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# **Level dependent annuities: Defaults of multiple degrees**

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# Level dependent annuities: Defaults of multiple degrees

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## Abstract

Motivated by the risk of stopped debt coupon payments from a leveraged company in financial distress, we value a level dependent annuity contract where the annuity rate depends on the value of an underlying asset-process. The range of possible values of the asset is divided into a finite number of regions. The annuity rate is constant within each region, but may differ between the regions. We consider both infinite and finite annuities, with or without bankruptcy risk, i.e., bankruptcy occurs if the asset value process hits an absorbing boundary. Such annuities are common in models of debt with credit risk in financial economics. Suspension of debt service under the US Chapter 11 provisions is one well-known real-world example. We present closed-form formulas for the market value of such multi-level annuities contracts when the market value of the underlying asset is assumed to follow a geometric Brownian motion.

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# 1 Introduction

Debt obligations are commonly defaulted on by companies in financial distress. A default is defined as stopped or reduced coupon payments, but may not in itself lead to a liquidation of the company. The reduction in coupon payments may even be contractual for particularly risky debt. Irrespective of the specific causes of these non-payments, they represent a challenge for the valuation of corporate debt. Chapter 11 in the US bankruptcy code is an important example of regulations that allow a company to default without necessarily being declared bankrupt and liquidated.

Broadie, Chernov, and Sundaresan (2007) determine debt and equity values in a model which distinguishes between default and liquidation, motivated by US legislation. They analyze conflicts of interest between debtholders and shareholders and solve their model numerically using the binomial approach of Broadie and Kaya (2007). An example of *contractual* non-payment of coupons is hybrid risk capital for financial institutions which incorporates elements of both equity and debt. One common feature of such claims is the issuer's right to omit coupon payments under certain conditions, see e.g, Mjøs and Persson (2005).

Motivated by the risk of lost coupon payments we define and value a multi-level annuity contract with a finite number of asset value levels which may be interpreted as states of 'financial health'. As such, the contract allows for different, but constant, coupon rates in the regions between the different asset levels. Both coupon rates and financial health levels are assumed to be exogenous. We derive closed form solutions for the market value of the multi-level annuity contract both in the cases of finite and infinite horizons. The choice of the market price process of a company's assets rather than a common financial market factor such as, e.g., an interest rate, as exogenous stochastic process is motivated by the assumption that distress is primarily caused by firm specific risk rather than general market risk.

Mathematically we solve a boundary value problem, see e.g., Øksendal (2005, Chapter 9). First, we find the market value of the multi-level annuity contract in the case of an infinite horizon using the standard assumption of *smooth-pasting*, see, e.g., Dixit and Pindyck (1994). The multi-level annuity contract can be considered as a portfolio of simpler annuities. The market value of the multi-level annuity is calculated as the sum of the market values of these annuities. In the case of a finite horizon we apply the standard argument that a finite annuity may be considered as an immediately starting infinite annuity from which another infinite annuity starting at the future time  $T$  is subtracted.

The bankruptcy asset level is modelled as an absorbing barrier in the

structural debt modelling framework of Black and Cox (1976), Leland (1994) and others. Both Broadie, Chernov, and Sundaresan (2007) and Mjøs and Persson (2005) apply one additional financial distress level, interpreted as the default level. In this paper we extend this idea to multiple, although exogenous, financial health levels, see Figure 1. In order to interpret these levels as various degrees of financial distress, the natural assumption is that the initial asset value is above all these levels. Our approach is general and some of our formulas are applicable for other assumptions regarding the initial asset level as well.

Our analytical solution may be applied to parts of the valuation problem of Broadie, Chernov, and Sundaresan (2007), although their model contains time-dependencies which severely complicate the use of closed-form solutions. Closed form solutions, as the ones we present, may increase computational speed, provide benchmarks for numerical solutions, and enhance economic understanding of the problems.

As an example of application of our results we divide the total value of an infinite annuity without bankruptcy risk into parts related to bankruptcy, specific regions of asset value, and a finite maturity. Table (1) shows the parameters used to calculate the value decomposition in Table (2).

Parameters	Values	Explanations
$\mu$	0.02	Drift of asset process
$\sigma$	0.20	Volatility of asset process
$r$	0.05	Riskfree interest rate
$A$	100	Initial total asset value at time 0
$B$	60	Barrier between annuity regions
$C$	30	Bankruptcy asset level
$c$	5	Annuity payment for all regions, $A_t > C$
$T$	10	Maturity of finite annuities and start of forward starting annuities

Table 1: Parameters used in example.

The time zero values in Table (2) are calculated using the formulas in this paper as follows. The values of infinite immediately starting above and below annuities with bankruptcy risk (lines (1) and (2)) are calculated using equations (10) and (11), respectively. The values of forward starting infinite above and below annuities are calculated using equations (13) and (14), respectively. The values of finite annuities maturing at time  $T$  are found in the conventional way by deducting the value of a forward starting infinite

	<b>Annuity</b>	<b>Finite</b>	<b>Forward starting</b>	<b>Infinite</b>
(1)	Above annuity w/ bankruptcy	35.25	39.97	75.22
(2)	Below annuity w/ bankruptcy	3.59	6.29	9.88
(3)	Above + Below annuity	38.84	46.26	85.10
(4)	Value of bankruptcy loss	0.51	14.39	14.90
(5)	Value of riskfree annuity	39.35	60.65	100.00

Table 2: Annuity value decomposition.

annuity starting at time  $T$  from an immediately starting infinite annuity, or calculated directly from expressions (15) and (16).

The values in line (3) are found by summing the region specific values in line (1) and (2). The value of the combined infinite annuity can be calculated directly as

$$\frac{c}{r}(1 - (\frac{A}{C})^{-\beta}),$$

where  $\beta$  is given in expression (3). This is a special case of the results of Black and Cox (1976). The value of the forward starting annuity in line (3) can be calculated as

$$\frac{c}{r}(e^{-rT}Q_g - (\frac{A}{C})^{-\beta}Q_g^\beta),$$

where  $Q_g$  and  $Q_g^\beta$  are given in expressions (24) and (32) in Appendix B. The first term is interpreted as the time zero value of a time  $T$  forward starting, infinite annuity. The second term is the time zero market value of a forward starting, infinite annuity, starting at the time of bankruptcy given that bankruptcy occurs after time  $T$ . The time zero value of the finite annuity in line (3) is then calculated as the difference between the infinite annuity and the forward starting annuity using these formulas.

The values of the loss due to bankruptcy (line (4)) have been calculated as the differences between the values of the respective annuities with (line (3)) and without (line (5)) risk of bankruptcy. In line (5) the infinite case, the value of an immediately starting riskfree annuity is simply  $\frac{c}{r}$ , whilst the value of a forward starting riskfree annuity is the same value discounted, i.e.,  $e^{-rT}(\frac{c}{r})$ . The finite annuity in line (5) is then calculated as the usual difference,  $\frac{c}{r}(1 - e^{-rT})$ .

Table (2) illustrates how the value of an immediately starting riskfree infinite annuity may be seen as the sum of value elements. The value of the riskfree infinite annuity is normalized to 100, and we apply a constant annuity

payment for the regions above and below the asset level  $B$ . Our formulas, in general, allow for multiple regions with possible different coupon payments in each region.

This paper is organized as follows: Section 2 contains the set-up and main result. Section 3 treats the case of immediately starting, infinite horizon annuities. Section 4 develops the results for the case of forward starting, infinite horizon annuities. In Section 5 results from Sections 3 and 4 are combined into finite horizon annuities. Whereas Sections 3 and 4 treat simpler annuities, Section 6 extends these results to infinite multi-levels annuities. The finite version of the multi-level annuities are developed in Section 7. Conclusions and areas for further research are indicated in Section 8. Some standard technical results are collected in two appendices.

## 2 Set-up and main result

A filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$  is given. In particular,  $Q$  represents a fixed *equivalent martingale measure*. We furthermore impose the standard frictionless, continuous time market assumptions of financial economics, see e.g., Duffie (2001).

We assume that the underlying asset process is given by a geometric Brownian motion

$$dA_t = \mu A_t dt + \sigma A_t dW_t,$$

where the initial value  $A_0 = A$  is a constant. Here the drift parameter  $\mu$  and volatility parameter  $\sigma$  are constants and  $W_t$  represents a standard Brownian motion.

Let  $T$  be the finite time horizon, let the constant  $C$  be an absorbing barrier, and define the stopping time  $\tau$  as

$$\tau = \inf\{t \geq 0, A_t = C\}.$$

We interpret  $C$  as the bankruptcy barrier, and  $\tau$  as the time of bankruptcy.

There are  $n$  additional constant levels or non-absorbing barriers  $B_1, \dots, B_n$  so that  $B_1 > \dots > B_n > C$ . For notational convenience we let  $B_0 = \infty$  and, in the case with bankruptcy risk  $B_{n+1} = C$ , or, in the case without bankruptcy risk  $B_{n+1} = 0$ , respectively. The constant annuity rate is  $c_1$  when  $A_t > B_1$ ,  $c_{i+1}$  when  $B_i > A_t > B_{i+1}$ ,  $i = 1, \dots, n-1$ , and  $c_{n+1}$  when  $B_n > A_t > B_{n+1}$ . All  $c_i$ 's are constants. The initial value of the asset process is by assumption above the highest barrier, i.e.,  $A > B_1$ .

Let  $r$  be the constant riskfree interest rate. Note that we allow  $\mu \leq r$ .

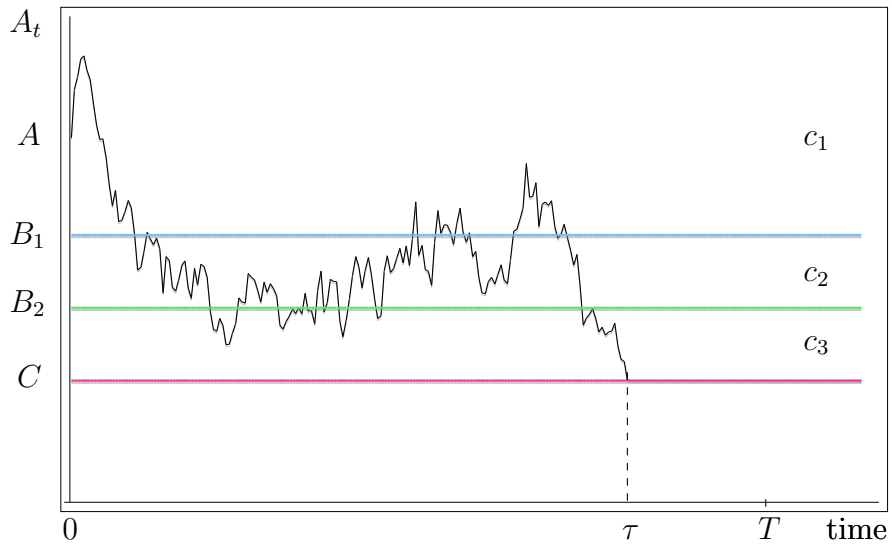


Figure 1: An illustration of a multi-level annuity where  $n = 2$ . The picture contains an example of a path of  $A_t$  and indicates in which regions the annuity rates are  $c_1$ ,  $c_2$ , and  $c_3$ , respectively. Also,  $A$ ,  $B_1$ ,  $B_2$ ,  $C$ ,  $T$ , and  $\tau$  are depicted.

We solve the valuation problem

$$M^T(A) = E \left[ \int_0^{\tau \wedge T} \sum_{i=0}^n c_{i+1} e^{-rs} 1\{B_i > A_s > B_{i+1}\} ds \right],$$

where  $1\{\cdot\}$  denotes the standard indicator function,  $E[\cdot]$  denotes the expectation under the equivalent martingale measure. Here  $B_{n+1} = C$ .

Our main result is that the time zero value of a finite, multi-level annuity with bankruptcy risk is

$$M^T(A) = \frac{c_1}{r} - \frac{c_{n+1}}{r} \left[ \left(\frac{A}{C}\right)^{-\beta} Q_l^\beta + e^{-rT} Q_g \right] + \sum_{i=1}^n \frac{c_i - c_{i+1}}{r} (\psi_i - Q_{gg}(B_i) e^{-rT}), \quad (1)$$

where

$$\psi_i = \frac{\alpha \left(\frac{A}{B_i}\right)^{-\beta} (Q_{gg}^\beta(B_i) - 1) - \beta \left(\frac{A}{B_i}\right)^\alpha Q_{lg}^\alpha(B_i) - \beta \left(\frac{A}{C}\right)^{-\beta} \left(\frac{C}{B_i}\right)^\alpha Q_l^\beta}{\alpha + \beta},$$

$$\alpha = \frac{\frac{1}{2}\sigma^2 - \mu + \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2 r}}{\sigma^2} \quad (> 1), \quad (2)$$

and

$$\beta = \frac{\mu - \frac{1}{2}\sigma^2 + \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2 r}}{\sigma^2} \quad \left( > \frac{2\mu}{\sigma^2} > 0 \right). \quad (3)$$

The probability  $Q_l^\beta = Q^\beta(\tau \leq T) = 1 - Q_g^\beta$ , where  $Q_g^\beta = Q^\beta(\tau > T)$  is given in expression (32) in Appendix B. Furthermore, the probabilities  $Q_g = Q(\tau > T)$ ,  $Q_{gg}^\beta(B_i) = Q^\beta(A_T > B_i, \tau > T)$ ,  $Q_{lg}^\alpha(B_i) = Q^\alpha(A_T \leq B_i, \tau > T)$ , and  $Q_{gg}(B_i) = Q(A_T > B_i, \tau > T)$  are given in expressions (24), (33), (30), and (25), respectively, in Appendix B. Here  $Q^\alpha$  and  $Q^\beta$  represent probability measures equivalent to  $Q$ , see Appendix A for details.

The first term of expression (1) represents the time zero value of an infinite annuity  $c_1$  without bankruptcy risk. The negative of the second term represents the time zero value of a forward starting infinite annuity  $c_{n+1}$  without bankruptcy risk, starting either at the time of bankruptcy  $\tau$  or time  $T$ , whichever comes first. Roughly interpreted, the remaining terms represent correction terms of the total time zero value due to the multiple annuity levels between  $c_1$  and  $c_{n+1}$ .



### 3 Immediately starting, infinite claims

In this section we consider immediately starting, infinite horizon claims, assuming that  $T = \infty$ . Let  $f$  be the time zero market value of an arbitrary infinite horizon claim on  $A_t$ , and denote the first and second order partial derivatives by  $f_A = \frac{\partial f}{\partial A}$  and  $f_{AA} = \frac{\partial^2 f}{\partial A^2}$ , respectively. Then the partial differential equation, see, e.g., Merton (1974)

$$\frac{1}{2}\sigma^2 A^2 f_{AA} + \mu A f_A - r f + c(A) = 0 \quad (4)$$

holds, subject to appropriate boundary conditions. Here  $c(A)$  represents the annuity payment rate (to be interpreted as dividends or coupons, depending on the nature of the claim) to the owner of the claim  $f$ . The general solution to the homogeneous part, obtained by letting  $c(A) = 0$ , of equation (4) is

$$f^*(A) = K_1 A^\alpha + K_2 A^{-\beta}, \quad (5)$$

where  $\alpha$  and  $\beta$  are given in expressions (2) and (3), respectively, and the constants  $K_1$  and  $K_2$  are determined by boundary conditions. The general solution to equation (4) is  $f(A) = f^*(A) + f^s(A)$ , where  $f^s(A)$  is any *special solution* of equation (4).

First we derive market values of some simpler claims, which subsequently will be used in the valuation of the multi-level annuities. We denote initial market values by capital letters, possibly with subscripts, e.g.,  $U$ , or  $U(A, B)$  to emphasize the dependence on the initial value of the process and on the barrier  $B$ .

#### 3.1 The value of 1 at the initial hit of a barrier

Let  $U$  denote the time 0 market price of a claim which pays 1 when  $A_t = B$  for the first time.

$$U(A, B) = \begin{cases} U^a = \left(\frac{A}{B}\right)^{-\beta} & \text{when } A \geq B, \\ U^b = \left(\frac{A}{B}\right)^\alpha & \text{when } A \leq B. \end{cases} \quad (6)$$

The superscripts  $a$  and  $b$  signify that  $A_t$  hits the barrier from *above* or *below*, respectively. These results are standard, but we include a proof for the completeness of the exposition.

**Proof 1**  $U$  does not pay any dividend so  $c(A) = 0$  in expression (4).  $U^a$  is calculated from equation (5) using the boundary conditions  $\lim_{A \rightarrow \infty} U^a = 0 \Rightarrow K_1 = 0$  and  $U^a(B) = 1$ .  $U^b$  is calculated from the boundary conditions  $\lim_{A \rightarrow 0} U^b = 0 \Rightarrow K_2 = 0$  and  $U^b(B) = 1$ .

We remark that  $U(\cdot, B)$  is continuous at  $B$ , but does not satisfy the *smooth pasting* condition at  $B$ .

## 3.2 Above and below annuities without bankruptcy risk

### 3.2.1 The value of an above-annuity without bankruptcy risk

Let  $V_A$  denote the time 0 market price of an annuity which pays the rate  $c$  when  $A_t > B$  (*above-annuity*).

$$V_A(A, B) = \begin{cases} V_A^a = \frac{c}{r} \left(1 - \frac{\alpha}{\alpha + \beta} \left(\frac{A}{B}\right)^{-\beta}\right) & \text{when } A \geq B, \\ V_A^b = \frac{c}{r} \frac{\beta}{\alpha + \beta} \left(\frac{A}{B}\right)^\alpha & \text{when } A \leq B. \end{cases} \quad (7)$$

Observe that  $V_A^b = 0$  when  $B = \infty$ .

**Proof 2**  $V_A$  pays  $c$  only when  $A_t > B$ , so in expression (4)  $c(A) = c$  when  $A_t > B$ , and  $c = 0$  otherwise. Observe that  $f^s(A) = \frac{c}{r}$  solves equation (4) when  $A > B$ . The relevant boundary conditions are  $\lim_{A \rightarrow \infty} V_A^a = \frac{c}{r} \Rightarrow K_1 = 0$  and  $\lim_{A \rightarrow 0} V_A^b = 0 \Rightarrow K_2 = 0$ . To determine  $K_2$  for  $V_A^a$  and  $K_1$  for  $V_A^b$  we require continuity and smooth pasting at  $B$ , i.e.,  $V_A^a(B) = V_A^b(B)$  and  $\frac{\partial}{\partial A} V_A^a(B) = \frac{\partial}{\partial A} V_A^b(B)$ .

### 3.2.2 The value of a below-annuity without bankruptcy risk

Let  $V_B$  denote the time 0 market price of an annuity which pays  $c$  when  $A_t < B$  (*below-annuity*).

$$V_B(A, B) = \begin{cases} V_B^a = \frac{c}{r} \frac{\alpha}{\alpha + \beta} \left(\frac{A}{B}\right)^{-\beta} & \text{when } A \geq B, \\ V_B^b = \frac{c}{r} \left(1 - \frac{\beta}{\alpha + \beta} \left(\frac{A}{B}\right)^\alpha\right) & \text{when } A \leq B. \end{cases} \quad (8)$$

Observe that  $V_B^b = \frac{c}{r}$  when  $B = \infty$ . Also observe that  $V_B^b = \frac{c}{r} - V_A^b$ , if  $A < B$ , an infinite annuity with payments below  $B$  equals an infinite annuity from which an annuity with payments only above  $B$  is subtracted. Also observe that  $V_A^a = \frac{c}{r} - V_B^a$ , if  $A > B$ , an annuity with payments above  $B$  equals an infinite annuity from which an annuity with payments only below  $B$  is subtracted.

**Proof 3**  $V_B$  pays  $c$  only when  $A_t < B$ , so in expression (4)  $c(A) = c$  when  $A_t < B$ , and  $c = 0$  otherwise. Observe that  $f^s(A) = \frac{c}{r}$  solves equation (4) when  $A < B$ . The relevant boundary conditions are  $\lim_{A \rightarrow \infty} V_B^a = 0 \Rightarrow K_1 = 0$  and  $\lim_{A \rightarrow 0} V_B^b = 0 \Rightarrow K_2 = 0$ . To determine  $K_2$  for  $V_B^a$  and  $K_1$  for  $V_B^b$  we also here require continuity and smooth pasting at  $B$ , i.e.,  $V_B^a(B) = V_B^b(B)$  and  $\frac{\partial}{\partial A} V_B^a(B) = \frac{\partial}{\partial A} V_B^b(B)$ .

### 3.3 Above and below annuities with bankruptcy risk

Let  $D_j$  denote the value of a claim  $V_j$  where  $j \in \{A, B\}$ , including bankruptcy risk. Using economic arguments we show in the proof that

$$D_j(A, B) = V_j(A, B) - V_j^b(C, B)U^a(A, C). \quad (9)$$

**Proof 4** Upon bankruptcy, i.e., at time  $\tau$ , the value of the claim  $V_j$  is  $V_j^b(C, B_i)$ . Because  $C < B_i$  for all  $i \leq n$ ,  $V_j = V_j^b$ .  $V_j^b(C, B_i)$  therefore represents the reduction in value of the claim  $V_j$  due to bankruptcy at the time of bankruptcy. The initial value of this claim is found by discounting by  $U = U^a$  because  $A > C$ .

#### 3.3.1 The value of an above annuity in the case with bankruptcy risk

$$D_A(A, B) = \begin{cases} D_A^a = \frac{c}{r} \left[ 1 - \left( \frac{\alpha}{\alpha+\beta} \left( \frac{C}{B} \right)^{-\beta} + \frac{\beta}{\alpha+\beta} \left( \frac{C}{B} \right)^\alpha \right) \left( \frac{A}{C} \right)^{-\beta} \right] & \text{when } A \geq B, \\ D_A^b = \frac{c}{r} \frac{\beta}{\alpha+\beta} \left[ \left( \frac{A}{B} \right)^\alpha - \left( \frac{C}{B} \right)^\alpha \left( \frac{A}{C} \right)^{-\beta} \right] & \text{when } C \leq A \leq B. \end{cases} \quad (10)$$

The two first terms in the case where  $A \geq B$ , and the first term in the case where  $C \leq A \leq B$  are identical to the corresponding annuities without bankruptcy risk. The final terms in both cases are identical and equals (the negative of) the value of an above annuity below the barrier when  $A = C$  multiplied by  $U^a(A, C)$ , the value of 1 upon bankruptcy. In the case where  $B = C$  the results collapse to the standard Black and Cox (1976) result for infinite horizon debt with bankruptcy risk.

#### 3.3.2 The value of a below annuity in the case with bankruptcy risk

$$D_B(A, B) = \begin{cases} D_B^a = \frac{c}{r} \left[ \frac{\alpha}{\alpha+\beta} \left( \frac{C}{B} \right)^{-\beta} + \frac{\beta}{\alpha+\beta} \left( \frac{C}{B} \right)^\alpha - 1 \right] \left( \frac{A}{C} \right)^{-\beta} & \text{when } A \geq B, \\ D_B^b = \frac{c}{r} \left[ 1 - \frac{\beta}{\alpha+\beta} \left( \frac{A}{B} \right)^\alpha - \left( 1 - \frac{\beta}{\alpha+\beta} \left( \frac{C}{B} \right)^\alpha \right) \left( \frac{A}{C} \right)^{-\beta} \right] & \text{when } C \leq A \leq B. \end{cases} \quad (11)$$

Similarly as for the above annuity, the last term in both these expressions can be interpreted as the value of a below annuity below the barrier when  $A = C$  multiplied by the value of 1 upon bankruptcy.

Both these results can alternatively be derived by solving equation (4) with appropriate boundary conditions.

## 4 Forward starting infinite annuities

In this section we calculate the time 0 market values of infinite annuities which start at a future time  $T > 0$ . For a general forward starting claim  $\nu_j(A_T, B)$  the time zero value  $\xi_j(A, B)$  is calculated as

$$\xi_j(A, B) = E[e^{-rT}\nu_j(A_T, B)].$$

### 4.1 Forward starting annuities without bankruptcy risk

#### 4.1.1 Forward starting above annuity without bankruptcy risk

Denote the time zero market value of a forward starting above annuity by  $W_A$ . Then

$$W_A(A, B) = \tag{12}$$

$$\frac{c}{r} \left( e^{-rT} Q_g(B) - \frac{\alpha}{\alpha + \beta} \left(\frac{A}{B}\right)^{-\beta} Q_g^\beta(B) + \frac{\beta}{\alpha + \beta} \left(\frac{A}{B}\right)^\alpha Q_l^\alpha(B) \right),$$

where  $Q_g(B) = Q(A_T > B) = 1 - Q_l(B)$ ,  $Q_g^\beta(B) = Q^\beta(A_T > B) = 1 - Q_l^\beta(B)$ . Here  $Q_l(B)$ ,  $Q_l^\beta(B)$ , and  $Q_l^\alpha(B) = Q^\alpha(A_T \leq B)$  are defined in expressions (23), (27), and (31), respectively, in Appendix B.

#### Proof 5

$$\begin{aligned} W_A &= E[e^{-rT}V_A(A_T, B)] \\ &= E \left[ e^{-rT} (V_A^a(A_T, B)1\{A_T > B\} + V_A^b(A_T, B)1\{A_T < B\}) \right], \\ &= E \left[ e^{-rT} \frac{c}{r} \left( \left(1 - \frac{\alpha}{\alpha + \beta} U^a(A_T, B)\right) 1\{A_T > B\} + \frac{\beta}{\alpha + \beta} U^b(A_T, B) 1\{A_T < B\} \right) \right], \\ &= \frac{c}{r} \left( e^{-rT} Q(A_T > B) - \frac{\alpha}{\alpha + \beta} P_1(A, B) + \frac{\beta}{\alpha + \beta} P_2(A, B) \right), \end{aligned}$$

where  $P_1(A, B)$  and  $P_2(A, B)$  are defined in Appendix A, and the event  $Z$  is specialized to  $\{A_T > B\}$  for  $P_1(A, B)$  and  $\{A_T \leq B\}$  for  $P_2(A, B)$ . The result follows from the expressions (21) and (22) in Appendix A.

#### 4.1.2 Forward starting below annuity without bankruptcy risk

Denote the time zero market value of a forward starting below annuity by  $W_B$ . Then

$$W_B(A, B) = \frac{c}{r} \left( e^{-rT} Q_l(B) + \frac{\alpha}{\alpha + \beta} \left(\frac{A}{B}\right)^{-\beta} Q_g^\beta(B) - \frac{\beta}{\alpha + \beta} \left(\frac{A}{B}\right)^\alpha Q_l^\alpha(B) \right).$$

**Proof 6**

$$\begin{aligned}
W_B &= E[e^{-rT} V_B(A_T, B)] \\
&= E \left[ e^{-rT} (V_B^a(A_T, B) 1\{A_T > B\} + V_B^b(A_T, B) 1\{A_T < B\}) \right], \\
&= E \left[ e^{-rT} \frac{c}{r} \left( \frac{\alpha}{\alpha + \beta} U^a(A_T, B) 1\{A_T > B\} + \left(1 - \frac{\beta}{\alpha + \beta}\right) U^b(A_T, B) 1\{A_T < B\} \right) \right].
\end{aligned}$$

using similar definitions of  $Z$  as in the previous proof. The result follows from the expressions (21) and (22) in Appendix A.

**4.2 Forward starting annuities with bankruptcy risk**

Denote by  $\xi(A, B)$  the time zero market value of a general forwarding starting annuity  $\nu(A_T, B)$  delivered at time  $T$  upon no prior bankruptcy. Then

$$\xi(A, B) = E[e^{-rT} \nu(A_T, B) 1\{\tau > T\}].$$

**4.2.1 Forward starting above annuity with bankruptcy risk**

Denote the time zero market value of a forward starting above annuity with bankruptcy risk by  $Y_A$ . Then

$$Y_A(A, B) = \tag{13}$$

$$\frac{c}{r} \left( e^{-rT} Q_{gg}(B) - \frac{\alpha}{\alpha + \beta} \left(\frac{A}{B}\right)^{-\beta} Q_{gg}^\beta(B) + \frac{\beta}{\alpha + \beta} \left[ \left(\frac{A}{B}\right)^\alpha Q_{lg}^\alpha(B) - \left(\frac{C}{B}\right)^\alpha \left(\frac{A}{C}\right)^{-\beta} Q_g^\beta \right] \right),$$

where  $Q_{gg}(B) = Q(A_T > B, \tau > T)$ ,  $Q_{gg}^\beta(B) = Q^\beta(A_T > B, \tau > T)$ ,  $Q_{lg}^\alpha(B) = Q^\alpha(A_T \leq B, \tau > T)$ , and  $Q_g^\beta = Q^\beta(\tau > T)$ , are given in (25), (33), (30), and (32), respectively, in Appendix B.

**Proof 7**

$$\begin{aligned}
Y_A &= E[e^{-rT} D_A(A_T) 1\{\tau > T\}] \\
&= E \left[ e^{-rT} (D_A^a(A_T, B) 1\{A_T > B\} + D_A^b(A_T, B) 1\{A_T < B\}) 1\{\tau > T\} \right].
\end{aligned}$$

### 4.2.2 Forward starting below annuity with bankruptcy risk

Here,

$$Y_B(A, B) = \frac{c}{r} \left( e^{-rT} Q_{lg}(B) + \frac{\alpha}{\alpha + \beta} \left(\frac{A}{B}\right)^{-\beta} Q_{gg}^\beta(B) + \frac{\beta}{\alpha + \beta} \left[ \left(\frac{C}{B}\right)^\alpha \left(\frac{A}{C}\right)^{-\beta} Q_g^\beta - \left(\frac{A}{B}\right)^\alpha Q_{lg}^\alpha(B) \right] - Q_g^\beta \left(\frac{A}{C}\right)^{-\beta} \right), \quad (14)$$

where  $Q_{lg}(B) = Q(A_T \leq B, \tau > T)$  is given in expression (26) in Appendix B.

#### Proof 8

$$\begin{aligned} Y_B &= E[e^{-rT} D_B(A_T) 1\{\tau > T\}] \\ &= E[e^{-rT} (D_B^a(A_T, B) 1\{A_T > B\} + D_B^b(A_T, B) 1\{A_T < B\}) 1\{\tau > T\}]. \end{aligned}$$

As in the introduction, we calculate the time zero market value of a forward starting annuity with bankruptcy risk as

$$Y_A(A, B) + Y_B(A, B) = \frac{c}{r} (e^{-rT} Q_g - \left(\frac{A}{C}\right)^{-\beta} Q_g^\beta).$$

The first term is interpreted as the time zero value of a time  $T$  forward starting, infinite annuity. The second term is the time zero market value of a forward starting, infinite annuity, starting at the time of bankruptcy given that bankruptcy occurs after time  $T$ . From Appendix A we know that  $P_3(A, C) = \left(\frac{A}{C}\right)^{-\beta} Q_g^\beta$  can be interpreted as the value of 1 unit account payable at bankruptcy, only if bankruptcy occurs after time  $T$ .

## 5 Finite above and below annuities

In this section we show how the previous infinite horizon annuities can be combined into annuities with finite horizon. Our results are based on the fact that a finite annuity may be considered as an immediately starting infinite annuity from which the time zero value of another infinite annuity starting at the future date  $T$ , is subtracted. We assume in this section that  $A > B$ .

## 5.1 Finite annuities without bankruptcy risk

### 5.1.1 Finite above annuity without bankruptcy risk

The time zero market price of a finite above annuity without bankruptcy risk is calculated as

$$\begin{aligned} V_A^T(A, B) &= V_A(A, B) - W_A(A, B) \\ &= \frac{c}{r} - \frac{c}{r} \left( e^{-rT} Q_g(B) + \frac{\alpha}{\alpha + \beta} \left(\frac{A}{B}\right)^{-\beta} Q_l^\beta(B) + \frac{\beta}{\alpha + \beta} \left(\frac{A}{B}\right)^\alpha Q_l^\alpha(B) \right). \end{aligned}$$

### 5.1.2 Finite below annuity without bankruptcy risk

The time zero market price of a finite below annuity without bankruptcy risk is calculated as

$$\begin{aligned} V_B^T(A, B) &= V_B(A, B) - W_B(A, B) \\ &= \frac{c}{r} \left( -e^{-rT} Q_l(B) + \frac{\alpha}{\alpha + \beta} \left(\frac{A}{B}\right)^{-\beta} Q_l^\beta(B) + \frac{\beta}{\alpha + \beta} \left(\frac{A}{B}\right)^\alpha Q_l^\alpha(B) \right). \end{aligned}$$

Observe that the time zero value of a finite annuity which pays both above and below  $B$  is  $V_A^T(A, B) + V_B^T(A, B) = \frac{c}{r}(1 - e^{-rT})$ , a familiar result.

## 5.2 Finite annuities with bankruptcy risk

### 5.2.1 Finite above annuity with bankruptcy risk

The time zero market price of finite above annuity with bankruptcy risk is calculated as

$$\begin{aligned} D_A^T(A, B) &= D_A(A, B) - Y_A(A, B) \\ &= \frac{c}{r} \gamma, \end{aligned} \tag{15}$$

where

$$\begin{aligned} \gamma &= 1 - \frac{\alpha}{\alpha + \beta} \left(\frac{A}{B}\right)^{-\beta} (1 - Q_{gg}^\beta(B)) \\ &\quad - \frac{\beta}{\alpha + \beta} \left( \left(\frac{A}{B}\right)^\alpha Q_{lg}^\alpha(B) + \left(\frac{C}{B}\right)^\alpha \left(\frac{A}{C}\right)^{-\beta} Q_l^\beta(B) \right) - e^{-rT} Q_{gg}(B). \end{aligned}$$

### 5.2.2 Finite below annuity with bankruptcy risk

The time zero market price of finite above annuity with bankruptcy risk is calculated as

$$\begin{aligned} D_B^T(A, B) &= D_B(A, B) - Y_B(A, B) \\ &= \frac{c}{r}\eta, \end{aligned} \tag{16}$$

where

$$\begin{aligned} \eta &= \frac{\alpha}{\alpha + \beta} \left(\frac{A}{B}\right)^{-\beta} (1 - Q_{gg}^\beta(B)) \\ &\quad + \frac{\beta}{\alpha + \beta} \left( \left(\frac{A}{B}\right)^\alpha Q_{lg}^\alpha(B) + \left(\frac{C}{B}\right)^\alpha \left(\frac{A}{C}\right)^{-\beta} Q_l^\beta \right) - e^{-rT} Q_{lg}(B) - \left(\frac{A}{C}\right)^{-\beta} Q_l^\beta. \end{aligned}$$

As indicated in the introduction, we calculate the time zero market value of immediately starting, finite annuity with bankruptcy risk as

$$D_A^T(A, B) + D_B^T(A, B) = \frac{c}{r} (1 - e^{-rT} Q_g - \left(\frac{A}{C}\right)^{-\beta} Q_l^\beta).$$

The first term represents the time zero market value of an immediately starting, infinite annuity without bankruptcy risk. The second term represents (the negative of) the time zero market value of a time  $T$  forward starting infinite annuity without bankruptcy risk. The final term represents (the negative of) the time zero market value of a forward starting infinite annuity, starting at the time of bankruptcy, but only if bankruptcy occurs before time  $T$ .

## 6 The value of an infinite multi-level annuity

In this section we consider annuities with multiple barriers and possibly different coupons in the regions defined by these barriers, as explained in Section 2. The results from Sections 3 and 4 are our building blocks. To incorporate level-dependent annuities we formally assume that the parameter  $c$  in the previous formulas equals 1 and multiply by  $c_i$ , the region specific coupon. This approach is without any loss of generality. For simplicity we only treat the case where  $A > B_1$ .



## 6.1 Immediately starting, infinite multi-level annuities

### 6.1.1 Immediately starting, infinite multi-level annuity without bankruptcy risk

The time zero market value  $\hat{M}^\infty(A)$  of an infinite multi-level annuity in the case of no bankruptcy risk is

$$\hat{M}^\infty(A) = \frac{c_1}{r} - \frac{\alpha}{\alpha + \beta} \sum_{i=1}^n \frac{c_i - c_{i+1}}{r} \left(\frac{A}{B_i}\right)^{-\beta}. \quad (17)$$

**Proof 9** *The time zero market value of an annuity  $c_{i+1}$  which only is paid when  $B_i \leq A_t \leq B_{i+1}$ , for  $i = 0, \dots, n$ , is  $(V_A(A, B_{i+1}) - V_A(A, B_i))c_{i+1}$ . The time zero market value of the multi-level annuity is found by simply adding such annuities, i.e.,*

$$\hat{M}^\infty(A) = \sum_{i=0}^n (V_A(A, B_{i+1}) - V_A(A, B_i)) c_{i+1}.$$

Observe that  $V_A(A, B_0) = 0$  and that  $V_A(A, B_{n+1}) = \frac{c_{n+1}}{r}$ . The formula follows by direct calculations using expression (7) with  $c = 1$ .

### 6.1.2 Immediately starting, infinite multi-level annuity with bankruptcy risk

The approach in the previous subsection holds when there is bankruptcy risk. Denote the time zero value of the infinite version of the multi-level annuity in the case of bankruptcy risk by  $M^\infty(A)$ .

The time zero market value of an infinite multi-level annuity in the case of bankruptcy risk is

$$M^\infty(A) = \frac{c_1}{r} - \frac{c_{n+1}}{r} \left(\frac{A}{C}\right)^{-\beta} - \sum_{i=1}^n \frac{c_i - c_{i+1}}{r(\alpha + \beta)} \left( \alpha \left(\frac{C}{B_i}\right)^{-\beta} + \beta \left(\frac{C}{B_i}\right)^\alpha \right) \left(\frac{A}{C}\right)^{-\beta}. \quad (18)$$

Observe that expression (18) is reduced to expression (17) for  $C = 0$ .

**Proof 10** *Similarly as in the previous proof we may write*

$$M^\infty(A) = \sum_{i=0}^n (D_A(A, B_{i+1}) - D_A(A, B_i)) c_{i+1},$$

where  $D_A(A, B_i)$  is given in expression (10) and  $B_{n+1} = C$ . Observe that  $D_A(A, B_0) = 0$  and that  $D_A(A, B_{n+1}) = D_A(A, C) = \frac{c_{n+1}}{r} \left(1 - \left(\frac{A}{C}\right)^{-\beta}\right)$ . The formula follows by direct calculations with  $c = 1$ .

## 6.2 Forward starting, infinite multi-level annuities

### 6.2.1 Forward starting, infinite multi-level annuity without bankruptcy risk

The time zero market value  $\hat{M}_T^\infty(A)$  of an infinite, time  $T$  forward starting, multi-level annuity in the case of no bankruptcy risk is

$$\hat{M}_T^\infty(A) = \frac{c_{n+1}}{r} e^{-rT} - \sum_{i=1}^n \frac{c_i - c_{i+1}}{r} (\lambda_i - Q_g(B_i) e^{-rT}), \quad (19)$$

where

$$\lambda_i = \frac{\alpha \left(\frac{A}{B_i}\right)^{-\beta} Q_g^\beta(B_i) - \beta \left(\frac{A}{B_i}\right)^\alpha Q_l^\alpha(B_i)}{\alpha + \beta}.$$

**Proof 11** *Similarly to the previous proofs*

$$\hat{M}_T^\infty(A) = \sum_{i=0}^n (W_A(A, B_{i+1}) - W_A(A, B_i)) c_{i+1},$$

where  $W_A(A, B_i)$  is given in expression (12) and  $B_{n+1} = 0$ . Observe that  $W_A(A, B_0) = 0$  and that  $W_A(A, B_{n+1}) = W_A(A, 0) = \frac{c_{n+1}}{r} e^{-rT}$ . The formula follows by direct calculations with  $c = 1$ .

### 6.2.2 Forward starting, infinite multi-level annuity with bankruptcy risk

The time zero market value  $M_T^\infty(A)$  of an infinite, time  $T$  forward starting, multi-level annuity in the case of bankruptcy risk is

$$M_T^\infty(A) = \frac{c_{n+1}}{r} [e^{-rT} Q_g - \left(\frac{A}{C}\right)^{-\beta} Q_g^\beta] - \sum_{i=1}^n \frac{c_i - c_{i+1}}{r} (\kappa_i - Q_{gg}(B_i) e^{-rT}), \quad (20)$$

where

$$\kappa_i = \frac{\alpha \left(\frac{A}{B_i}\right)^{-\beta} Q_{gg}^\beta(B_i) - \beta \left(\frac{A}{B_i}\right)^\alpha Q_{lg}^\alpha(B_i) + \beta \left(\frac{A}{C}\right)^{-\beta} Q_g^\beta\left(\frac{C}{B_i}\right)^\alpha}{\alpha + \beta}$$

**Proof 12** *Similarly to the previous proofs*

$$D_{0,T}^{\infty}(A) = \sum_{i=0}^n (Y_A(A, B_{i+1}) - Y_A(A, B_i)) c_{i+1},$$

where  $Y_A(A, B_i)$  is given in expression (13) and  $B_{n+1} = C$ . Observe that  $Y_A(A, B_0) = 0$  and that  $Y_A(A, B_{n+1}) = Y_A(A, C) = \frac{c_{n+1}}{r}[e^{-rT}Q(\tau > T) - (\frac{A}{C})^{-\beta}Q^{\beta}(\tau > T)]$ . The formula follows by direct calculations with  $c = 1$ .

Also here observe that expression (20) is reduced to expression (19) for  $C = 0$ .

## 7 Finite multi-level annuities

Also in this section we calculate the values of the finite multi-level annuities as the difference between infinite annuities and forward starting annuities.

### 7.1 Finite multi-level annuity without bankruptcy risk

The time zero market value  $\hat{M}^T(A)$  of an finite multi-level annuity in the case of no bankruptcy risk is

$$\begin{aligned} \hat{M}^T(A) &= \hat{M}^{\infty}(A) - \hat{M}_T^{\infty}(A), \\ &= \frac{c_1}{r} - \frac{c_{n+1}}{r}e^{-rT} + \sum_{i=1}^n \frac{c_i - c_{i+1}}{r} (\phi_i - Q_g(B_i)e^{-rT}), \end{aligned}$$

where

$$\phi_i = \frac{-(\alpha(\frac{A}{B_i})^{-\beta}Q_l^{\beta}(B_i) + \beta(\frac{A}{B_i})^{\alpha}Q_l^{\alpha}(B_i))}{\alpha + \beta}.$$

### 7.2 Finite multi-level annuity with bankruptcy risk

The time zero market value  $M^T(A)$  of an finite multi-level annuity in the case of bankruptcy risk is

$$\begin{aligned} M^T(A) &= M^{\infty}(A) - M_T^{\infty}(A) \\ &= \frac{c_1}{r} - \frac{c_{n+1}}{r} \left[ \left(\frac{A}{C}\right)^{-\beta}Q_l^{\beta} + e^{-rT}Q_g \right] + \sum_{i=1}^n \frac{c_i - c_{i+1}}{r} (\psi_i - Q_{gg}(B_i)e^{-rT}), \end{aligned}$$

where

$$\psi_i = \frac{\alpha(\frac{A}{B_i})^{-\beta}(Q_{gg}^{\beta}(B_i) - 1) - \beta(\frac{A}{B_i})^{\alpha}Q_{lg}^{\alpha}(B_i) - \beta(\frac{A}{C})^{-\beta}(\frac{C}{B_i})^{\alpha}Q_l^{\beta}}{\alpha + \beta}.$$

This result is already presented in expression (1) in Section 2, but is also included here for completeness.

## 8 Conclusions and areas of further research

We present closed form solutions for the market value of multi-level annuities applicable to debt type contracts with bankruptcy risk. Possible applications with varying annuity levels include US Chapter 11 regulations, strategic debt service and hybrid capital for financial institutions. In the introduction we showed how our results may be used to decompose the total value of an annuity into region specific values. Our results may also be applied to more general models including endogenous coupons and financial health levels.

It is also straightforward to generalize our results to the case where all barriers, including the bankruptcy barrier, are time dependent and exponential, i.e., on the form  $B_t = Be^{\gamma t}$  for a constant  $\gamma$ , identical for all barriers.

## A Some standard valuation results

In this appendix we apply the change of measure technique introduced in finance by Geman, El Karoui, and Rochet (1995).

Let  $Z$  be any  $\mathcal{F}_T$ -measurable event. Denote its associated indicator function by  $1_Z$ .

First, the time zero market value of a claim with time  $T$  market value  $U^a(A_T, B)$ , given in expression (6), receivable at time  $T$  only if the event  $Z$  occurs is

$$\begin{aligned}
 P_1(A, B) &= \mathbb{E}[e^{-rT}U^a(A_T, B)1_Z], \\
 &= U^a(A, B)\mathbb{E}[1_Ze^{-\frac{1}{2}\sigma^2\beta^2T-\sigma\beta W_T}], \\
 &= U^a(A, B)Q^\beta(Z), \\
 &= \left(\frac{A}{B}\right)^{-\beta}Q^\beta(Z),
 \end{aligned} \tag{21}$$

where the probability measure  $Q^\beta$  is defined by  $\frac{\partial Q^\beta}{\partial Q} = \exp(-\frac{1}{2}\sigma^2\beta^2T - \sigma\beta T)$ , and the dynamics of  $A_t$  under  $Q^\beta$  is  $dA_t = (\mu - \sigma^2\beta)A_t dt + \sigma A_t dW_t$  (abusing notation by letting  $W_t$  also denote a standard Brownian motion under  $Q^\beta$ ).

Similarly, the time zero market value of a claim with time  $T$  market value  $U^b(A_T, B)$ , given in expression (6), receivable at time  $T$  only if the event  $Z$

occurs is

$$\begin{aligned}
P_2(A, B) &= \mathbb{E}[e^{-rT}U^b(A_T, B)1_Z], \\
&= U^b(A, B)\mathbb{E}[1_Z e^{-\frac{1}{2}\sigma^2\alpha^2T + \sigma\alpha W_T}], \\
&= U^b(A, B)Q^\alpha(Z), \\
&= \left(\frac{A}{B}\right)^\alpha Q^\alpha(Z), \tag{22}
\end{aligned}$$

where the probability measure  $Q^\alpha$  is defined by  $\frac{\partial Q^\alpha}{\partial Q} = \exp(-\frac{1}{2}\sigma^2\alpha^2T + \sigma\alpha T)$  and the dynamics of  $A_t$  under  $Q^\alpha$  is  $dA_t = (\mu + \sigma^2\alpha)A_t dt + \sigma A_t dW_t$  (repeatedly abusing notation by letting  $W_t$  also denote a standard Brownian motion under  $Q^\alpha$ ).

The time 0 market value of a claim which pays 1 upon bankruptcy (when  $A_t$  hits  $C$ ) if bankruptcy occurs after time  $T$  is

$$\begin{aligned}
P_3(A, C) &= \mathbb{E}[e^{-rT}U^a(A_T, C)1\{\tau > T\}], \\
&= U^a(A, C)\mathbb{E}[e^{-\frac{1}{2}\sigma^2\beta^2T - \sigma\beta W_T}1\{\tau > T\}], \\
&= U^a(A, C)Q^\beta(\tau > T), \\
&= \left(\frac{A}{C}\right)^{-\beta}Q^\beta(\tau > T).
\end{aligned}$$

Finally, the time 0 market value of a claim which pays 1 upon bankruptcy (when  $A_t$  hits  $C$ ) if bankruptcy occurs before time  $T$  is

$$\begin{aligned}
P_4(A, C) &= U^a(A, C)Q^\beta(\tau \leq T), \\
&= \left(\frac{A}{C}\right)^{-\beta}Q^\beta(\tau \leq T).
\end{aligned}$$

## B Some standard probability results

In this appendix we consider the process below under different probability measures. Consider

$$X_t = \ln(A_t) = \ln(A) + \hat{\mu}t + \sigma W_t,$$

where  $W_t$  is defined under a fixed probability measure  $P$ , and  $\ln(A)$ ,  $\hat{\mu}$ , and  $\sigma$  are constants. The process  $X_t$  represents the logarithmic version of the process  $A_t$  used in the paper. Define the stopping time

$$\tau = \inf\{t : A_t = C\}.$$

The following results are standard

$$P_g = P(\tau > T) = N\left(\frac{\ln(\frac{A}{C}) + \hat{\mu}T}{\sigma\sqrt{T}}\right) - \left(\frac{A}{C}\right)^{\frac{-2\hat{\mu}}{\sigma^2}} N\left(-\frac{\ln(\frac{A}{C}) - \hat{\mu}T}{\sigma\sqrt{T}}\right).$$

$$P_{gg}(B) = P(A_T > B), \tau > T) = N\left(\frac{\ln(\frac{A}{B}) + \hat{\mu}T}{\sigma\sqrt{T}}\right) - \left(\frac{A}{C}\right)^{\frac{-2\hat{\mu}}{\sigma^2}} N\left(-\frac{\ln(\frac{A}{C}) + \ln(\frac{B}{C}) - \hat{\mu}T}{\sigma\sqrt{T}}\right).$$

Observe that  $\lim_{B \downarrow C} P(X_t > \ln(B), \tau > T) = P(\tau > T)$ . Trivially,

$$P_{lg}(B) = P(A_T < B), \tau > T) = N\left(\frac{\ln(\frac{A}{C}) + \hat{\mu}T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln(\frac{A}{B}) + \hat{\mu}T}{\sigma\sqrt{T}}\right) + \left(\frac{A}{C}\right)^{\frac{-2\hat{\mu}}{\sigma^2}} \left( N\left(-\frac{\ln(\frac{A}{C}) + \ln(\frac{B}{C}) - \hat{\mu}T}{\sigma\sqrt{T}}\right) - N\left(-\frac{\ln(\frac{A}{C}) - \hat{\mu}T}{\sigma\sqrt{T}}\right) \right).$$

Here  $N(\cdot)$  denotes the cumulative standard normal distribution function. The notation  $P_g$  is used for the univariate distribution of the stopping time  $\tau$ , the  $g$  signifies that  $\tau$  is *greater* than  $T$ . The notation  $P_{gg}(B)$  is used for the joint distribution between  $A_T$  and  $\tau$ , footscript  $gg$  indicates that  $A_T$  is *greater* than the value in the paranthesis  $B$  and that  $\tau$  is *greater* than  $T$ . Similarly, an occurrence of  $l$  in the footscript signifies that the relevant variable is *lower* than some value. For example the notation  $P_l(B)$  is used for the univariate distribution of  $A_T$ , the  $l$  signifies the probability of the event  $A_T$  is *lower* than  $B$ . A similar notation is used throughout.

## B.1 Probability measure $Q$

Under the probability measure  $Q$

$$\hat{\mu} = \mu - \frac{1}{2}\sigma^2.$$

$$Q_l(B) = Q(A_T \leq B) = N(-d_3), \quad (23)$$

where

$$d_3 = \frac{\ln(\frac{A}{B}) + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

$$Q_g = Q(\tau > T) = N(d_1) - \left(\frac{A}{C}\right)^{\alpha-\beta} N(-d_2), \quad (24)$$

where

$$d_1 = \frac{\ln(\frac{A}{C}) + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

and

$$d_2 = \frac{\ln(\frac{A}{C}) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

Also,

$$Q_{gg}(B) = Q(A_T > B, \tau > T) = N(d_3) - \left(\frac{A}{C}\right)^{\alpha-\beta} N(-d_4), \quad (25)$$

$$d_4 = \frac{\ln(\frac{A}{C}) + \ln(\frac{B}{C}) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

and

$$Q_{lg}(B) = Q(A_T < B, \tau > T) = N(d_1) - N(d_3) + \left(\frac{A}{C}\right)^{\alpha-\beta} (N(-d_4) - N(-d_2)). \quad (26)$$

## B.2 Probability measure $Q^\alpha$

Under the probability measure  $Q^\alpha$

$$\hat{\mu} = \mu + \sigma^2\alpha - \frac{1}{2}\sigma^2.$$

$$Q_l^\alpha(B) = Q^\alpha(A_T \leq B) = N(-d_3^\alpha), \quad (27)$$

where

$$d_3^\alpha = \frac{\ln(\frac{A}{B}) + (\mu + \sigma^2\alpha - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

$$Q_g^\alpha = Q^\alpha(\tau > T) = N(d_1^\alpha) - \left(\frac{A}{C}\right)^{-(\alpha+\beta)} N(-d_2^\alpha), \quad (28)$$

where

$$d_1^\alpha = \frac{\ln(\frac{A}{C}) + (\mu + \sigma^2\alpha - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

and

$$d_2^\alpha = \frac{\ln(\frac{A}{C}) - (\mu + \sigma^2\alpha - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

Also,

$$Q_{gg}^\alpha(B) = Q^\alpha(A_T > B, \tau > T) = N(d_3^\alpha) - \left(\frac{A}{C}\right)^{-(\alpha+\beta)} N(-d_4^\alpha), \quad (29)$$

where

$$d_4^\alpha = \frac{\ln(\frac{A}{C}) + \ln(\frac{B}{C}) - (\mu + \sigma^2\alpha - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

and

$$Q_{ig}^\alpha(B) = Q^\alpha(A_T < B, \tau > T) = N(d_1^\alpha) - N(d_3^\alpha) + (\frac{A}{C})^{-(\alpha+\beta)}(N(-d_4^\alpha) - N(-d_2^\alpha)). \quad (30)$$

### B.3 Probability measure $Q^\beta$

Under the probability measure  $Q^\beta$

$$\hat{\mu} = \mu - \sigma^2\beta - \frac{1}{2}\sigma^2.$$

$$Q_i^\beta(B) = Q^\beta(A_T \leq B) = N(-d_3^\beta), \quad (31)$$

where

$$d_3^\beta = \frac{\ln(\frac{A}{B}) + (\mu - \sigma^2\beta - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

$$Q_g^\beta = Q^\beta(\tau > T) = N(d_1^\beta) - (\frac{A}{C})^{(\alpha+\beta)}N(-d_2^\beta), \quad (32)$$

where

$$d_1^\beta = \frac{\ln(\frac{A}{C}) + (\mu - \sigma^2\beta - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

and

$$d_2^\beta = \frac{\ln(\frac{A}{C}) - (\mu - \sigma^2\beta - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

Also,

$$Q_{gg}^\beta(B) = Q^\beta(A_T > B, \tau > T) = N(d_3^\beta) - (\frac{A}{C})^{(\alpha+\beta)}N(-d_4^\beta), \quad (33)$$

where

$$d_4^\beta = \frac{\ln(\frac{A}{C}) + \ln(\frac{B}{C}) - (\mu - \sigma^2\beta - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

and

$$Q_{ig}^\beta(B) = Q^\beta(A_T < B, \tau > T) = N(d_1^\beta) - N(d_3^\beta) + (\frac{A}{C})^{(\alpha+\beta)}(N(-d_4^\beta) - N(-d_2^\beta)). \quad (34)$$



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