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Welfare Foundations of Discounting

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Abstract

We investigate whether temporal preferences expressed as a sum of discounted consumption utilities can be derived from a welfare representations in the form of a sum of discounted total utilities. We find that a consumption-based representation in the usual exponential form corresponds to one-period "altruism" towards one's future selves: the current self gives positive weight to her total utility in the next period, and weight zero to her total utility in all subsequent periods. We also find that a consumption-based representation in the quasi-exponential (β, δ)-form suggested by Phelps and Pollak (1968) and Laibson (1997) correspond to quasi-exponential altruism towards one's future selves. For $\beta = 1/2$, the welfare weights are exponential, while for $\beta < 1/2$ they are biased in favor of the current self, and for $\beta > 1/2$ in favor of one's future selves. More generally, we establish a functional equation which relates welfare weights to consumption-utility weights. We also postulate five desiderata for consumption-utility weights. None of the usual formalizations satisfy all desiderata, but we propose a simple formalization which does.

JEL codes: D11, D64, D91, E21.

Keywords: Altruism, discounting, dynamic inconsistency, time inconsistency, welfare.

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1 Introduction

Time preferences have come into the foreground again in the economics literature, this time in macroeconomics. This recent strand of the literature is focused on non-exponential or "hyperbolic" preferences, which may lead to dynamic consistency problems, see for example Laibson (1997), Barro (1999), Krusell and Smith (1999) and Harris and Laibson (2001). This phenomenon was noted already by Ramsey (1928) and analyzed by Strotz (1956), Pollak (1968), Phelps and Pollak (1968), and Peleg and Yaari (1973). In all these studies, preferences are represented by utility functions in the form of a sum of discounted instantaneous utilities from consumption. We here analyze the question whether such preferences are consistent with the assumption of forward-looking agents who care about their future total utility, not only about their future instantaneous utility from consumption.

In order to high-light the dynamic consistency issue, consider a decision-maker who is to choose a sequence $x = (x_0, x_1, x_2, ...)$ of consumption vectors x_t to be consumed at dates $t = 0, 1, 2, ...$ In Phelps and Pollak (1968) and Laibson (1997), the consumer's preferences at any decision time τ are represented by a utility function of the form

$$U_\tau (x) = u (x_\tau) + \beta \sum_{t=1}^{\infty} \delta^t u (x_{\tau+t}) \ , \tag{1}$$

where $\beta > 0$ and $0 < \delta < 1$. The term $u (x_t)$ is interpreted as the instantaneous utility of consumption in period t. Dynamic inconsistency may arise if $\beta \neq 1$: a sequence $x^* = (x_0^*, x_1^*, x_2^*, ...)$ which maximizes U_0 need not maximize U_τ for some $\tau > 0$, given the "history" $x_0^*, x_1^*, x_2^*, ..., x_{\tau-1}^*$ preceding period τ. The reason is simple: once a future period τ has become the present, the rate of substitution between the instantaneous utilities in periods $\tau + 1$ and τ has changed from its original value δ to the new value $\beta\delta$. We will call discount functions in the above (β, δ)-form *quasi-exponential*, with exponential discounting as the special case $\beta = 1$.

In the cited studies, the function U_τ is decision theoretic in the usual sense of revealed preferences: it determines the actual choice made by the consumer in period τ (with due regard to the presence or absence of commitment possibilities). From a normative viewpoint, $U_\tau (x)$ represents the *welfare* of the individual in period τ: the higher this function value is, the "better off" is the individual in that period. Current welfare (or "total utility"), so defined, does not stem only from current consumption but also from (the anticipation of) the stream of future consumption. But, by assumption, this is true for the welfare in all future periods as well. In particular, the welfare in a future decision period $\tau' > \tau$ will in part depend on (anticipation in period τ' of) consumption in periods $t > \tau'$. However, formula (1) does not explicitly account for future welfare. For example, a marginal increase in consumption two periods ahead from some decision period τ by an infinitesimal amount $\varepsilon > 0$ will add $\beta\delta^2\varepsilon u'(x_{\tau+2})$ to current welfare, but it will also add $\beta\delta\varepsilon u'(x_{\tau+2})$ to welfare in the next period - an effect not explicitly accounted for in equation (1).

We argue that a rational and forward-looking decision maker should respect the preferences of his or her future selves. In particular, if also future selves are forward-looking, then this should not be neglected by the current self.[1] Alternatively phrased: a rational decision maker who cares about his or her own welfare in future periods should strive to maximize some increasing function of her welfare in those periods. By contrast, an individual who in each period strives to maximize U_τ, as defined in equation (1), appears to suffer from *second-order myopia*: she cares today about her future instantaneous utility from consumption, but not about her future total utility (which also includes caring about her future total utility etc.).[2] Does this matter for the resulting behavior? Or are preferences of the form (1) behaviorally equivalent with preferences that explicitly care about one's future welfare? We here investigate this question - whether consumption-based preferences have a welfare-theoretic foundation in this sense.

In particular, we find that a consumption-based representation (1) in the "classical" exponential form, that is with $\beta = 1$, corresponds to one-period altruism towards one's future selves: the individual attaches weight δ to her welfare in the next period and weight zero to all later periods (but her next self attaches weight δ to the welfare two periods ahead, etc. in an infinite chain). Such preferences are sometimes assumed in intergenerational (dynastic) macroeconomic models, see for example Barro (1974) and Barro and Becker (1988). We also find that consumption-utility based representations (1) in the quasi-exponential form, that is with $\beta < 1$, correspond to quasi-exponential altruism towards one's future selves. The case $\beta = 1/2$ plays a special role. For such quasi-exponential consumption-utility weights, the welfare weights are in fact exponential; such individuals attach exponentially declining weight to their welfare in all future periods. For $\beta < 1/2$, the welfare weights are quasi-exponential with a bias in favor of the current self ("myopia"), while for $\beta > 1/2$ the welfare weights are biased in favor of one's future selves ("longsightedness").

We also find that exponential welfare weights attached to the next two periods - and weight zero to all future periods - yield consumption-utility weights that are based on the so-called Fibonacci sequence, and we show that these consumption weights need not decrease monotonically over time. Indeed, such an individual may attach more weight to his consumption two periods ahead than to the consumption next period. We also show, by way of examples, that certain consumption-based preferences imply "spite" rather than

[1] This point was made already by Zeckhauser and Fels (1968), see below.

[2] A decision maker could be said to be *first-order myopic* if she does not even care about her future instantaneous utility from consumption, that is, if $\beta\delta = 0$ in eq. (1).

"altruism" towards one's future selves, i.e., a preference for lower future welfare in certain periods. For example, if the parameter β in equation (1) would instead apply to periods $t = 2, 3, \ldots$, rather than to periods $t = 1, 2, \ldots$, then the associates welfare weight two periods ahead would be negative (and this would be repeated each decision period).

We are not the first to search for a welfare foundation of the consumption-based formulation in equation (1). Already Zeckhauser and Fels (1968) raised the issue. They showed that the welfare-based formulation behind the quasi-exponential consumption-based representation (1) is also quasi-exponential. They also claimed that the boundary case $\beta = \delta = 1$ ("perfect altruism" as considered by Ramsey (1928)) has no welfare-based counterpart.[3] As indicated above, and shown below, this claim is not entirely correct. There does exist a welfare representation, though not of the quasi-exponential form, namely the above-mentioned one-period altruism which attaches welfare weight 1 to the next period and zero to all other periods.

The present investigation may also have some bearing on a related modelling issue in macroeconomics, namely whether it matters, in models of sequences of altruistic generations, if each generation cares about the next generation's total utility, consumption utility, consumption or wealth. Since the latter two cases are analytically considerably simpler than the first, and therefore more commonly used, our analysis might help identify circumstances under which the two latter models are behaviorally equivalent with the first model.

The remainder of the paper is organized as follows. Section 2 provides the model and establishes a one-to-one relationship between welfare weights and consumption-utility weights. Section 3 analyzes a few examples from the literature, and section 4 postulates some desiderata for discounting functions. None of the usual formulations satisfy all desiderata, but in section 5 we propose a (three-dimensional) parametric family of discount functions which meet the desiderata. Section 6 suggest how forward-looking discounting models can be generalized to include memory of past consumption. Mathematical proofs are collected in an appendix at the end of the paper.

2 The model

Consider an infinitely lived individual who makes decisions over a sequence of periods $t \in \mathbb{N} = \{0, 1, 2, \ldots\}$. In each period t, the individual consumes some vector $x_t \in X$, where $X \subset \mathbb{R}^n$ is a set of consumption alternatives and $n \in \mathbb{N}_+ = \{1, 2, \ldots\}$. A *consumption*

[3] "With $\delta = 1$, equation (3) [our equation (1)] is meaningless. This rules out perfect altruism [consumption-based altruism with $\delta = 1$] in a forward-looking model that relates altruistic preferences to total utilities rather than felicities [instantaneous utilities]. Here, total altruism [welfare-based altruism] and perfect altruism are incompatible with each other." (p. 4, Zeckhauser and Fels, 1968)

stream x is an infinite sequence of consumption vectors x_t, and we write $x = (x_0, x_1, ...) \in X^\infty$. Let $\succcurlyeq_\tau$be the preferences of the decision maker in period τ over consumption streams $x \in X^\infty$. A *preference profile* $\succcurlyeq$ for the individual is a sequence $(\succcurlyeq_\tau)_{\tau\in\mathbb{N}}$ of preferences, one for each "self τ".

We here study preference profiles that can be represented by stationary and additively separable utility functions of the type used in the macroeconomics literature. More exactly, we focus on preference profiles $\langle\succcurlyeq_\tau\rangle_{\tau\in\mathbb{N}}$ for which there exists functions $U_\tau : X^\infty \to \mathbb{R}$, one for each decision period $\tau \in \mathbb{N}$, such that $x \succcurlyeq_\tau y$ if and only if $U_\tau(x) \geq U_\tau(y)$, where

$$U_\tau(x) = f(0)\, u(x_\tau) + \sum_{t=1}^{\infty} f(t) u(x_{\tau+t}) \tag{2}$$

for some for some $u : X \to \mathbb{R}_+$ and $f : \mathbb{N} \to \mathbb{R}_+$ with $f(0) = 1$. Here $u(x_s)$ will be called the *instantaneous (sub)utility* from consumption in period s, and $f(t)$ the *weight* that the decision maker assigns to her instantaneous consumption utility t periods later.

We will say that a sequence $\langle U_\tau\rangle_{\tau\in\mathbb{N}}$ of such utility functions admits a (stationary and additively separable) *welfare representation* if for all $\tau \in \mathbb{N}$ and $x \in X^\infty$,

$$U_\tau(x) = u(x_\tau) + \sum_{t=1}^{\infty} f^*(t)\, U_{\tau+t}(x) \ , \tag{3}$$

for some $f^* : \mathbb{N}_+ \to \mathbb{R}$. Here $f^*(t)$ is the weight that the decision maker places on her welfare (total utility) t periods later.

A negative weight attached to another individual's welfare expresses "spite" rather than "altruism." Such welfare weights appear pathological in the present context.[4] We will hence call a welfare-weight function f^* *regular* if it is nonnegative. In this case we will say that $\langle U_\tau\rangle_{\tau\in\mathbb{N}}$ admits a *regular* welfare representation.

2.1 The functional equation

This study originated with the following question: Does the sequence $\langle U_\tau\rangle_{\tau\in\mathbb{N}}$ defined in equation (1) admit a regular welfare representation? If so, which?[5]

A key result for answering this and related questions is the observation that every sequence $\langle U_\tau\rangle_{\tau\in\mathbb{N}}$ in the more general form (2) admits a welfare representation of the form (3), and, moreover, $f^* : \mathbb{N}_+ \to \mathbb{R}$ is uniquely determined by the following system of recursive equations:

[4] We do not deny that the excluded possibility may sometimes be psychologically relevant, but it appears not to be typical for consumers.

[5] We were then ignorant of Zeckhauser and Fels (1968).

$$f^*(1) = f(1) \quad \text{and} \quad f^*(t) = f(t) - \sum_{s=1}^{t-1} f(t-s) f^*(s) \quad \forall t > 1. \tag{4}$$

Proposition 1: *If $\langle U_\tau \rangle$ satisfies equation (2) for some $u : X \to \mathbb{R}$* and *$f : \mathbb{N} \to \mathbb{R}$, then $\langle U_\tau \rangle$ admits the welfare representation (3), where $f^* : \mathbb{N}_+ \to \mathbb{R}$ is the unique solution to (4).*

(See Appendix for a proof.)

Conversely, the consumption-weight function f may be obtained from the welfare-weight function f^* via equation (4), since this equation implies that, for all positive integers t,

$$f(t) = f^*(t) + \sum_{s=1}^{t-1} f^*(t-s) f(s), \tag{5}$$

a recursive equation system which uniquely determines f from f^*, given the initial value $f(0) = 1$. This equation states that the consumption-utility weight $f(t)$ can be computed as the sum of that period's instantaneous utility's contributions to the decision maker's welfare in all interim periods: $f(1) = f^*(1)$, $f(2) = f^*(2) + [f^*(1)]^2$, $f(3) = f^*(3) + 2f^*(2)f^*(1) + [f^*(1)]^3$ etc.

It is immediate from equation (5) that if f^* is non-negative, so is f. However, Proposition 1 does not claim that the welfare representation necessarily be regular, even if f is non-negative. Indeed, the welfare-weight function f^* may well take negative values although all consumption weights are positive. To see this, note that (4) gives $f^*(1) = f(1)$, $f^*(2) = f(2) - f^2(1)$, $f^*(3) = f(3) - 2f(1)f(2) + f^3(1)$ etc. Hence, in order for the welfare weight $f^*(2)$ to be negative it suffices that $f(2) < f^2(1)$. This is the case, for example, if $f(1) = \delta$ and $f(2) = \beta\delta^2$ for some $\beta < 1$. Another example is $f(t) = 1/(0.5 + t)$ for $t = 1, 2$ - since then $f^2(1) = (2.25)^{-1} > f(2) = (2.5)^{-1}$. A third counter-example is $f(t) = 1/(1 + t^2)$, yielding $f^2(1) = 1/4 > f(2) = 1/5$. In all three cases, the decision maker is constantly spiteful to his future self two periods ahead.

A sufficient condition for all welfare weights to be non-negative, and hence for the welfare representation to be regular, is that all consumption weights are positive and that the ratio between successive consumption weights - the intertemporal substitution rate between instantaneous utilities from consumption - be non-decreasing over time. This result is due to Ulf Persson (private communication, see Appendix for a proof):

Proposition 2: Suppose *$f : \mathbb{N} \to \mathbb{R}_{++}$ and let $q : \mathbb{N}_+ \to \mathbb{R}$ be defined by $q(t) = f(t)/f(t-1)$. If q is non-decreasing, then $f^* \geq 0$. If q is strictly increasing, then $f^* > 0$.*

With $f, f^* \geq 0$, we clearly have $0 \leq f^*(t) \leq f(t)$ for all positive integers, by (4). If, moreover, $f(t) \to 0$ as $t \to \infty$, then so does $f^*(t)$.

In the following section we analyze examples of consumption-based and welfare-based discount functions.

3 Examples

3.1 Exponential consumption weights

Suppose the consumption weights decline exponentially: $f(t) = \delta^t$ for all t, for some $\delta \in (0,1)$. This is the standard case in macroeconomic modelling, corresponding to the special case $\beta = 1$ in equation (1). It is not difficult to verify that equation (4) then gives $f^*(1) = \delta$ and $f^*(t) = 0$ for all integers $t > 1$.

To see this, first note that equation (4) gives $f^*(1) = \delta$ and $f^*(2) = 0$. Suppose that $f^*(1) = \delta$ and $f^*(s) = 0$ for all $s = 1, 2, .., t-1$. Then (4) gives

$$f^*(t) = \delta^t - \sum_{s=1}^{t-1} \delta^{t-s} f^*(s) = \delta^t - \delta^{t-1}\delta = 0 \ .$$

Hence, by induction this holds for all positive integers t. Note that this derivation also applies to the boundary case $\delta = 1$. Hence, "perfect altruism" in this sense (Ramsey, 1928) is behaviorally equivalent with one-period welfare-based altruism, where next period's welfare is given weight 1 and the welfare in all future periods are given weight zero.

Conversely, suppose that the decision maker cares only about her utility from current consumption and her welfare in the next period. Then $f^*(1) = \alpha$, for some $\alpha > 0$, and $f^*(t) = 0$ for all integers $t > 1$. An application of equation (5) immediately gives $f(t) = \alpha^t$ for all t. Hence, the reduced form (2) for such an individual is indeed exponential:

$$U_\tau(x) = \sum_{t=0}^{\infty} \alpha^t u(x_{\tau+t}) \ , \tag{6}$$

where the discount factor equals the weight that the decision maker attaches to his or her welfare in the next period.

In sum: exponential consumption weights have a regular welfare foundation. Zero weight is given to the welfare in all future periods except the next.

3.2 Exponential welfare weights

Suppose instead that it is the welfare weights $f^*(t)$ that decrease exponentially over future periods t. What are then the associated consumption weights? More exactly, suppose that

$f^*(t) = \alpha^t$ for some $\alpha \in (0,1)$ and for all $t \in \mathbb{N}_+$. Equation (5) gives $f(1) = \alpha$, $f(2) = 2\alpha^2$, and $f(3) = 4\alpha^3$. One may thus conjecture that

$$f(t) = \frac{1}{2}(2\alpha)^t \qquad \forall t > 0. \tag{7}$$

This conjecture is easily proved to be true induction, see appendix. Substituting (7) in (2) we thus obtain

$$U_\tau(x) = u(x_\tau) + \frac{1}{2}\sum_{t=1}^{\infty} \delta^t u(x_{\tau+t}) \ , \tag{8}$$

for $\delta = 2\alpha$. Hence, exponential welfare weights assigned to all future periods imply the Phelps-Pollak-Laibson reduced form (1) with $\beta = 1/2$.

Note that in the special case when $\alpha = 1/2$, $\delta = 1$ and thus $f(t) = 1/2$ for all positive integers t. Hence, in this case the same weight is given to the instantaneous consumption utility in all time periods. This special case is relevant from a biological viewpoint, since the genetic kinship between any pair of successive generations is precisely $1/2$.

3.3 Quasi-exponential consumption-utility weights

We found that exponential welfare weights imply quasi-exponential consumption weights (β, δ) with $\beta = 1/2$. What welfare weights correspond to quasi-exponential consumption weights (β, δ) when $\beta \neq 1/2$?

Suppose, thus, that $f(0) = 1$ and $f(t) = \beta\delta^t$ for all positive integers, where $\beta, \delta \in (0,1)$. Then $f^*(1) = \beta\delta$ and $f^*(2) = \beta(1-\beta)\delta^2$. It is not hard to prove by induction that

$$f^*(t) = \frac{\beta}{1-\beta}\left[(1-\beta)\delta\right]^t \qquad \forall t \in \mathbb{N}_+ \tag{9}$$

(see appendix). In other words: a representation in the Phelps-Pollak-Laibson form (1) is the reduced form of a welfare representation (3) in the quasi-exponential form

$$U_\tau(x) = u(x_\tau) + \beta^* \sum_{t=1}^{\infty} (\delta^*)^t U_{\tau+t}(x) \ , \tag{10}$$

where

$$\beta^* = \beta/(1-\beta) \qquad \text{and} \qquad \delta^* = (1-\beta)\delta. \tag{11}$$

Quasi-exponential consumption weights thus do have a regular welfare foundation, namely quasi-exponential welfare weights, and *vice versa*, and these weights are related as in equation (11). As mentioned above, this result was obtained in Zeckhauser and Fels (1968, eq. (4)). We note that the "welfare myopia" factor β^* is an increasing function of the “consumption myopia” factor β, such that β^* reaches the value 1 - hence exponential welfare weights - precisely when β reaches $1/2$, an observation that is consistent with our earlier finding in the case of exponential welfare weights. At $\beta = 1/2$, welfare weights switch from being biased toward “myopia” to being biased toward “longsightedness.”

Laibson et al (2001) made the following estimate of the parameter pair (β, δ) in the Phelps-Pollak-Laibson model, based on annual US data: $\beta = 0.55$ and $\delta = 0.96$. The associated welfare representation is thus slightly biased toward “longsightedness”: $\beta^* = 1.22$ and $\delta^* = 0.43$. In other words, individuals place relatively more weight on their future welfare, in comparison with exponential weights: $f^*(1) = \beta^*\delta^* = 0.52$, $f^*(2) = \beta^*(\delta^*)^2 = 0.23$ etc., see Figure 1 below.

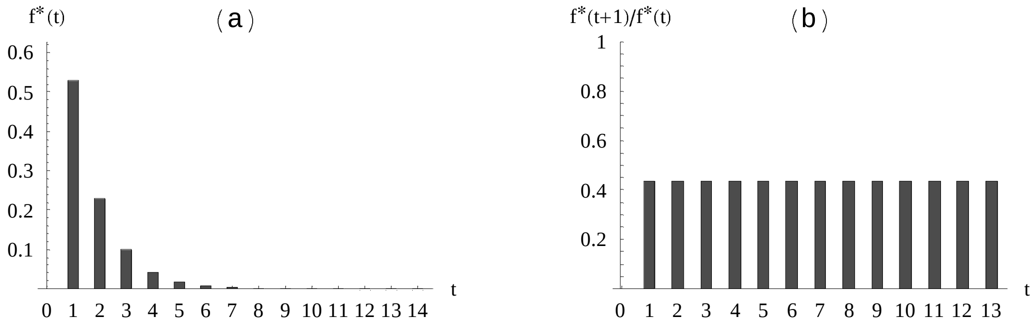

Figure 1: (a) Altruistic weights, $f^*(t)$, corresponding to quasi-exponential consumption weights (β, δ), for $\beta = 0.55$ and $\delta = 0.96$. (b) The ratio $f^*(t+1)/f^*(t)$ for the same parameters.

3.4 Finite-horizon exponential welfare weights

We next consider the intermediate cases between one-period altruism and infinite-horizon exponential altruism, namely when the welfare weight decreases exponentially over a finite number of time periods, beyond which all weights are zero. What is the corresponding reduced form (2)? More exactly, suppose $f^*(t) = \alpha^t$ for some $\alpha \in (0, 1)$ and for all positive integers $t \leq T < +\infty$, and suppose $f^*(t) = 0$ for all integers $t > T$. It is then easily verified that f can be written as

$$f(t) = m_T(t)\alpha^t, \tag{12}$$

where

$$m_T(t) = \sum_{s=1}^{\min\{t,T\}} m_T(t-s) \tag{13}$$

for all positive integers t, and $m_T(0) = 1$ (see appendix). It follows from (13) that, for any finite horizon T,

$$1 \leq m_T(t) \leq m_{T+1}(t) \leq 2^{t-1} \tag{14}$$

and

$$\alpha^t \leq f_T(t) \leq f_{T+1}(t) \leq \frac{1}{2}(2\alpha)^t \tag{15}$$

for all t. Hence, the longer the altruism horizon T is, the higher is the weight given to each future instantaneous utility term. Moreover, it follows from a well-known result for recursive equations that the ratio between the m-weights assigned to two consecutive periods t and $t+1$ converges as t goes to infinity (see e.g. Weisstein, 1999):

$$\lim_{t \to \infty} \frac{m_T(t+1)}{m_T(t)} = \lambda_T \ , \tag{16}$$

where λ_T is the unique solution $\lambda > 1$ of $\lambda = 2 - \lambda^{-T}$. Notice that λ_T is increasing in T and $\lim_{T\to\infty} \lambda_T = 2$.

In particular the sequence $m_2(t)$ is the Fibonacci sequence, 1, 1, 2, 3, 5, 8 etc. (each term being the sum of the two preceding terms). The ratio between successive Fibonacci numbers is known to converge to the so-called golden number (Kelley and Peterson, 1991):

$$\frac{m_2(t+1)}{m_2(t)} \to \lambda_2 = \frac{1+\sqrt{5}}{2}.$$

Note also that the induced weight function f, need not be monotonic. In fact, for all $T \geq 2$ and $\alpha > 1/2$: $f(1) < f(2) < f(0)$. Figure 2 illustrates this feature for $T = 2$ and $\alpha = 0.55$.[6]

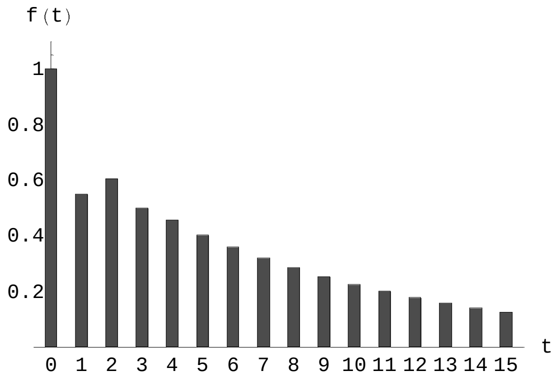

Figure 2: Consumption weights, $f(t)$, generated from two-period-horizon exponential altruism, with $\alpha = 0.55$.

[6]The number $f(0) = 1$ has been inserted for the sake of mathematical completeness when solving the recursive equation (5).

3.5 Hyperbolic consumption-utility weights

Empirical work suggests that the consumption weights $f(t)$ be hyperbolic, rather than exponential, in t. Hence, Ainslie (1992) suggests $f(t) = (\lambda + \alpha t)^{-1}$ for some $\alpha, \lambda > 0$ (op. cit. eq. (3.7)). A similar hyperbolic expression, $(1 + \alpha t)^{-\beta/\alpha}$ is suggested by Loewenstein and Prelec (1992).

As noted above, for certain α and λ, the first form may correspond to negative welfare weights - in which case the representation does not have a *regular* welfare representation. In particular, such hyperbolic preferences express spite against oneself two periods ahead (after each current period) if and only if $\lambda + 2\alpha > (\lambda + \alpha)^2$. In Figure 3 below, this is the area below the curve.[7]

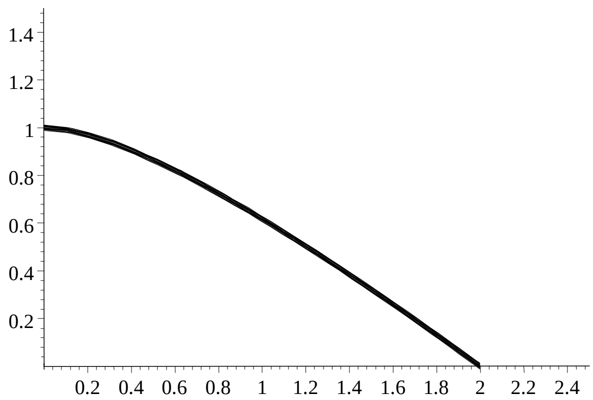

Figure 3: Points below the curve are parameter combinations (α, λ) for which the welfare weight $f^*(2)$ is negative.

We also note that this implies that the conditions in proposition 2 are violated if $\lambda + 2\alpha > (\lambda + \alpha)^2$. In fact, under the reversed inequality, and the assumption that $f(0) = 1$, all welfare weights are non-negative. To see this, let $f(0) = 1$ and $f(t) = (\lambda + \alpha t)^{-1}$ for all positive t. Then $q(t) \geq q(t-1)$ for all $t \geq 1$ if and only if $q(2) \geq q(1)$, which is equivalent to $\lambda + 2\alpha \leq (\lambda + \alpha)^2$.

Without any loss of generality, we relabel the parameters in the second, and study

$$f(t) = (1 + at)^{-b} \qquad \forall t \in \mathbb{N}, \tag{17}$$

for $a, b > 0$. It follows from proposition 2 that the corresponding welfare-weight function f^* is everywhere positive, since $f > 0$ and the ratio between successive consumption-utility weights is strictly increasing over time:

$$q(t) = \frac{f(t)}{f(t-1)} = \left[1 - \frac{a}{1 + at}\right]^b .$$

[7] Recall that $f^*(2) = f(2) - f^2(1)$. Also note that $f^*(1) = f(1) \leq 1$ iff $\lambda + \alpha \geq 1$.

We do not have an explicit formula for f^*, though. Instead, using equation (4) we have generated the welfare weights $f^*(t)$, for $t = 1, 2, ..., 50$, corresponding to the consumption weights $f(t) = (1 + at)^{-b}$, for different combinations of a and b, and fitted the function

$$\tilde{f}(t) = \frac{\theta}{(1 - \alpha + \alpha t)^{\beta}} \tag{18}$$

to the data. (Note that $\tilde{f}(1) = \theta$.)

Figure 4 below shows the welfare weights $f^*(t)$ (dots) obtained from equation (5), for $a = 100$ and $b = 1$, the estimated function-values $\tilde{f}(t)$, as well as the ratio $\tilde{f}(t)/f^*(t)$.

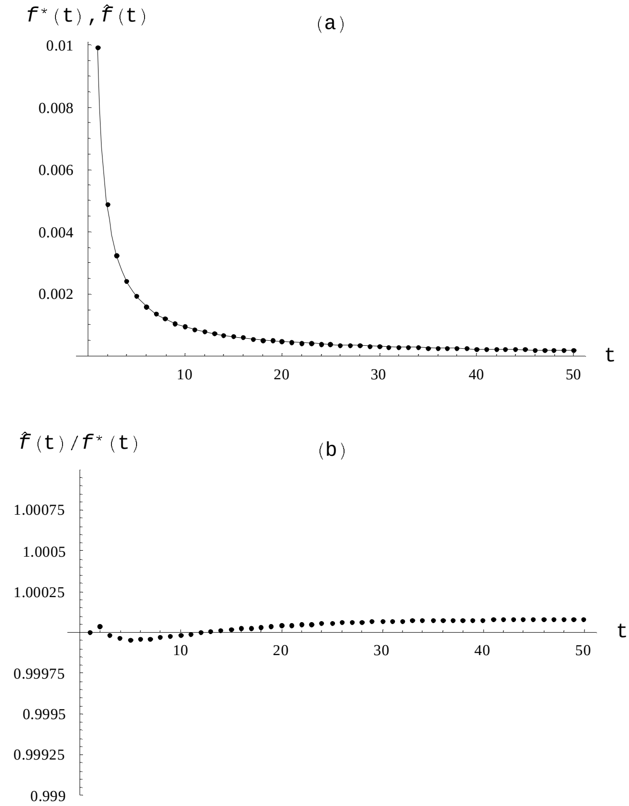

Figure 4: (a) Altruistic weights $f^*(t)$ obtained from equation $f(t) = (1 + 100t)^{-1}$, t=1,2,..50 (dots) and the estimated function $\tilde{f}(t)$ (solid line). (b) The ratio $\tilde{f}(t)/f^*(t)$, for t=1,2,..50.

Table 1 reports the estimates α, β and θ, as well as the maximum absolute and relative errors in the first 50 periods. We note that, for fixed a, α is decreasing and β increasing in b. Moreover, $\alpha \approx 1$ and $\beta \approx b$ when a is large.

Table 1: Estimates of f^* based on computer simulations.

a	b	θ	α	β	$\max_{t\leq 50}\left\Vert \tilde{f}(t)\text{-}f^*(t)\right\Vert$	$\max_{t\leq 50}\left\Vert \tilde{f}(t)/f^*(t)\text{-}1\right\Vert$
1	1	0.49999	3.02432	1.28221	0.00079	0.09922
1	2	0.24999	1.74474	1.61539	0.00047	0.44643
1	5	0.03125	0.78373	3.97018	0.00001	4.31526
10	1	0.09091	1.07430	1.14852	0.00003	0.01001
10	2	0.00826	0.95289	1.97737	$9.9 \cdot 10^{-7}$	0.02664
10	5	$6.2 \cdot 10^{-6}$	0.90933	4.99927	$4.3 \cdot 10^{-7}$	0.00149
100	1	0.00990	1.00327	1.01909	$1.9 \cdot 10^{-7}$	0.00008
100	2	0.00009	0.99069	1.99968	$1.7 \cdot 10^{-10}$	0.00041
100	5	$9.5 \cdot 10^{-11}$	0.99009	4.99999	$1.1 \cdot 10^{-22}$	$2.9 \cdot 10^{-8}$

4 Desiderata for stationary discount functions

Having examined circumstances under which utility functions U_τ in the consumption-based form (2) have a regular welfare foundation in the form (3), we now turn to a discussion of some more desiderata for consumption-based discount functions f.

Our first desideratum is that the representation (2) should be invariant with respect to periodization, in the sense that there should exist a continuous-time discount function from which the discrete-time consumption discount factor in each period t can be derived for any given period length $\Delta > 0$. The Phelps-Pollack-Laibson model (1), to be referred to as the *PPL model*, is unclear in this respect, since it states that discounting kicks in from period 1 on, without specifying for what lengths Δ of the time period this should hold, or, more generally, how the parameters β and δ should be adjusted if the time period is changed. In exponential discounting models one usually assumes $\delta = \exp(-r\Delta)$ for some real-time discount rate r, but what about β?

Secondly, empirical work suggests that the considered class of discount functions should contain some form of hyperbolic discounting as a special case. As mentioned above, hyperbolic discounting of instantaneous utilities of consumption has been shown to fit the data better than exponential consumption discounting. It therefore seems desirable that the model contain such hyperbolic consumption discounting as a special case. Clearly the quasi-exponential PPL model does not meet this second desideratum - it only *approximates* hyperbolic discounting over the first few periods.

Third, exponential discounting has traditionally been the main approach in economics, and should therefore be a special case of the model. The PPL model clearly meets this desideratum (just set $\beta = 1$ in equation (1)).

If a random variable T is exponentially distributed, then its conditional probability distribution, given $T \geq t$, is identical to the original, for any t. It is precisely this time homogeneity property that guarantees dynamic consistency. As a weaker requirement, in the present context of discount functions, our fourth desideratum is that the class of discount functions considered should be "closed under truncation" in the sense that the normalized consumption discount factors, from any given future date on should belong to the class. When currently contemplating a future decision point, in a dynamic decision problem, it should not be necessary to step outside the model. The quasi-exponential (β, δ) model evidently satisfies this desideratum: the decision maker's models of his future selves are exponential.

Finally, the model, given in the consumption form (2), should have a regular welfare foundation. We saw above that the PPL model (1) satisfies this last desideratum.

Formally, we consider stationary preferences over infinite consumption streams x represented in the form

$$U_\tau(x) = \sum_{t=0}^{\infty} \varphi(t, \Delta) u(x_{\tau+t}), \tag{19}$$

where $\varphi(t, \Delta)$ is the discount factor that the decision maker in period $\tau \in \mathbb{N}$ assigns to his or her instantaneous utility of consumption in period $\tau + t$, if the length of each period is $\Delta > 0$.

Let F be any family of functions $\varphi : \mathbb{N} \times \mathbb{R}_+ \to [0, 1]$ such that for all $\Delta > 0$, $\varphi(0, \Delta) = 1$ and $\varphi(t, \Delta)$ is non-increasing in t. Our desiderata are:

Desideratum 1 *(invariance w.r.t. periodization): There exists a function $f : \mathbb{R}_+ \to [0, 1]$ such that $\varphi(t, \Delta) = f(t\Delta)$ for all $t \in \mathbb{N}$ and $\Delta > 0$.*

Desideratum 2 *(hyperbolic discounting allowed): Every function φ of the form $\varphi(t, \Delta) = (1 + \alpha t\Delta)^{-\beta}$ for some $\alpha > 0$ and $\beta > 1$, belongs to F.*

Desideratum 3 *(exponential discounting allowed): Every function φ of the form $\varphi(t, \Delta) = \exp(-\gamma t\Delta)$ for some $\gamma > 0$, belongs to F.*

Desideratum 4 *(algebraic closure under truncation): If $\varphi \in F$, then also $\varphi_\tau \in F$ for any $\tau \in \mathbb{N}_+$, where $\varphi_\tau : \mathbb{N} \times \mathbb{R}_+ \to [0, 1]$ is defined by*

$$\varphi_\tau(t, \Delta) = \frac{\varphi(\tau + t, \Delta)}{\varphi(\tau, \Delta)} \qquad \forall t \in \mathbb{N}.$$

Desideratum 5 *(regular welfare foundation): If $\varphi \in F$ and $f : \mathbb{N} \to [0, 1]$ is defined by $f(t) = \varphi(t, 1)$ for all t, then the associated welfare weights $f^*(t)$ are all non-negative.*

5 Hyperbolic-exponential discount functions

One family F which meets all five desiderata are the functions φ of the form

$$\varphi(t, \Delta) = (1 + at\Delta)^{-b} \exp(-ct\Delta) \tag{20}$$

for some $a, b, c > 0$. This family F is three-dimensional, the minimal parametric dimensionality for the PPL model to hold across different time discretization. Hence, we have not added any real degree of freedom above and beyond that of the PPL model.

It is not difficult to see that all five desiderata indeed hold. Desideratum 1 is given by construction. Also Desiderata 2 and 3 are self-evident; one obtains exponential discounting by setting $b = 0$, and hyperbolic discounting by setting $c = 0$. That desideratum 4 holds follows from

$$\varphi_\tau(t, \Delta) = (1 + a' t\Delta)^{-b} \exp(-ct\Delta),$$

where $a' = a/(1 + \Delta\tau a) > 0$. In other words, $\varphi_\tau \in F$. Note that the parameters b and c are unaffected by such truncation of the past, while the parameter a changes. Dynamic inconsistency in time preferences arises from the single fact that this parameter decreases with the number τ of past periods, for any fixed period length Δ. Desideratum 5, finally, also holds. To see this, note that we have

$$f(t) = (1 + at)^{-b} \exp(-ct),$$

which implies that all welfare weights $f^*(t)$ are positive, see proposition 2.

In order to compare this family of discount functions with the PPL model, we consider the successive ratios between discount values. Recall that in the PPL model these rates are $q(1) = f(1)/f(0) = \beta\delta$ and $q(t) = f(t)/f(t-1) = \delta$ for all positive integers t, while the corresponding ratios for discount functions f from the family F are

$$q(t) = \frac{f(t)}{f(t-1)} = \left(1 - \frac{a}{1 + at}\right)^b \delta,$$

where $\delta = e^{-c}$. In particular, $q(1) = \beta\delta$, for $\beta = (1 + a)^{-b}$, and $q(t) \to \delta$ as $t \to \infty$, just as in the PPL model. See Figure 5 below, plotted for the values $\beta = 0.55$ and $\delta = 0.96$, with a and b such that $(1 + a)^{-b} = \beta$. As a increases, the curve point-wise approaches the PPL-values.

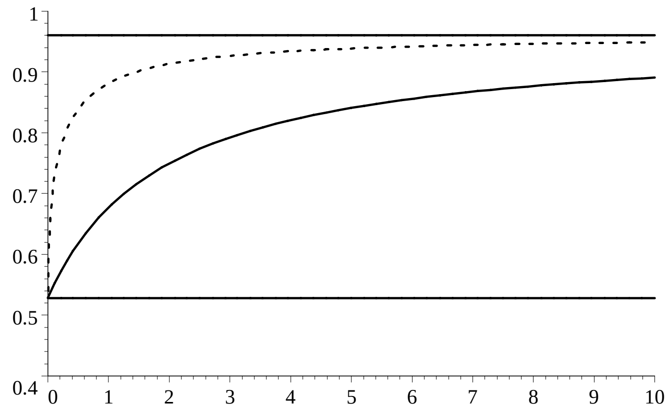

Figure 5: The ratios $q(t)$, for $a = 1$ (solid curve) and 125 (dotted curve), where $b = -(\ln\beta)/\ln(1+a)$, and $c = -\ln\delta$, for $\beta = 0.55$ and $\delta = 0.96$.

As a final remark, we note that the present family of discount functions seems to be sufficiently rich to fit a wide range of empirical observations. Frederick, Loewenstein and O'Donoghue (2001) report empirical estimates of discount rates from no less than 40 studies (Table 2, *op. cit.*) Their general finding is that the average discount rate over longer time intervals is lower than the average discount rate over shorter time intervals. Figure 6 below is their Figure 1, with the addition of the dotted curve. The points are their data points, and the solid curve has been fitted by them, while the dotted curve has been fitted by us, from a discount function in the present family F.[8] This fitting was made by way of "eye econometrics," resulting in the following estimates: $a = 10$, $b = 0.3$ and $c = 0$.

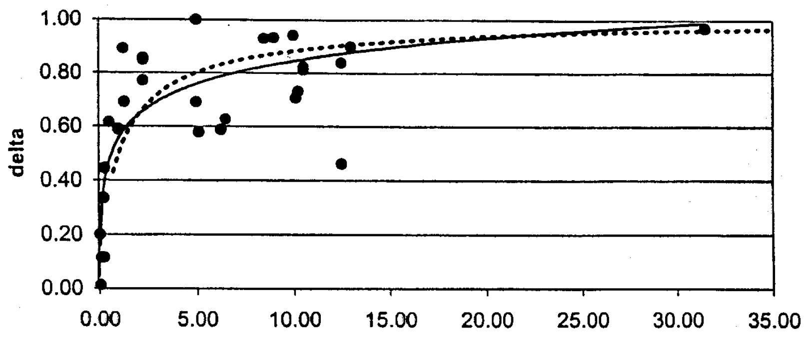

Figure 6: Fitting a discount function f (dotted curve) from the family F to the data in Figure 1 of Frederick, Loewenstein and O'Donoghue (2001).

[8]The dotted curve is the graph of $y(t) = [f(t)]^{1/t} = (1+at)^{-b/t}e^{-c}$, for $a = 10$, $b = 0.3$ and $c = 0$. Note that $\lim_{t\to 0} y(t) = \exp[-(ab+c)]$ and.$\lim_{t\to+\infty} y(t) = \exp(-c)$.

Figures 7 and 8 compare the consumption weights $f(t)$ and the welfare weights $f^*(t)$ corresponding to our estimate, $f(t) = (1+10t)^{-0.3}$ (gray bars), with the Laibson et al (2001) estimate (black bars in figure 7) and with exponential discounting with an annual discount rate of 5% (black bars in figure 8). The latter is the estimate of Cooley and Prescott (1995).[9]

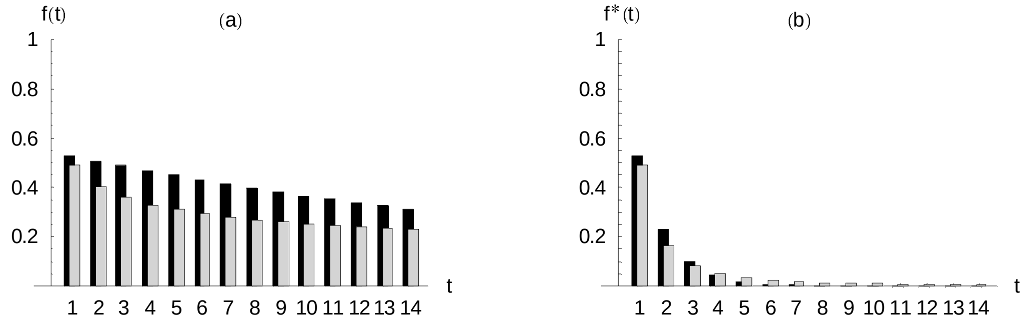

Figure 7: (a) $f(t) = (1+10t)^{-0.3}$ (black) and quasi-exponential consumption weights (β, δ), for $\beta = 0.55$ and $\delta = 0.96$ (gray).(b) The corresponding welfare weights $f^*(t)$

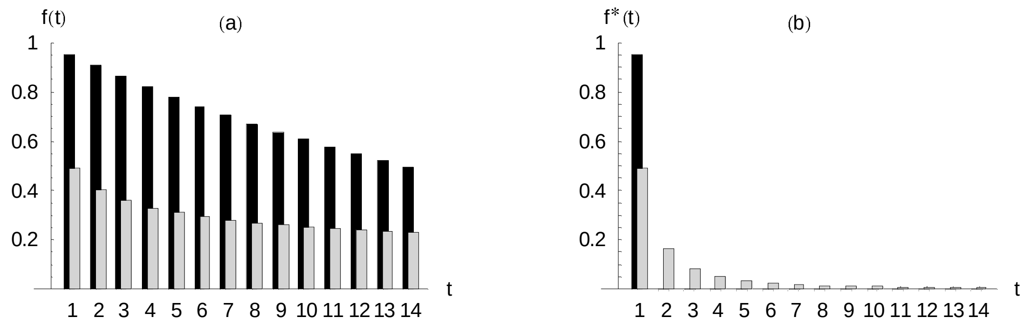

Figure 8: (a) $f(t) = (1+10t)^{-0.3}$ (black) and $f(t) = e^{-0.05t}$ (gray).(b) The corresponding welfare weights $f^*(t)$.

6 Extension

The class of models studied here contains as special cases the traditional exponential discounting as well as the quasi-exponential discounting models which are currently under investigation in the macroeconomics literature (see e.g. Laibson (1997), Barro (1999), Krusell and Smith (1999) and Harris and Laibson (2001)). However, from the viewpoint

[9] To be more precise, they give the estimate 0.987 of the quarterly discount factor.

of psychology and experimental economics, all these models are quite special, as is apparent from reading, for example, Frederick, Loewenstein and O'Donoghue (2001) and Kahneman (2000). Given the accumulated knowledge of discounting models, it might be wise to proceed step-wise when generalizing these, by adding only one new element at a time, so as better understand what assumption leads to what conclusion. At each such step, the task then is to change little and yet add a lot in terms of richer conclusions. The mentioned recent work on quasi-exponential models can be viewed as a successful such step.

We feel that a major aspect of intertemporal preferences and decision making which is missing from current economics models is memory. At first sight, one might argue that although our memories certainly do affect our well-being, this is irrelevant for decision making, since the memory of the decision maker is fixed and given. However, if the decision maker is forward-looking, and cares about his or her future well-being, then memory is relevant, since a decision today may influence future memories. Can the forward-looking models considered here be generalized in an operational way so as to include memory? We believe they can.

Consider a decision maker who is born in period 0 and lives through a sequence of time periods. In each period τ, let his or her memory be a function of the "history" $h_\tau = (x_0, x_1, ..., x_{\tau-1})$ preceding that period. Assume that the preferences $\succcurlyeq_\tau$ in each decision period τ are represented by a utility function $U_\tau : X^\infty \to \mathbb{R}$ of the following form:

$$U_\tau(x) = f(0)\, v(x_\tau, h_\tau) + \sum_{t=1}^{\infty} f(t) v(x_{\tau+t}, h_{\tau+t})$$

for some $v : X \times H \to \mathbb{R}$ and $f : \mathbb{N} \to \mathbb{R}$, where $H = \{h_0\} \cup_{t \in \mathbb{N}_+} X^t$ and h_0 is the "null" history at birth. Here $v(x_\tau, h_\tau)$ may still be interpreted as the *instantaneous utility* in period τ, arising from current consumption and, now, also from current memory. Formally, also this generalized model falls into the category of models covered by the above analysis, so one may speak of welfare foundations and other desiderata in precisely the manner done above.

A key concern, however, is to find an operational and behaviorally justifiable functional form for v. In the light of Kahneman (2000) it seems desirable to allow v to account for the "peak" and "end" effects, that is, v should depend on the maximum value of pleasure or pain in the history and on the most recent elements of that history. Roughly speaking, the idea is that one's best and worst meals play a prominent role in one's memory. One formalization which captures this is:

$$\begin{aligned} v(x_\tau, h_\tau) &= (1-\mu)\,u(x_\tau) + \mu\left[\alpha \max_{t<\tau} u(x_t) + \beta \min_{t<\tau} u(x_t)\right] \\ &\quad + \mu(1-\alpha-\beta)\,\frac{1-(1-\nu)^\tau}{\nu} \sum_{t=0}^{\tau-1} (1-\nu)^{\tau-t-1}\, u(x_t)\ , \end{aligned}$$

where $u : X \to \mathbb{R}_+$ as before represents the instantaneous utility from current consumption, and where $\alpha, \beta, \gamma, \mu, \nu \in [0,1]$. Here μ represents the importance of memory in comparison with current consumption, and α and β the relative importance for memory of the maximal and minimal experienced consumption utility, respectively, and the last term is the exponentially discounted mean value of past consumption utilities, where ν is the emphasis on the most recent past (only the most recent period matters when $\nu = 1$).

We leave these investigations for future research.

7 Appendix

7.1 Proof of proposition 1

Suppose $\langle U_\tau \rangle$ satisfies equation (2) for some $u : X \to \mathbb{R}$ and $f : \mathbb{N} \to \mathbb{R}$ with $f(0) = 1$. Let $f^* : \mathbb{N}_+ \to \mathbb{R}$ be defined by (4). Then

$$f(t) = \sum_{s=1}^{t} f^*(s) f(t-s) \qquad \forall t \in \mathbb{N}_+\ .$$

Hence,

$$\begin{aligned} U_\tau(x) &= u(x_\tau) + \sum_{t=1}^{\infty} \sum_{s=1}^{t} f^*(s) f(t-s) u(x_{\tau+t}) = \\ &= u(x_\tau) + \sum_{s=1}^{\infty} f^*(s) \left[\sum_{t=s}^{\infty} f(t-s) u(x_{\tau+t})\right] \\ &= u(x_\tau) + \sum_{s=1}^{\infty} f^*(s) \left[\sum_{k=0}^{\infty} f(k) u(x_{\tau+s+k})\right] = u(x_\tau) + \sum_{s=1}^{\infty} f^*(s) U_{\tau+s}(x) \end{aligned}$$

Since this holds for all τ, this proves the claim.

7.2 Proof of proposition 2[10]

Suppose first that q is non-decreasing. We know that $f^*(1) = f(1) > 0$. Suppose $f^*(s) \geq 0 \quad \forall s < t$. Then

$$\begin{aligned}
f^*(t) &= f(t) - f(1)f^*(t-1) - \sum_{s=1}^{t-2} f^*(s)f(t-s) \\
&= q(t)f(t-1) - f(1)f^*(t-1) - \sum_{s=1}^{t-2} q(t-s)f^*(s)f(t-s-1) \\
&\geq q(t)\left[f(t-1) - \sum_{s=1}^{t-2} f^*(s)f(t-s-1)\right] - f(1)f^*(t-1) \\
&= q(t)f^*(t-1) - f(1)f^*(t-1) = [q(t) - f(1)]\, f^*(t-1) \geq 0.
\end{aligned}$$

The last inequality follows from the assumption that q is non decreasing and $f(1) = q(1)$.

Secondly, suppose that q is strictly increasing. Suppose $f^*(s) > 0 \; \forall s < t$. The same reasoning as above then leads to $f^*(t) > [q(t) - q(1)]\, f^*(t-1) > 0$.

7.3 Proof of equation (7)

Suppose $f(s) = 2^{s-1}\alpha^s$ for $s = 1, 2, .., t$, for some positive integer t. Then (5) gives

$$f(t+1) = \alpha^{t+1} + \sum_{s=1}^{t} 2^{s-1}\alpha^s\alpha^{t+1-s} = \alpha^{t+1}\left[1 + \sum_{s=1}^{t} 2^{s-1}\right] \tag{21}$$

$$= \alpha^{t+1}\left[1 + (2^t - 1)\right] = 2^t\alpha^{t+1}. \tag{22}$$

By induction in t, this establishes (7).

7.4 Proof of equation (9)

Equation (9) may be established by induction over t, as follows. First note that $f(1) = f^*(1)$. Suppose that equation (9) holds for all $s < t$ for some t. Equation (5) then gives

[10] Due to Ulf Persson, Department of Mathematics, Chalmers University of Technology (Gothenburg, Sweden).

$$
\begin{aligned}
f(t) &= f^*(t) + \sum_{s=1}^{t-1} f^*(s) f(t-s) \\
&= \beta^* \left(\delta^*\right)^t + \sum_{s=1}^{t-1} \beta^* \left(\delta^*\right)^s \left(\delta^*/\delta\right)^s \beta\delta^t \\
&= \beta^* \left(\delta^*\right)^t + \beta\delta^t \left[\sum_{s=1}^{t-1} \beta^* (1-\beta)^s\right] \\
&= \beta^* \left(\delta^*\right)^t + \beta\delta^t \left[1 - (1-\beta)^{t-1}\right] \\
&= \beta\delta^t
\end{aligned}
$$

7.5 Proof of equation (13)

Under equation (13), we have

$$
\begin{aligned}
f(t) &= \alpha^t \sum_{s=1}^{\min\{t,T\}} m(t-s) = \alpha^t \sum_{s=1}^{\min\{t,T\}} \alpha^{s-t} f(t-s) \\
&= \sum_{s=1}^{\min\{t,T\}} \alpha^s f(t-s).
\end{aligned}
$$

Setting $f^*(s) = \alpha^s$ for all positive integers $s \leq T$ and otherwise $f^*(s) = 0$, we obtain (5).

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