ ($\mathbb{N}$ stands for the natural numbers) and that the quota is defined by

$$\mu = \left\lceil \frac{w(N)+1}{2} \right\rceil \tag{2.3}$$

where the brackets stand for the the ceiling operator (that is the smallest integer greater than or equal to $\frac{w(N)+1}{2}$). If this is the case, that the $w_j \in \mathbb{N}$ and q are defined by equation (2.3), then the game will be called a **voting game**. In voting games the numbers w_j represent the 'votes' that player j possesses. To present the main result concerning the number of winning coalitions in voting games a few results need to be presented which are requisite for its proof.

Lemma 2.6. *Let $G = (N, W)$ be a weighted majority game. If $\mu > \frac{w(N)}{2}$ then G is proper.*

Proof: If $\mu > \frac{w(N)}{2}$ then there cannot exist a coalition S such that $w(S) \geq \mu$ and $w(N \backslash S) \geq \mu$. As otherwise

$$w(N) = w(S) + w(N \backslash S) \geq 2\mu > w(N).$$

q.e.d.

Corollary 2.7. *All voting games are proper.*

The following Lemma is also important.

Lemma 2.8. *Let $G = (N, W)$ be a weighted majority game satisfying $\mu > \frac{w(N)}{2}$ then G is strong if and only if there does not exist S such that*

$$w(S) = w(N \backslash S).$$

Proof: First of all assume that there exists a coalition S such that

$$w(S) = w(N \backslash S).$$

Then if $w(S) \geq \mu$ the game is no longer proper contradicting Lemma (2.6) and if $w(S) < \mu$ then the game is not strong. If the game is strong then $w(S) < \mu$ implies that $w(N \backslash S) \geq \mu$ and if $w(S) \geq \mu$ then as the game is proper $w(N \backslash S) < \mu$. Hence in both cases $w(S) \neq w(N \backslash S)$. **q.e.d.**

Finally the following is a corollary of Lemma (2.8).

Corollary 2.9. *Let $G = (N, W)$ be a voting game. If $w(N)$ is odd then G is strong.*

Proof: If $w(N)$ is odd then there cannot exist a coalition S with

$$w(S) = w(N \backslash S).$$

Hence by Lemma (2.8) G is strong. **q.e.d.**

Before the main result in this section can be presented the following result is necessary.

Lemma 2.10. *Let $G = (N, W)$ be a strong and proper simple game with $|N| = n$. Then $|W| = 2^{n-1}$.*

Proof: This result follows from the fact that either a set S is in W or its compliment hence half of the 2^n subsets of N are winning. **q.e.d.**

The following result is then the main result in this section concerning voting games.

Theorem 2.11. *Let $G = (N, W)$ be a voting game with $|N| = n$. Then $|W| = 2^{n-1}$ if and only if G is strong.*

Proof: The only if direction follows from Lemma (2.10) and Corollary (2.7). So the if direction will now be proven. Assume that the game is not strong. This implies that the $w(N)$ is even by Corollary (2.9) and that there is a coalition S with $w(S) = \mu - 1$ and $w(N\backslash S) = \mu - 1$ by Lemma (2.8). Now consider the game $G' = (N, W')$ where one adds 1 to w^j for a player $j \in S$. Note that in this new game G', $\mu' = \mu$. In G' one has that $S \in W'$. In addition, by Corollary (2.9), the game G' is strong as $w(N)$ is odd. Hence $|W'| = 2^{n-1}$. As well, for all $T \in W$ one has that $T \in W'$ however there exists a S with $S \in W'$ such that $S \notin W$. Hence

$$|W| < |W'| = 2^{n-1}.$$

q.e.d.

This result also holds for weighted majority games with $\mu > \frac{w(N)}{2}$ and not just for voting games. The reason is because such weighted majority games are proper and hence the maximum number of winning coalitions is 2^{n-1}. This is realised if and only if the game is strong. In such a case one speaks of a zero sum game because it then follows for all $S \subseteq N$ that $v(S) + v(N\backslash S) = v(N) = 1$. In the next section on apportionment methods it is the voting games such as those investigated in this section that form the focus of the analysis.

3 Apportionment methods

In this section the idea of an apportionment method will be introduced. Afterwards results relating to a specific apportionment method will be presented. As was mentioned in the last section, voting games form the basis of the investigation to be presented here and hence before the main definitions regarding apportionment methods are given the notation regarding voting games needs to explained. So given a vector $\mathbf{m} \in \mathbb{N}^n$ the game represented by $\mathbf{m}$ is the voting game $G = (N, W)$ (with $|N| = n$ as usual) defined by $G = [\,\mu\,; m_1, \ldots, m_n]$ whereby

$$\mu = \left\lceil \frac{m(N) + 1}{2} \right\rceil$$

(as usual $m(N) = \sum_{i=1}^{n} m_i$). To demonstrate that a game $G = (N, W)$ is represented via a integer vector $\mathbf{m}$ the notation $v_\mu^{\mathbf{m}}$ will be used. Hence such a game $G = (N, W)$ will be identified with the coalition function $v_q^{\mathbf{m}}$ arising from the weighted majority game $G = [\,\mu\,; m_1, \ldots, m_n]$, so that the two notations will be used interchangeably. As all the games in this section are voting games the subscript μ will not be necessary and hence as long as there is no chance of confusion it will be omitted.

The definition of an apportionment is as follows.

Definition 3.1. *An* ***apportionment*** *(also called an apportionment method) is a mapping*

$$H : \mathbb{N}^n \times \mathbb{N} \longrightarrow \mathbb{N}^n.$$

This mapping represents the situation where there are n states (or parties) with populations (or sizes) in $\mathbb{N}$. An apportionment is a way of assigning seats within a parliament in accordance with the population (size) proportions. The rule by which one apportions the seats is the mapping H. The first component in the mapping H is a vector representing the population (party) sizes in the states. The second component represents the size of parliament (house size), that is the number of seats, h, to be distributed amongst the states. The mapping H provides at least one vector in $\mathbb{N}^n$, given a vector of populations, $\mathbf{g}$, and a house size h, as an apportionment. Hence the notation $H(\mathbf{g}, h)(N)$ can be applied in the normal fashion, as above, for $H(\mathbf{g}, h)$. One should note that an apportionment mapping may generate more than one vector as a solution. Therefore the notation $\mathbf{a} \in H(\mathbf{g}, h)$ will be used when discussing the apportionment(s) that the mapping H defines. If only the mapping H is referred to in a particular sentence and not $\mathbf{a} \in H(\mathbf{g}, h)$ then this will imply that all apportionments arising via the mapping H are meant.

In the following, properties of a new apportionment method based on the number of winning coalitions of voting games will be investigated. Firstly, however, a general property to be satisfied by all apportionment methods ensuring that all possible seats are distributed is encapsulated by the next definition.

Definition 3.2. *An apportionment method* ***preserves the house*** *if it satisfies*

$$H(\boldsymbol{g}, h)(N) = h, \ \ (\boldsymbol{g} \in \mathbb{N}^n,\ h \in \mathbb{N}).$$

It will always be assumed that the apportionment method under question preserves the house. This is a fundamental requirement of any apportionment method. The second definition forms the basis of the investigation in this section.

Definition 3.3. *An apportionment method* ***preserves the coalition function*** *if for all* $\boldsymbol{a} \in H(\boldsymbol{g}, h)$ *it satisfies*

$$v_\alpha^{\boldsymbol{a}} = v_\beta^{\boldsymbol{g}}$$

where, as usual,

$$\alpha := \left\lceil \frac{H(\boldsymbol{g},h)(N)+1}{2} \right\rceil \text{ and } \beta := \left\lceil \frac{\boldsymbol{g}(N)+1}{2} \right\rceil. \tag{3.1}$$

In the following the subscripts α and β will be omitted. The question that will be investigated in this section can now be outlined. Given a population vector $\mathbf{g}$ and the game it represents $v^{\mathbf{g}}$, then for a given house size h, what is the apportionment method that preserves the coalition function when possible and when not, provides the "closest" voting game to the given voting game $v^{\mathbf{g}}$? What is meant by "closest" will be explained shortly, however the basic intuitive idea is clear. The goal is to find an apportionment method that preserves the power proportions in the voting game represented by population vector $\mathbf{g}$ according to either the winning coalitions or minimal winning coalitions.

To specify what was meant by closest in the previous paragraph one needs to be able to talk about the distance between voting games. The normal mathematical procedure for determining the distance between two mathematical objects is to define a metric on the set of objects under consideration. The following definition will be used to provide a metric to measure the distance between voting games.

Definition 3.4. *Let A and B be two finite sets and define*

$$d(A,B) := |A \backslash B| + |B \backslash A|.$$

Lemma 3.5. *$d(A,B)$ is a metric, that is $d(A,B)$ satisfies the following three properties:*
i) $d(A,B) = 0$ if and only if $A = B$
ii) $d(A,B) = d(B,A)$
iii) $d(A,B) \leq d(A,C) + d(C,B)$ for all finite sets A, B and C.

Proof: First of all for $i)$ if $d(A,B) = 0$ then it follows that

$$|A \backslash B| + |B \backslash A| = 0.$$

Hence $|A \backslash B| = 0$ and $|B \backslash A| = 0$. This then implies that $A = B$. If $A = B$ then it follows straightaway that $d(A,B) = 0$. For $ii)$

$$d(A,B) := |A \backslash B| + |B \backslash A| = |B \backslash A| + |A \backslash B| = d(B,A).$$

Finally $iii)$ will now be proven. So let A, B and C be finite sets and so one has to show that

$$|A \backslash B| + |B \backslash A| \leq |A \backslash C| + |C \backslash A| + |C \backslash B| + |B \backslash C|. \tag{3.2}$$

First of all the left hand side of equation (3.2) can be written as

$$|A| + |B| - 2|A \cap B|.$$

Whereas the right hand side can be written as

$$|A| + |C| - 2|A \cap C| + |B| + |C| - 2|B \cap C|.$$

Hence it must just be shown that

$$-2|A \cap B| \leq 2|C| - 2|A \cap C| - 2|B \cap C|$$

or that

$$2|A \cap B| \geq 2|A \cap C| + 2|B \cap C| - 2|C|.$$

Reformulating this once more it becomes equivalent to

$$|A \cap B| + |C| \geq |A \cap C| + |B \cap C|. \tag{3.3}$$

Now one always has the following equality between two finite sets A and B,

$$|A| + |B| = |A \cap B| + |A \cup B|. \tag{3.4}$$

Using equation (3.4) one can write the right hand side of equation (3.3) as

$$|(A\cap C)\cup(B\cap C)|+|A\cap C\cap B\cap C| = |(A\cap C)\cup(B\cap C)|+|A\cap B\cap C|. \tag{3.5}$$

However the claim now follows because in equation (3.5)

$$|A \cap B| \geq |A \cap B \cap C|$$

and

$$|C| \geq |(A \cap C) \cup (B \cap C)|.$$

Hence the distance $d(A, B)$ is a metric. **q.e.d.**

This metric can be applied to simple games as follows.

Definition 3.6. *Let $G_1 = (N_1, W_1)$ and $G_2 = (N_2, W_2)$ be two simple games then define*

$$d_W(G_1, G_2) := |W_1 \backslash W_2| + |W_2 \backslash W_1|.$$

This metric can now be used to "measure" the distance between two voting games. Also important to note is that using winning coalitions in the definition is not essential. One could also take minimal winning coalitions and define the following metric. This metric would be equivalent (in the sense that for given $\mathbf{g}$, $M_v(d_{min}(v, v^{\mathbf{g}})) = M_v(d_W(v, v^{\mathbf{g}}))$, see the following definitions)in the case that the games are monotonic (see Lemma (2.3)).

Definition 3.7. *Let $G_1 = (N_1, W_1)$ and $G_2 = (N_2, W_2)$ be two simple games with corresponding minimal winning coalitions W_1^{min} and W_2^{min} then define*

$$d_{min}(G_1, G_2) := |W_1^{min} \backslash W_2^{min}| + |W_2^{min} \backslash W_1^{min}|.$$

Interesting results, not just relating to voting games, can be proven using the metric $d_W(G_1, G_2)$. The following Lemma is an example thereof and describes the distance between arbitrary strong and proper simple games.

Lemma 3.8. *Let $G_1 = (N_1, W_1)$ and $G_2 = (N_2, W_2)$ be two proper and strong simple games with $N_1 = N_2 := N$. Then $d_W(G_1, G_2)$ is even.*

Proof: First of all let $S \in W_1$ however $S \notin W_2$. As the game is strong this then implies that $N \backslash S \in W_2$. As the game is proper $N \backslash S \notin W_1$ and hence for each set $S \in W_1$ such that $S \notin W_2$ there exists a $N \backslash S \in W_2$ such that $N \backslash S \notin W_1$ and vice versa. This implies that there exists a bijection between the $S \in W_1 \backslash W_2$ and the $T \in W_2 \backslash W_1$ and hence that

$$d_W(G_1, G_2) := 2|W_1 \backslash W_2| = 2|W_2 \backslash W_1| \tag{3.6}$$

q.e.d.

Although the previous Lemma will not be used in the following, it demonstrates a useful application of the metric d_W.

In the following an apportionment method which minimises the distance between voting games will be discussed. As was mentioned at the beginning the goal of this section is to investigate properties of an apportionment method that, for a given house size h, preserves the coalition function when possible. If that is not possible then the method should minimise the distance, with respect to the metric d_W or d_{min}, to the original game $v^{\mathbf{g}}$, for a given population vector $\mathbf{g}$. Therefore the definition of the apportionment method in question could be described by the following. To define the apportionment method a few definitions are required. In the following let $\mathcal{V}$ be the set of voting games. For a general function f define the following two sets.

$$m_v(f) := \min_{v \in \mathcal{V}} f(v).$$

$$M_v(f) := \{\hat{v} \mid f(\hat{v}) = m_v(f)\}.$$

Now by letting $f = d_{min}(v, v^{\mathbf{g}})$ for $v \in \mathcal{V}$, one has the set

$$M_v(d_{min}(v, v^{\mathbf{g}})) := \{\hat{v} \mid (\hat{v}, v^{\mathbf{g}}) = m_v(d_{min}(v, v^{\mathbf{g}}))\}.$$

Definition 3.9. *The σ method is the set of vectors $H(\boldsymbol{g}, h)$ defined by the following equation.*

$$H(\boldsymbol{g}, h) := \{\boldsymbol{m} \in \mathbb{N}^n \mid v^{\boldsymbol{m}} \in M_v(d_{min}(v, v^{\boldsymbol{g}}))\}$$

The reasons for taking d_{min} instead of d_W are the following. First of all the use of d_{min} allows for simpler calculations as there are less minimal winning coalitions that winning coalitions. Secondly in a parliament it is the minimal winning coalitions that determine which party gains the control over the parliament. Coalitions that are already winning would never invite new superfluous members to join the coalition because they would then have to share their power with the new otiose members. This would reduce the power and influence of all members in the original coalition. This implies that the coalitions of interest are the minimal winning coalitions.

However due to the following result, for the current case of voting games it is equivalent (that is they generate the same set of apportionments $H(\mathbf{g}, h)$) if one uses the metric d_{min} or d_W (see Lemma (2.3)).

Lemma 3.10. *Voting games are monotonic*

Proof: The result is clear because if given a set of weights $\mathbf{w} \in \mathbb{N}^n$ and $S \subseteq N$ satisfying $w(S) \geq \mu$ then for all $T \subseteq N$ satisfying $S \subseteq T$ it follows that $w(T) \geq w(S) \geq \mu$. **q.e.d.**

Another possibility for the above is to consider the l^1 metric instead of the d_{min} metric. It is defined as follows.

Definition 3.11. *Let (N, v) and (N, w) be two cooperative TU games. Then the l^1 distance between the two games (N, v) and (N, w) is defined as follows.*

$$l^1(v, w) := \sum_{S \subseteq N} |v(S) - w(S)|.$$

However as the following result demonstrates they provide the same results.

Proposition 3.12. *Let (N, v) be a voting game and $\boldsymbol{g}$ a population vector. Let*

$$H_M(\boldsymbol{g}, h) := \{\boldsymbol{m} \in \mathbb{N}^n \mid v^{\boldsymbol{m}} \in M_v(d_{min}(v, v^{\boldsymbol{g}}))\}$$

and

$$H_L(\boldsymbol{g}, h) := \{\boldsymbol{m} \in \mathbb{N}^n \mid v^{\boldsymbol{m}} \in M_v(l^1(v, v^{\boldsymbol{g}}))\}.$$

Then $H_M(\boldsymbol{g}, h) = H_L(\boldsymbol{g}, h)$.

Proof: Assume per absurdum that there exists an $\mathbf{m} \in H_M(\mathbf{g}, h)$ such that $\mathbf{m} \notin H_L(\mathbf{g}, h)$. First of all, all $\mathbf{m} \in H_M(\mathbf{g}, h)$ are such that the game generated by $\mathbf{m}$ has the same set of winning coalitions and hence the same set of coalitions $S \subseteq N$ such that $v^{\mathbf{m}}(S) = 1$. Hence it follows that if one of the m is not in $H_L(\mathbf{g}, h)$ then none of the $\mathbf{m} \in H_M(\mathbf{g}, h)$ are in $H_L(\mathbf{g}, h)$. However for all of the $\mathbf{m} \in H_M(\mathbf{g}, h)$ the number of coalitions S such that $v^{\mathbf{g}}(S) = 1$ and $v^{\mathbf{m}}(S) = 0$ and vice versa is a minimum as the vectors of the set $M_v(d_{min}(v, v^{\mathbf{g}}))$ minimise the number of winning coalitions in $v^{\mathbf{m}}$ not in $v^{\mathbf{g}}$ and vice versa. This implies that for all $\mathbf{n} \in H_L(\mathbf{g}, h)$ and $\mathbf{m} \in H_M(\mathbf{g}, h)$

$$l^1(v^{\mathbf{g}}, v^{\mathbf{m}}) \leq l^1(v^{\mathbf{g}}, v^{\mathbf{n}}).$$

This, however, implies that $\mathbf{m} \in H_L(\mathbf{g}, h)$ a contradiction. Hence $H_L(\mathbf{g}, h) = H_M(\mathbf{g}, h)$. **q.e.d.**

The previous result is useful as it is often simpler to program with the l^1 metric than the d_{min} metric when calculating examples.

The σ method will now be discussed, in particular its suitability for applications to apportionment problems. Although an exact apportionment method will not be given, properties of the σ method will be discussed here. Without giving an exact apportionment method that guarantees the closest game for a given h does, however, leave a level of freedom in the voting game chosen as the σ apportionment. This freedom of choice will be important, in particular, when discussing house monotonicity. This also means that by applying certain criteria one could select one of the σ method's apportionments based on certain desirable criteria. This possibility will be examined at the end of this section.

In order to discuss the suitability of the σ method general properties of apportionment methods need to introduced. The first is usually referred to as house monotonicity and arose in connection with the so called "Alabama paradox". An apportionment method displays the "Alabama paradox" when it does not fulfill house monotonicity.

Definition 3.13. *An apportionment method H is called **house monotone** if it always satisfies*

$$h \geq h' \Rightarrow H(\boldsymbol{g}, h) \geq H(\boldsymbol{g}, h'), \ (\boldsymbol{g} \in \mathbb{N}^n, \, h, h' \in \mathbb{N}).$$

Were an apportionment method not to satisfy house monotonicity then it would mean that when the size of parliament increases, then, counterintuitively, one or more states could lose seats. Such an occurrence would make no sense what so ever and in the words of Balinski and Young, ([1]), on p.42 in their authoritative book on the subject of apportionments, *Fair Representations*,

"No apportionment method is reasonable that gives some state fewer seats when there are more seats to go around".

Hence the requirement that an apportionment method fulfills house monotonicity is essential. The problem is, however, that the σ method is not necessarily house monotone. An example demonstrating this is the following.

Example 3.14. Let $\mathbf{g}$ be the following population vector

$$\mathbf{g} := (16, 14, 20, 21, 21, 25).$$

Then $\mathbf{g}$ provides the following representation for a voting game $v^{\mathbf{g}}$

$$[\,59\,; 16, 14, 20, 21, 21, 25].$$

Now consider the case when the size of the house is $h = 46$. Then an apportionment $\mathbf{a}$ satisfying

$$W^{min}_{w^{\mathbf{a}}} = W^{min}_{v^{\mathbf{g}}}$$

is the following

$$\mathbf{a} = (4, 4, 7, 9, 9, 13).$$

Note that this is one of numerous voting games, w, with the weights summing to 46 that satisfy

$$W^{min}_{w} = W^{min}_{v^{\mathbf{g}}}.$$

If one considers the case $h = 47$ then one has as a possible apportionment $\mathbf{a}$ satisfying

$$W^{min}_{v^{\mathbf{a}}} = W^{min}_{v^{\mathbf{g}}}$$

the following

$$\mathbf{a} = (3, 3, 8, 10, 10, 13).$$

This example demonstrates that the σ method does not necessarily satisfy house monotonicity.

Although in the previous example the two chosen apportionments were two from numerous possibilities, an important fact is that for $h = 46$ there does not exist an apportionment H that assigns to player one and two a value of three that then also satisfies

$$W_{v^{\mathbf{a}}}^{min} = W_{v^{\mathbf{g}}}^{min},$$

unlike the case for $h = 47$.

A second important property for apportionment methods is that they are population monotone. The exact definition of population monotonicity is as follows. Before the definition is given a notion utilised in the definition needs to be explained. Let $\mathbf{a} \in H(\mathbf{g}, h)$ and $\mathbf{a}' \in H(\mathbf{g}, h)$, then a'_i can be substituted for a_i in $\mathbf{a}$ means that when one replaces a_i with a'_i in $\mathbf{a}$ then it follows that the new apportionment $\mathbf{a^*} \in H(\mathbf{g}, h)$.

Definition 3.15. *An apportionment method H is population monotone, if for any two population vectors $\boldsymbol{g}$, $\boldsymbol{g}' > 0$ and $\boldsymbol{a} \in H(\boldsymbol{g}, h)$, $\boldsymbol{a}' \in H(\boldsymbol{g}', h')$ it follows that*

$$\frac{g'_{i'}}{g'_{j'}} \geq \frac{g_i}{g_j} \text{ implies } \begin{cases} a'_{i'} \geq a_i \text{ or } a'_{j'} \leq a_j \\ \text{or} \\ \frac{g'_{i'}}{g'_{j'}} = \frac{g_i}{g_j} \text{ and } a'_{i'}, a'_{j'} \\ \text{can be substituted for } a_i, a_j \text{ in } \boldsymbol{a}. \end{cases}$$

The following Corollary is proven in ([1]).

Corollary 3.16. *If an apportionment method is population monotone then it is house monotone*

This result then implies that the σ method is not necessarily population monotone. An example demonstrating this is the following.

Example 3.17. Consider the following games v and w generated by the population vectors

$$\mathbf{g} := (3, 3, 3, 3, 13) \text{ and } \mathbf{g}' := (2, 3, 3, 3, 10).$$

$$v := [\,13\,; 3, 3, 3, 3, 13] \text{ and } w := [\,11\,; 2, 3, 3, 3, 10].$$

Now consider the case that $h = 7$. Then an apportionment $\mathbf{a}$ satisfying

$$W_v^{min} = W_{v^{\mathbf{a}}}^{min}$$

is the following

$$\mathbf{a} = (0, 1, 1, 1, 4)$$

and hence could be suggested as the σ method's apportionment. An apportionment $\mathbf{a}'$ satisfying

$$W_w^{min} = W_{w^{\mathbf{a}'}}^{min}$$

is the following

$$\mathbf{a}' = (1, 1, 1, 1, 3).$$

However one sees that

$$\frac{10}{2} > \frac{13}{3}$$

and hence the σ method's apportionment does not satisfy population monotonicity.

In the words of Balinski and Young, ([1]), p. 68

"No method can be considered acceptable for either proportional representation or federal systems that forces one state to give up seats to another that has become proportionally smaller, i.e. that suffers from the population paradox [i.e. is not population monotone]"

Unfortunately this is another blow for the σ method. The fact that the σ method is not population monotone and displays the "Alabama paradox" are two strong arguments as to why one should not consider it as a serious apportionment method. However the σ method also displays another deficit known as the new states paradox (in other words, it does not satisfy uniformity, see ([1])). This occurs under the following situation. Say a new state (or player), j, enters the game with a population of say g_j. One adds to h the nearest integer to

$$\frac{g_j}{g(N')} h \tag{3.7}$$

(where N' is the new population with player j) to create a new house size h'. If the method is not to display the new states paradox, with this new state, and the corresponding increase in h to h', the original states should be distributed the same allotments (with the new state receiving $h' - h$) under the apportionment method as if the new state was not there. This requirement represents a type of stability amongst the players. It is clear that if a new state/player comes and receives his fair share of seats defined by the nearest integer to the expression in equation (3.7), then there should be no need to redistribute seats amongst the states. Balinski and Young term this property uniformity and demonstrate the following result (the reader is referred to ([1]) for an exact definition of uniformity).

Proposition 3.18. *Every apportionment method which is uniform is house monotone.*

Hence the σ method cannot be uniform as the following example demonstrates.

Example 3.19. Consider the following voting games representing the following population vectors

$$\mathbf{g} := (1, 1, 4, 7) \text{ and } \mathbf{g}' := (1, 1, 4, 6, 7).$$

$$v := [\,7\,; 1, 1, 4, 7] \text{ and } w := [\,10\,; 1, 1, 4, 6, 7].$$

Then for $h = 9$ one would have one of the following games as the σ method's apportionment for the game defined by v.

$$\mathbf{z} = (1, 1, 2, 5) \text{ or } \mathbf{z} = (1, 1, 1, 6).$$

However when one now adds a new state as in the game w and also in proportion adds 3 to h, so that $h = 12$, then one has that the σ method's apportionment should be

$$\mathbf{z}' = (1, 1, 3, 3, 4).$$

In this case both players 3 and 4, in the original game defined by $\mathbf{g}$ or $\mathbf{g}'$, have exchanged one seat, demonstrating that the σ method also suffers from the new states paradox.

Balinski and Young, ([1]), describe the logic behind why an apportionment method should not demonstrate the new states paradox, p. 44,

"If an estate is divided up fairly among heirs, then there should be no reason for them to want to trade afterwards. Anyone should be able to compare his share with anyone else and remain satisfied."

Finally it is of interest to see if the σ method at least satisfies what is known as quota.

Definition 3.20. *Define the vector $\boldsymbol{x}$ as follows*

$$x_i = \frac{g_i}{g(N)} h$$

for a population vector $\boldsymbol{g}$ and house size h. Then an apportionment method H satisfies quota if for all $\boldsymbol{a} \in H(\boldsymbol{g}, h)$

$$\lfloor x_i \rfloor \leq a_i \leq \lceil x_i \rceil$$

where $\lfloor x_i \rfloor$ stands for the largest integer smaller than or equal to x_i and $\lceil x_i \rceil$ the smallest integer greater than or equal to x_i.

Unfortunately the σ method also does not satisfy quota as the following example demonstrates.

Example 3.21. Consider the following voting game generated by the population vector $\mathbf{g} := (2, 2, 2, 3)$,

$$v := [\,5\,; 2, 2, 2, 3].$$

Consider the case when $h = 6$ then the σ method would recommend the following vector $\mathbf{a}$ as its unique apportionment satisfying $W_v^{min} = W_{w^{\mathbf{a}}}^{min}$

$$\mathbf{a} = (1, 1, 1, 3).$$

However for player 4,

$$\frac{g_4}{g(N)} h = 2$$

and hence the σ method does not satisfy quota.

The fact that the σ method also does not satisfy quota is not truly a major deficit in the apportionment method. Most methods applied today do not satisfy quota. However, as a trade off, they satisfy population monotonicity or house monotonicity or other important properties. The fact that the σ method does not satisfy quota and does not satisfy population monotonicity or house monotonicity and hence also falls victim to the new states paradox is however a major inadequacy in the apportionment method. This would suggest that a method such as the σ method could not be taken seriously as an apportionment method to be applied in reality.

As nearness (minimising the distance) to the original population game does not guarantee desirable properties such as population monotonicity and house monotonicity one could take a class of apportionment methods that first of all fulfill certain properties and then ask, which of these apportionment methods are nearest (with respect to d_{min}) to the original voting game. However one needs to be careful when selecting axioms. As is shown in ([1]) for certain desirable values of the house size, h, and the number of states s, there does not exist an apportionment method that satisfies both quota and population monotonicity (see page 129 in ([1]) for more details). As well in ([1]), Balinski and Young show that there is one unique method that satisfies both quota and house monotonicity. Hence the only remaining cases to consider concerning the previously mentioned properties are the apportionment methods that satisfy house monotonicity or population monotonicity. Then one could investigate if from this class of apportionment methods if there is a single one that is nearest (i.e. is closest, with respect to the metric d_{min}) to the original population game. However before this case can be analysed the definition of a divisor method needs to be given.

Definition 3.22. *Let $\boldsymbol{p}$ be a population vector with s entries. An apportionment method H is a* ***divisor method*** *if for a given $\alpha \in [0, 1]$ one can select a $\lambda \in \mathbb{R}$ such that λ satisfies the following equation*

$$\sum_{i=1}^{s} \left\lfloor \frac{p_i}{\lambda} + \alpha \right\rfloor = h,$$

and the entries x_i of the vector(s) $\boldsymbol{x}$ resulting from the apportionment method H satisfy

$$\left\lfloor \frac{p_i}{\lambda} + \alpha \right\rfloor = x_i.$$

In the case of ties between entries x_i for a seat (and by giving each state a seat would result in the sum of all entries in $\boldsymbol{x}$ being greater than h) then one requires a tie breaking rule. Hence there exists a divisor method for each value of α.

In the traditional apportionment nomenclature the method whereby $\alpha = 0$ is known as the **method of Jefferson**, the method whereby $\alpha = 0.5$ is known as the **method of Webster**, as well as numerous other names for other values of α. Due to the earlier critic on methods that did not satisfy population monotonicity it is desirable to develop apportionment methods that satisfy population monotonicity. In ([1]) the authors classify all such apportionment methods via the following theorem.

Theorem 3.23. *An apportionment method is population monotone if and only if it is a divisor method.*

Hence what one is then looking for is a divisor method that minimises the distance between the original voting game and the apportioned voting game. However there exists no one divisor method as the following example demonstrates.

Example 3.24. Let $\mathbf{g}$ be the following population vector

$$\mathbf{g} := (35, 27, 19, 12, 11, 9).$$

Then $\mathbf{g}$ provides the following representation for a voting game $v^{\mathbf{g}}$

$$[\,57\,; 35, 27, 19, 12, 11, 9].$$

Now consider the case when the size of the house is $h = 21$. Then Jefferson's apportionment method supplies the following vector ($\lambda = \frac{90}{19}$)

$$\mathbf{j} := (7, 5, 4, 2, 2, 1)$$

and Webster's method apportions the following vector ($\lambda = \frac{220}{41}$)

$$\mathbf{w} := (6, 5, 4, 2, 2, 2).$$

As can be seen by calculating the minimal winning coalitions

$$d_{min}(v^{\mathbf{g}}, v^{\mathbf{j}}) > d_{min}(v^{\mathbf{g}}, v^{\mathbf{w}}).$$

Hence Webster's method preserves the coalition function better than Jefferson's method in this example. However for an example showing that the other direction can also occur consider the following example. Let $\mathbf{g}$ be the following population vector

$$\mathbf{g} := (2, 2, 2, 5).$$

Then $\mathbf{g}$ provides the following representation for a voting game $v^{\mathbf{g}}$

$$[\,6\,;2,2,2,5].$$

Now consider the case when the size of the house is $h = 4$. Then Jefferson's apportionment method generates the following vector (or one where the first three players share two seats according to some tie breaking rule, this however does not effect the calculation for $d_{min}(v^{\mathbf{g}}, v^{\mathbf{j}})$) ($\lambda = \frac{11}{5}$)

$$\mathbf{j} := (0, 1, 1, 2)$$

and Webster's method apportions the following vector ($\lambda = \frac{11}{4}$)

$$\mathbf{w} := (1, 1, 1, 1).$$

As can be seen by calculating the minimal winning coalitions

$$d_{min}(v^{\mathbf{g}}, v^{\mathbf{j}}) < d_{min}(v^{\mathbf{g}}, v^{\mathbf{w}}),$$

Jefferson's method is closer to the original voting game.

Finally as it has been shown that their is no one unique divisor method that is always closest to the σ method's apportionments, a final possibility is to also consider the σ method's apportionment that is closest to the proportions

$$\frac{g_j}{g(N)}h. \tag{3.8}$$

The idea of finding a solution nearest to the proportions, and hence one that satisfies quota, is very intuitive. The proportions represent what each state should receive were it possible to exactly allocate these proportions. Hence methods that are nearest to the proportions are very intuitive and possess a very desirable property. One of the well-known methods that grants such an apportionment is the Hare (also known as the Hamilton) procedure. However the Hare method also lacks desideratum of apportionment methods. The method fulfills neither house monotonicity nor population monotonicity. The idea of the coming investigation is to combine the Hare procedure with the σ method and to see if one can create a new apportionment method fulfilling some of the aforementioned desideratum. However as will be shown, this new method not only does not amend the problems with the Hare method but also fails to satisfy quota.

Again in the following no specific algorithm will be given to calculate the new method, however properties of the new method will be, nevertheless, investigated. Let $H_\sigma(\mathbf{g}, h)$ be the σ apportionment method as defined earlier. Then the σ-Hare method is defined as follows.

Definition 3.25. *The σ-Hare method is the set of apportionments $\boldsymbol{m} \in H_\sigma(\boldsymbol{g}, h)$ so that*

$$\sum_{j\in N} |\boldsymbol{m}_j - \frac{g_j}{g(N)}h| \leq \sum_{j\in N} |\boldsymbol{n}_j - \frac{g_j}{g(N)}h| \;\forall\; \boldsymbol{n} \in H_\sigma(\boldsymbol{g}, h)$$

The properties of the σ-Hare method will now be investigated via a number of examples. First of all it will be demonstrated that the method does not satisfy house monotonicity. This has important consequences and means in particular that the method is not population monotone and also demonstrates the new states paradox (proposition (3.18)). The numbers chosen for the following example are due to ([1]).

Example 3.26. Let $\mathbf{g}$ be the following population vector

$$\mathbf{g} := (501, 394, 156, 149).$$

Then for $h = 11$ the σ-Hare method has as its unique apportionment

$$\mathbf{m} := (5, 4, 1, 1).$$

However for $h = 12$ the σ-Hare method has as its unique apportionment

$$\mathbf{n} := (5, 3, 2, 2).$$

This demonstrates that the method is not house monotone.

Finally an example will be given showing that the method also does not satisfy quota. This is a serious problem as it was exactly this desideratum that led to the definition of the σ-Hare method and hence it cannot be seriously considered.

Example 3.27. Let $\mathbf{g}$ be the following population vector

$$\mathbf{g} := (1, 2, 6, 7, 10, 13, 18).$$

Then for $h = 22$ the σ-Hare method has as its unique apportionment

$$\mathbf{m} := (0, 0, 2, 3, 5, 5, 7).$$

However the proportions for $h = 22$ are

$$(0.38, 0.77, 2.3, 2.7, 3.86, 5.01, 6.95).$$

Hence the σ-Hare method does not fulfill quota.

As the general case did not provide any results which led to the justification of the σ-Hare method, a special case will now be investigated. The reason for this is due to the desirability of both the preservation of the coalition function and an apportionment which is near to the proportions. In the following, a special class of games is given for which there always exists a σ apportionment which stays within quota. To do this the definition of a homogeneous game need to be introduced.

Definition 3.28. *Let (N, v) be a voting game generated by $\boldsymbol{m}$ and W the set of winning coalitions. Then (N, v) is homogeneous if for all $T \in W$ there exists $S \subseteq T$ such that $m(S) = \mu_m$, whereby $\mu_m = \left\lceil \frac{m(N)+1}{2} \right\rceil$.*

From here on it will be assumed, w.l.o.g., that the voting games in question do not contain dummy players (a dummy player is a player j such that j is not contained in any minimum winning coalitions). Also the players will be arranged in decreasing order within the vector $\mathbf{m}$ which generates the given voting game. In ([3]) (Proof of Theorem 4.13, p. 77) a method is given by which one can generate a homogenous representation of a zero sum voting game given the smallest (non dummy) player. It implies, *inter alia*, that for the smallest player (i.e. the last in the vector $\mathbf{m}$), player n, that all other players $j \in \{1, \ldots, n-1\}$ can be written as a positive integer multiple of player n. That is

$$m_j = k(j)m_n$$

for some $k(j) \in \mathbb{N}$. Then the main result is as follows and assumes that all games in question are zero sum.

Proposition 3.29. *Let $\boldsymbol{g}$ be a population vector generating a homogenous game $v^{\boldsymbol{g}}$. If h is a house number such that there exists a vector $\boldsymbol{m}$ generating a homogenous game $v^{\boldsymbol{m}}$ with*

$$W_{v^{\boldsymbol{g}}}^{min} = W_{v^{\boldsymbol{m}}}^{min}$$

and

$$\frac{\mu_g}{g_n} = \frac{\mu_m}{m_n}$$

then there exists a σ apportionment that satisfies quota.

Proof:
1$^{\text{st}}$STEP :
First of all as $\mathbf{m}$ satisfies,

$$W_{v^{\mathbf{g}}}^{min} = W_{v^{\mathbf{m}}}^{min}$$

and $\mathbf{m}(N) = h$ it follows that $\mathbf{m}$ is a σ apportionment. So it just remains to be shown that $\mathbf{m}$ fulfills quota. As was stated above all g_j and m_j can be written as $g_j = k'(j)g_n$ and $m_j = k(j)m_n$. Due to the condition

$$\frac{\mu_g}{g_n} = \frac{\mu_m}{m_n}$$

it follows that for all $S \in W_{v^{\mathbf{g}}}^{min} = W_{v^{\mathbf{m}}}^{min}$ that

$$\sum_{i \in S} k'(i)g_n = \mu_g$$

and

$$\sum_{i \in S} k(i)m_n = \mu_m$$

hence

$$\sum_{i \in S} k(i) = \sum_{i \in S} k'(i). \tag{3.9}$$

It will now be shown, via induction, that $k'(j) = k(j)$ for all $j \in \{1, \ldots, n\}$. So for $j = n$ it follows as $k'(n) = 1 = k(n)$. So assume that it holds true for all $n \geq j \geq l+1$. Let T be a minimal winning coalition containing $l+1$ and consider $S = (T \cup l) \backslash l+1$. Then there exists a subset $R \subseteq \{l+1, \ldots, n\}$ such that $S \backslash R$ is minimal winning for both $v^{\mathbf{g}}$ and $v^{\mathbf{m}}$ (as l is not a dummy, and via Theorem 4.13, ([3]), it follows that the difference between the weights of the players l and $l+1$ is equal to a sum of players j with $j \geq l+1$). Now as $k'(j) = k(j)$ for all $j \in \{l+1, \ldots, n\}$, from induction, it follows that

$$\sum_{i \in S \backslash (R \cup l)} k(i) = \sum_{i \in S \backslash (R \cup l)} k'(i).$$

Now assume, per absurdum, that $k'(l) > k(l)$. Then it follows that

$$\sum_{i \in S \backslash R} k(i) < \sum_{i \in S \backslash R} k'(i).$$

which contradicts equation (3.9). The case $k'(l) < k(l)$ is handled identically and hence $k'(l) = k(l)$ for all $l \in \{1, \ldots, n\}$.

2$^{\text{nd}}$STEP :

Now $g(N) = K' g_n$ and $m(N) = K m_n$ for some $K', K \in \mathbb{N}$. As $k'(l) = k(l)$ for all $l \in \{1, \ldots, n\}$ it follows that $K' = K$. It will now be shown that $\mathbf{m}$ fulfills quota. To that end assume the contrary for m_n, that is

$$m_n > \left\lceil \frac{g_n h}{g(N)} \right\rceil$$

this implies that (using $K' = K$)

$$m(N) > \sum_{i \in N} k(i) \left\lceil \frac{g_n h}{g(N)} \right\rceil \geq \sum_{i \in N} \left\lceil \frac{k(i) g_n h}{g(N)} \right\rceil = \left\lceil \frac{K' g_n h}{g(N)} \right\rceil = h.$$

This contradicts the fact that $m(N) = h$. A similar argument shows that

$$m_n < \left\lfloor \frac{g_n h}{g(N)} \right\rfloor$$

also cannot hold. Therefore

$$\left\lceil \frac{g_n h}{g(N)} \right\rceil \geq m_n \geq \left\lfloor \frac{g_n h}{g(N)} \right\rfloor$$

Now m_n must be equal to one of these two numbers so first of all let

$$\left\lceil \frac{g_n h}{g(N)} \right\rceil = m_n.$$

Then it follows for all $j \in \{1, \ldots, n\}$ that

$$m_j = k(j) \left\lceil \frac{g_n h}{g(N)} \right\rceil \geq \left\lceil k(j) \frac{g_n h}{g(N)} \right\rceil.$$

However one has

$$m(N) = \sum_{i \in N} k(i) \left\lceil \frac{g_n h}{g(N)} \right\rceil \geq \sum_{i \in N} \left\lceil \frac{k(i) g_n h}{g(N)} \right\rceil = \left\lceil \frac{k'(i) g_n h}{g(N)} \right\rceil = h = m(N).$$

Hence it follows for all $j \in \{1, \ldots, n\}$ that

$$m_j = \left\lceil \frac{k(j) g_n h}{g(N)} \right\rceil = \left\lceil \frac{k'(j) g_n h}{g(N)} \right\rceil = \left\lceil \frac{g_j h}{g(N)} \right\rceil .$$

This implies that $\mathbf{m}$ fulfills quota for all j. The other case

$$\left\lfloor \frac{g_n h}{g(N)} \right\rfloor = m_n$$

is treated similarly. Hence in both cases $\mathbf{m}$ fulfills quota. **q.e.d.**

The author conjectures that the condition

$$\frac{\mu_g}{g_n} = \frac{\mu_m}{m_n}$$

utilised in the previous proposition is not necessary. Finally a simple example demonstrating the applicability of the proposition is presented.

Example 3.30. Let $\mathbf{g}$ be the following population vector

$$\mathbf{g} := (4, 2, 2, 2)$$

and for $h = 5$, let $\mathbf{m}$ be the following apportionment

$$\mathbf{m} := (2, 1, 1, 1).$$

Then it follows that

$$W^{min}_{v^{\mathbf{g}}} = W^{min}_{v^{\mathbf{m}}}$$

and

$$\frac{\mu_g}{g_n} = \frac{\mu_m}{m_n}$$

and both games are homogenous. Hence for $h = 5$ there is an apportionment which preserves the coalition function and satisfies quota.

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