Core and Bargaining Set of Shortest Path Games

Sofia Grahn[1]

Abstract: In this paper it is shown that the core and the bargaining sets of Davis-Maschler and Zhou coincide in a class of shortest path games

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[1]Department of Economics, Uppsala University, P.O.Box 513, 75120 Uppsala, Sweden. Sofia.Grahn@nek.uu.se. The author acknowledges the hospitality of the Department of Econometrics at Tilburg University during part of the research. Helpful comments by Mark Voorneveld are greatfully appreciated.

1 Introduction

In the operations research literature, numerous algorithms and heuristics for network optimization problems are offered. In game theoretic approaches to this type of problems, different agents control the elements of the network. Therefore, next to finding optimal solutions, this adds the problem of dividing the costs or benefits generated by such solutions over the involved agents. An overview of several classes of games arising from problems in operations research is provided by Curiel (1997).

Only recently two game theoretical approaches have been initiated to shortest path problems: Fragnelli, García-Jurado, and Méndez-Naya (2000) and Voorneveld and Grahn (2000). The present paper follow the model of Voorneveld and Grahn (2000). In this model each player owns arcs or connections in a finite network. There are costs associated to the use of each arc. For each player there is a reward (possibly equal to zero) if he transports his goods from source to sink.The value of a coalition is the maximal profit it can generate by transporting its goods from source to sink via a shortest path owned by this coalition. In these shortest path games, which are briefly reviewed in Section 3, it is shown that the core and the bargaining set of Davis and Maschler (1963) and Zhou (1994) coincide.

Solymosi (1999) provided necessary and sufficient conditions for the core and the Davis-Maschler bargaining set to coincide. According to these conditions, certain games, induced by imputations in the bargaining set, have to be balanced. In practice however, it is very hard to prove that a certain class of games satisfy these conditions. Therefore this paper gives a more direct approach.

Basic game theoretical definitions are given in Section 2. In Section 3 shortest path games are defined. Section 4 contain the main result of the paper: the core and the bargaining sets of Davis-Maschler and Zhou are shown to coincide in shortest path games.

2 Preliminaries

This section settles matters of notation and defines basic game theoretic concepts.

For a finite set N, $2^N = \{S : S \subseteq N\}$ denotes the collection of all subsets of N. $\mathbb{R}$ denotes the set of reals, $\mathbb{R}_+ = [0, \infty)$ the set of nonnegative reals. For $(x_i)_{i \in N}$, where $x_i \in \mathbb{R}$ for each $i \in N$, and for a subset $S \subseteq N$, we denote $x(S) = \sum_{i \in S} x_i$. The end of proofs is indicates with the symbol $\square$, the end of definitions, examples, and remarks with the symbol $\triangleleft$.

Recall that a cooperative game with transferable utility (TU-game) is a tuple (N, v), where N is a finite set of players and $v : 2^N \setminus \{\emptyset\} \to \mathbb{R}$ is a function that assigns to each coalition $S \in 2^N \setminus \{\emptyset\}$ its value $v(S) \in \mathbb{R}$.

Definition 2.1 Let (N, v) be a TU-game. A vector $(x_i)_{i \in S}$ of real numbers is an S-feasible payoff vector if $x(S) = v(S)$. $\triangleleft$

An N-feasible payoff vector is refered to as a feasible payoff profile.

Definition 2.2 An imputation of (N,v) is a feasible payoff profile $x \in \mathbb{R}$ for which $x_i \geq v(\{i\})$ for all $i \in N$. The set $I(N,v)$ is the set of all imputations of (N,v) :

$$I(N,v) = \left\{x \in \mathbb{R}^N : x(N) = v(N) \text{ and } x_i \geq v(\{i\}) \ \forall i \in N\right\}$$

$\triangleleft$

The bargaining set is a solution concept for cooperative games and consists of those imputations for which every objection is refuted by a counterobjection.

Let x be an imputation of a TU-game (N,v). Following Davis and Maschler (1963) objections and counterobjections are defined as follows:

- A pair (y,S), where S is a coalition and y is an S-feasible payoff vector, is an objection of i against j to x if S includes i but not j and $y_k > x_k$ for all $k \in S$.
- A pair (z,K), where K is a coalition and z is a K-feasible payoff vector, is a counterobjection to the objection (y,S) of i against j if K includes j but not i, $z_k \geq x_k$ for all $k \in K \backslash S$, and $z_k \geq y_k$ for all $k \in K \cap S$.

The bargaining set is defined as follows:

Definition 2.3 The bargaining set $B(N,v)$ of a TU-game (N,v) is the set of all imputations x with the property that for every objection (y,S) of any player i against any other player j to x there is a counterobjection to (y,S) by j. $\triangleleft$

Definition 2.4 The core of a game (N,v) is the set

$$C(N,v) = \{x \in \mathbb{R}^N \mid x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for each } S \in 2^N \setminus \{\emptyset\}\}.$$

$\triangleleft$

Note that an imputation is in the core if and only if no player has an objection against any other player; hence the core is a subset of the bargaining set.

3 Shortest path games

This section summarizes the model of shortest path games of Voorneveld and Grahn (2000). In this model there is a finite set of players. Each player owns arcs or connections in a finite network. There are costs associated to the use of each arc. Each player receives a nonnegative reward if he manages to transport a good from the source of the network to its sink.

Definition 3.1 A *shortest path problem* is a tuple $\langle N, V, (A_i)_{i\in N}, w, (r_i)_{i\in N}\rangle$, where

- N is a finite set of players;

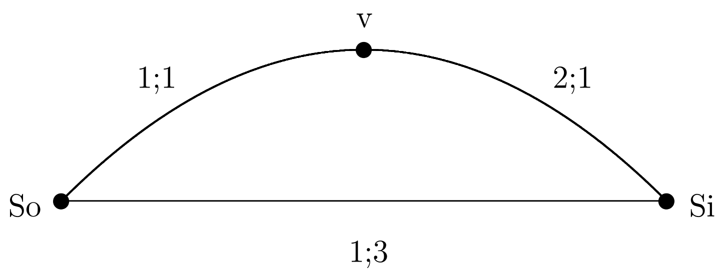

Figure 1: A shortest path problem

- V is a finite set of vertices with two special elements: the *source* So and the *sink* Si;
- each player $i \in N$ owns a set $A_i \subseteq V \times V$ of directed arcs in the network;
- the function $w : \cup_{i \in N} \{i\} \times A_i \to \mathbb{R}_+$ assigns a cost (or length, or weight) to all the arcs owned by the players. The cost assigned to arc (a, b) owned by player $i \in N$ is $w(i, (a, b)) \in \mathbb{R}_+$;
- each player $i \in N$ receives a reward $r_i \in \mathbb{R}_+$ for transporting his goods from the source to the sink.

◁

Notice that more than one player can own an arc between two vertices, and that the costs of an arc can depend on its owner. The following example illustrates the definition of a shortest path game.

Example 3.2 The shortest path problem with player set $N = \{1, 2, 3\}$, vertex set $V = \{So, Si, v\}$, the players respectively owning arc sets $A_1 = \{(So, v), (So, Si)\}, A_2 = \{(v, Si)\}, A_3 = \emptyset$, and costs $w(1, (So, v)) = w(2, (v, Si)) = 1, w(1, (So, Si)) = 3$ is depicted in Figure 1 (where it is assumed that arcs are directed from left to right). The numbers $2; 1$ next to the arc (v, Si), for instance, indicate that this arc is owned by player 2 and that the costs of this arc owned by player 2 equal 1. Take the rewards equal to $r_1 = r_2 = r_3 = 2$. ◁

Let $S \in 2^N \setminus \{\emptyset\}$ be a coalition of players. A path owned by the players in S is a sequence $(v_1, i_1, v_2, i_2, \ldots, i_{m-1}, v_m)$ of vertices v_k and players i_k such that $v_1 = So$, $v_m = Si$, and for each $k \in \{1, \ldots, m-1\}$ the arc (v_k, v_{k+1}) is owned by player $i_k \in S$. Let $P(S)$ denote the collection of all paths owned by coalition S.

The costs associated to a path $p = (v_1, i_1, v_2, i_2, \ldots, i_{m-1}, v_m) \in P(S)$ are defined as the sum of the costs of its arcs:

$$cost(p) = \sum_{k=1}^{m-1} w(i_k, (v_k, v_{k+1})).$$

Obviously, if a coalition S has to go from source to sink, it will choose among its alternatives in $P(S)$ the path with minimal costs. Define for each $S \in 2^N \setminus \{\emptyset\}$:

$$c(S) = \begin{cases} \min_{p \in P(S)} \ cost(p) & \text{if } P(S) \neq \emptyset, \\ \infty & \text{otherwise.} \end{cases}$$

Shortest paths in directed networks can be determined, for instance, by the algorithm of Dijkstra (1959).

From now on, when we refer to 'a shortest path' without explicitly stating a coalition owning it, we mean a shortest path owned by the grand coalition N.

The cooperative game associated with a shortest path problem reflects the following intuition: if a coalition $S \in 2^N \setminus \{\emptyset\}$ transports its goods from source to sink, it will receive a total reward $r(S) = \sum_{i \in S} r_i$ and incur costs $c(S)$, the costs of the cheapest alternative S has to go from source to sink. If $r(S) - c(S) > 0$, coalition S makes a profit. If $r(S) - c(S) \leq 0$, coalition S can generate profit zero by simply doing nothing. Therefore, coalition S can make a profit $\max\{r(S) - c(S), 0\}$.

Definition 3.3 Let $\langle N, V, (A_i)_{i \in N}, w, (r_i)_{i \in N} \rangle$ be a shortest path problem. The associated *shortest path game* (N, v) is defined as follows:

$$\forall S \in 2^N \setminus \{\emptyset\} \quad : \quad v(S) = \max\{r(S) - c(S), 0\}.$$

◁

Let $S, T \in 2^N \setminus \{\emptyset\}, S \subseteq T$. Then $r(S) \leq r(T)$ and $c(S) \geq c(T)$, so $v(S) \leq v(T)$: the shortest path game (N, v) is monotonic.

Example 3.4 In the shortest path problem of Example 3.2, coalition $\{2\}$and $\{3\}$ do not own a path: $c(\{i\}) = \infty$ for $i = 2, 3$. The cooperative game associated with the shortest path problem is given by $v(\{1\}) = \max\{r_1 - c(\{1\}), 0\} = \max\{2 - 3, 0\} = 0, v(\{2\}) = v(\{3\}) = \max\{2 - \infty, 0\} = 0, v(\{1, 2\}) = \max\{r(\{1, 2\}) - c(\{1, 2\}), 0\} = 2, v(\{1, 3\}) = 1, v(\{2, 3\}) = 0, v(N) = 4.$ ◁

Voorneveld and Grahn (2000) prove that shortest path games are totally balanced, indicate easy ways to construct core elements and provide the existence of population monotonic allocation schemes (Sprumont, 1990). Moreover, they provide a cost allocation mechanism taking into account opportunity cost and the role of players that are crucial to the construction of the shortest path.

4 Core and bargaining set coincide

In this section, the bargaining set is shown to coincide with the core for shortest path games. We already know that $C(N, v) \subseteq B(N, v)$ for each TU-game (N, v). The intuition behind the proof of $C(N, v) \supseteq B(N, v)$ is roughly as follows:

Let x be an imputation that is not in the core. Let T be the set of players who get less then their reward by allocation x. Let S be the "best" coalition for T to join if the players in T have to pay $c(S)$ and $x(S \backslash T) - r(S \backslash T)$ (the cost of the shortest path owned by S and the cost of players in S who get more then their reward). The players in T are the only players who contribute to the cost of the used path and to the cost of players who according to allocation x, get more than their reward. S also is the "best" coalition for every subset $P \subset T$ to join. Let $j \in \arg\max_{t \in N \backslash S} x_t - r_t$, i.e., the player outside S who is most "costly". For each $i \in T$ there is an objection (S, y) against player j. For player j to counterobject there has to be a counterobjection (K, z) such that $i \notin K$. We prove that $P = K \cap T \neq \emptyset$, i.e. to counterobject player j has to use players in T. Since S is the best coalition for $P \subset T$ to join, we know that the cost of P for joining coalition K is equal or larger to the cost of joining coalition S. But since $i \notin K$ and $x_i < r_i$ it is clear that the cost of P joining K is larger then the cost of joining S. This means that at least one player in P must be worse off in (K, z) than in (S, y). Hence, there is no counterobjection (K, z), so $x \notin B(N, v)$.

Lemma 4.1 *Let $\langle N, V, (A_i)_{i \in N}, w, (r_i)_{i \in N} \rangle$ be a shortest path problem and (N, v) the associated shortest path game. Let $x \in I(N, v) \backslash C(N, v)$ and $T = \{i \in N : x_i < r_i\}$. Let $S \in \min_{P \in 2^N} x(P) - v(P)$, then*

i) $T \neq \emptyset$;

ii) $S \neq N$, $x(S) - v(S) < 0$ and $T \subseteq S$;

iii) $S \backslash T$ solves $\min_{P \subseteq N \backslash T} x(P) - r(P) + c(P \cup T)$.

Proof. i) Assume that $T = \emptyset$. Then $x_i \geq r_i \; \forall i \in N$. Since $x(N) = v(N) \leq r(N)$ we have that $x_i = r_i \; \forall i \in N$. But if $x \in I(N, v) \backslash C(N, v)$, then $r = x \in C(N, v)$, since $x(R) = r(R) \geq v(R)$ for each $R \subseteq N$, contradicting the assumption that $x \notin C(N, v)$.

ii) $x \in I(N, v) \backslash C(N, v)$ implies that $x(P) - v(P) < 0$ for some coalition P. Since $S \in \min_{P \in 2^N} x(P) - v(P)$ and $x(N) = v(N)$ it follows that $S \neq N$, and $x(S) - v(S) < 0$. Assume that $i \in T \backslash S$, then

$$\begin{aligned} x(S) - v(S) &> x_i - v(S) - r_i + x(S) \\ &\geq x(S \cup \{i\}) - v(S \cup \{i\}), \end{aligned}$$

contradicting that $S \in \arg\min_{P \in 2^N} x(P) - v(P)$. Hence $T \backslash S = \emptyset$ so $T \subseteq S$.

iii) $T \subseteq S$ and $S \in \min_{P \in 2^N} x(P) - v(P)$ gives that $S \backslash T$ solves

$$\begin{aligned} & \min_{P \subseteq N \backslash T} x(P) + x(T) - r(P) - r(T) + c(P \cup T) \\ = & \left[\min_{P \subseteq N \backslash T} x(P) - r(P) + c(P \cup T) \right] + x(T) - r(T), \end{aligned}$$

so $S \backslash T \in \min_{P \subseteq N \backslash T} x(P) - r(P) + c(P \cup T)$. □

Lemma 4.2 *Let $\langle N, V, (A_i)_{i\in N}, w, (r_i)_{i\in N}\rangle$ be a shortest path problem and (N, v) the associated shortest path game. Let $x \in I(N, v) \backslash C(N, v)$ and sets S and T as in Lemma 4.1. Let $i \in T$ object to*

$$j \in \arg\max_{t\in N\backslash S} x_t - r_t$$

via an objection (S, y). Assume that (K, z) is a counterobjection of j to (S, y), i.e. z is a K-feasible payoff vector such that:

$$\begin{aligned} z_k &\geq x_k && \forall k \in K\backslash S \\ z_k &\geq y_k > x_k && \forall k \in K \cap S. \end{aligned}$$

then

i) $v(K) = r(K) - c(K)$;

ii) $P = K \cap T \neq \emptyset$.

Proof. i) Assume that $r(K) - c(K) < 0$. Then $v(K) = 0$ which gives that $z(K) = 0$. By the definition of a counterobjection and individual rationality of x, it follows that $0 = z(K) \geq x(K) \geq \sum_{i\in K} v(\{i\}) \geq 0$ so $x(K) = 0$. This gives that $x_j = r_j = 0$, and since $j \in \arg\max_{t\in N\backslash S} x_t - r_t$ it follows that $x_k = r_k = 0$ $\forall k \in N\backslash S$. This gives that $x(S) - v(S) = x(N) - v(S) \geq x(N) - v(N) = 0$ contradicting Lemma 4.1 ii). Hence $v(K) = r(K) - c(K) \geq 0$

ii) Discern two cases:

CASE 1: $c(K) > 0$
If $K \cap T = \emptyset$, then $z_k \geq x_k \geq r_k$ $\forall k \in K$ implies $z(K) \geq r(K) > r(K) - c(K) = v(K)$, contradicting that z is K-feasible payoff vector. Hence $K \cap T \neq \emptyset$.

CASE 2: $c(K) = 0$
$c(K) = 0$ gives that $x_j - r_j > 0$. Otherwise, $r_k = x_k$ $\forall k \in N\backslash S$. But then $c(N) = 0$ implies that $x(N) = v(N) = r(N)$, so $x(S) = r(S) \geq v(S)$, contradicting $v(S) - x(S) > 0$. Assume that $K \cap T = \emptyset$, then $z_k \geq x_k \geq r_k$ $\forall k \in K$ and $x_j - r_j > 0$ implies $z(K) > r(K) = v(K)$, contradicting that z is K-feasible payoff vector. Hence $K \cap T \neq \emptyset$. $\square$

Lemma 4.3 *Let $\langle N, V, (A_i)_{i\in N}, w, (r_i)_{i\in N}\rangle$ be a shortest path problem and (N, v) the associated shortest path game. Let $x \in I(N, v) \backslash C(N, v)$. Let S and T be defined as in Lemma 4.1 and take K, P, y and z as in Lemma 4.2. Define*

$$M_S := x(S\backslash T) - r(S\backslash T) + c(S)$$

and

$$M_K := z(K\backslash P) - r(K\backslash P) + c(K).$$

Then $M_K \geq M_S$.

Proof.

$$\begin{aligned} M_K &= z(K\backslash P) - r(K\backslash P) + c(K) \\ &\geq x(K\backslash P) - r(K\backslash P) + c(K) \\ &= x(K\backslash T) - r(K\backslash T) + c(K) \\ &\geq x(K\backslash T) - r(K\backslash T) + c(K \cup T) \\ &\geq x(S\backslash T) - r(S\backslash T) + c(S\backslash T \cup T) \\ &= M_S, \end{aligned}$$

where the first inequality follows from the definition of a counterobjection, the second from decreasingness of the cost function, and the third inequality from Lemma 4.1 iii). □

Proposition 4.4 *Let $\langle N, V, (A_i)_{i\in N}, w, (r_i)_{i\in N}\rangle$ be a shortest path problem and (N, v) the associated shortest path game. Then the core and the bargaining set of (N, v) coincide.*

Proof. We already know that $C(N, v) \subseteq B(N, v)$. To prove that $C(N, v) \supseteq B(N, v)$. Let $x \in I(N, v)\backslash C(N, v)$ and $T = \{i \in N : x_i < r_i\}$. Then lemma 4.1 i) gives $T \neq \emptyset$. Let

$$S \in \arg\max_{P\in 2^N} v(P) - x(P).$$

Then lemma 4.1 iii) gives that $S\backslash T$ solves

$$\min_{P\subseteq N\backslash T} x(P) - r(P) + c(P \cup T).$$

Let (S, y) be an objection of player $i \in T$ against player $j \in N\backslash S$ with

$$j \in \arg\max_{t\in N\backslash S} x_t - r_t.$$

Suppose that (K, z) is a counterobjection of $j \in N\backslash S$ i.e. z is a K-feasible payoff vector such that:

$$\begin{aligned} z_k &\geq x_k \quad \forall k \in K\backslash S \\ z_k &\geq y_k > x_k \quad \forall k \in K \cap S. \end{aligned}$$

Let $P := K \cap T$, so $P \subset T$. Lemma 4.2 ii) gives that $P \neq \emptyset$. Define

$$M_S := x(T\backslash S) - r(T\backslash S) + c(S)$$

and

$$M_K := z(K\backslash P) - r(K\backslash P) + c(K).$$

Discern five cases:

CASE I: $c(S) > 0$ and $S = T$
Define

$$y_k = x_k + \frac{r_k - x_k}{r(T) - x(T)} (v(S) - x(S)) \;\; \forall k \in S = T.$$

Then $y(S) = v(S)$ and $r_k > y_k > x_k$ $\forall k \in S = T$, since $r_k > x_k$ for all $k \in S = T$ and $0 < v(S) - x(S) = r(S) - c(S) - x(S) < r(S) - x(S) = r(T) - x(T)$ implies that $\frac{v(S)-x(S)}{r(T)-x(T)} \in (0,1)$. So (S, y) is an objection. Then:

$$\begin{aligned} M_K &\geq M_S \\ &= c(S) \\ &= r(T) - y(T) \\ &= r(T\backslash\{i\}) - y(T\backslash\{i\}) + r_i - y_i \\ &> r(T\backslash\{i\}) - y(T\backslash\{i\}) \\ &\geq r(P) - y(P) \\ &\geq r(P) - z(P) \\ &= M_K. \end{aligned}$$

where the first inequality follows from Lemma 4.3. The first equality follows from $S = T$ and $y(T) = v(T) = r(T) - c(T)$. The second inequality follows from $r_i > y_i$ when $c(S) > 0$. The third inequality from $P \subset T$ and $r_k > y_k$ $\forall k \in T$. The last inequality follows from $z(P) \geq y(P)$ by definition of a counterobjection. But $M_K > M_K$ is a contradiction. Hence there is no counterobjection (K, z).

CASE II: $c(S) > 0$, $S \neq T$
Then

$$\begin{aligned} 0 &< v(S) - x(S) \\ &= r(S) - c(S) - x(S) \\ &= r(S\backslash T) - x(S\backslash T) - c(S) + r(T) - x(T) \\ &< r(T) - x(T), \end{aligned}$$

where the last inequality follows from $r(S\backslash T) - x(S\backslash T) \leq 0$ and $c(S) > 0$. Hence $\frac{v(S)-x(S)}{r(T)-x(T)} \in (0,1)$.

Let $\varepsilon \in \left(0, (r_i - x_i)\frac{v(S)-x(S)}{r(T)-x(T)}\right)$ and define

$$\begin{aligned} y_k &= x_k + \frac{r_k - x_k}{r(T) - x(T)} (v(S) - x(S)) \quad \forall k \in T\backslash i \\ y_i &= x_i + \frac{r_i - x_i}{r(T) - x(T)} (v(S) - x(S)) - \varepsilon \\ y_k &= x_k + \frac{\varepsilon}{|S\backslash T|} \quad \forall k \in S\backslash T. \end{aligned}$$

Then $y(S) = v(S)$ and $r_k > y_k > x_k\ \forall k \in T$, $y_k > x_k\ \forall k \in S \backslash T$, so (S, y) is an objection of i against j. Then

$$\begin{aligned}
M_K &\geq M_S \\
&= x(S\backslash T) - r(S\backslash T) + c(S) \\
&= r(T) - x(T) + x(S) - r(S) + c(S) \\
&= r(T) - x(T) + [x(S) - v(S)] \\
&= r(T) - y(T) - \varepsilon \\
&= r(T\backslash\{i\}) - y(T\backslash\{i\}) + r_i - y_i - \varepsilon \\
&> r(T\backslash\{i\}) - y(T\backslash\{i\}) \\
&\geq r(P) - y(P) \\
&\geq r(P) - z(P) \\
&= M_K,
\end{aligned}$$

where the first inequality follows from Lemma 4.3. The fourth equality follows from $y(T) = x(T) + v(S) - x(S) - \varepsilon$. The second inequality follows from $r_i - x_i > 0$ and $\frac{v(S)-x(S)}{r(T)-x(T)} \in (0,1)$. But $M_K > M_K$ is a contradiction. Hence there is no counterobjection (K, z).

CASE III: $c(S) = 0$, $S = T$
Let $y_k = r_k\ \forall k \in S = T$. Then (S, y) is an objection. By monotonicity $c(N) = 0$. Hence $x(N) = v(N) = r(N)$ and $v(S) = r(S) > x(S)$ implies $r(N\backslash S) < x(N\backslash S)$, so $r_j < x_j$. Then

$$\begin{aligned}
z_k &\geq x_k \geq r_k \ \ \forall k \in K\backslash S,\ k \neq j \\
z_j &\geq x_j > r_j \\
z_k &\geq y_k = r_k \ \ \forall k \in K \cap S.
\end{aligned}$$

This gives $z(K) > r(K) \geq v(K)$, contradicting that $z(K) = v(K)$. Hence there is no counterobjection (K, z).

CASE IV: $c(S) = 0$, $S \neq T$, $r_k = x_k\ \forall k \in S\backslash T$
Let

$$\begin{aligned}
y_k &= r_k \ \text{ if } k \in T\backslash\{i\} \\
y_i &= r_i - \varepsilon \\
y_k &= r_k + \frac{\varepsilon}{|S\backslash T|} \ \text{ if } k \in S\backslash T.
\end{aligned}$$

with $\varepsilon \in (0, r_i - x_i)$. Then $y(S) = r(S) = v(S)$ and $y_k > x_k\ \forall k \in S$, so (S, y) is an objection. As in CASE III $r_j < x_j$. Then

$$\begin{aligned}
z_k &\geq x_k \geq r_k \ \ \forall k \in K\backslash S,\ k \neq j \\
z_j &\geq x_j > r_j \\
z_k &\geq y_k = r_k \ \ \forall k \in K \cap S.
\end{aligned}$$

So $z(K) > r(K) \geq v(K)$, a contradiction. Hence there is no counterobjection (K, z).

CASE V: $c(S) = 0$, $S \neq T$, $r_k < x_k$ for some $k \in S \backslash T$
Then

$$\begin{aligned} 0 &< v(S) - x(S) \\ &= r(S) - c(S) - x(S) \\ &= r(S \backslash T) - x(S \backslash T) - c(S) + r(T) - x(T) \\ &< r(T) - x(T), \end{aligned}$$

where the last inequality follows from $r(S \backslash T) - x(S \backslash T) \leq 0$ and $c(S) > 0$. Hence $\frac{v(S)-x(S)}{r(T)-x(T)} \in (0, 1)$.

Proceeding as in Case II yields a contradiction. Hence there is no counterobjection (K, z).

Since the five cases are exhaustive, there exists an uncountered objection for each $x \in I(N, v) \backslash C(N, v)$. This shows that $C(N, v) \supseteq B(N, v)$, finishing the proof. $\square$

Notice that case II and IV are almost identical and that case III and IV are similar to each other.

The bargaining set of Zhou (1994) imposes additional conditions on the counterobjection (K, z) in the proof above, namely:

- $K \backslash S \neq \emptyset$, which is true since $j \in K \backslash S$;
- $S \backslash K \neq \emptyset$, which is true since $i \in S \backslash K$;
- $S \cap K \neq \emptyset$, which is true since $S \cap K = P \neq \emptyset$.

Consequently the proof also indicates that the core coincides with Zhou's bargaining set.

It is easy to see that the core and bargaining set do not necessarily coincide in the shortest path games of Fragnelli et al. (2000); their games are not necessarily balanced, while the bargaining set always is nonempty.

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