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Two support results for the Kalai-Smorodinsky solution in small object division markets

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Abstract

We discuss two support results for the Kalai-Smorodinsky bargaining solution in the context of an object division problem involving two agents. Allocations of objects resulting from strategic interaction are obtained as a demand vector in a specific market. For the first support result games in strategic form are derived that exhibit a unique Nash equilibrium. The second result uses subgame perfect equilibria of a game in extensive form. Although there may be multiple equilibria, coordination problems can be removed.

Keywords: support result, object division, market, Kalai-Smorodinsky solution
JEL Classification: C78, D40

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1 Introduction

The task to obtain a non-cooperative foundation for a cooperative solution concept is widely described by the term *Nash program.*[2] Starting with an underlying cooperative game, one needs to derive a non-cooperative game in normal or extensive form in such a way that equilibrium payoffs coincide with (or belong to) the cooperative solution. The most prominent example for a support result arguably is Nash's *simple demand game* (see Nash (1953)). There, each player picks a coordinate. If the resulting vector belongs to the players' bargaining problem, then it is the final payoff. Otherwise the status quo is enforced. Equilibrium payoffs of this game are exactly the Pareto efficient utility allocations of the feasible set. Hence, this game supports the (set-valued) solution that always selects the Pareto boundary in Nash equilibria.

In this paper, we demonstrate how to *support* the Kalai-Smorodinsky (hereafter KS) bargaining solution[3]. In effect, the bargaining solution can as well be obtained by strategic interaction instead of signing an agreement. Regarded from the perspective of a social planner, one is interested in formulating a universal way to derive such non-cooperative games that is independent of the underlying bargaining problem.

Apparently, there are striking similarities between the Nash program and the theory of implementation, although the foci are slightly different. However, Trockel (2002) shows that for solution concepts from cooperative game theory, a support result as discussed above can be "transformed" into an implementation result in the spirit of implementation theory (see also Serrano (1997), Dagan & Serrano (1998) and Bergin & Duggan (1999)). This is the content of the so-called *embedding principle.* In particular, any support result for a bargaining solution readily yields an implementation result for this solution as well (see also discussion at the end of Section 4).

Roughly, there are two ways, in which we could think of supporting a bargaining solution. One way is to follow a purely welfaristic approach, which means that one only considers the possible allocations of utility provided in the bargaining situation and takes this set (together with the status quo point) as the primitives of the cooperative model. Along this line, Trockel (1999) discusses support results for a class of bargaining solutions including the KS solution (cf. also Haake (2000)).

In the second direction, there is an additional entity considered in the model; a set of physical outcomes. This set may, for example, consist of allocations of goods or (lotteries over) alternatives. A bargaining problem is induced by evaluation of outcomes with individual utility functions. Therefore, supporting a bargaining solution in such a setup means achieving a

[2]See Serrano (2005) for a recent survey.

[3]see Kalai & Smorodinsky (1975).

certain outcome through strategic interaction and utilities in (Nash) equilibrium coincide with the given bargaining solution. Such a non-welfaristic approach can cure an important drawback: In practice it is not necessarily clear, how a specific utility allocation is actually obtained, whereas it appears to be a much simpler task to select a certain (physical) outcome (see also Serrano (2005)). In a seminal paper, Moulin (1984) discusses an implementation of the KS solution, using a mechanism, in which "fractions of dictatorship are auctioned off". The winning bid in the auction later constitutes a probability distribution on the set of (physical) alternatives, so that the KS solution is the expected payoff from this lottery. In a similar spirit, Miyagawa (2002) obtains a subgame-perfect implementation result for a class of bargaining solutions including the KS solution.

As Moulin's (1984) work does, we follow the second approach. We investigate an object division problem, in which a finite number of (divisible) objects may be distributed among two agents. Our approach differs from Moulin's in two respects. First, we obtain an exact support result in the sense that deterministic payoffs in equilibrium coincide with the KS solution. And second, instead of using an auction mechanism, we set up an exchange market. Payoffs (in equilibrium) are the result of individual demand. Loosely speaking, a strategy choice in the supporting games determines prices and income, that in turn yield utility maximizing allocations of objects. So, we use the Walrasian equilibrium concept to first derive an allocation as the result from strategic interaction and then evaluation of this allocation with individual utility functions constitutes the payoffs in the game.

We present two support results in Section 3: First, we derive from any object division problem a non-cooperative game in strategic form, which has a unique Nash equilibrium. This game gives rise to a second game in extensive form. There the payoff in any subgame perfect equilibrium coincides with the KS solution. Although we cannot guarantee uniqueness of equilibria, no coordination problem occurs, as the resulting outcomes are (essentially) unique. Section 4 discusses the results.

2 Notation and Framework

We investigate a market, in which finitely many desirable objects are distributed among two agents. Let $I = \{1, 2\}$ denote the set of agents and $N := \{1, \ldots, n\}$ the set of objects. There is exactly one unit of each object in the market. Denote by $e := (1, \ldots, 1) \in \mathbb{R}^n$ the vector of total endowments. We assume that objects are divisible, so that an *allocation* (of objects) is described by a pair $x = (x^1, x^2) \in [0,1]^n \times [0,1]^n$ satisfying $x^1 + x^2 \leq e$ and denote the set of allocations by $\mathcal{A}$. Neither agent is initially endowed with some object. We assume that agents' preferences are linear over divisions of an object and additively separable across objects. That means, agent i's preferences may be represented by a vector $u^i = (u^i_1, \ldots, u^i_n) \gg 0$.[4] The quantity u^i_j may be interpreted as agent i's willingness to pay for object j. Agent i's utilities over bundles are given by a function $U^i : [0,1]^n \longrightarrow \mathbb{R}$, defined by $U^i(x^i) := \sum_{j \in N} x^i_j \cdot u^i_j$ $(i \in I)$ and we denote the set of all such pairs of utility functions (U^1, U^2) by $\mathcal{U}$. For presentational reasons we will assume that for each pair in $\mathcal{U}$ the corresponding utility vectors u^1, u^2 satisfy $u^1_j/u^2_j > u^1_{j+1}/u^2_{j+1}$ for all $j = 1, \ldots, n-1$, which means in particular that no two objects exhibit the same rate of substitution between the two agents. To sum, we describe a market by a tuple $\mathcal{M} = (I, N, U)$ with $U \in \mathcal{U}$.

In the sequel, we want to allocate objects by having agents demand the objects according to specific income and prices. A *price system* is a vector $p \in \mathbb{R}^n_+$. For given price system p and income level $m \in \mathbb{R}_+$, we define the budget set $B(m,p) := \{y \in [0,1]^n \,|\, p\,y \leq m\}$ and for $i \in I$ agent i's demand correspondence $D^i(m,p) := \operatorname{argmax}\{U^i(y) \,|\, y \in B(m,p)\}$. In the remainder, we make use of two specific price systems. Due to our assumption that utility functions are representable by a vector $u^i \in \mathbb{R}^n_{++}$, we may also view u^i as a linear function to evaluate (bundles of) objects; hence, we may consider u^i as a specific price system. Throughout the paper, we set $\bar{p}^i := u^i$ $(i \in I)$. Observe that a relatively "high" price $u^i_k = \bar{p}^i_k$ for object k on the one hand means that it is highly valued by agent i and on the other hand it is highly expensive, hence less attractive, to agent $3 - i$.

Any market (I, N, U) induces a two person bargaining game in the following way. The set of feasible utility allocations is given by $V^U := \operatorname{compH}(\{U^1(y^1), U^2(y^2) \in \mathbb{R}^2 \,|\, y \in \mathcal{A}\})$.[5] Status quo utilities are (always) given by the origin $0 \in \mathbb{R}^2$ and so we identify the game with its utility possibility set V^U. It is easy to see that V^U can be written as a sum of utility possibility sets; one for each object separately, i.e., $V^U = \sum_{j \in N} V^U_j$ with $V^U_j := \operatorname{compH}(\{\lambda(u^1_j, 0) + (1-\lambda)(0, u^2_j) \,|\, 0 \leq \lambda \leq 1\})$. Hence, the class $\mathcal{V} := \{V^U \,|\, U \in \mathcal{U}\}$ of bargaining games generated by an object division problem is the class of games with compactly generated, polyhedral utility possibility sets, which is dense[6] in the class of all two person bargaining games. For $U \in \mathcal{U}$ define $M^i = M^{i,U^i} := U^i(e)$. M^i reflects agent i's maximal possible utility in V^U, as it is his utility of the entire set of objects. The *Kalai-Smorodinsky* bargaining solution is the mapping $K : \mathcal{V} \longrightarrow \mathbb{R}^2$ that takes each V^U to its unique Pareto optimal utility allocation, in which both agents obtain the same fraction of their maximal utility, i.e., $K^1(V^U)/M^{1,U} = K^2(V^U)/M^{2,U} =: \tau(V^U)$.

We close with some important observation on the demand correspondence. Define a parametrization $h = h^U : [0,1] \longrightarrow \mathbb{R}^2$ of the Pareto boundary of V^U through $h(\delta) := (\bar{t}(\delta), \delta \cdot M^2)$, with $\bar{t}(\delta) := \max\{t \in \mathbb{R} \,|\, (t, \delta \cdot M^2) \in V^U\}$. That means, to each $\delta \in [0,1]$ the

[4]We use the following notation for vector inequalities in $\mathbb{R}^n$: $x \gg y$ means $x_j > y_j (j \in N)$; $x > y$ means $x_j \geq y_j (j \in N)$ and $x \neq y$; $x \geq y$ means $x > y$ or $x = y$.

[5]Here $\operatorname{compH}(\cdot)$ denotes the comprehensive hull operator.

[6]with respect to the Hausdorff metric on feasible sets

point $h(\delta)$ is the unique Pareto efficient point in V^U, in which agent 2 receives a δ share of his maximal utility. As one easily sees, h_1^U is a concave, strictly decreasing function. The following lemma shows that (certain) demand sets $D^i(m,p)$ are singletons. In such cases we will identify the set with its single element.

Lemma 1. *Let $\mathcal{M} = (I, N, U)$ be a market satisfying the above assumptions.*

1. *For $i \in I$ and $m \in [0, M^{3-i}]$, the demand set $D^i(m, \bar{p}^{3-i})$ is a singleton.*
2. *To each Pareto optimal utility allocation $(v^1, v^2) \in V^U$ there exists a unique (Pareto optimal) allocation $(z^1, z^2) \in \mathcal{A}$ with $U^i(z^i) = v^i$. Clearly, $z^1 + z^2 = e$.*
3. *Any Pareto optimal allocation $a = (a^1, a^2)$ takes the form $a^1 = (1, \ldots, 1, \lambda, 0, \ldots, 0)$, $a^2 = (0, \ldots, 0, 1 - \lambda, 1, \ldots, 1)$ with $a^1 + a^2 = e$.*
4. *$U^i(e - D^{3-i}(m, \bar{p}^i)) = M^i - m$ holds for $i \in I$ and $m \in [0, M^i]$.*
5. *For $m \in [0, M^1]$ we have $U^1(D^1(m, \bar{p}^2)) = h_1(1 - \frac{m}{M^2})$.*

Proof:
1) First note that for a price system p, $D^i(m,p)$ is obtained as follows. Order the objects according to their utility/price ratio, i.e., u_j^i/p_j. Then agent i first demands the object with the highest ratio, then the one with the second highest a.s.o. until his budget m is used up. For $p = \bar{p}^{3-i} = u^{3-i}$ this means, we order the objects according to u_j^i/u_j^{3-i}, i.e., according to their substitution rates. With our assumptions that objects are already ordered in this way and different objects have different substitution rates, we conclude that $D^1(m, \bar{p}^2)$ consists of a unique vector of the form $(1, \ldots, 1, \lambda, 0, \ldots, 0)$ and $D^2(m, \bar{p}^1)$ is of the form $(0, \ldots, 0, \mu, 1, \ldots, 1)$ with $\lambda, \mu \in [0, 1]$.
2) This immediately follows from our assumption that substitution rates are different across objects. Therefore, to any allocation z of objects, the utility allocation of which is Pareto optimal in V^U, there does not exist a redistribution of z, so that every agent is equally well off.
3) Note that any Pareto efficient utility allocation $v \in V^U = \sum_{j \in N} V_j^U$ can be uniquely written as the sum of utility allocations in V_j^U, i.e., $v = \sum_j v_j$. All utility allocations v_j as well as v have to exhibit the same normal vector of a supporting hyperplane and, with our assumption on different substitution rates, it follows that for all but at most one j we have either $v_j = (u_j^1, 0)$ or $v_j = (0, u_j^2)$. So, all but at most one object are completely allocated to some agent. With the assumption on the order of objects according to substitution rates, we conclude that the allocation of objects corresponding to v takes the asserted form.
4) We prove the case $i = 1$. Let $z^2 := D^2(m, \bar{p}^1)$. At price system $\bar{p}^1$ we have $u_j^1/\bar{p}_j^1 = 1$ for all $j \in N$. Hence, $D^1(m', \bar{p}^1) = \{x \in [0,1]^n \mid \bar{p}^1 x = m'\}$ collects all bundles, the worth of which

under $\bar{p}^1$ is exactly m'. Since $z^2 \in B(m, \bar{p}^1)$ (and preferences are strictly monotonic), we know that $\bar{p}^1 z^2 = m$, hence $\bar{p}^1 (e - z^2) = M^1 - m$ and therefore $e - z^2 \in D^1(M^1 - m, \bar{p}^1)$. Note that $(e - z^2, z^2)$ is a Pareto efficient allocation, since agent 2 maximizes his utility on the budget set and agent i can only be better off, when obtaining a higher income than $M^1 - m$.[7] Therefore $U^1(e - z^2) = M^1 - m$.
5) With part 4, we obtain $U^2(e - D^1(m, \bar{p}^2)) = M^2 - m = (1 - \frac{m}{M^2})\, M^2$ and since $(D^1(m, \bar{p}^2), e - D^1(m, \bar{p}^2))$ is Pareto efficient, we conclude $U^1(D^1(m, \bar{p}^2)) = h_1(1 - \frac{m}{M^2})$. □

Note that as a consequence of part 1 of the lemma, we can conclude $D^i(m, \bar{p}^{3-i}) \geq D^i(m', \bar{p}^{3-i})$ (vector inequality), if $m \geq m'$.

Another consequence of the lemma (part 3) is that, although objects are assumed to be divisible, at most one object has to be split in any Pareto efficient allocation.

3 Support Results

Achieving a support result for a bargaining solution means performing the following task: Derive from each bargaining game V^U a non-cooperative game Γ^U, so that payoffs in equilibrium coincide with the bargaining solution applied to the bargaining game at hand. For this, one shall specify general rules that describe *how* Γ^U is derived from V^U. Clearly, the "strength" of a support result is connected to the strength of the equilibrium concept that is used.

In this section, we provide two support results for the KS solution in the present context. First, we describe rules for deriving a game in strategic form having exactly one Nash equilibrium. The equilibrium payoff coincides with the KS solution of the underlying bargaining game. Second, we derive a game in extensive form, which has multiple subgame perfect equilibria. However, there are no coordination problems involved, as all equilibria have s the same payoff; again the KS solution.

Payoffs in both games, and hence the final allocations of objects, are achieved by endowing agents with a specific amount of money and assuming that they behave as utility maximizers in a market for the objects with prices $\bar{p}^1$ or $\bar{p}^2$. Thereby, incomes are determined by players' strategy choices. Put in another way, payoff functions can be decomposed into (a) an allocation function that describes the assigned bundles resulting from strategic interaction and (b) the evaluation of these bundles with utility functions.

Sections 3.1 and 3.2 present the two supporting results, which are then discussed in Section 4.

[7] In fact, the pair $(\bar{p}^1; (e - z^2, z^2))$ constitutes a Walrasian equilibrium of the underlying economy w.r.t. to the given income distribution. See Haake (2006) for further details.

3.1 Unique Nash equilibrium support

For $U \in \mathcal{U}$ we construct a game $\Gamma^U = (S^1, S^2, F^1, F^2)$ as follows: Strategy spaces are $S^1 = S^2 = [0,1]$. For any pair of strategies $\gamma = (\gamma^1, \gamma^2) \in S^1 \times S^2$, we first determine an allocation $g(\gamma) = (g^1(\gamma), g^2(\gamma)) \in \mathcal{A}$ of objects as follows:

($g1$) Determine for $i \in I$ agent i's demand at income level $(1-\gamma^{3-i}) \cdot M^{3-i}$, at prices $\bar{p}^{3-i} = u^{3-1}$, i.e., determine $x^i(\gamma) := D^i((1-\gamma^{3-i}) \cdot M^{3-i}, \bar{p}^{3-i})$.
Set $z^i(\gamma) := e - x^{3-i}(\gamma)$. If $z^1(\gamma) + z^2(\gamma) \le e$ holds, then define $g(\gamma) := (z^1(\gamma), z^2(\gamma))$.

($g2$) Otherwise ($z^1(\gamma) + z^2(\gamma) \not\le e$), determine $\hat{z}^i(\gamma) := D^i((1-\gamma^i) \cdot M^{3-i}, \bar{p}^{3-i})$ and set

$$g^i(\gamma) := \begin{cases} z^i(\gamma) & \text{, if} \quad z^i(\gamma) + (e - \hat{z}^i(\gamma)) \le e \\ \hat{z}^i(\gamma) & , \quad \text{otherwise.} \end{cases}$$

Payoffs in Γ^U are given by evaluation of the resulting allocation, i.e., $F^i(\gamma) = U^i(g^i(\gamma))$ $(i \in I)$.

Proposition 1. *For each $U \in \mathcal{U}$ the game Γ^U has a unique Nash equilibrium $\bar{\gamma}$. This is given by $\bar{\gamma}^1 = \bar{\gamma}^2 = \tau(V^U)$. Furthermore, $F(\bar{\gamma}) = K(V^U)$.*

Proof: Fix $U \in \mathcal{U}$ and set $\tau := \tau(V^U)$, hence we examine $\bar{\gamma} = (\tau, \tau)$.
Step 1: Note first, that $g(\bar{\gamma})$ is determined according to ($g1$). With part 4 of Lemma 1, we know $U^i(z^i(\bar{\gamma})) = U^i(e - D^{3-i}((1-\tau)M^{3-i}, \bar{p}^{3-i})) = \tau M^i$ $(i = 1, 2)$ and hence $(U^1(z^1(\bar{\gamma})), U^2(e - z^1(\bar{\gamma}))) = K(V^U) = (U^1(e - z^2(\bar{\gamma})), U^2(z^2(\bar{\gamma})))$, which implies $z^1(\bar{\gamma}) = e - z^2(\bar{\gamma})$. Therefore $g(\bar{\gamma}) = (z^1(\bar{\gamma}), z^2(\bar{\gamma}))$ and $F^i(g^i(\bar{\gamma})) = K^i(V^U)$ $(i \in I)$.
Step 2: Next, we show that $\bar{\gamma}$ is a Nash equilibrium in Γ^U. Suppose agent 1 deviates to $\sigma^1 < \tau$. Since $e - D^2((1-\tau) \cdot M^1, \bar{p}^1) \ge e - D^2((1-\sigma^1) \cdot M^1, \bar{p}^1)$ holds[8], the function g is still determined according to ($g1$) and hence $F^1(\sigma^1, \tau) = U^1(g^1(\sigma^1, \tau)) \le U^1(g^1(\tau, \tau)) = F^1(\sigma^1, \tau)$.
If agent 1 deviates to $\rho^1 > \tau$, we conclude that g is determined by ($g2$). Suppose $z^1(\rho^1, \tau) + (e - \hat{z}^1(\rho^1, \tau)) \le e$ were true. Then $(z^1(\rho^1, \tau), e - \hat{z}^1(\rho^1, \tau)) \in \mathcal{A}$ is a feasible allocation. Using Lemma 1 we compute its utility allocation $(U^1(z^1(\rho^1, \tau)), U^2(e - \hat{z}^1(\rho^1, \tau))) = (\rho^1 M^1, \tau M^2) > (\tau M^1, \tau M^2) = K(V^U)$, which contradicts Pareto efficiency of the KS solution. It follows again with part 5 of Lemma 1 that $F^1(\rho^1, \tau) = U^1(\hat{z}^1(\rho^1, \tau)) = h_1^U(\rho^1) < h_1^U(\tau) = \tau M^1 = F^1(\bar{\gamma})$ has to hold. Analogous arguments apply for agent 2.
Step 3: Step 2 shows that agent i can assure himself a payoff of τM^i by choosing $\bar{\gamma}^i = \tau$. Therefore, the payoff in any other equilibrium has to be at least this amount for both agents. But the only utility allocation in V^U that does satisfy this condition is $K(V^U)$. It is then immediate that $\bar{\gamma}$ is the only strategy profile with payoff $K(V^U)$ and therefore the only Nash equilibrium in Γ^U. □

[8] Use part 1 of Lemma 1 and the fact that demand is increasing in income.

3.2 Subgame Perfect support

Next, we derive an extensive form game Σ^U from a bargaining game $V^U \in \mathcal{V}$. Again, we first obtain an allocation as the result of strategy choices. The rules are as follows:

Stage 1 Agent 1 selects $\eta \in [0,1]$.

Stage 2 Agent 2 either chooses a bundle $z^2 = z^2(\eta) \in B((1-\eta)M^1, \bar{p}^1)$ or passes to the next stage. In the former case, the final allocation is $(e-z^2, z^2)\in\mathcal{A}$.

Stage 3 Agent 1 chooses a bundle $z^1 = z^1(\eta) \in B((1-\eta)M^2, \bar{p}^2)$. The final allocation is $(z^1, e-z^1)\in\mathcal{A}$.

Again, payoffs in Σ^U are determined by evaluation of the final allocation with $U^i(\cdot)$.

Proposition 2. *Let $\bar{z} = (\bar{z}^1, \bar{z}^2)$ be the final allocation and $\bar{\eta}$ be the chosen parameter at Stage 1 in a subgame-perfect equilibrium of Σ^U. Then we have $\bar{\eta} = \tau(V^U)$ and $(U^1(\bar{z}^1), U^2(\bar{z}^2)) = K(V^U)$.*

Proof:
First, in any subgame perfect equilibrium, if either agent 2 or 1 chooses a bundle from the budget set (at stage 2 or 3), he will choose $z^2 = D^2((1-\eta)M^1, \bar{p}^1)$ or $z^1 = D^1((1-\eta)M^2, \bar{p}^2)$, respectively. By parts 4 and 5 of Lemma 1, we know that $U^1(e-z^2) = \eta\cdot M^1$ and $U^2(e-z^1) = \eta \cdot M^2$, which means $U^1(z^1) = h_1^U(\eta)$.
At stage 2, agent 2 compares his payoff from choosing z^2 himself with $U^2(e-z^1)$, which he anticipates when passing to the next round. Since the respective allocations $(e-z^2, z^2)$ and $(z^1, e-z^1)$ are Pareto efficient, we have that $U^2(z^2) > U^2(e-z^1)$, if and only if $U^1(e-z^2) < U^1(z^1)$. so agent 2 takes his decision as to minimize agent 1's payoff.
Therefore, at stage 1, agent 1 faces a payoff of $\min(U^1(z^1), U^1(e-z^2)) = \min(h_1^U(\eta), \eta M^1)$. To maximize this expression, agent 1 chooses $\bar{\eta}$ to equate $h_1^U(\bar{\eta}) = \bar{\eta}M^1$.[9] Hence, $h^U(\bar{\eta}) = (\bar{\eta}M^1, \bar{\eta}M^2)$, so $\bar{\eta} = \tau(V^U)$ and the equilibrium payoff coincides with $K(V^U)$. □

Note that in equilibrium agent 2 is indifferent between choosing himself or having agent 1 choose at stage 3. Nonetheless, there is no coordination problem at all, since with the unique equilibrium parameter $\bar{\eta} = \tau(V^U)$ the resulting allocations yield the same payoffs. With our assumption on different substitution rates, agent 1 and 2 will even choose the same allocation of objects.

[9] Recall that $h_1^U(\eta)$ is decreasing in η.

4 Discussion and Further Results

Although the game Γ^U is a one-shot game, we may think of payoff functions as to be determined in several steps. First, according to strategy choice $\gamma^i \in [0,1]$ of agent i, agent $3-i$ is equipped with an income of $(1-\gamma^i)M^i$, thus with a $1-\gamma^i$ share of the value of the total object set under prices $\bar{p}^i$. Next, agent $3-i$ selects a bundle w.r.t. his income and leaves the remaining objects to agent i (the bundle $z^i(\gamma)$ in $(g1)$). Assuming that $3-i$ chose the utility maximizing bundle, we have $U^i(z^i(\gamma)) = \gamma^i M^i$. But this will be the final payoff, only if the two bundles $z^1(\gamma), z^2(\gamma)$ constitute a feasible allocation. In this sense, we check compatibility of γ^1 and γ^2. Intuitively, on the one hand a "higher" γ^i claims a higher utility, but on the other hand makes it less likely that the we have a feasible allocation in $(g1)$. Now, what happens, if $(z^1(\gamma), z^2(\gamma))$ fails to be an allocation? Then we examine each γ^i once again. Now agent i himself will be endowed with a $(1-\gamma^i)$ share of M^{3-i} and is supposed to choose a utility maximizing bundle at prices $\bar{p}^{3-i}$ (this is $\hat{z}^i(\gamma)$). Observe that $U^{3-i}(e - \hat{z}^i(\gamma)) = \gamma^i M^{3-i}$. According to $(g2)$, agent i still obtains $z^1(\gamma)$, only if it is possible to provide a γ^i share of total utilities to both agents (in that case $(z^1(\gamma), e - \hat{z}^1(\gamma))$ is feasible). But, if γ^i is so "high" that this would not be possible, then agent i gets away with $\hat{z}^i(\gamma)$, i.e., the bundle chosen w.r.t. the "low" income $(1-\gamma^i)M^{3-i}$.
To sum, the idea behind the payoff functions is that in that players' claims are incompatible, an agent with a bold claim has to take what is left over in the situation, in which this claim is realized for the other player.

Whereas we assumed utility maximizing choice in Γ^U, we let agents freely choose in Σ^U. But, since the bundle chosen by either agent 2 or 1 is the final bundle for that agent, utility maximizing prevails in any subgame perfect equilibrium. Inspecting the decision to be taken by agent 2 at stage 2 reveals that he has to decide between two (Pareto efficient) allocations. Suppose, η was chosen at stage 1. Then either agent 1 obtains an η share of his maximal utility M^1 (which is the case, if agent 2 chooses himself), or agent 2 passes and obtains ηM^2 (agent 1 chooses at stage 3). As in Γ^U a "high" η can backfire for agent 1, as agent 2 may pass to stage 3, where agent 1 is endowed with a "low" income of $(1-\eta)M^2$. Conversely, a "low" η gives incentives for agent 2 to choose himself at stage 2 with a "high" income. In effect, there remains a "max-min problem" for agent 1 similar to the one in the definition of the KS solution. This kind of tradeoff was also observed by Trockel (2000) in a welfaristic context.

We should add a note on the role of prices. Although one can formally show that the two price systems $\bar{p}^i$ $(i \in I)$ constitute Walrasian equilibrium prices[10], we shall not regard them as competitive prices. Put in other words, these prices shall not be viewed as being formed

[10] see Haake (2006) for details.

under perfect competition, but result from application of the (normative) solution concept of Walrasian equilibrium. Therefore, we rather use equilibrium prices as an instrument to achieve certain allocations. So, price taking is not understood as a consequence of perfect competition but belongs to the rules of the game.

Both games Γ^U and Σ^U satisfy a property that was termed *full range property* by Trockel (2000). This means that the rules of the game allow for any feasible allocation in $\mathcal{A}$ as the outcome of the game, or, put in other words, any utility allocation V^U is attainable as payoff from strategic interaction. So, by playing the game instead of signing an agreement, the agents do not lose any of their allocation possibilities. For example, in Γ^U a high γ^1 can be compatible with a "low" γ^2, so that a high payoff for agent 1 is (in principle) possible.

Observe that the assumption of ordered and mutually different substitution rates does not constitute a material restriction. In case there are different objects having the same substitution rate, we lose the one-to-one correspondence between Pareto efficient utility and object allocations. Moreover, the demand sets in part 1 of Lemma 1 are no longer singletons. However, we will always be able to select an appropriate allocation, so that its utility allocation satisfies the required properties.

The reader may dislike that there are two different price systems involved to obtain payoffs. As Haake (2006) shows the price system $\bar{p}$ with $\bar{p}_j = (u_j^1 \cdot u_j^2)^{1/2}$ is always an equilibrium price system and therefore individual demand leads to an efficient allocation. As it is basically shown there, we get support results for the superadditive bargaining solution introduced by Perles & Maschler (1981). If one replaces $\bar{p}^i$ by $\bar{p}$ in Γ^U (Σ^U), the results remain valid with $K(V^U)$ replaced by the Perles-Maschler solution. The equilibrium strategies in Γ^U are given by $\bar{\gamma} = (1/2, 1/2)$ as well as $\bar{\eta} = 1/2$ in Σ^U.

So far, there is no taxonomy on support results in the literature that sharply divides them into "plausible" and "implausible" ones. For instance, consider the game with two strategies for each player, meaning that he can accept the KS solution or not. The latter is the final payoff, only if both players chose to accept it. Otherwise, disagreement utilities prevail. This game, though formally a "correct" support result, can hardly be called a reasonable way to support the KS solution as it makes direct use of the solution. Observe that in our approach, it is not necessary to compute an allocation of objects that yields a given profile of payoffs. We always employ a market to arrive at an allocation. Players' utilities only enter in so far as they determine prices according to which we compute demand. The fairness consideration incorporated in the KS solution is reflected in the assignment of income.

One should note that, in contrast to the theory of implementation (under complete information), the Nash program is purely game theoretic. There is no social planner involved. In effect, games are derived for specific underlying bargaining situations and we may assume common knowledge over both the data of the bargaining problem and the data of the non-cooperative game. We close by briefly commenting on how the support results discussed in this paper yield (weak) implementation results for the KS solution as discussed in Trockel (2002) (see also Haake & Trockel (2007)). The key is a proper definition of the implementation environment. For instance, the set of outcomes is the set of bargaining solutions for two person bargaining games. Any preference profile over outcomes (bargaining solutions) can be identified with a specific bargaining problem along the notions of *effectivity* and *supportability* discussed in Bergin & Duggan (1999). Then, the social choice correspondence that is implemented assigns to each profile of preferences (here: a bargaining problem) a set of outcomes (bargaining solutions) that coincide with the KS solution on this particular bargaining problem. So, strictly speaking it is not the KS solution itself that is implemented, but a social choice correspondence that takes the KS solution as benchmark to define desirable outcomes.

References

Bergin, J. & J. Duggan (1999): "An Implementation-Theoretic Approach to non-cooperative Foundations", *Journal of Economic Theory*, 86, 50–76.

Dagan, N. & R. Serrano (1998): "Invariance and Randomness in the Nash Program for coalitional Games", *Economics Letters*, 58, 43–59.

Haake, C.-J. (2000): "Support and Implementation of the Kalai–Smorodinsky Bargaining Solution", in *Operations Research Proceedings 1999*, ed. by K. Inderfurth, G. Schwödiauer, W. Domschke, F. Juhnke, P. Kleinschmidt & G. Wäscher, pp. 170–175. Springer.

——— (2006): "Dividing by Demanding: Object Division through Market Procedures", *International Game Theory Review*, forthcoming.

Haake, C.-J. & W. Trockel (2007): "On Maskin monotonicity of solution based social choice rules", Institute of Mathematical Economics, Bielefeld University, Working Paper No. 393.

Kalai, E. & M. Smorodinsky (1975): "Other Solutions to Nash's Bargaining Problem", *Econometrica*, 43, 513–518.

Miyagawa, E. (2002): "Subgame-perfect implementation of bargaining solutions", *Games and Economic Behavior*, 41(2), 292–308.

Moulin, H. (1984): "Implementing the Kalai–Smorodinsky Bargaining Solution", *Journal of Economic Theory*, 33, 32–45.

Nash, J. F. (1953): "Two–Person Cooperative Games", *Econometrica*, 21, 128–140.

Perles, M. & M. Maschler (1981): "The Super-Additive Solution for the Nash Bargaining Game", *International Journal of Game Theory*, 10(3/4), 163–193.

Serrano, R. (1997): "A Comment on the Nash Program and the Theory of Implementation", *Economics Letters*, 55, 203–208.

——— (2005): "Fifty years of the Nash program, 1953–2003", *Invest. Economicas*, 29, 219–258.

Trockel, W. (1999): "Unique Nash Implementations for a Class of Bargaining Solutions", *International Game Theory Review*, 1, 267–272.

——— (2000): "Implementations of the Nash Solution Based on its Walrasian Characterizations", *Economic Theory*, 16, 277–294.

——— (2002): "Integrating the Nash Program into Mechanism Theory", *Review of Economic Design*, 7(1), 27–43.