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An Arbitrary Benchmark CAPM: One Additional Frontier Portfolio is Sufficient

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Abstract

The benchmark CAPM linearly relates the expected returns on an arbitrary asset, an arbitrary benchmark portfolio, and an arbitrary MV frontier portfolio. The benchmark is not required to be on the frontier and may be non-perfectly correlated with the frontier portfolio. The benchmark CAPM extends and generalizes previous CAPM formulations, including the zero beta, two correlated frontier portfolios, riskless augmented frontier, and inefficient portfolio versions. The covariance between the off-frontier benchmark and the frontier portfolio affects the systematic risk of any asset. Each asset has a composite beta, derived from the simple betas of both the asset and the benchmark.

JEL classifications: G12, G11, G10

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1. Introduction

A mean-variance (hereafter MV) frontier portfolio minimizes risk for a given expected return. According to the two fund separation theorem, any frontier portfolio of risky assets may be generated by a pair of arbitrary frontier portfolios. Traditional mean-variance asset pricing is concerned with the expected returns on an arbitrary asset (portfolio or security) related to two basic portfolios. Quite often the two basic portfolios are both assumed to be on the frontier, whether uncorrelated as in the Black (1972) zero beta CAPM or correlated as extended by Roll (1977). On the other hand, benchmarks and benchmark portfolios are essential for delegated or active portfolio management, but play no role in traditional CAPM models. Benchmarks are frequently used for anchoring performance evaluations, without particular regard as to whether the benchmarks are on the MV frontier. Many designated strategic portfolio benchmarks, say, with fixed and steady asset class weights over time, may be off the MV frontier by design.

The benchmark CAPM presented here expresses the expected total return on any arbitrary asset as an exact linear function of the expected returns on an arbitrary and possibly non-frontier benchmark portfolio and on an arbitrary MV frontier portfolio with a different mean. Its distinct features are that the benchmark portfolio is not required to be on the frontier, and that it may be non-perfectly correlated with the frontier portfolio. This formulation extends and generalizes previous formulations, including the zero beta CAPM, which drop out as special cases of the benchmark CAPM. The systematic risk of the arbitrary asset then depends not only on the covariance between the returns on the asset in question and the frontier portfolio, but also on the covariance between the returns on the off-frontier benchmark and the frontier portfolio. In a similar beta linear risk-return representation, each asset now has a composite beta, reflecting not only its own traditional simple beta, but also the benchmark's simple beta, both with respect to the frontier portfolio.

The two basic portfolios used for MV pricing will here be referred to as a primary portfolio and a secondary portfolio. Different CAPM formulations differ in the choice of primary and secondary portfolios. The primary portfolio is assumed to be on the MV frontier, but is different from to the global mean variance portfolio. Otherwise, the primary portfolio is arbitrary and may vary across applications. In particular, it is consistent with being the value-weighted market portfolio, but equilibrium is not required, and in general the primary portfolio may even be located on the downward sloping lower portion of the MV frontier. Asset simple betas are computed as the covariance between the returns on the asset and on the primary portfolio, divided by the variance of the primary portfolio's return. The secondary portfolio is arbitrary and may also vary across applications. Its mean return is different from the mean return of the primary portfolio. A benchmark portfolio is here used as a generic term for a secondary portfolio that is not required to be on the MV frontier, but may be so.

The paper is organized as follows: Section 2 presents a useful lemma for the expected return of any arbitrary asset, derived from properties of a frontier primary portfolio and a possibly non-frontier and correlated secondary portfolio. The benchmark CAPM follows in section 3 by a reformulation of the systematic risk in terms of betas, and interpreting the secondary portfolio as an arbitrary benchmark. Section 4 illustrates how various previous models are special cases of the benchmark CAPM. Section 5 concludes. Mathematical proofs of the lemma are relegated to appendices.

2. A useful lemma

All CAPM-like expressions do not necessarily have any solid economic basis. Consider any arbitrary asset j, and a weighted combination of any pair of arbitrary portfolios P and S with different means, with an exogenously given weight $\beta_j \equiv \dfrac{E(r_j) - E(r_S)}{E(r_P) - E(r_S)}$ for

portfolio P. Trivially, the expected asset return is then the weighted expected portfolio

returns: $E(r_j) = (1 - \beta_j) E(r_S) + \beta_j E(r_P)$. The expression may be rearranged into

$E(r_j) = E(r_S) + [E(r_P) - E(r_S)] \beta_j$, which resembles the linear Security Market Line (SML)

formulation. However, both these expressions may simply be tautologies, lacking any

meaningful economic substance. The expected return expression does not require that the

stochastic returns satisfy a similar relation (possibly with a noise term added), like

$r_j = (1 - \beta_j) r_S + \beta_j r_P + \varepsilon_j$. No particular systematic risk interpretation of the weight β_j is

implied for the linear mean-beta relation.

A sound economic model would imply some reasonable foundations, preferably such

that the weight has some interesting economic interpretation. On the other hand, commonly

used assumptions may impose unnecessarily strong structure, such as assuming two

uncorrelated MV frontier portfolios in the zero beta CAPM. The following lemma shows

how the mean of the arbitrary asset may be expressed in terms of stochastic properties of two

basic portfolios, when assuming portfolio optimality:

Lemma: Let P denote an arbitrary frontier primary portfolio, S an arbitrary, non-

perfectly correlated, and possibly non-frontier secondary portfolio with a different mean, and

j some arbitrary asset. Then the means are exactly related according to

$$E(r_j) = E(r_S) + [E(r_P) - E(r_S)] \frac{\text{Cov}(r_j, r_P) - \text{Cov}(r_S, r_P)}{\text{Var}(r_P) - \text{Cov}(r_S, r_P)} \tag{1}$$

At the risk of overkill, the lemma will be demonstrated in three different ways.

Appendix 1 shows how to derive the lemma from portfolio optimality conditions. Appendix 2

mimics the CAPM tangency approach to get the lemma. Appendix 3 verifies the lemma from

variance and covariance relations, when using the parsimonious "efficient set constants" of

the "fundamental matrix of information" approach developed by Merton (1972) and Roll

(1977) in their seminal papers. The reader is then free to make his or her pick of a favorite procedure.

A similar expression to Equation (1) has appeared in various editions of the Bodie, Kane and Miller *Investments* textbook, but there the two basic portfolios were both stated as "efficient-frontier portfolios". The restriction to frontier portfolios is not discussed in the textbook. Furthermore the references have changed over time[1].

Actually, Roll (1977) is somewhat ambiguous with respect to the lemma. His Corollary 6A and statement (S.6) basically contain the lemma, for the restricted case when both portfolios are arbitrary frontier portfolios. He furthermore showed that for two frontier portfolios, the fraction $\dfrac{\mathrm{Cov}\left(r_j, r_P\right) - \mathrm{Cov}\left(r_S, r_P\right)}{\mathrm{Var}\left(r_P\right) - \mathrm{Cov}\left(r_S, r_P\right)}$ is the bivariate regression coefficient for P, when regressing r_j on r_P and r_S. The similar property also holds for portfolio S, with the P and S subscripts interchanged, when both portfolios are assumed to be on the frontier. The two bivariate regression coefficients then sum to one, consistent with the asset mean being a weighted average of the two portfolio means. Roll's formulation is consistent with, but does not require, that the asset return is generated by a two-factor process like

$$r_j = \left(1 - \beta_j\right) r_S + \beta_j r_P + \varepsilon_j.$$

3.   The benchmark CAPM

The zero-beta portfolio is no longer appropriate in an extended version, when allowing for correlated primary and secondary portfolios, and where the secondary portfolio is no longer required to be on the MV frontier. With a combined perspective on both absolute and

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[1] Up through Bodie *et al*. (2006:Equation (9.9)), the result was allegedly shown by Black (1972), but it does not explicitly appear there. The most current edition Bodie *et al*. (2008:Equation (9.11)) changed the attribution to Merton (1972) and Roll (1977), without further details.

relative performance, benchmarks become interesting candidates as a secondary portfolio. So in this section the arbitrary secondary portfolio is renamed as a benchmark.

As in traditional CAPM formulations, the lemma implies that the required compensation for carrying risk different from that of the benchmark, may be formulated as the product of a "price of risk" and a "systematic risk" term. The "price of risk" is proportional to the difference in mean returns. From the fraction in (1), the "systematic risk" is proportional to the difference in covariance between the returns on the asset and on the benchmark, both covariances computed with the return on the primary portfolio (possibly the "market" portfolio). The fraction's denominator may be included in the "price of risk", in the "systematic risk", or partly in both.

In the beta representation, the whole fraction is interpreted as systematic risk. The betas in traditional CAPMs are now referred to as simple betas. The asset's simple beta is the ratio of return covariance to variance: $\beta_{jP} \equiv \dfrac{\mathrm{Cov}\left(r_j, r_P\right)}{\mathrm{Var}\left(r_P\right)}$, where the second beta subscript is often omitted as in β_j. Similarly, the benchmark's simple beta is $\beta_{BP} \equiv \dfrac{\mathrm{Cov}\left(r_B, r_P\right)}{\mathrm{Var}\left(r_P\right)} = \beta_B$. It is well known that such simple betas are consistent with univariate regression coefficients, with the primary portfolio as explanatory variable.

Applying the beta representation, the lemma may be rearranged, by dividing through by the variance of the primary portfolio, and letting the benchmark be the secondary portfolio:

The benchmark CAPM proposition: Let P denote an arbitrary frontier primary portfolio, B an arbitrary, non-perfectly correlated, and possibly non-frontier benchmark with a different mean, and j some arbitrary asset. Then the means are exactly related according to

$$E\left(r_j\right) = E\left(r_B\right) + \left[E\left(r_P\right) - E\left(r_B\right)\right]\beta_j^c \tag{2}$$

where the composite beta depends on simple betas according to

$$\beta_j^c \equiv \frac{\beta_{jP} - \beta_{BP}}{1 - \beta_{BP}} \qquad (3)$$

Whereas just one simple beta β_{jP} for each asset j is needed for the traditional

CAPMs, a common additional simple beta β_{BP} is required for the composite beta β_j^c of the

benchmark CAPM. The additional computational burden is negligible.

4.   Special cases of the Benchmark CAPM

The seminal contribution by Roll (1977) is the one closest to the current benchmark

model, but Roll focused on the case where both the primary and the secondary portfolio were

MV frontier portfolio. From his Corollary 6A, expressions (2) and (3) formally carry over

unchanged, but the benchmark portfolio is then restricted to be a frontier portfolio. In Roll's

setting, the composite beta β_j^c and the complementary composite beta $1 - \beta_j^c$ correspond to

the bivariate regression coefficients, with the asset return as dependent variable and the two

frontier portfolio returns as independent variables. Equation (3) may alternatively be

considered as a recipe for building bivariate regression coefficients from univariate ones.

Among reasonable candidates as a benchmark, Admati and Pfleiderer (1997) suggest

using the global mean variance portfolio (GMVP). The GMVP would work fine as a

benchmark in the benchmark CAPM. It cannot be used as a primary portfolio, as its

covariance with any asset is constant and equal to its variance, causing all assets to have an

identical simple beta of unity, and division by zero in the expression for composite beta. A

somewhat related extended CAPM formulation, using the market portfolio with the GMVP,

can be found in van Zijl (1987).

The Black (1972) zero beta CAPM applies to an environment with no riskless security,

where the secondary portfolio is uncorrelated and hence zero beta with the primary portfolio.

The benchmark portfolio may then be written as $B = Z(P)$, with the covariance

$\text{Cov}\left(r_P, r_{Z(P)}\right) = 0$. The benchmark CAPM then specializes into the standard zero beta CAPM:

$$E\left(r_j\right) = E\left(r_{Z(P)}\right) + \left[E\left(r_P\right) - E\left(r_{Z(P)}\right)\right]\beta_j \qquad (4)$$

With uncorrelated primary and secondary portfolio, $\beta_{Z(P)P} = 0$, and the composite beta β_j^c

collapses into the simple beta $\beta_{jP} = \beta_j$. Note that the zero beta benchmark portfolio $Z(P)$ is

not required to be on the MV frontier. Any portfolio with the same mean as the zero beta

frontier portfolio will do, as they are all uncorrelated with P. Elton *et al.* (2007:310) simply

comment that it makes sense to use the least risky zero beta portfolio[2]. Furthermore, the

primary portfolio P may be the market portfolio M, but market equilibrium is not necessary,

as long as portfolio optimality holds. Any primary portfolio on the frontier will do for CAPM

pricing, even if it should be on the inefficient downward sloping part of the frontier. Cochrane

(2001:91) selects the portfolio with the minimum second moment gross return, which lies on

the lower segment of the MV frontier. He comments that it is initially surprising that this is

the location of the most interesting return on the frontier, implying an unusual negative risk

premium. In any case, the primary frontier portfolio P and the corresponding zero-beta

secondary portfolio $Z(P)$ are on MV frontier segments with differently signed slopes.

A degenerate case of the zero beta CAPM occurs when picking as primary portfolio

the frontier portfolio N whose zero beta portfolio $Z(N)$ has an expected return $E\left(r_{Z(N)}\right) = 0$.

This frontier portfolio may be referred to as the null orthogonal frontier portfolio. The null

orthogonal CAPM is thus simply

$$E\left(r_j\right) = E\left(r_N\right)\beta_{jN} \qquad (5)$$

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[2] In their figure illustrating the set of portfolios uncorrelated with a frontier portfolio, the alleged zero beta portfolios visually appear to have different means.

This null orthogonal frontier portfolio has several interesting properties. By definition, it is uncorrelated with all mean zero assets. In mean-standard deviation space, the tangent to the efficient frontier at N intercepts the mean-return axis in origo. In mean-variance space, a ray from origo through N will pass through the GMVP. It is the only frontier portfolio, for which the ratio of expected return to beta is the same constant for all assets, which furthermore equals the expected return of this frontier portfolio. Its mean, variance and composition have simple closed form expressions[3].

Suppose a riskless security exists, with the same rate r_f for both lending and borrowing. Assume the riskless rate is different from the expected return on the risky GMVP. In mean-SD space, the riskless augmented portfolio frontier for all assets now consists of two half lines, generated by the riskless security and its tangency portfolio T to the frontier of risky assets only. Both half lines originate at r_f and have differently signed slopes with the same absolute value, depending on whether the investment proportion in T is nonnegative or nonpositive. Selecting an arbitrary portfolio P on the riskless augmented frontier as the primary portfolio, and using the riskless security as the secondary portfolio, yields the riskless augmented frontier CAPM:

$$E\left(r_j\right) = r_f + \left[E\left(r_P\right) - r_f\right]\beta_j \qquad (6)$$

This result holds, regardless of whether the portfolio P is on the upper or lower half line, and whether the riskless rate is above or below the expected return of the GMVP[4]. As the portfolio P moves along either half line of the riskless augmented frontier, the effects on the price of risk and on beta exactly cancel. Traditional special cases are where the primary

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[3] See e.g. Roll (1977:165) or Roll (1992:20).
[4] It appears as Equation (3.19.1) in the Huang and Litzenberger (1988) textbook. Feldman and Reisman (2003) state a similar result in their Lemma 1, but there it is restricted to arbitrary portfolios P on the upper half line only.

portfolio is the tangency portfolio T, or the market portfolio M, both usually on the upper

half line, with both expected returns exceeding the riskless rate.

The framework may also be applied to cases where simple asset betas are computed

against an arbitrary and non-frontier or inefficient portfolio I. Let the primary portfolio P be

the frontier portfolio having the same mean and a smaller standard deviation than the

inefficient portfolio I. As the secondary portfolio, use the zero beta portfolio $Z(P) = Z(I)$.

From the lemma, using the equal mean and zero beta properties of the primary and the

inefficient portfolios, $E(r_j) = E(r_{Z(I)}) + \left[E(r_I) - E(r_{z(I)}) \right] \dfrac{\mathrm{Cov}(r_j, r_P)}{\mathrm{Var}(r_P)}$. Let e be an arbitrage

portfolio, with weights summing to zero, mean zero, and being uncorrelated with the frontier

portfolio P. Diacogiannis and Feldman (2007) decompose the return on the inefficient

portfolio as $r_I = r_P + r_e$, in the current notation. Hence, $\mathrm{Cov}(r_j, r_P) = \mathrm{Cov}(r_j, r_I) - \mathrm{Cov}(r_j, r_e)$.

The asset simple betas are $\beta_{jP} \equiv \dfrac{\mathrm{Cov}(r_j, r_P)}{\mathrm{Var}(r_P)}$, $\beta_{jI} \equiv \dfrac{\mathrm{Cov}(r_j, r_I)}{\mathrm{Var}(r_I)}$ and $\beta_{je} \equiv \dfrac{\mathrm{Cov}(r_j, r_e)}{\mathrm{Var}(r_e)}$, when

computed against the frontier portfolio P, the inefficient portfolio I, and the arbitrage

portfolio e, respectively. Progressing to a linear mean-beta representation thus requires

variance adjustments in the betas.

With some reformulations, the Diacogiannis and Feldman (2007) inefficient portfolio

CAPM becomes

$$E(r_j) = E(r_{Z(I)}) + \left[E(r_I) - E(r_{z(I)}) \right] \beta_j^I \tag{7}$$

where the composite inefficient beta is

$$\beta_j^I \equiv \frac{\mathrm{Var}(r_I)\beta_{jI} - \mathrm{Var}(r_e)\beta_{je}}{\mathrm{Var}(r_P)} \tag{8}$$

The standard zero beta CAPM (4) and the inefficient portfolio CAPM (7) only differ in the beta terms, respectively the asset simple beta $\beta_j = \beta_{jP}$ and the composite inefficient beta β_j^I. Consistency requires that these two betas are equal, whereas (8) shows that in general the simple asset beta β_{jI} is different from the composite inefficient beta β_j^I. This is a reminder of fallacies in using a non-frontier proxy I for a frontier portfolio P in an otherwise standard zero beta CAPM.

Arbitrary and presumably non-frontier benchmarks also appear in MV models, which not are direct special cases of the current benchmark CAPM. The delegated agent pricing model of Cornell and Roll (2005) is a recent example. Roll (1992) and Jorion (2003) are contributions studying the effects of applying MV analyses to differential returns relative to an arbitrary benchmark, rather than to total returns or excess return above the riskless rate.

5.   **Conclusions**

The benchmark CAPM generalizes and extends previous CAPM models. This further refinement preserves the general structure and adjusted properties of relating the return of any arbitrary asset to the returns of a primary and secondary portfolio, but in a less restrictive setting.

The primary portfolio is a frontier portfolio, but may be on the lower non-efficient portion of the MV frontier. The benchmark interpretation of the secondary portfolio is convenient, given the widespread use of benchmarks in modern portfolio management. The benchmark is literally quite arbitrary, except for having a mean return different from the primary portfolio. Thus the benchmark is neither required to be on the frontier nor to be uncorrelated with the primary portfolio.

The asset return remains a weighted average of the two portfolio returns. The linear risk-return CAPM relationship is maintained. The weights reflect systematic risk depending

on covariances. Each asset has its own composite beta, to be used as systematic risk in a SML context. The composite betas are easily computed from simple betas for the asset and for the benchmark, both computed against the primary frontier portfolio. The simple betas may still be interpreted as univariate regression coefficients. The composite betas and their complements may be interpreted as bivariate regression coefficients, if the benchmark is also on the frontier.

Previous CAPM formulations drop out as special cases. The benchmark CAPM is thus consistent with, among others, any secondary frontier portfolio imperfectly correlated with the primary frontier portfolio, the global mean variance portfolio, the frontier zero beta portfolio, any non-frontier portfolio being uncorrelated with the primary portfolio, and a riskless security if it exists. The framework may also be adapted to the inefficient portfolio CAPM, where an asset's composite beta is a linear function of the asset's simple beta computed against an arbitrary and non-frontier inefficient portfolio. Further extensions may require more than simple modifications of existing assumptions.

Classical CAPM models were introduced and developed into maturity decades ago, through a series of path breaking papers. Financial practice is still heavily influenced by its MV heritage, whereas innovative financial research has mostly moved on to different areas. Current assent pricing theory is quite sophisticated, with complex models and advanced methods, possibly requiring state-of-the-art software[5]. In contrast, the benchmark CAPM is a unifying extension of traditional CAPM models along mostly familiar lines, representing an evolution but not a revolution. It may be derived by mimicking standard approaches, with some creative adjustments. General familiarity with fundamental finance concepts and financial models, basic probability theory, elementary optimization and simple matrix

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[5] Cochrane (2001) is an advanced and challenging graduate level textbook, surveying modern asset pricing.

operations may be sufficient background. Whereas it may be surprising that additional niche results are still obtainable, it is encouraging that old tricks still seem to work!

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Appendix 1: Portfolio optimality conditions

Consider $n \geq 2$ linearly independent and thus non redundant risky securities, where at least two securities have different expected returns. The vector of the securities' expected returns is $\boldsymbol{\mu}$. Their variance-covariance return matrix $\mathbf{V}$ is symmetric and positive definite, such that the inverse covariance matrix $\mathbf{V}^{-1}$ exists. A portfolio of risky assets is defined by its weight vector $\mathbf{w}$ of proportions invested in the risky assets, summing to unity, such that $\mathbf{w}'\mathbf{1} = 1$, where $\mathbf{1}$ is a summation vector of ones, and primes denote vector or matrix transposition. Short selling is allowed, such that some securities may have negative weights in a portfolio. Subscripts identify different portfolios. An arbitrary portfolio P fully invested in risky assets has mean $\mu_P = \mathbf{w}_P'\boldsymbol{\mu}$ and variance $\sigma_P^2 = \mathbf{w}_P'\mathbf{V}\mathbf{w}_P$. The covariance between arbitrary portfolios P and S is $\sigma_{PS} = \mathbf{w}_P'\mathbf{V}\mathbf{w}_S$.

A frontier portfolio is the risky portfolio that minimizes (one half of) the variance among all portfolios having the same targeted expected return $\bar{\mu}$. It satisfies the portfolio optimality necessary and sufficient condition

$$\mathbf{V}\mathbf{w}_P = \lambda\boldsymbol{\mu} + \gamma\mathbf{1} \tag{A1}$$

where λ and γ are Lagrange multipliers associated with the portfolio mean and weight sum constraints, respectively. A frontier portfolio therefore has the weight vector in risky assets of

$$\mathbf{w}_P = \lambda\mathbf{V}^{-1}\boldsymbol{\mu} + \gamma\mathbf{V}^{-1}\mathbf{1} \tag{A2}$$

Premultiplying the optimality condition (A1) with the weight vectors $\mathbf{w}_P$, $\mathbf{w}_S$ and $\mathbf{w}_j$ of , respectively, the primary frontier portfolio P, the secondary portfolio S, and the arbitrary asset j, gives three linear equations in the two Lagrange multipliers: $\sigma_P^2 = \lambda\mu_P + \gamma$, $\sigma_{SP} = \lambda\mu_S + \gamma$, and $\sigma_{jP} = \lambda\mu_j + \gamma$. Subtracting the second equation from the first gives

$\sigma_P^2 - \sigma_{SP} = \lambda\left(\mu_P - \mu_S\right)$, implying $\dfrac{1}{\lambda} = \dfrac{\mu_P - \mu_S}{\sigma_P^2 - \sigma_{SP}}$. Subtracting the second equation from the

third gives $\sigma_{jP} - \sigma_{SP} = \lambda(\mu_j - \mu_S)$, implying $\dfrac{1}{\lambda} = \dfrac{\mu_j - \mu_S}{\sigma_{jP} - \sigma_{SP}}$. Equating the two expressions

for the inverse of the Lagrange multiplier λ and solving for μ_j yields

$\mu_j = \mu_S + (\mu_P - \mu_S)\dfrac{\sigma_{jP} - \sigma_{SP}}{\sigma_P^2 - \sigma_{SP}}$. Equation (1) of the lemma follows by writing out the moments

more explicitly.

Appendix 2: The CAPM tangency approach

This is really an exercise in "back to basics" portfolio analysis with two risky assets.

Consider a portfolio Q of assets A and B, with a weight x in A. The stochastic portfolio

return $r_Q = x r_A + (1-x) r_B$ has mean $\mu_Q = x\mu_A + (1-x)\mu_B$ and variance

$\sigma_Q^2 = x^2\sigma_A^2 + (1-x)^2\sigma_B^2 + 2x(1-x)\sigma_{AB}$. In mean-SD space, the portfolio frontier slope is

$\dfrac{d\mu_Q}{d\sigma_Q} = \dfrac{d\mu_Q/dx}{(d\sigma_Q/d\sigma_Q^2)(d\sigma_Q^2/dx)}$. Differentiating, collecting terms, and evaluating the derivative

at $x = 1$, the portfolio frontier slope when fully invested in asset A, turns out as

$$\frac{d\mu_Q}{d\sigma_Q}\Big|\, x = 1 = (\mu_A - \mu_B)\frac{\sigma_A}{\sigma_A^2 - \sigma_{BA}} \tag{A3}$$

First consider a portfolio of a primary frontier portfolio P and an arbitrary secondary

portfolio S. Next consider a portfolio of the same primary frontier portfolio P and an

arbitrary asset j. When both portfolios are fully invested in the primary frontier portfolio,

they must have the same slope: $(\mu_P - \mu_S)\dfrac{\sigma_P}{\sigma_P^2 - \sigma_{SP}} = (\mu_P - \mu_j)\dfrac{\sigma_P}{\sigma_P^2 - \sigma_{jP}}$ from (A3).

Cancelling σ_P, cross multiplying, and adding and subtracting $\mu_S\sigma_{SP}$, the expression can be

rearranged as $\mu_j\left(\sigma_P^2 - \sigma_{SP}\right) = \mu_S\left(\sigma_P^2 - \sigma_{SP}\right) + \mu_P\left(\sigma_{jP} - \sigma_{SP}\right) - \mu_S\left(\sigma_{jP} - \sigma_{SP}\right)$. Solving for μ_j

gives $\mu_j = \mu_S + \left(\mu_P - \mu_S\right)\dfrac{\sigma_{jP} - \sigma_{SP}}{\sigma_P^2 - \sigma_{SP}}$, which is equivalent to equation (1) of the lemma.

Appendix 3: Variance and covariance relations for frontier portfolios

Following Merton (1972) and Roll (1977), it will be useful to introduce the "efficient

set constants" of the "fundamental matrix of information" $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, with $a \equiv \mathbf{\mu}'\mathbf{V}^{-1}\mathbf{\mu} > 0$,

$b \equiv \mathbf{\mu}'\mathbf{V}^{-1}\mathbf{1}$, $c \equiv \mathbf{1}'\mathbf{V}^{-1}\mathbf{1} > 0$ and $d \equiv ac - b^2 > 0$ [6]. Premultiplying the frontier weight vector

from (A2) first by the transposed mean vector $\mathbf{\mu}'$ and next by the summation vector $\mathbf{1}'$, give

two linear equations for the Lagrange multipliers: $a\lambda + b\gamma = \mu_P$ and $b\lambda + c\gamma = 1$. The

Lagrange multipliers are then solved as $\lambda = \dfrac{c\mu_P - b}{d}$ and $\gamma = \dfrac{a - b\mu_P}{d}$, which may be

substituted back into the frontier weight expression (A2). Premultiplication by the transposed

primary portfolio weight vector $\mathbf{w}_P$ gives the variance relation to be satisfied by all frontier

portfolios:

$$\sigma_P^2 = \frac{a - 2b\mu_P + c\mu_P^2}{d} \tag{A4}$$

From premultiplying (A2) by the transposed weight vector $\mathbf{w}_S$ of any secondary portfolio, the

covariance relation

$$\sigma_{SP} = \frac{a - b\mu_P - b\mu_S + c\mu_P\mu_S}{d} \tag{A5}$$

holds for any frontier portfolio P and any arbitrary portfolio S [7].

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[6] The notation may vary. Merton and his followers generally write A for Roll's b, B for Roll's a, and C for Roll's c.

[7] The variance and covariance relations are found in Roll (1977) as Equations (A.11) and (A.16).

Apply the covariance relation (A5) first to the arbitrary asset j, and then to the

arbitrary secondary portfolio S. By subtraction and cancellations, the covariance difference

$\sigma_{jP} - \sigma_{SP} = \left(\dfrac{1}{d}\right)(\mu_j - \mu_S)(-b + c\mu_P)$. Subtracting the covariance relation (A5) from the

variance relation (A4), yields $\sigma_P^2 - \sigma_{SP} = \left(\dfrac{1}{d}\right)(\mu_P - \mu_S)(-b + c\mu_P)$. By division of these to

expressions, $\dfrac{\mu_j - \mu_S}{\mu_P - \mu_S} = \dfrac{\sigma_{jP} - \sigma_{SP}}{\sigma_P^2 - \sigma_{SP}}$. Rearranging, $\mu_j = \mu_S + (\mu_P - \mu_S)\dfrac{\sigma_{jP} - \sigma_{SP}}{\sigma_P^2 - \sigma_{SP}}$. Hence, the

lemma has been verified from the variance and covariance relations as well.