Piecewise Linear Bertrand Oligopoly

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Abstract

We describe a modell of price competition between firms with piecewise linear cost functions. Thus, we consider "Bertrand oligopoly", an n-person noncooperative game in which players choose prices and the market, reflected by a decreasing demand function, reacts discontinuously as total demand concentrates on those firms that offer minimal prices. Firms do not have to be identical. But a notion of similarity between firms is necessary in order to prove the existence of a Nash (-Bertrand) equilibrium. Here we are only interested in an equilibrium involving all firms – the case of subgroups with "similar" members deserves an additional study.

1 Bertrand Oligopoly

Within this paper we discuss the existence of equilibria within a certain type of Bertrand Oligolpoly. [1] The main feature is the structure of the cost functions of the firm, these are supposed to be piecewise linear and convex. Such cost functions appear naturally in the context of network flow structures, where flows passing through capacity limited nodes and edges generate costs depending on the choice of edges as well. We think of such kind of flow as electricity or data material on an electronic net. Routing the flow optimally (cost minimizing) results in a linear programming problem, the solution of which yields a piecewise linear cost function. See [6] for a detailed model of this type.

The technique is not far away from standard procedures. However, apart from missing differentiability assumptions we also do not assume symmetric firms.

Most of the literature seems to rely on at least one of these assumptions. DASTIDAR[2] discusses the asymmetric case as well (assuming that costfunctions are twice differentiable), however the assumptions imposed on the model vary during the presentation. HOERNIG[4] constructs in addition to the continuum of pure equilibria existing a host of mixed ones. See also MASKIN[5] for mixed equilibria. Symmetry is also assumed in HEHENKAMP–LEININGER [3], who discuss evolutionary Bertrand equilibria.

It would seem that none of the properties derived in the context of this literature suffers when differentiability is sacrificed and firms are just “similar”.

As frequently, it is assumed that firms have a limited capacity of production. Yet they are supposed to meet market demand at the level required. The game in which firms may plan to sell less than required has different strategies and payoffs. Yet it seems that the type of equilibrium exhibited would constitute an equilibrium in the extended game as well. Within our present framework, we will not attend to this question.

The model is specified essentially by a set of piecewise linear costfunctions for the firms and a demand function of the market. We specify this data as follows.

For any nonnegative convex, monotone function D on the reals we denote by D' the derivative of a linear support function of D at t. This derivative is unique up to at most countably many points.

A decreasing function is *slowly decreasing* if it does not decrease faster than

$1/t$, i.e., if $\frac{D(t)}{t} \geq -D'(t)$ holds true for all t in the domain of definition. Economically this reflects nonneagtive marginal expenditure.

Given positive real numbers d_0 and p_0, we call a function

$$D \;:\; [0, p_0] \to [0, d_0]$$

a ***demand function*** if it is continuous at 0, convex, and slowly decreasing. A demand function is hence continuous and differentiable with the exception of at most countably many points.

On the other hand, let for $K \in \mathbb{N}$

$$\boldsymbol{C}^{(0)} \;:=\; (\boldsymbol{A}^{(0)}, \boldsymbol{B}^{(0)}) \in \mathbb{R}^{2K} \tag{1.1}$$

be such that $\boldsymbol{A}^{(0)} = (A_k^{(0)})_{k=0,\dots K}$ and $\boldsymbol{B}^{(0)} = (B_k^{(0)})_{k=0,\dots K}$ are real numbers *strictly increasing* in k and satisfy $A_0^{(0)} = 0$, $B_0^{(0)} = 0$. We put

$$\Delta_0^{(0)} \;:=\; 0,\; \Delta_k^{(0)} \;:=\; \frac{\Delta B_k^{(0)}}{\Delta A_k^{(0)}} \;:=\; \frac{B_k^{(0)} - B_{k-1}^{(0)}}{A_k^{(0)} - A_{k-1}^{(0)}}. \tag{1.2}$$

We assume that $\Delta_k^{(0)}$ is as well *strictly increasing* in k and satisfies

$$\Delta_K = d_0,\; A_K \Delta_K - B_K \leq p_0. \tag{1.3}$$

Given these conditions, we identify the data (1.1)with the strictly increasing piecewise linear function $C^{(0)}$ given by

$$\begin{aligned} C^{(0)} \;&:\; \quad [0, d_0] \to [0, p_0], \\ C^{(0)}(t) \;&:\; \quad \max\left\{ A_k^{(0)} t - B_k^{(0)} \,\middle|\, k = 0, \dots K \right\} \quad (t \in [0, d_0]). \end{aligned} \tag{1.4}$$

As a consequence, the numbers $\Delta_k^{(0)}$ describe the arguments at which the function shows kinks: it is seen that

$$C^{(0)}(t) \;=\; A_k^{(0)} t - B_k^{(0)} \quad (t \in [\Delta_k^{(0)}, \Delta_{k+1}^{(0)}]) \tag{1.5}$$

holds true (cf. Figure 1.1). Thus E.g. (1.3) shows that

$$C^{(0)}(d_0) \leq p_0$$

is satisfied, thus the domain of definition is indeed $[0, d_0]$ and the range is contained in $[0, p_0]$.

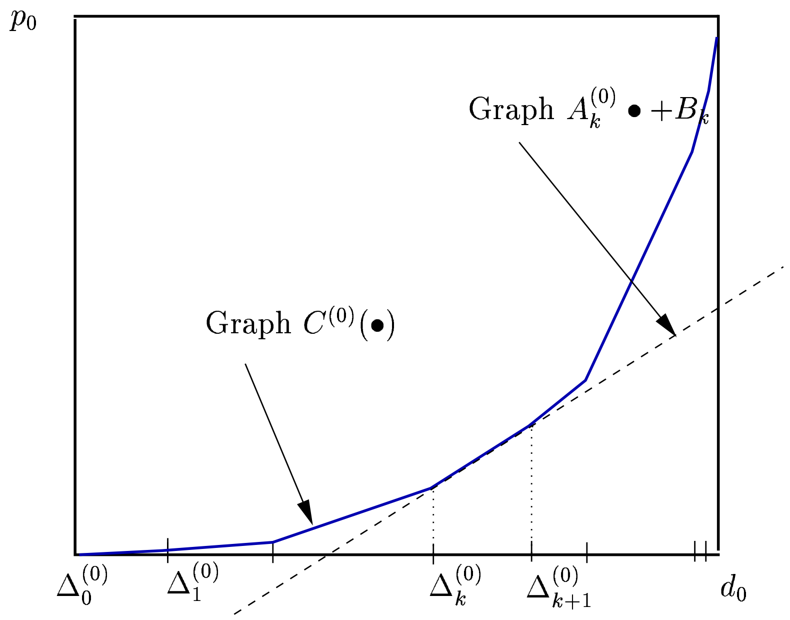

Figure 1.1: A Cost Function

Given p_0 and d_0, we call $\boldsymbol{C}^{(0)}$ or $C^{(0)}$ a ***cost function*** if, in addition, we have

$$A_K^{(0)} = C^{(0)\prime}(d_0) > p_0, \quad p_0 d_0 < C^{(0)}(d_0) = A_K^{(0)} d_0 + B_K^{(0)}. \tag{1.6}$$

The first inequality shows that marginal cost at maximum production exceeds maximal prices. The latter reads also $p_0 d_0 - C^{(0)}(d_0) < C^{(0)}(0) = 0$, meaning that, at maximal prices, a firm's profit at the maximal possible demand is less than at zero production. The advantage of dealing with this simple setup is provided by the topology available for cost functions; this is given by the Euclidean metric on $\mathbb{R}^{2K}$.

Definition 1.1. *A **Piecewise Linear Bertrand Oligopoly (PLBO)** for a set of players $\boldsymbol{I} := \{1, \ldots n\}$ is a set of data*

$$\mathcal{O} := (p_0, d_0, D, (\boldsymbol{C}^{(i)})_{i \in I}) \tag{1.7}$$

such that p_0, d_0 are the domain of definitions, D is a demand function and $\boldsymbol{C}^{(i)}$ represents the cost function for player $i \in \boldsymbol{I}$.

Given some price $p \in [0, p_0]$, we first consider the function

$$\begin{aligned} G^{(i)} &= G_p^{(i)} \; : \; [0, d_0] \to \mathbb{R} \\ G^{(i)}(t) &:= pt - C^{(i)}(t) \quad (t \in [0, d_0]). \end{aligned} \tag{1.8}$$

This function describes the profit of player $i \in \boldsymbol{I}$ if the total demand of the market would accumulate at this player (the monopolistic profit function at fixed price).

However, the actual payoff within the N-person game resulting from price competition is defined via the modified demand function customary in Bertrand oligopoly as follows:

Definition 1.2. *1. For any (price) vector* $\boldsymbol{p} \in \mathbb{R}^n$ *let*

$$M_{\boldsymbol{I}}(\boldsymbol{p}) \ := \ arg\ min_I\ \boldsymbol{p} \ = \ \left\{ i \in \boldsymbol{I} \,\middle|\, p_i = \min_{k \in \boldsymbol{I}} p_k \right\} \tag{1.9}$$

denote the set of minimizing arguments or ***minimizers*** *of* p.

2. *Let* $D : [0, p_0] \to \mathbb{R}$ *be a demand function. Then, for* $i \in \boldsymbol{I}$ *the function*

$$\begin{aligned} \boldsymbol{D}^i \ &: \quad [0, p_0]^{\boldsymbol{I}} \to \mathbb{R} \\ \boldsymbol{D}^i(\boldsymbol{p}) \ &:= \quad \begin{cases} 0 & i \notin M_{\boldsymbol{I}}(\boldsymbol{p}) \\ \frac{D(p_i)}{|M_{\boldsymbol{I}}(\boldsymbol{p})|} & i \in M_{\boldsymbol{I}}(\boldsymbol{p}) \end{cases} \qquad (\boldsymbol{p} \in [0, p_0]^I) \end{aligned} \tag{1.10}$$

is the ***Bertrand demand*** *function resulting from* D.

3. *Finally, let* $\boldsymbol{C}^i$ *or* $C^{(i)}$ *respectively be the costfunction of firm or player* i*. Then the (oligolpolistic Bertrand)* ***profit function*** *of player* i *is the function*

$$\begin{aligned} \boldsymbol{G}^i \ &:= \quad [0, p_0]^{\boldsymbol{I}} \to \mathbb{R} \\ \boldsymbol{G}^i(\boldsymbol{p}) \ &:= \quad p_i \boldsymbol{D}^i(\boldsymbol{p}) - C^{(i)}(\boldsymbol{D}^i(\boldsymbol{p})) \end{aligned} \tag{1.11}$$

We note that $\Gamma = \Gamma^{\mathcal{O}} =: ([0, p_0]^{\boldsymbol{I}}, (\boldsymbol{G}^i)_{i \in \boldsymbol{I}})$ is the n–person game based on the data of $\mathcal{O}$, the Nash equilibria of which we are concerned with. These Nash equilibria are referred to as ***Bertrand equilibria***.

For a beginning, we attempt to establish a Bertrand equilibrium in which all players participate. Then we are dealing with a price vector (strategy n–tupel) $\boldsymbol{p} = (p, \ldots, p)$, in which case some definitions simplify e.g. to

$$\boldsymbol{D}^i(\boldsymbol{p}) \ = \ \frac{D(p)}{n}, \quad \boldsymbol{G}^i(\boldsymbol{p}) \ = \ p\frac{D(p)}{n} - C^{(i)}\left(\frac{D(p)}{n}\right).$$

Now we wish to discuss some *necessary* conditions for equilibria and, if possible, establish a situation in which these conditions turn out to be *sufficient* as well.

Fix $\overline{\boldsymbol{p}} = (\overline{p}, \ldots, \overline{p})$ in order to tentatively denote an equilibrium. Let $\overline{\xi} := D(\overline{p})$ so that

$$\boldsymbol{G}^i(\overline{\boldsymbol{p}}) = \frac{\overline{p}\overline{\xi}}{n} - C^{(i)}(\frac{\overline{\xi}}{n}) = G^{(i)}\left(\frac{\overline{\xi}}{n}\right)$$

is player i's payoff in equilibrium. Here we refer to the function $G^{(i)} = G^{(i)}_{\overline{p}}$ defined via (1.8) for the fixed price $\overline{p}$.

First of all, suppose that player i wants to deviate in a way that all market demand is concentrated at his firm. That is, the player lowers his price to $\overline{p} - \varepsilon$. The equilibrium condition can be formulated to be

$$\begin{aligned} G^{(i)}\left(\frac{\overline{\xi}}{n}\right) &= \boldsymbol{G}^i(\overline{\boldsymbol{p}}) \geq \boldsymbol{G}^i(\overline{\boldsymbol{p}} - \varepsilon e^i) \\ &= (\overline{p} - \varepsilon)D(\overline{p} - \varepsilon) - C^{(i)}(D(\overline{p} - \varepsilon)) \quad (\varepsilon > 0) \end{aligned}$$

which implies for $\varepsilon \to 0$

$$G^{(i)}\left(\frac{\overline{\xi}}{n}\right) \geq \overline{p}\overline{\xi} - C^{(i)}(\overline{\xi}). \tag{1.12}$$

Thus,

$$\overline{p}\frac{\overline{\xi}}{n} - C^{(i)}(\frac{\overline{\xi}}{n}) \geq \overline{p}\overline{\xi} - C^{(i)}(\overline{\xi}). \tag{1.13}$$

is a necessery condition for a Bertrand equilibrium involving all players. The condition indicates that it should not be profitable for player i to draw all the market demand $\overline{\xi}$ *at equilibrium* on himself compared to share of $\frac{1}{n}\overline{\xi}$ he obtains when the equilibrium is sustained.

Essentially we would like to estabish a situation in which (1.13) is part of a *sufficient* condition as well. To this end we prove a standart lemma which is based on concavity of the cost functions and on slowly decreasing demand.

Lemma 1.3. *Let* $\mathcal{O}$ *be a* PLBO *(Definition 1.1) and let* $\boldsymbol{G}^i$ *be the resulting Bertrand profit function of player i (Definition 1.2). Then, for all* $\bar{\boldsymbol{p}} \in [0, p_0]^I$ *and* $t > 0$ *such that* $\bar{\boldsymbol{p}} - t\boldsymbol{e}^i \in [0, p_0]^I$ *is true, we have*

$$\boldsymbol{G}^i(\bar{\boldsymbol{p}} - t\boldsymbol{e}^i) \leq \bar{p}_i D(\bar{p}_i) - C^{(i)}\left(D(\bar{p}_i)\right). \tag{1.14}$$

Proof: Assume first of all that D is differentiable.

Note that $-C^{(i)} \circ D$ is a decreasing function. Also, D is convex and decreasing, hence we have for positive and suitable arguments ξ and η

$$D(\xi - \eta) \leq D(\xi) - \eta D'(\xi - \eta).$$

Applying this we find

$$\begin{aligned}
\boldsymbol{G}^i(\bar{\boldsymbol{p}} - t\boldsymbol{e}^i) &= (\bar{p}_i - t)D(\bar{p}_i - t) - C^{(i)}\Big(D(\bar{p}_i - t)\Big) \\
&\leq (\bar{p}_i - t)\Big[D(\bar{p}_i) - tD'(\bar{p}_i - t)\Big] \\
&\quad - C^{(i)}\Big(D(\bar{p}_i)\Big) \\
&= \bar{p}_i D(\bar{p}_i) - C^{(i)}\Big(D(\bar{p}_i)\Big) \\
&\quad - t\Big[D(\bar{p}_i) + \bar{p}_i D'(\bar{p}_i - t)\Big] \\
&\qquad + \underbrace{t^2 D'(\bar{p}_i - t)}_{\leq 0} \\
&\leq \bar{p}_i D(\bar{p}_i) - C^{(i)}\Big(D(\bar{p}_i)\Big) - t\Big[D(\bar{p}_i) + \bar{p}_i D'(\bar{p}_i)\Big] \\
&\leq \bar{p}_i D(\bar{p}_i) - C^{(i)}\Big(D(\bar{p}_i)\Big);
\end{aligned} \tag{1.15}$$

the last inequality uses the requirement that D is slowly decreasing,

q.e.d.

We note that, in the particular case of $\overline{\boldsymbol{p}} = (\overline{p}, \ldots, \overline{p})$, equation (1.14) reads $\boldsymbol{G}^i(\bar{\boldsymbol{p}} - t\boldsymbol{e}^i) \leq \bar{p}_i D(\bar{p}_i) - C^{(i)}(D(\bar{p}_i)) =: \overline{p}\overline{\xi} - C^{(i)}(\overline{\xi}) = G^{(i)}(\bar{\xi})$. Therefore the lemma shows that equation (1.13) indeed implies that player i cannot profitably deviate by decreasing his price arbitrarily, i.e., (1.13) is sufficient in order to establish (part of) the equilibrium condition.

There is a second type of deviation of a player from equilibrium that we have to take into account. At this version, player i inserts a price exceeding the common equilibrium price $\overline{p}$. Naturally, it is much easier to see that this is not profitable. For, if $\widehat{\boldsymbol{p}} = (\overline{p}, \ldots, \overline{p} + te^i, \ldots, \overline{p})$ for some $t > 0$ denotes the resulting strategy n-tuple, then the Bertrand demand accumulating at player i is $\boldsymbol{D}^i(\widehat{\boldsymbol{p}}) = 0$, hence player i's payoff is

$$\boldsymbol{G}^i(\widehat{\boldsymbol{p}}) = G^{(i)}(0) = -C^{(i)}(0)(= 0), \tag{1.16}$$

(if we are assuming zero fixed costs). Combining these ideas we obtain

Corollary 1.4. *Let $\mathcal{O}$ be a PLBO. Also, let $\bar{p} \in [0, p_0]$ and $\bar{\xi} := D(\bar{p})$. Suppose that, for all $i \in I$, the inequalities*

$$\bar{p}\frac{\bar{\xi}}{n} - C^{(i)}(\frac{\bar{\xi}}{n}) \geq \bar{p}\bar{\xi} - C^{(i)}(\bar{\xi}) \tag{1.17}$$

and

$$\bar{p}\frac{\bar{\xi}}{n} - C^{(i)}(\frac{\bar{\xi}}{n}) \geq -C^{(i)}(0) \tag{1.18}$$

are satisfied. Then $\bar{\boldsymbol{p}} := (\bar{p}, \ldots, \bar{p})$ is a Bertrand equilibrium in $\mathcal{O}$ involving all players.

The problem is that the quantity $\overline{\xi}$ does not depend on i, it has to be chosen simultaneously for all players. The simple idea to generate this quantity is described as follows.

If, for some $i \in \boldsymbol{I}$ and $\overline{p} \in [0, p_0]$ the inequality

$$\overline{p} \geq C^{(i)\prime} = A_0^{(i)} \tag{1.19}$$

is satisfied, then $G^{(i)}$ has either a second zero in $[0, p_0]$ or is nonnegative within all of the interval. Let

$$\xi_0 = \max\left\{x \in [0, d_0] \,\middle|\, \xi > 0, G^{(i)}(\xi) = 0\right\} \tag{1.20}$$

with the understanding that $\xi_0 = d_0$ whenever the *max* has to be extended over the empty set. Also, let

$$\widehat{\xi} \;:=\; \min\left\{x \in [0, d_0] \,\middle|\, G^{(i)}(\xi) \geq G^{(i)}(\eta) \quad)\eta \in [0, d_0]\right\} \tag{1.21}$$

denote the first maximizer of the function $G^{(i)}$. Then we have

Corollary 1.5. *Let $\bar{p} \in [0, p_0]$ and write $\bar{\xi} := D(\bar{p})$. Suppose the corresponding function $G^{(i)} = G^{(i)}_{\overline{p}}$ with respect to its zeros and maximizers satisfies*

$$\widehat{\xi^i} \leq \frac{\overline{\xi}}{n} \leq \xi_0^i \tag{1.22}$$

for all $i \in \boldsymbol{I}$. Then $\overline{\boldsymbol{p}} = (\bar{p}, \ldots, \bar{p})$ is a Bertrand equilibrium.

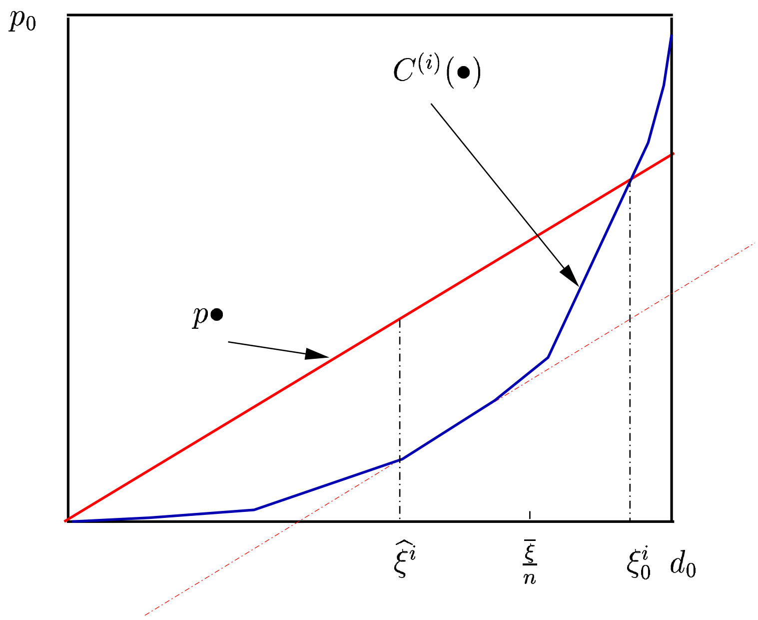

Figure 1.2: The Location of Equilibrium Demand

Proof: As $\frac{\overline{\xi}}{n}$ is located to the right of the maximizer of $G^{(i)}$, we have (1.17). Furthermore, as $\frac{\overline{\xi}}{n}$ is located to the left of the zero of the same function, we have (1.18).

q.e.d.

An illustrating picture is obtained as follows (Figure 1). The

linear function with slope p and a cost function $C^{(i)}$ are depicted simultaneously. The maximizer $\widehat{\xi^i}$ is obtained as the first point admitting of a tangent p at the cost function. The zero ξ_0^i is the last point at which the graphs of both functions intersect. If we can find $\overline{\xi}$ such that $\frac{\overline{\xi}}{n}$ is *simultaneously* contained within the interval spanned by both points, then an equilibrium prevails. Slightly more formal: the function $G^{(i)}$ depends on p and so do the maximizer and the zero. Define interval–valued correspondences on $[0, p_0]$ by

$$\Xi^i(p) \;:=\; [\widehat{\xi^i}(p), \xi_0^i(p)] \quad (p \in [0, p_0] \;\;, i \in \boldsymbol{I}). \tag{1.23}$$

and

$$\Xi(p) = \bigcap_{i \in \boldsymbol{I}} \Xi^i(p) \quad (p \in [0, p_0]). \tag{1.24}$$

Then we have

Theorem 1.6. *Suppose* $\overline{p} \in [0, p_0]$ *satisfies*

$$\frac{\overline{\xi}}{n} = \frac{D(\overline{p})}{n} \in \Xi(\overline{p}). \tag{1.25}$$

Then $\overline{\boldsymbol{p}} = (\bar{p}, \ldots, \bar{p})$ *is a Bertrand equilibrium.*

Proof: Obviously this theorem is just a reformulation of the previous corollary. **q.e.d.**

Now in order to supply an existence theorem, some preparations are necessary in order to understand the behavior of the interval–valued correspondence Ξ. This is the topic of the next section.

2 The DMP Correspondence

We start out discussing some properties of the correspondence Ξ^i that results from player i's monopolistic profit function $G^{(i)}$. For this purpose, the generic index $^{(i)}$ is tentatively omitted for the sake of more translucence.

So we consider a cost function $\boldsymbol{C}$ or C and the derived profit function G.

Let $p \geq A_0$. For any k with $A_k > p$, the function $A_k \bullet - B_k$ intersects the function $p\bullet$ at $\frac{B_k}{A_k - p}$; therefore, if the graph of C intersects the graph of $p\bullet$ (i.e., the straight line with slope p) beyond the origin, then this occcurs at the point η_0 given by

$$\eta_0 = \eta_0(p) = \min\left\{ \frac{B_k}{A_k - p} \,\middle|\, A_k > p \right\} =: \frac{B_{k^0}}{A_{k^0} - p} \geq \frac{B_{k^0}}{A_{k^0} - A_{k^0-1}} \tag{2.1}$$

If we agree on $k_0 := K$ for the empty set in (2.1), then the index k_0 is uniquely defined. In particular, if it so happens that $p = A_0$ is the case, then $k_0 = 1$ and $\eta_0 = \Delta_1$ follow at once. Now, η_0 defines the point at which profit is zero (apart from the origin), provided there is such point located within the admissible interval. Hence we put

$$\xi_0 = \xi_0(p) := \min\left\{ d_0, \frac{B_{k^0}}{A_{k^0} - p} \right\} \quad (p \geq A_0). \tag{2.2}$$

This way we have defined the function

$$\xi_0 : [A_0, d_0] \to \mathbb{R} \tag{2.3}$$

which depends continuously on the data $\boldsymbol{C}$ and on p. The function ξ_0 is closely related to the ***average cost function*** which is given by

$$\begin{aligned} M_k := & \frac{C(\Delta_k)}{\Delta_k} \\ = & \frac{A_k \Delta_k - B_k}{\Delta_k} \\ = & A_k - \frac{B_k}{\Delta_k} \quad (k = 1, \ldots, K). \end{aligned} \tag{2.4}$$

If p represents a slope between the average slope at Δ_k and Δ_{k+1}, then the graph of $p\bullet$ intersects the one of C just within the interval $[\Delta_k, \Delta_{k+1}]$. Hence we find that ξ_0 can be represented via

$$(2.5)\qquad \xi_0(p) \;=\; \frac{B_k}{A_k - p} \qquad (M_k \leq p \leq M_{k+1}, \quad k = 1,\ldots,K, \quad p \geq A_0).$$

Next, the (profit-) *maximizer correpondence* derived from $\boldsymbol{C}$ is described by

$$(2.6)\qquad \widehat{\Xi}(p) \;:=\; \begin{cases} \{\Delta_k\} & A_{k-1} < p < A_k \;, \\ [\Delta_k, \Delta_{k+1}] & p = A_k \end{cases} \qquad (p \geq A_0)$$

Obviously, this correpondence is interval–valued and upper hemi continuous (*uhc*) in p. The construction shows, however, that it is also *uhc* in $\boldsymbol{C}$. Accordingly, the ***smallest profit maximizer*** is given by

$$(2.7)\qquad \widehat{\xi}(p) \;:=\; \begin{cases} \Delta_k & \text{if} \quad A_{k-1} < p \leq A_k, \\ 0 & \text{if} \qquad p = A_0 \quad . \end{cases}$$

this quantity is either the singleton contained in $\widehat{\Xi}(p)$ or the minimum of the interval defining this correspondence. Therefore, we obtain the correspondence $\Xi \;:\; [0, p_0] \to \mathcal{P}(\mathbb{R})$ which is given by

$$(2.8)\qquad \Xi(p) \;:=\; \begin{cases} [\widehat{\xi}(p), \xi_0(p)] & p \in [A_0, p_0] \\ [\Delta_0, \Delta_1] = [0, \Delta_1] & p = A_0 \\ \emptyset & p \in [0, A_0) \end{cases}$$

which we call the ***dmp–correspondence*** derived from $\boldsymbol{C}$. This is motivated as it describes the interval of non–positive or *decreasing marginal profit* for a player whose cost function is described by $\boldsymbol{C}$. From our construction it follows easily that we have

Lemma 2.1. *Whenever* $p \geq A_0$*, then the dmp–correspondence* Ξ *is nonempty, interval–valued and uhc in* p *and* $\boldsymbol{C}$*.*

Our next task is to estimate the length of the interval describing Ξ. We claim that this is a positive constant depending in an *uhc* way on our data To make this more precise, we claim

Lemma 2.2. *Given* p_0, d_0*, let* C *be a cost function. Then there is a lower bound* $\beta > 0$ *such that, whenever* $p \geq A_1$*, it follows that*

$$|\Xi(p)| = \xi_0(p) - \widehat{\xi}(p) \geq \beta$$

holds true.

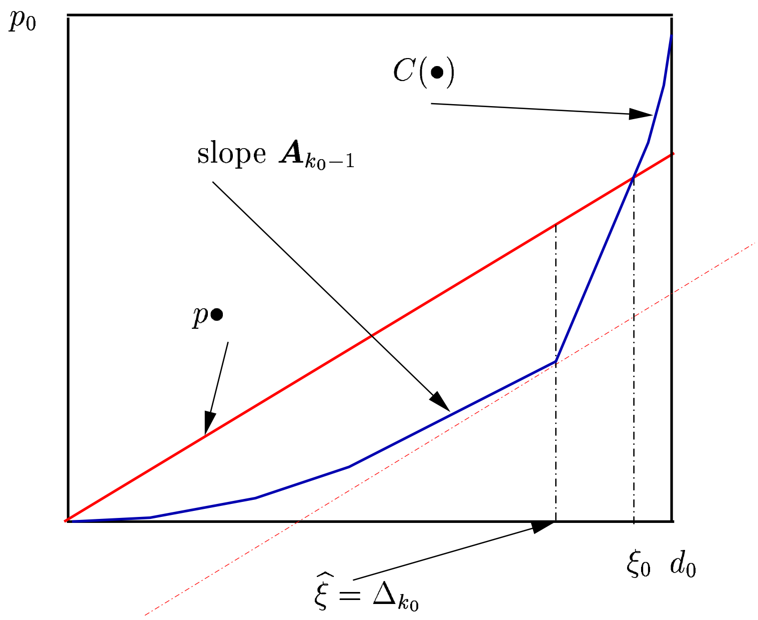

Figure 2.1: The Worst Case

Proof:

First of all assume that the graphs of C and $p\bullet$ intersect beyond the origin. Choose the index $\widehat{k}$ such that

$$\widehat{\xi} \;=\; \Delta_{\widehat{k}}$$

holds true. As $p \geq A_1$, we have $\widehat{k} \geq 1$ Also, let k^0 be given as in (2.2). Then clearly $\widehat{k} \leq k^0$ (both quantities depending on p). Note that $k^0 \geq 2$ follows from our assumption $p \geq A_1$. Now, if ("in the worst case"), it so happens that we have $k^0 = \widehat{k}$, then from (2.2) we deduce

$$\begin{aligned} \xi_0(p) - \widehat{\xi}(p) &= \frac{B_{k^0}}{A_{k^0} - p} - \frac{\Delta B_{\widehat{k}}}{\Delta A_{\widehat{k}}} \\ &\geq \frac{B_{k^0}}{A_{k^0} - A_{k^0-1}} - \frac{\Delta B_{\widehat{k}}}{\Delta A_{\widehat{k}}} \\ &= \frac{B_{k^0-1}}{\Delta A_{k^0}}, \end{aligned} \tag{2.9}$$

(see Figure 2.1). If $k^0 > \widehat{k}$, then the estimate is even stronger. Therefore, (2.9) provides the desired lower bound.

If the graphs of C and $p\bullet$ do not intersect at some point apart from the origin, then

$$\xi_0(p) - \widehat{\xi}(p) = d_0 - \Delta_{K-1} = \Delta_K - \Delta_{K-1} \tag{2.10}$$

follows from the fact that we have $p \leq p_0 \leq A_K$ (cf. equation (1.6)).

q.e.d.

Remark 2.3. *In the above situation, if $p = A_0$ happens to be true, then we obtain*

$$\xi_0(p) - \widehat{\xi}(p) = \Delta_1 = \Delta_1 - \Delta_0 \quad . \tag{2.11}$$

For $A_0 \leq p < A_1$ the estimate provided by (2.9) does not yield a lower bound. The width of $\Xi(p)$ is arbitrarily small when p approaches A_0. Yet, if we fix some $\varepsilon > 0$ and the argument p avoids the intervall $[A_0, A_0 + \varepsilon]$, then again the width of the width of $\Xi(p)$ is bounded away from 0 by some constant $\beta = \beta(\varepsilon, C)$. Thus the statement of the above theorem can be sharpened accordingly. This version may be preferable if the interval $[\Delta_1, \Delta_2]$ is relatively large.

On the other hand, as p increases, so does $\xi_0(p)$ and hence k^0. Imagine that the slopes A_k are about equally distributed so that the differences ΔA_k are about equal. Then (2.9) suggests that, as B_{k^0-1} increases, the interval $\Xi(p)$ also increases with p. A more refined analysis shows that this depends on a relation between marginal and mean cost. The graph of $\Xi(\bullet)$ is an area bounded by the piecewise constant function $\widehat{\xi}(\bullet)$ from below and by the minimum of certain hyperbola from above (in view of (2.2)). We claim that, depending on the curvature of $\boldsymbol{C}$, this graph "widens" with increasing p. Essentially, the idea is that marginal cost increases faster than mean cost. We clarify the precise meaning as follows.

Definition 2.4. *We shall say that a cost function $\boldsymbol{C}$ admits of **proper MM increments** if, for any $L \in \mathbb{N}$ there exists $K \in \mathbb{N}$ such that for all $k \geq l \geq K$ with $\Delta_k - \Delta_l \leq L$ it follows that $A_{l-1} \geq M_{k+1}$ is true.*

We have

Theorem 2.5. *If $\boldsymbol{C}$ amits of proper MM increments, then $|\xi_0(p) - \widehat{\xi}(p)|$ is arbitrary large for increasing p.*

Proof: Note that the statement of Definition 2.4 can be equivalently given in the following version:

> For any $L \in \mathbb{N}$ there exists $K \in \mathbb{N}$ such that for all $k \geq l \geq K$ with $A_{l-1} < M_{k+1}$ it follows that $\Delta_k - \Delta_l > L$ is true.

Now let $L \in \mathbb{N}$ be a ("large") integer and choose K accordingly. Pick $l \geq K+1$ and consider a price $p \in (A_{l-1}, A_l]$. Then in view of (2.7) we find

$$\widehat{\xi}(p) = \Delta_l \ . \tag{2.12}$$

As ξ_0 increases in a strictly monotone way, we have

$$\xi_0(p) - \widehat{\xi}(p) > \xi_0(A_{l-1}) - \Delta_l. \tag{2.13}$$

Now choose k such that for $p = A_{l-1}$ one has

$$M_k \leq p = A_{l-1} < M_{k+1} \tag{2.14}$$

and hence

$$\xi_0(p) = \xi_0(A_{l-1}) = \frac{B_k}{A_k - p} \geq \frac{B_k}{A_k - M_k} = \Delta_k. \tag{2.15}$$

We know that $A_{k+1} \geq M_{k+1}$, hence $k+1 > l-1$ is necessarily true. Therefore we have $k \geq K$.

Combining (2.13) and (2.15) we obtain

$$\xi_0(p) - \widehat{\xi}(p) \geq \Delta_k - \Delta_l > L$$

which exceeds L in view of the above version of Definition 2.4,

q.e.d.

Example 2.6. Let c be a positive constant and let

$$A_k = 2ck, \quad B_k = ck(k+1) \quad (k = 0, \ldots, K) \tag{2.16}$$

such that

$$\Delta_0 = 0, \quad \Delta_k \ = \ k \quad , M_k \ = \ c(k-1) \quad (k = 1, \ldots, K) \tag{2.17}$$

is computed at once.

In view of (2.7) we find

$$\widehat{\xi}(p) = k \quad (2c(k-1) < p \leq 2ck)$$

i.e.,

$$\widehat{\xi}(p) = k \quad (\frac{p}{2c} \leq k < \frac{p}{2c} + 1).$$

Similarly, formula (2.5) shows

$$\xi_0(p) = \frac{k(k+1)}{2k - \frac{p}{c}} \quad (c(k-1) \leq p \leq ck)$$

i.e.,

$$\xi_0(p) = \frac{k(k+1)}{2k - \frac{p}{c}} \quad (\frac{p}{c} \leq k \leq \frac{p}{c} + 1).$$

If we write $t := \frac{p}{c}$ for the moment, then we obtain

(2.18)
$$\begin{aligned} \xi_0(t) &= \frac{k(k+1)}{2k-t} && (t \leq k \leq t+1) \\ \widehat{\xi}(t) &= k && (\frac{t}{2} \leq k < \frac{t}{2} + 1). \end{aligned}$$

From this we derive an estimate

(2.19)
$$\begin{aligned} \xi_0(t) - \widehat{\xi}(t) &> \frac{t(t+1)}{2(t+1)-t} - \left(\frac{t}{2} + 1\right) \\ &= \frac{t^2 - 2t - 4}{2t+4}, \end{aligned}$$

which increases like $\frac{t}{2}$ for increasing t.

Now, if we rewrite (2.18) as

(2.20)
$$\begin{aligned} \xi_0(t) &= \frac{k(k+1)}{2k-t} && (k-1 \leq t \leq k, \quad k = 1, \ldots, K) \\ \widehat{\xi}(t) &= k && (2k-2 < t \leq 2k), \end{aligned}$$

then we can provide a sketch of the correspondence Ξ as in Figure 2.2.

Note that this figure represents the correspondence with $t = \frac{p}{c}$ as the independent variable. If we want to represent it as a correspondence in p, then the above sketch has to be rescaled by a factor c, it is shrinking for $c > 1$ and expanding for $c < 1$.

Definition 2.7. *Let $\mathcal{O}$ be a* PLBO. *Then, for every player $i \in \boldsymbol{I}$, the correspondence $\Xi^{(i)}$ derived from $\boldsymbol{C}^{(i)}$ is the* ***dmp–correpondence*** *of player i. The correspondence*

(2.21)
$$\begin{aligned} \Xi : [0, p_0] &\rightarrow \mathcal{P}(\mathbb{R}) \\ \Xi(p) &:= \bigcap_{i \in I} \Xi^{(i)}(p) \end{aligned}$$

is called the ***dmp–correspondence*** *of $\mathcal{O}$.*

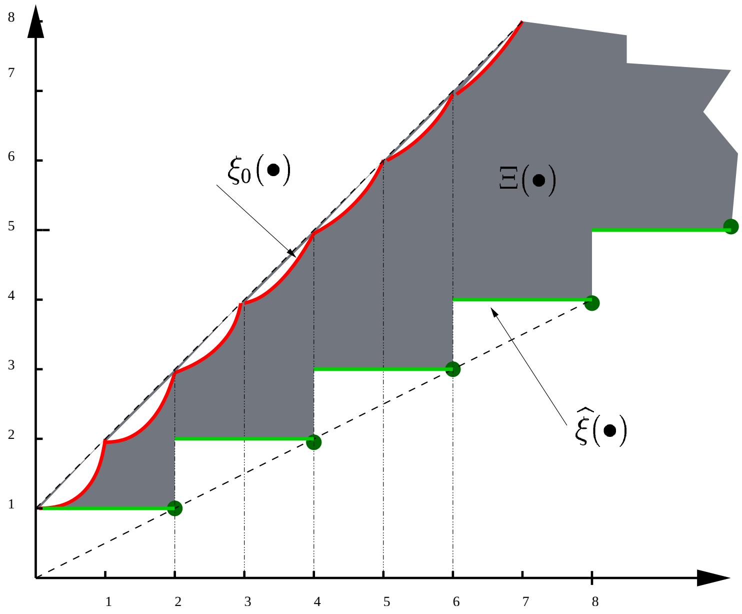

Figure 2.2: A DMP Correspondence

Corollary 2.8. *Let $\bar{\boldsymbol{C}}$ be a costfunction. Then there is a neighborhood of $\bar{\boldsymbol{C}}$ such that for any* PLBO $\mathcal{O}$ *with costfunctions $\boldsymbol{C}^{(i)}$ located within this neighborhood and for any $p \geq \bar{\boldsymbol{A}}_{\mathbf{1}}^{(\boldsymbol{i})}$ $(i \in \boldsymbol{I})$, the dmp–correspondence is nonempty, interval–valued, and* uhc *in p as well as in the data of $\mathcal{O}$.*

In other words, if the cost functions of the firms are similar, then the dmp–correspondence is nonempty provided the argument p is not too small. Similarly as in Remark 2.3, the dmp–correpondence "widens" in a sense with increasing p and this all the more with the curvature of the costfunction (which is similar for all of them) increases. Figure 2.2 provides the intuition: suppose that constants c_i $(i \in \boldsymbol{I})$ describe the various cost functions of players given as in Example 2.6. This amounts to a variation of the rescaling factors to be applied to Figure 2.2. As there is a minimum width of the correpondence to the right of $A_1 = 1$, a moderate rescaling will provide a nonempty intersection of *all* correpondences of the players.

Definition 2.9. *For any PLBO $\mathcal{O}$ the price $\bar{A}_1 := \max\{A_1^{(i)} \mid i \in \boldsymbol{I}\}$ is called the* ***max–min marginal price*** *of $\mathcal{O}$. We shall say that $\mathcal{O}$ is a PLBO with* ***similar firms*** *if there is a cost function $\bar{\boldsymbol{C}}$ and a neighborhood of this function such that the conclusions of Corollary 2.8 are satisfied. That is, for $p \geq \bar{A}_1$ the DMP correspondence satisfies*

$$\Xi(p) \neq \emptyset. \tag{2.22}$$

The above theorem is a *local* one. It calls for cost functions in a joint neighborhood. A *global* theorem can be constructed in the spirit of Theorem 2.5. We require that the relations between marginal costs and mean cost *globally* do not vary to much between the members of the oligopoly.

Definition 2.10. *We shall say that a PLBO $\mathcal{O}$ has* ***uniform MM increments****, if there is $K \in \mathbb{N}$ such that for $k, l \geq K$ and any pair of players $i, j \in \boldsymbol{I}$ the following two conditions are satisfied:*

1. $A^i_k \geq M^j_k$,
2. *If $\Delta^i_k \leq \Delta^j_l$ holds true, then $A^i_{l-1} \geq M_{k+1}$ follows.*

Theorem 2.11. *Let $\mathcal{O}$ be an PLBO with uniform MM increments. Then there is $K \in \mathbb{N}$ such that for $p > A^i_K$ $(i \in \boldsymbol{I})$ we obtain $\Xi(p) \neq \emptyset$.*

Proof: We prove that, for any pair $i, j \in \boldsymbol{I}$ and for sufficiently large p the relation $\xi^i_0 > \widehat{\xi}^j$ is satisfied. If so, then we see that

$$\underline{\xi}_0 \;:=\; \min_{i \in \boldsymbol{I}} \xi^i_0 \geq \max_{j \in \boldsymbol{I}} \widehat{\xi}^j \tag{2.23}$$

is satisfied. It follows at once that $\underline{\xi}_0 \in \Xi(p)$ holds true. The proof follows exactly the path led in the proof of Theorem 2.5.

Now fix i and j. Let $p > \max_{r \in \boldsymbol{I}} A^r_K$ and choose l such that $p \in (A^j_{l-1}, A^j_l]$ holds true. Then necessarily we obtain $l - 1 \geq K$. Also, we know that $\widehat{\xi}^j(p) = \Delta^j_l$ holds true (from (2.7), cf. the corresponding step in Theorem 2.5). Next, choose k such that

$$M^i_k \leq A^j_{l-1} < M^i_{k+1} \tag{2.24}$$

is true. From condition *1.* above (which in this context is an assumption and in Theorem 2.5 was a result) we know that $l - 1 < k + 1$, hence $k \geq K$. Similarly as in (2.15) we obtain

$$\xi^i_0(A^j_{l-1}) \geq \Delta^i_k.$$

Finally, the (reverse formulation of) condition *2.* implies

$$\xi^i_0(p) - \widehat{\xi}^j(p) \geq \xi^i_0(A^j_{l-1}) - \Delta^j_l \geq \Delta^i_k - \Delta^j_l > 0, \tag{2.25}$$

q.e.d.

3 An Existence Theorem

We start out with some auxiliary theorems.

Theorem 3.1. *Let* $\Theta : [0, p_0] \to \mathcal{P}([0, d_0])$ *be an* uhc *and convex valued correpondence and let* $F : [0, p_0] \to [0, d_0]$ *be a continuous function. Assume that* $\Theta(p) \neq \emptyset \quad (p \geq \alpha)$ *holds true for some* $\alpha \in (0, p_0)$*. Now, if*

$$\Theta(\alpha) \cap [0, F(\alpha)] \neq \emptyset \tag{3.1}$$

and

$$\Theta(p_0) \cap [F(p_0), d_0] \neq \emptyset \tag{3.2}$$

holds true, then there exists $\bar{p} \in [0, p_0]$ *with* $F(\bar{p}) \in \Theta(\bar{p})$*.*

Proof: This is an obvious generalization of the intermediate value theorem. It can be proved by the same procedure or by a suitable application of the Kakutani fixed point theorem.

q.e.d.

Definition 3.2. *Let* $\mathcal{O}$ *be an PLOB. Let* $\boldsymbol{\alpha} = \bar{\boldsymbol{A}}_1$ *be the* max–min *marginal price and let* $\Xi = [\widehat{\xi}, \xi_0]$ *be the* dmp *correspondence. If*

$$\widehat{\xi}(\boldsymbol{\alpha}) \leq \frac{D(\boldsymbol{\alpha})}{n} \tag{3.3}$$

and

$$\xi_0(p_0) \geq \frac{D(p_0)}{n} \tag{3.4}$$

holds true, then we shall say that demand and supply are ***intersecting****. If* $\boldsymbol{\alpha}$ *is some other quantitiy, then we will use the definition accordingly.*

Corollary 3.3. *Let* $\mathcal{O}$ *be an PLOB with similar costfunctions (cf. Definition 2.9). Assume that demand and supply are intersecting (Definition 3.2). Then there exists* $\bar{p} \in [\boldsymbol{\alpha}, p_0]$ *satisfying* $\frac{D(\bar{p})}{n} \in \Xi(\bar{p})$*.*

Proof: Put $\Theta := \Xi$ and $F := \frac{D}{n}$. Then apply Theorem 3.1.

q.e.d.

Thus, we require that at the *max–min* marginal price the total production (averaged out in a sense) is not sufficient to satisfy the demand and that, on the other hand, at the maximal price the demand is below of the possibilities of total production. If so, then there is a price at which per capita demand is located within the interval of decreasing profits.

Combining Corollary 3.3 and Theorem 1.6 we obtain

Theorem 3.4. *Let $\mathcal{O}$ be an PLOB with similar costfunctions (Definition 2.9). Assume that demand and supply are intersecting (Definition 3.2). Then there exists a Bertrand equilibrium. Within a certain neighborhood, the Bertrand equilibrium correspondence is* uhc *in the data of* $\mathcal{O}$.

The global versions are obtain in a quite similar fashion. However, with respect to Definition 3.2, the role of $\boldsymbol{\alpha}$ has to be changed.

Corollary 3.5. *Let $\mathcal{O}$ be an PLOB with uniform MM increments. (cf. Definition 2.10). Let K be defined accordingly and let $\boldsymbol{\alpha} = \max_{i \in \boldsymbol{I}} A^i_K$. Assume that demand and supply are intersecting. Then there exists $\bar{p} \in [\boldsymbol{\alpha}, p_0]$ satisfying $\frac{D(\bar{p})}{n} \in \Xi(\bar{p})$.*

Proof: In view of Theorem 2.11 we know that the dmp correspondence is nonempty for $p > \alpha$. Therefore we can again apply Theorem 3.1 and obtain the analogous result.

q.e.d.

Theorem 3.6. *Let $\mathcal{O}$ be an PLOB with uniform MM increments (Definition 2.10). Let $\boldsymbol{\alpha}$ be as in Corollary 3.5 and assume that demand and supply are intersecting. Then there exists a Bertrand equilibrium. Within a certain neighborhood, the Bertrand equilibrium correspondence is* uhc *in the data of* $\mathcal{O}$.

Remark 3.7. *The framework of the model can be relaxed with respect to the uniform domain of definition required in Definition 1.1. It is sufficient to require that the costfunctions are mappings*

$$C^i \; : \; [0, \bar{d}^i] \to \mathbb{R} \quad (i \in \boldsymbol{I}).$$

Thus, firms may have varying capacities. The role of p_0 can be played by any real number satisfying $p_0 \geq \max\{C^i(\bar{d}^i) | i \in \boldsymbol{I}\}$. The property of similarity can at once be formulated in this framework (and leads to capacity boundaries that are close to each other in a well defined sense). The intersecting property has to be slightly reformulated, e.g., w.r.t. $d_0 \; := \; \min\{\bar{d}^i | i \in \boldsymbol{I}\}$.

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