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INSTITUTT FOR FORETAKSØKONOMI

DEPARTMENT OF FINANCE AND MANAGEMENT SCIENCE

FOR 4 2009

ISSN: 1500-4066

MAY 2009

Discussion paper

A Model of Deferred Callability in Defaultable Debt

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Abstract. Banks and other financial institutions raise hybrid capital as part of their risk capital. Hybrid capital has no maturity, but, similarily to most corporate debt, includes an embedded issuer's call option. To obtain acceptance as risk capital, the first possible exercise date of the embedded call is contractually deferred by several years, generating a *protection period*. The existence of this call feature affects the issuer's optimal bankruptcy decision, in addition to the value of debt. We value the call feature as a European option on perpetual defaultable debt. We do this by first modifying the underlying asset process to incorporate a time dependent bankruptcy level before the expiration of the embedded option. We identify a call option on debt as a fixed number of put options using a modified exercise price on a modified asset, which is lognormally distributed, as opposed to the market value of debt. To include the possibility of default before the expiration of the option we apply barrier options results. The formulas are quite general and may be used for valuing both embedded and third-party options. All formulas are developed in the seminal and standard Black-Scholes-Merton model and, thus, standard analytical tools such as 'the greeks', are immediately available.

1. Introduction

Banks and other financial institutions issue risk capital to meet regulatory capital requirements, reflecting the risk of their business. This risk capital may, in addition to equity, include subordinated debt, and some intermediary instruments, commonly denoted *hybrid capital.* The purpose of such risk capital is to protect customers/depositors by reducing the issuer's probability of default and bankruptcy. Similar to debt, hybrid capital has a fixed principal and coupon, and priority before equity. Also, similar to most corporate bonds, hybrid capital is issued with an embedded issuer's call option. To qualify as risk capital, however, hybrid capital cannot have any fixed expiration date, and has to absorb losses without the right to declare bankruptcy, see Committee on Banking Supervision (1988). To obtain acceptance as risk

Date: This version: April 28, 2009.

Key words and phrases. callable perpetual debt, barrier options, JEL classifications G12, G13, G33.

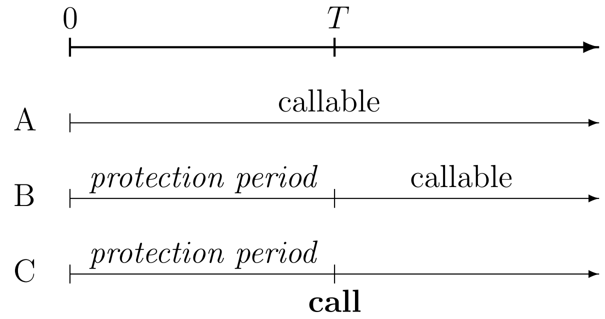

FIGURE 1. Illustrative call structures in debt contracts.

capital, the first possible exercise date of the embedded call is contractually deferred by several years. We refer to this deferral period as the *protection period*.

Regular corporate debt usually includes callability for issuer from issue until maturity, see case A in Figure 1. This call may be exercised by the issuer at any coupon date and is thus of *Bermudian* type, although commonly modeled as an *American* option. Such callability has been analyzed extensively in the literature, see, e.g., Ingersoll (1977a), Ingersoll (1977b), Brennan and Schwartz (1977), Acharya and Carpenter (2002).

As mentioned above, in the case of hybrid capital the first possible call date is contractually deferred for the instrument to qualify as risk capital, as illustrated by case B in Figure 1. As far as we know, no debt-like instruments with a protection period has been studied in the literature before.

For parsimony we simplify the structure further and disregard the conventional callability also after time T, as illustrated in case C in Figure 1. Although hybrid capital contracts typically include callability after the protection period, such contracts also include incentives for exercising the call at the first possible date. In addition, market practice[1] indicates that these calls actually are exercised at the first exercise date. Disregarding the callability feature after time T is therefore not likely to have a major impact on the market value of this contract, and we believe that our model yields reasonable values.

The scope of this paper is to provide valuation formulas for embedded options including default risk in the underlying defaultable perpetual

[1]Due to the extraordinary market conditions in 2008/2009, however, there are some examples of contracts which have not been called at the first possible date.

debt before the exercise date. We use defaultable perpetual continuous coupon paying Black and Cox (1976)-debt as the underlying asset. Infinite horizon corresponds to the contractual horizon of hybrid capital. We could alternatively include callability in the underlying asset, but, as argued above, we do not expect this property to have a significant valuation impact on the option. The exercise date of the option coincides with the end-date of the protection period, and is given in the contract, and reflects relevant risk capital regulation. The option is naturally of European type. We account for the time-varying influence of the embedded option on the issuer's bankruptcy decision during the protection period by allowing the issuer's bankruptcy barrier to be time dependent during the protection period. The economic explanation is that the value of a finitely lived option changes as it approaches its maturity (this time effect is called the option's *theta* using financial jargon). For analytical tractability we assume an exponential structure of the bankruptcy barrier. The parameters of the bankruptcy barrier are exogenous in our formulas. The *optimal* bankruptcy barrier is a somewhat subtle topic, which, e.g., requires a full specification of the issuer's capital structure and, thus, is outside the scope of this paper. Note that our formulas are valid for any choice of input parameters, also the parameters representing the optimal barrier.

For a given capital structure our formulas allow for calculation of the optimal bankruptcy barrier. In this paper we develop the option formulas without imposing assumptions regarding a company's capital structure. Applications, including such assumptions, are addressed in another paper, Mjøs and Persson (2005).

The fundamental source of uncertainty in our model is an EBIT (earnings before interest and taxes) process which determines the value of firm assets. The underlying instrument of the option is classified as debt, whose value then is determined by the value of the firm assets. By assumption the EBIT process is lognormal, causing the firm asset value to be lognormal as well. However, the market value of debt is not lognormal, and debt may not be used as the underlying asset in the standard Black-Scholes-Merton option formulas. We show how one call option on defaultable perpetual debt must be valued as a fixed number of put options on a modified (solves time dependency), different ('reinstates' lognormality) asset in order to fit into the standard model. Analogously, one put option on debt must be valued as a fixed number of call options on the modified asset. Furthermore, results from barrier options are used to include default risk before the expiration of the option similar to the approach by Toft and Prucyk (1997).

We assume a constant riskfree interest rate. When applied to, e.g., hybrid capital, our formulas may be used to calculate coupon rates including bankruptcy risk. Market practise indicates that issuers typically pay a fixed credit margin on top of a market reference rate, and may thus hedge their exposure to the nominal interest rate. Subtracting the riskfree rate from the coupon rate yields a credit spread. Even though this spread is based on a constant interest rate model, and disregards potential correlation effects between the company's EBIT process and more general interest rate dynamics, it may serve as a benchmark.

We develop the option pricing formulas sequentially: We first derive formulas for European plain vanilla options on defaultable perpetual debt, disregarding default-risk before expiration. Secondly, comparable to Johnson and Stulz (1987) and Jarrow and Turnbull (1995), we acknowledge the bankruptcy risk of the issuer of the option and, implicitly, of the underlying security at time T by including a bankruptcy asset barrier. Such options are denoted *vulnerable options* in the literature. Finally, we also include bankruptcy risk before time T by including a time-dependent bankruptcy barrier B_t before time T, using a barrier option approach following Björk (2004). These formulas for embedded options allow us to study the effect of the protection period on prices and bankruptcy decisions when the counterparty risk of the option and the default risk in the underlying security are inseparable.

This paper is organized as follows: Section 2 presents the model and basic results. In Section 3 we present the valuation formulas for different European options. Section 4 contains some numerical examples. Finally, Section 5 concludes.

2. The model and basic results

We consider the standard Black and Scholes (1973) and Merton (1973) economy and impose the usual perfect market assumptions:

- All assets are infinitely separable and continuously tradeable.
- No taxes, transaction cost, bankruptcy costs, agency costs or short-sale restrictions. All agents have costless and immediate access to all information.
- There exists a known constant riskless rate of return r.

2.1. **The EBIT-based market value process.** We analyze options on debt issued by a limited liability company with financial assets. In line with Goldstein, Ju, and Leland (2001), we assume that the assets generate an EBIT (earnings before interest and tax) cashflow denoted δ_t given by the stochastic differential equation

$$d\delta_t = \mu\delta_t dt + \sigma\delta_t dW_t, \tag{1}$$

where μ and σ are constants representing the drift and volatility parameters respectively, and δ_0 is the fixed initial cashflow level. Here W_t is a standard Brownian motion under a fixed equivalent martingale measure. The total time t market value $\hat{A}_t$ of the assumed perpetual

EBIT stream from the assets equals

$$\hat{A}_t = E_t^Q \left[\int_t^\infty e^{-r(s-t)} \delta_s ds \right]$$

$$= \frac{\delta_t}{r - \mu} \tag{2}$$

This formula is some places known as *Gordon's formula* (where the discount rate naturally equals the riskfree interest rate using a risk neutral set-up). The market value of this EBIT stream is the solution to the stochastic differential equation

$$d\hat{A}_t = (r\hat{A}_t - \delta_t)dt + \sigma \hat{A}_t dW_t$$

$$= \mu \hat{A}_t dt + \sigma \hat{A}_t dW_t. \tag{3}$$

The quantity $\hat{A}$ is elsewhere in the literature referred to as *the unlevered value of the firm's assets.*

In this setting there is a level of $\hat{A}_t$ where it is optimal for the company to stop paying debt coupons and declare bankruptcy. In the classic case this level is independent of time, i.e., constant.

2.2. **The standard Black and Cox (1976) results.** The time 0 market value of defaultable perpetual debt with continuous constant coupon payment is

$$D(A) = \frac{cD}{r} - (\frac{cD}{r} - \bar{A})(\frac{A}{\bar{A}})^{-\beta}, \tag{4}$$

where c is the constant coupon rate, D is the par value of the debt-claim and cD is the continuous coupon payment rate. The term $(\frac{A}{\bar{A}})^{-\beta}$ may be interpreted as the current market value of one monetary unit paid upon bankruptcy, i.e., when the process $\hat{A}_t$ hits the bankruptcy level $\bar{A}$. Here β solves the ordinary partial differential equation

$$\frac{1}{2}\sigma^2 \beta(\beta + 1) - \beta\mu - r = 0, \tag{5}$$

and is given by

$$\beta = \frac{\mu - \frac{1}{2}\sigma^2 + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2} > 0. \tag{6}$$

Expression (4) for the market value of debt carries a nice intuition. Observe that $\frac{cD}{r}$ is the current market value of perpetual default-free debt. Upon bankruptcy the debtholder looses infinite stream of coupon payments which at the time of bankruptcy have market value $\frac{cD}{r}$. On the other hand the debtholder, under the absolute priority rule, receives the remaining assets with a value equal to $\bar{A}$. We can therefore interpret $(\frac{cD}{r} - \bar{A})$ as the debtholder's *net loss* upon bankruptcy. The time 0 market value of this net loss, $(\frac{cD}{r} - \bar{A})(\frac{A}{\bar{A}})^{-\beta}$, therefore represents the reduction of the time 0 total market value of debt due to default risk. The default risk is the only source of risk for debt in this model.

The value of equity as the residual claim on the assets is determined by

$$E(A) = A - D(A) = A - \frac{cD}{r} + (\frac{cD}{r} - \bar{A})(\frac{A}{\bar{A}})^{-\beta} \tag{7}$$

In the classic case of a constant bankruptcy level, Black and Cox (1976) determine the *optimal* bankruptcy level for a given capital structure (E, D) from the perspective of the equityholders (found by differentiating expression (7) with respect to $\bar{A}$) as

$$\bar{A}^* = \frac{\beta}{\beta + 1} \frac{cD}{r}, \tag{8}$$

where the star $(^*)$ indicates optimality.

2.3. **The exogenous time dependent bankruptcy barrier and the modified market value process.** Due to the finite horizon of the embedded option, the bankruptcy level of the issuer depends on remaining time to maturity of the option. In order to include this aspect we assume a time-dependent exponential bankruptcy asset level B_t,

$$B_t = Be^{\gamma t},$$

for a given time 0 level B and a constant γ. The time of bankruptcy is given by the stopping time τ defined as

$$\tau = \inf\{t \geq 0, \hat{A}_t = B_t\}$$

where $\hat{A}_t$ is given in expression (2).

By modifying the asset process this stopping time can equivalently be expressed as

$$\tau = \inf\{t \geq 0, A_t = B\},$$

where A_t is

$$dA_t = (\mu - \gamma)A_t dt + \sigma A_t dW_t, \tag{9}$$

Compared to equation (2), the modified process has a drift adjustment of γ. Although γ determines the curvature of the time dependent exponential bankruptcy level, it can formally be interpreted as a constant dividend yield on A_t. Again formally, this transformation allows us to analyze the simpler setting of a constant bankruptcy level B, although no economic fundamentals are changed.

3. Option formulas for finite options on defaultable perpetual debt

We develop formulas for European options applying the standard approach from financial economics. We denote the maturity date of the options by T.

3.1. **The generalized debt dynamics.** We rewrite the market value of debt in terms of the modified process as

$$D_t = \frac{cD}{r} - JF_t, \tag{10}$$

where

$$F_t = (\frac{e^{\gamma t} A_t}{\bar{A}})^{-\beta}$$

and J represents the net loss upon bankruptcy. Furthermore, A_t is given in expression (9). Before we explain this expression, observe that the parameter choices $J = \frac{cD}{r} - \bar{A}$ and $t = 0$ yield the time zero value of standard Black and Cox (1976) debt, identical to expression (4).

To incorporate the time dependent bankruptcy level, as explained above, we work with a modified asset value process $\{A_t, t \geq 0\}$ in expression (9). Observe that $\hat{A}_T = e^{\gamma T} A_T$, thus the parameter γ allows us to express the time T *actual* option payoff in terms of the *modified* asset value process.

We study finitely lived European options embedded in perpetual debt contracts. The general time T payoff of such a *call* option with exercise price K is

$$(\frac{cD}{r} - JF_T - K)^+ 1\{e^{\gamma T} A_T > \bar{A}\} 1\{\tau > T\}.$$

The first factor represents the payoff $(D_T - K)^+$ of a plain vanilla option disregarding any default risk. The first indicator function cancels the time T payoff when $e^{\gamma T} A_T$ is less than the time T bankruptcy level $\bar{A}$. The second indicator function cancels the time T payoff upon earlier default, i.e., if $\inf_{0,T} A_t \leq B$. The similar payoff for a put option is

$$(K - \frac{cD}{r} + JF_T)^+ 1\{e^{\gamma T} A_T > \bar{A}\} 1\{\tau > T\}.$$

In some applications it is natural to require continuity of the bankruptcy barrier at time T, i.e., $B_T = Be^{\gamma T} = \bar{A}$. In this case the first indicator function in the two above expressions is redundant.

3.2. **Properties of D_t and F_t.** Applying Itô's lemma on expression (10) shows that

$$\frac{dD_t}{D_t} = (r - \frac{cD}{D_t})dt + \sigma\beta(\frac{cD}{rD_t} - 1)dW_t \tag{11}$$

which is not a geometric Brownian motion (the right-hand side depends on D_t), and is thus not lognormally distributed. Options on D_t can therefore not be valued directly using standard option pricing formulas.

By applying Itô's lemma on F_t using expressions (9) and (5) we get

$$dF_t = rF_t dt - \beta\sigma F_t dW_t, \tag{12}$$

which we recognize as a geometric Brownian motion. It has drift parameter r and volatility parameter $-\beta\sigma$. Furthermore, F_t is a function of A_t, and can therefore also be interpreted as a tradable asset.

The use of F_T as underlying asset allows us to use the standard option approach even in the case of options on debt and is, as such, fundamental to our results.

3.3. **European call and put options.** First we consider the 'plain vanilla' version of standard European put and call options. These call and put options have time T payoffs

$$(D_T - K)^+ = (\frac{cD}{r} - JF_T - K)^+ = J(X - F_T)^+,$$

and

$$(K - D_T)^+ = (K - \frac{cD}{r} + JF_T)^+ = J(F_T - X)^+$$

respectively, where the *modified exercise price* is

$$X = \frac{\frac{cD}{r} - K}{J}.$$

We have shown that the payoff from one call option on debt with exercise price K is equivalent to the payoff from J put options on F_T with a modified exercise price X. Similarly, the payoff from one put option on debt with exercise price K is equivalent to the payoff from J call options on F_T with a modified exercise price X.

From the above expression for the payoffs of the plain vanilla call and put options, we see that the value of A_T

$$\dot{A} = \theta e^{-\gamma T} \bar{A} \tag{13}$$

where

$$\theta = \left(\frac{J}{\frac{cD}{r} - K} \right)^{\frac{1}{\beta}}$$

produces payoffs of zero for both the plain vanilla put and call options. Note that K represents the exercise price relative to D_T, and $\dot{A}$ similarly can be interpreted as the exercise price relative to A_T, see Figure 2. The factor $e^{-\gamma T}$ scales $\bar{A}$ down to the modified A_t-process. If $\theta > 1$, the exercise price $\dot{A}$ is greater than the 'discounted' bankruptcy level $e^{-\gamma T}\bar{A}$. If $\theta < 1$, the exercise price $\dot{A}$ is less than the 'discounted' bankruptcy level.

In general our valuation formulas depend on

- four asset process parameters (μ, γ, σ, δ_0),
- four debt parameters (c, D, J, $\bar{A}$),
- three option parameters (K, T, B),

in addition to r, in total 12 parameters.

For notational simplicity we write the pricing formulas as functions of A and K only.

Proposition 1. *The time zero market prices of European plain vanilla call and put options on defaultable perpetual continuous coupon paying debt claims as described above are*

$$C_0^D(A,K) = JP_0^F(F_0,X) = (\frac{cD}{r} - K)e^{-rT}N(-d_2) - J(\frac{A}{\bar{A}})^{-\beta}N(-d_1), \tag{14}$$

and

$$P_0^D(A,K) = JC_0^F(F_0,X) = J(\frac{A}{\bar{A}})^{-\beta}N(d_1) - (\frac{cD}{r} - K)e^{-rT}N(d_2) \tag{15}$$

where

$$d_1 = \frac{\ln(\frac{\bar{A}}{A}) - \frac{1}{\beta}(\ln(\frac{cD}{r} - K) - \ln J) - (\mu - \frac{1}{2}\sigma^2 - \sigma^2\beta)T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\beta\sqrt{T}$$

and $A = \frac{\delta_0}{r-\mu}$.

Proof. We have shown how the payoff at maturity of one call [put] option on D_T equivalently can be seen as the payoff at maturity of J put [call] options on F_T with a modified exercise price. Under no-arbitrage assumptions the market value of one call [put] option must be equal to the market value of J put [call] options on F_T at any point in time before expiration. Options on F_T can immediately be calculated by the Black-Scholes-Merton formulas, using $F_0 = (\frac{A}{\bar{A}})^{-\beta}$ as the time 0 market value of the underlying asset, $|-\beta\sigma| = \beta\sigma$ as the volatility parameter[2], X as the exercise price, and expression (5). □

Compared to the payoffs from regular options, the payoffs at maturity T from options on defaultable perpetual debt are non-linear, not piecewise linear, functions of A_T. The payoffs at maturity T for plain vanilla options are illustrated in Figure 2.

These option pricing formulas do not take into account that the issuer of the underlying security may be bankrupt at time T, i.e., if A_T is below $e^{-\gamma T}\bar{A}$ or if A_t has hit B during the protection period, i.e., before time T. The formulas are still useful building-blocks in the following formulas which include both types of default risk.

[2]Option prices on assets with negative volatility, as F_t, are, in this setting, calculated by inserting the absolute value of the volatility parameter into the option pricing formula, see e.g., Aase (2004).

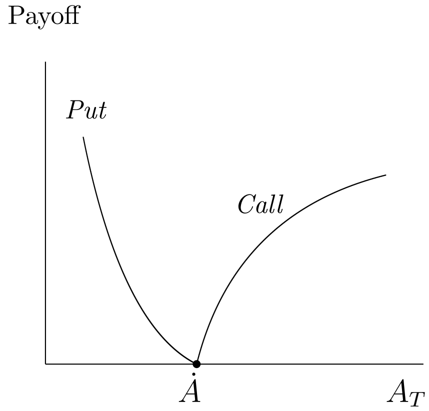

FIGURE 2. The payoff at maturity for plain vanilla put and call options on defaultable perpetual debt as functions of the time T market value of the firm A_T.

3.4. **European call and put options with time T default risk.** Denote the time T cash flow of a European *call* option on D_T with exercise price K and expiration at time T by $C_T^D(A_T, K)$. The option only has positive payoff if the issuer of the underlying security is *not* bankrupt at time T, i.e., if $A_T > e^{-\gamma T}\bar{A}$. Similarly to the plain vanilla case, the time T call option cashflow is

$$C_T^D(A_T, K) = (D_T - K)^+ 1\{A_T > e^{-\gamma T}\bar{A}\} \\ = J(X - F_T)^+ 1\{A_T > e^{-\gamma T}\bar{A}\} = JP_T^F(F_T, X). \tag{16}$$

The time T cash flow of a European *put* option on D_T with exercise price K and expiration at time T is

$$P_T^D(A_T, K) = (K - D_T)^+ 1\{A_T > e^{-\gamma T}\bar{A}\} = \\ = J(F_T - X)^+ 1\{A_T > e^{-\gamma T}\bar{A}\} = JC_T^F(F_T, X). \tag{17}$$

To develop option pricing formulas which reflect that the issuing company may be bankrupt at time T, it is useful to distinguish between the cases where $\theta > 1$ and $\theta < 1$, cf. equation (13).

Proposition 2. *In the case when $\theta > 1$ the time zero market prices of European call and put options on defaultable perpetual continuous coupon paying debt claims, with positive payoff only when $A_T > e^{-\gamma T}\bar{A}$, are*

(18)
$$C_0(A, K)^\theta = C_0^D(A, K) = (\frac{cD}{r} - K)e^{-rT}N(-d_2) - J(\frac{A}{\bar{A}})^{-\beta}N(-d_1),$$

where $C_0^D(A, K)$ is given in expression (14)*, and*

(19)

$$P_0(A,K)^\theta = P_0^D(A,K) - \left(J(\frac{A}{\bar{A}})^{-\beta} N(f_1) - (\frac{cD}{r} - K)e^{-rT}N(f_2) \right),$$

$$f_1 = \frac{\ln(\frac{\bar{A}}{A}) - (\mu - \frac{1}{2}\sigma^2 - \sigma^2\beta)T}{\sigma\sqrt{T}},$$

$$f_2 = f_1 - \sigma\beta\sqrt{T},$$

and $P_0^D(A,K)$ *is given in expression* (15).

Proof. In the case of the put option we must calculate the time 0 market value of the 'chopped' claim with the payoff

$$(K - D_T)^+ 1\{A_T > e^{-\gamma T}\bar{A}\}.$$

First observe that

$$(K - D_T)^+ 1\{A_T > e^{-\gamma T}\bar{A}\} =$$
$$(K - D_T)^+ - (K_1 - D_T)^+ - (K - K_1)1\{A_T \leq e^{-\gamma T}\bar{A}\},$$

i.e., as a difference between two plain vanilla put options from which a constant is subtracted for values of A_T less than $e^{-\gamma T}\bar{A}$. See Figure 4. Here K_1 is a modified exercise price calculated as follows: The second put option must have zero payoff for values of $A_T > e^{-\gamma T}\bar{A}$, and we therefore choose the exercise price, denoted by K_1, so that $\dot{A} = e^{-\gamma T}\bar{A}$. From expression (13) this is

$$K_1 = \frac{cD}{r} - J.$$

The constant $K - K_1$ represents the net difference in the payoff of a long position in the first and a short position in the second option for values of A_T less than $e^{-\gamma T}\bar{A}$. The above identity is then verified.

The market value of the above claim is easily calculated and the result given by the formula $P_0(A,K)^\theta$ above.

The call formula has a strictly positive payoff for values of $A_T > \dot{A}$. In this case $\theta > 1$, so $\dot{A} > e^{-\gamma T}\bar{A}$, thus the inclusion of time T default risk has no effect on the payoff, see Figure 3. □

Proposition 3. *In the case when* $\theta < 1$ *the time zero market prices of European call and put options with positive payoff only when* $A_T > \bar{A}e^{-\gamma T}$ *on defaultable perpetual continuous coupon paying debt claims are*

(20) $$C_0(A,K)_\theta = (\frac{cD}{r} - K)e^{-rT}N(-f_2) - J(\frac{A}{\bar{A}})^{-\beta}N(-f_1).$$

(21) $$P_0(A,K)_\theta = 0,$$

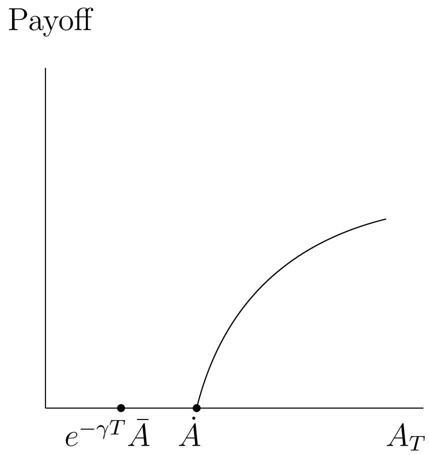

FIGURE 3. The payoff at maturity for a *call* option on defaultable perpetual debt when $e^{-\gamma T}\bar{A} < \dot{A}$ ($\theta > 1$), as a function of the firm value A_T. In this case the payoff is identical to a plain vanilla call option.

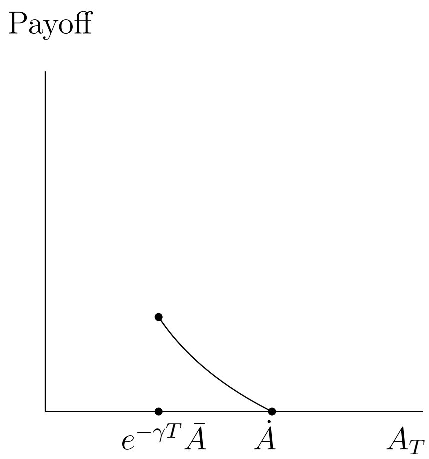

FIGURE 4. The payoff at maturity for a *put* option on defaultable perpetual debt when $e^{-\gamma T}\bar{A} < \dot{A}$ ($\theta > 1$), as a function of the firm value A_T. Compared to the plain vanilla put option this payoff is chopped for values of A_T below $e^{-\gamma T}\bar{A}$.

Proof. In the case when $\theta < 1$, $\dot{A} < e^{-\gamma T}\bar{A}$, so the chopped put option does not have positive payoff for any values of A_T.

The time T payoff of the chopped call option is

$$(D_T - K)^+ 1\{A_T > e^{-\gamma T}\bar{A}\}.$$

This can be written as

$$(D_T - K_1)^+ + (K_1 - K)1\{A_T > e^{-\gamma T}\bar{A}\},$$

where K_1 is given in the proof of Proposition 2. See also Figure 5.

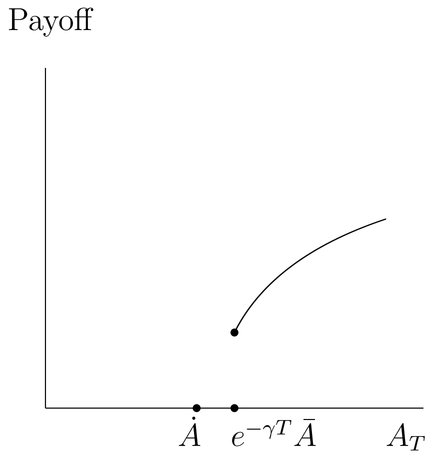

FIGURE 5. The payoff at maturity for a *call* option on defaultable perpetual debt when $e^{-\gamma T}\bar{A} > \dot{A}$ ($\theta < 1$), as a function of the firm value A_T. Compared to the plain vanilla call option this payoff is chopped for values of A_T below $e^{-\gamma T}\bar{A}$.

The market value of the above claim is easily calculated and is given by the formula $C_0(A, K)_\theta$ above. □

3.5. **Down-and-out barrier call and put options.** The previous section includes the possibility of default at the exercise time T. In this section we also include the possibility of an earlier default. We assume that the issuing company defaults if the market value process A_t drops below the constant B before time T.

We treat the following cases separately. We assume that $B < e^{-\gamma T}\bar{A}$.

- Case 1: $\theta > 1$, $B < e^{-\gamma T}\bar{A} < \dot{A}$.
- Case 2: $\theta < 1$, $B < \dot{A} < e^{-\gamma T}\bar{A}$ or $\dot{A} < B < e^{-\gamma T}\bar{A}$.

The time T cashflows of down-and-out barrier call and put options on defaultable perpetual debt with barrier B for the asset-process A_t and exercise price K are

(22)
$$C_T^{do}(A_T, K) = (\frac{cD}{r} - J(\frac{e^{\gamma T}A_T}{\bar{A}})^{-\beta} - K)^+ 1\{m_T^A > B\} 1\{A_T > e^{-\gamma T}\bar{A}\},$$

and

(23)
$$P_T^{do}(A_T, K) = (K - \frac{cD}{r} + J(\frac{e^{\gamma T}A_T}{\bar{A}})^{-\beta})^+ 1\{m_T^A > B\} 1\{A_T > e^{-\gamma T}\bar{A}\},$$

where $1\{\cdot\}$ represents the usual indicator function and the minimum function $m_T^A = \min\{A_t; 0 \leq t \leq T\}$.

The payoff at maturity from barrier options is not only dependent on the asset level A_T as plain vanilla options, but also on the two relevant bankruptcy barriers, B for $t < T$ and $e^{-\gamma T}\bar{A}$ for $t = T$.

3.6. **Case 1: Down-and-out barrier options when $\theta > 1$.**

Proposition 4. *The time zero market values of the down-and-out barrier call and put options on defaultable perpetual continuous coupon paying debt claims, with $B < e^{-\gamma T}\bar{A} < \dot{A}(\theta > 1)$ and exercise price K are, respectively*

$$C_1^{do}(A,K) = C_0^D(A,K) - (\frac{B}{A})^{(\frac{2(\mu-\gamma)}{\sigma^2}-1)} C_0^D(\frac{B^2}{A},K) \tag{24}$$

and

$$P_1^{do}(A,K) = P_0(A,K)^\theta - (\frac{B}{A})^{(\frac{2(\mu-\gamma)}{\sigma^2}-1)} P_0(\frac{B^2}{A},K)^\theta. \tag{25}$$

Proof. The results follow immediately from Theorem 18.8 in Björk (2004). □

3.7. **Case 2: Down-and-out barrier call and put options when $\theta < 1$.**

Proposition 5. *The time zero market values of down-and-out barrier call and put options on defaultable perpetual continuous coupon paying debt claims, with $B < \dot{A} < e^{-\gamma T}\bar{A}$ and exercise price K are*

$$C_2^{do}(A,K) = C_0(A,K)_\theta - (\frac{B}{A})^{(\frac{2(\mu-\gamma)}{\sigma^2}-1)} C_0(\frac{B^2}{A},K)_\theta, \tag{26}$$

and

$$P_2^{do}(A,K) = P_0(A,K)_\theta = 0. \tag{27}$$

Proof. The formulas follow immediately from Theorem 18.8 in Björk (2004). □

The option formulas in the this section are also applicable in situations where third parties trade options on corporate defaultable perpetual debt. In such situations the existence of an option contract will neither influence the pricing of the debt nor the issuing company's own optimal choice of bankruptcy level. The option pricing formulas above can thus be applied by third parties using $B = e^{-\gamma T}\bar{A}$ and $\gamma = 0$. Recall that $\bar{A}$ represents the constant bankruptcy level, for the original asset process in expression (3), in the case of defaultable perpetual debt claims with no embedded call option.

δ_0	3	Initial value of EBIT
μ	2 %	Drift of EBIT
σ	25%	Volatility of EBIT
r	5 %	Riskfree interest rate
γ	0.002	Curvature of protection period barrier
D	70	Face value of debt
c	8%	Coupon rate
T	10	Expiration date of option
A_0	100	Total asset value at time 0, see Proposition 1
$\bar{A}$	57.91	Long-run bankruptcy level, from expression (8)
B	56.76	Initial protection period bankruptcy level, $B = e^{-\gamma T}\bar{A}$
θ	1.27	From expression (13)

TABLE 1. Base case parameters, all rates are annualized.

Option type	Eq.	Call		Eq.	Put	
Plain vanilla	(14)	6.95		(15)	11.61	
Time T risk	(18)	6.95	0%	(19)	0.45	−96%
Barrier options	(24)	5.85	−16%	(25)	0.06	−99%

TABLE 2. Numerical examples of option values using input parameters from Table 1.

4. Numerical examples

Below we present some numerical valuation examples.

Table 2 demonstrates how the market values of the options decrease compared to the plain vanilla options when, first, time T default risk, and, subsequently, also protection period default risk, are included. Consistent with Figures 4 and 3 the value reduction from including time T default risk of the put is larger than for the call option.

Figure 6 shows the market values of the six options from Table 2 for increasing EBIT volatility σ. The put values are increasing, whereas the call values are decreasing. In our model the EBIT volatility enters three of the 'Black-Scholes parameters', see the proof of Proposition 1. An increased σ leads to 1) decreased Black-Scholes volatility, 2) decreased initial price of the underlying asset F_0 for smaller values of σ and increased initial price of the underlying asset F_0 for larger values of σ, 3) reduced exercise price X. For the call option on debt, which is priced using a Black-Scholes put option formula, each of these effects reduce the value of the option for large and increasing values of σ (but the overall effect, for smaller σ, is an increasing value given our choice of parameters). For the put option on debt, which is priced using a Black-Scholes call option formula, effect 2) for larger σ and 3) increase

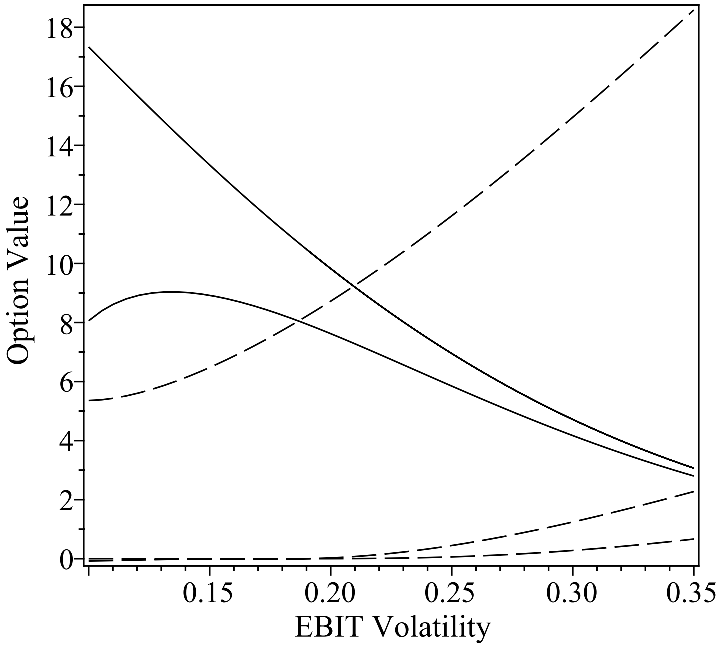

FIGURE 6. The graph shows the market values of the six options from Table 2 for increasing EBIT volatility. The solid lines represent the call options. The dashed lines represent the put options. The upper lines represent the 'plain vanilla' options, the middle lines (coincides with the plain vanilla option for the call) represent options including time T default risk, the lower lines represent barrier options. In the graph $\bar{A}$, B and θ are recalculated for each value of σ. The remaining parameters are given in Table 1.

the value of the option, whereas effect 1) and 2) for smaller σ also in this case reduces the value of the option. The combined effect given our choice of parameters for the put option is an overall increase in value as σ increases.

5. Concluding remarks

We have developed some option formulas for embedded options where the underlying asset is defaultable perpetual debt. The formulas are motivated by the embedded options in hybrid capital which have a contractually defined long protection period, necessary for approval as risk capital. Our main contribution is to quantify the effect of a protection period on the value of embedded options and the bankruptcy decision of the issuer. We do this by first modifying the underlying asset process to incorporate a time dependent bankruptcy level before the expiration of the embedded option. We then circumvent the non-lognormality of the market value of debt by considering a call option on debt as a fixed number of put options with a modified exercise price on a different asset, which possesses the required lognormal property. To include the possibility of default before the expiration of the option we apply barrier option results. The formulas are quite general and may be used for valuing both embedded and third-party options. All formulas are developed in the seminal and standard Black-Scholes-Merton model, thus, standard analytical tools such as 'the greeks', are immediately available.

Our results may be applied to value the embedded options in a large class of perpetual financial instruments which incorporates elements of both debt and equity, collectively denoted "hybrid capital". Issuances in the USD 260 Bn (2005, source: Lehman Brothers) global market of hybrid capital are often motivated by capital requirements for financial institutions. The Bank for International Settlements (BIS) has devised the fundamental requirements for how hybrid capital may qualify as a part of core ("Tier 1") regulatory capital for banks. Our results facilitate both pricing of these instruments and provide insights regarding the bankruptcy impact of embedded options and are also relevant in a policy perspective. Our formulas may also be used to value third party options.

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