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Unique Supply Function Equilibrium with Capacity Constraints

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Abstract

Consider a market where producers submit supply functions to a procurement auction — e.g. an electric power auction — under uncertainty, before demand has been realized. In the Supply Function Equilibrium (SFE), every firm commits to the supply function maximizing his expected profit given the supply functions of the competitors. The presence of multiple equilibria is one basic weakness of SFE. This paper shows that with (i) symmetric producers, (ii) inelastic demand, (iii) a reservation price, and (iiii) capacity constraints that bind with a positive probability, there is a unique symmetric SFE.

Keywords: supply function equilibrium, auction, oligopoly, capacity constraint, wholesale electricity market

JEL codes: D43, D44, L11, L13, L94

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1. INTRODUCTION

The Supply Function Equilibrium (SFE) was introduced by Klemperer & Meyer in 1989 [10]. The equilibrium concept assumes that producers submit supply functions simultaneously in a one-shot game. In the non-cooperative Nash Equilibrium, each producer commits to the supply function that maximizes his expected profit given the bids of the competitors and the properties of the uncertain demand. The equilibrium is often used when modeling bidding behavior in electric power auctions. This useful application was first observed by Green & Newbery [9] and Bolle [5]. Although there are few papers with other applications, SFE can be applied to any uniform price auction where bidders have common knowledge, quantity discreteness is negligible and the demand/supply of the auctioneer is uncertain. The multiplicity of equilibria is one basic weakness of SFE. This paper shows that under certain conditions, which are reasonable for electric power markets, and especially so for balancing markets, there is in fact a unique SFE.

Supply Function Equilibria are traditionally found by making the following observation: each producer submits a supply function, such that, for each demand outcome, the price in the market is optimized with respect to his residual demand. Intuitively, the optimal price of a producer is given by the optimal monopoly response. Hence, the mark-up percentage should be inversely proportional to the elasticity of the residual demand curve for every outcome. The elasticity of the residual demand comprises derivatives of the supply functions of the competitors. Thus the optimal response of each producer is given by a differential equation. In equilibrium all producers make optimal bids; hence, the SFE is given by the solution to a system of differential equations. For symmetric producers, one can show that only symmetric equilibria exist [10] and the system can be reduced to a single differential equation. However, there is no end-point condition, so the solution normally has one arbitrary constant.

The arbitrary constant allows for a continuum of symmetric equilibria; from Nash-Cournot down to Bertrand, if marginal costs are constant. The continuum can intuitively be understood from the monopoly mark-up rule. When the supply functions of the competitors are very elastic, i.e. they have low mark-ups at every supply, the best response is to have a low mark-up at every supply. When the competitors have a very inelastic supply, i.e. they have high mark-ups at every supply, the best response is to have a large mark-up at every supply. Multiple equilibria make it difficult to predict outcomes with SFE. Further, it is a nuisance for comparative statics. How can one be sure that the arbitrary constant associated with an equilibrium does not change, when market conditions are changed? Hence, the multiplicity of equilibria is a considerable drawback for SFE.

I consider a market with symmetric producers, *inelastic demand* and *capacity constraints* that bind with a positive probability. I show that under these conditions, there is a unique symmetric SFE[3]. A *price cap*, i.e. reservation price, is needed to limit the equilibrium price and guarantee the existence of the equilibrium. The unique symmetric equilibrium price reaches the price cap precisely when the capacity constraints bind. Hence, it turns out that the arbitrary constant, in the solution of the differential equation, is pinned down by the price cap and the total production capacity. The assumptions leading to uniqueness and existence are very reasonable for electric power markets. In the short-term, *demand is very inelastic* in the electric power market. Thus inelastic demand is often assumed for spot markets [2,7,8,13]. It is even more realistic for real-time and balancing markets.

Capacity constraints reduce the set of SFE in the electric power market. This has been observed in previous work [4,8,9]. In particular Genc & Reynolds have recently shown that pivotal suppliers may drastically limit the range of SFE [8]. But that there is a *risk of power shortage* is a new assumption in the literature. It is a plausible assumption; even if there are several years, or even decades, between power shortages, they are not zero-probability events. The risk of power shortage may have increased after the deregulation of the power market, as power producers have often trimmed their reserve capacities. A profit-maximizing producer will not pay for reserve capacity that is used with a negligible probability.

Price caps are used in most deregulated power markets and are considered in some previous models of electric power markets [4,7,8]. One argument for having them is that even if consumers do not switch off their equipment for extremely high electricity prices, it does not mean that they have an extremely high marginal benefit of power. The reason they do not switch off is that they do not have this option or that they, due to long-term contracts, do not face the short-term price. To maximize social welfare, it is sometimes better to randomly disconnect consumers (consumer rationing) than to force supply to meet demand at an extremely high price.

The uniqueness of the symmetric equilibrium can intuitively be understood from the following reasoning (see Fig. 1): When the capacity constraints of the competitors bind, a producer faces an inelastic residual demand. Following the monopoly response, his optimal price for this outcome should equal the price cap, otherwise there would be a profitable

[3] Inelastic demand and symmetry simplify the analysis, but intuitively these assumptions are not critical to get uniqueness.

deviation. Further, there are profitable deviations from equilibrium candidates hitting the price cap before the capacity constraints bind. The reason is that it is profitable to slightly undercut horizontal supply of the competitors when the price exceeds the marginal cost, á la Bertrand.

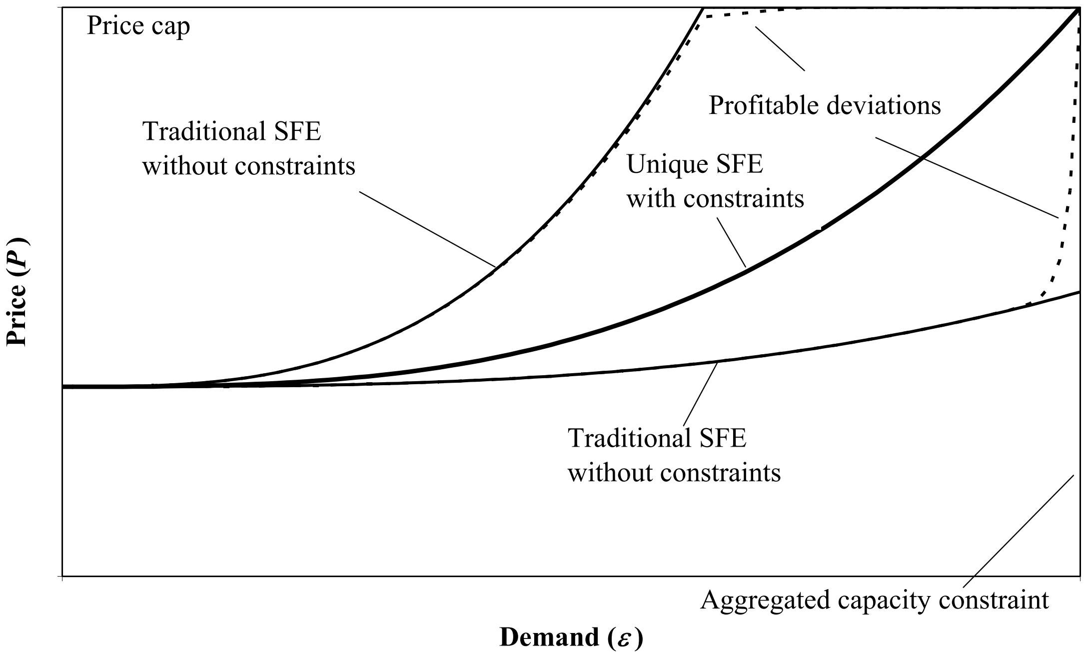

Fig. 1. *Capacity constraints and a price cap rule out all traditional SFE, but one.*

However, just deviating at the point where the capacity constraint binds does not change expected profits. To get a profitable deviation, a producer must deviate before his capacity constraint binds. This means that he will lose profits for some demand outcomes. The same is true for producers deviating by slightly undercutting the price cap. Thus intuitive reasoning and optimization of the price for one demand outcome at a time is not satisfactory. To make the analysis rigorous, it is carried out using optimal control theory.

Many papers in the SFE literature try to single out a unique equilibrium. Klemperer & Meyer show that if outcomes with infinite demand occur with positive probability, and if demand can be met with non-binding capacity constraints — not realistic for the electric power market — then there is a unique SFE. With a price cap and capacity constraints, Baldick & Hogan [4] actually single out the same equilibrium as in this paper, but with a weaker motivation. Price caps are seen as a public signal that coordinates the bids of the producers. In some papers, the equilibrium with the highest profit — the worst case — is used [9]. For some specific combinations of demand and capacity constraints, the worst case turns out to be a unique equilibrium [9]. Newbery gets a unique SFE by considering entry and assuming bid-coordination; incumbent firms coordinate their bids to the most profitable equilibrium that deters entry [12]. Baldick & Hogan [4] and Anderson & Xu [3] find a unique equilibrium, in some cases, by ruling out unstable equilibria. Stability is tested assuming an infinite speed of adjustment when there are small deviations from best-response bids. One could argue, however, that with a sufficiently slow speed of adjustment, other equilibria might be stable as well. The stability literature predicts that high mark-ups tend to be unstable. Inspired by these results, Rudkevich et al. [13] assume that the least profitable equilibrium should be closest to the reality.

In Section 2, I present the notation and assumptions used in the analysis of this paper. The unique SFE is derived in several steps in Section 3. A first order condition is derived for smooth and monotonically increasing segments of a symmetric SFE by means of optimal control theory. The result is the first-order condition derived for unconstrained production by Klemperer & Meyer [10]. Next various equilibrium candidates that are not completely smooth are ruled out. Symmetric equilibria with vertical or horizontal segments can be ruled out by using optimal control theory with final values and their associated transversality conditions. Some candidates cannot be ruled out by means of optimal control theory; instead they are excluded by the observation of a profitable deviation. To avoid horizontal and vertical segments in the supply, the equilibrium price must reach the price cap exactly when the capacity constraint binds. It is shown that there is exactly one smooth symmetric SFE candidate that fulfills this end-condition and the first-order condition. It is verified that the unique candidate is an equilibrium, i.e. it fulfills a second order condition.

In Section 4, the unique SFE is characterized. Comparative statics show that the equilibrium has intuitive properties, e.g. mark-ups are reduced if there are more competitors. An important implication of the analysis is that the price cap and capacity constraints affect the equilibrium price also when the constraints do not bind. The assumptions leading to the unique SFE are realistic for electric power auctions, but even more so for balancing markets. Such a market is considered in Section 5. In Section 6, the unique equilibrium is illustrated by an example with a quadratic cost function. The paper is concluded in Section 7.

2. NOTATION AND ASSUMPTIONS

Assume that there are N symmetric producers. The bid of each producer i consists of a supply function $S_i(p)$, where p is the price. $S_i(p)$ is required to be non-decreasing. The aggregate supply of the competitors of producer i is denoted $S_{-i}(p)$ and the total supply $S(p)$.

In the original work by Klemperer & Meyer [10], the analysis is confined to twice continuously differentiable supply functions[4]. In this paper the set of admissible bids is extended to include piece-wise twice continuously differentiable supply functions, see Fig. 2. The extension allows for supply functions with vertical and horizontal segments, i.e. binding slope constraints. Allowing for deviations with partly horizontal or vertical segments is very useful, when ruling out SFE candidates. $S_i(p)$ is not necessarily differentiable at every price, but it is required that it is differentiable on the left and right at every price. Further, it is required that all supply functions are left-hand continuous. From the requirements of the supply functions, it follows that, a sufficiently large p_- can always be found for every p such that all supply functions are twice continuously differentiable in the interval $[p_-, p]$.

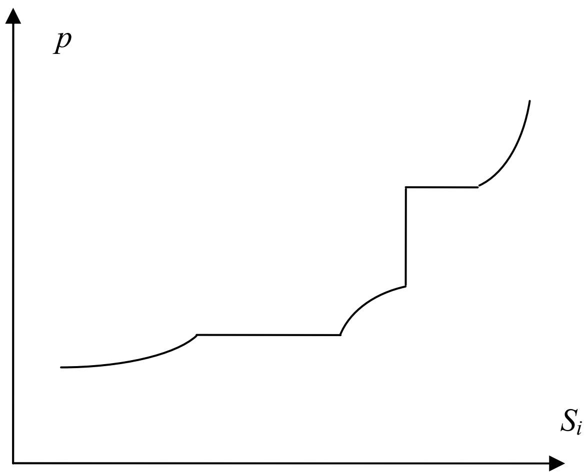

Fig. 2. *The considered supply functions are piece-wise twice continuously differentiable.*

Denote the inelastic demand by ε and its probability density function by $f(\varepsilon)$. The density function is continuously differentiable and has a convex support set, which includes ε=0. Let the capacity constraint of each producer be $\bar{\varepsilon}/N$, so that $\bar{\varepsilon}$ is the total capacity of the producers. A key assumption is that the capacity constraints of all producers will bind with a positive probability, i.e. there are extreme outcomes, for which $\varepsilon > \bar{\varepsilon}$.

The demand is assumed to be zero above the reservation price. In the electricity market this is achieved by means of forced disconnection of consumers, when the price threatens to rise above the price cap. Thus the market price for extreme outcomes equals the price cap. Allowing for extreme outcomes and rationing is different from the traditional SFE view on

[4] There is an alternative model that considers stepped supply functions [7].

market clearing. The new assumption is crucial to get a unique equilibrium. It is realistic for electric power markets, especially real-time and balancing markets.

In case the total supply has an inelastic segment that coincides with the inelastic demand, it is assumed that the market design is such that the lowest price is chosen[5]. This means that the equilibrium price as a function of the demand is left-hand continuous.

Let $q_i(\varepsilon,p)$ be the residual demand that a producer i faces for $p < \overline{p}$. Provided that the supply functions of his competitors are non-horizontal at p, his residual demand is given by:

$$q_i(\varepsilon, p) = \varepsilon - S_{-i}(p) \quad \text{if } p < \overline{p}. \tag{1}$$

All firms have identical cost functions $C(q_i)$, which are increasing, strictly convex, twice continuously differentiable, and fulfill $C'(\overline{\varepsilon} / N) < \overline{p}$. Thus marginal costs are monotonically increasing.

If more than one producer has a supply with a perfectly elastic segment at some price p_0, supply rationing at this price is necessary for some demand outcomes. The perfectly elastic supply of producer i at this price is given by $\Delta S_i(p_0) \equiv S_i(p_0 +) - S_i(p_0)$, where $S_i(p_0 +) \equiv \lim_{p \downarrow p_0} S_i(p)$[6]. Similarly, the total perfectly elastic supply of his competitors is $\Delta S_{-i}(p_0) \equiv S_{-i}(p_0 +) - S_{-i}(p_0)$. I assume that the rationed supply of producer i at p_0 is given by: $S_i(p_0) + R(\varepsilon - S(p_0), \Delta S_i(p_0), \Delta S_{-i}(p_0))$. It is assumed that the rationing mechanism has the following properties: $R_1 \geq 0$, $R_2 \geq 0$, $R_3 \leq 0$, and $R(0, \Delta S_i(p_0), \Delta S_{-i}(p_0)) = 0$. Further, if $\Delta S_{-i}(p_0) > 0$:

$$\begin{aligned} &0 \leq R_1 + R_2 < 1 \quad \text{if } S(p_0) \leq \varepsilon < S(p_0 +) \\ &R_1 + R_2 = 1 \quad \text{if } \varepsilon = S(p_0 +). \end{aligned} \tag{2}$$

The intuition of this assumption is as follows. Consider a case where rationing is needed at p_0. Assume that producer i increases the price up to p_0, for one unit that was previously offered below p_0. Then his accepted total supply will decrease. The assumption can, for example, be verified for a rationing mechanism, where all producers gets a ration proportional to their perfectly elastic supply at p_0.

I also assume that if the total supply has perfectly elastic segments at the price cap, all of these bids are accepted before demand is rationed.

[5] The same assumption is used by Baldick & Hogan [4].
[6] Recall that supply functions are left-hand continuous for positive demand.

3. THE UNIQUE SYMMETRIC SFE

In previous work, except for the recent paper by Genc & Reynolds [8], optimization for one demand outcome at a time has been used in the derivation of Supply Function Equilibria. Here optimal control theory is used instead. Optimizing the price for one demand at a time is much more straightforward, but when allowing for generalized supply functions, optimal control theory is useful when ruling out irregular SFE.

Vertical and horizontal segments are very useful deviation strategies, when ruling out SFE candidates. Nevertheless, allowing for such segments complicates the analysis, as SFE with vertical and horizontal segments have to be ruled out as well. To ensure that optimal control theory is applicable when testing whether a supply function of a producer is the best response, we have to ensure that the supply functions of his competitors are continuously differentiable in the integrated price range. Further, the control variable needs to be finite. These technicalities imply that supply functions of a potential equilibrium have to be studied piece-by-piece.

In Section 3.1, optimal control theory is used to derive conditions that must necessarily be fulfilled for all *smooth and monotonically increasing* segments of a symmetric supply function equilibrium. These conditions are simplified to a differential equation, the first-order condition. This is the standard first-order condition used in the SFE literature and there is an analytic solution for inelastic demand.

In Sections 3.2-3.4, irregular SFE are ruled out. In Section 3.2, it is shown that there are no symmetric equilibria where supply functions have *perfectly elastic segments*. This can be shown by means of optimal control theory with a final value[7]. The result of Section 3.2 also rules out perfectly elastic segments at the price cap. In Section 3.3, equilibria with *discontinuities in the equilibrium price* are ruled out using optimal control theory with a final value. To avoid a discontinuity in the price when all bids have been accepted, the total supply must be elastic up to the price cap. Section 3.4 rules out kinks, and finally Section 3.5 shows that no capacity is withheld in equilibrium.

Thus in equilibrium, all supply functions must fulfill the first order condition in the whole price range. The end-condition is that the symmetric supply function must reach the price cap exactly when all capacity constraint binds. In Section 3.6 it is observed that there is a unique SFE candidate that fulfills the first-order condition and the end-condition. In Section 3.7, it is shown that, given that the competitors follow the unique candidate, the equilibrium price of

[7] In the special case when the total supply is inelastic just below the perfectly elastic segment, this is proven with a profitable deviation.

the unique candidate globally maximizes the profit of an arbitrary producer for every demand outcome. Thus it is a Nash-equilibrium and a SFE.

3.1. The optimal control problem for smooth segments of a SFE

In equilibrium, an arbitrary producer *i* submits his best supply function out of the class of allowed supply functions, given the bids of his competitors. Now consider *a segment* of a symmetric equilibrium candidate $\hat{S}_i(p)$, where supply functions are monotonically increasing and twice continuously differentiable in the range $p_- \le p \le p_+$. Assume that the competitors of producer *i* follow the equilibrium candidate. Will it be a best response of producer *i* to follow as well? In this section, only local deviations in the range $p_- \le p \le p_+$ are considered. His bids outside this range are unchanged. Considering such deviations give a necessary, but not sufficient, condition. Let the set $\mathbb{S}$ be defined by $\hat{S}_i(p)$ and all considered deviations. We note that by choosing his supply function, producer *i* can control the total supply function, *S(p)*.

$$S(p) \equiv S_i(p) + \hat{S}_{-i}(p) = \varepsilon . \tag{3}$$

As competitors follow the equilibrium candidate and $S_i(p)$ is required to be non-decreasing, the total supply, *S(p),* is monotonically increasing in the interval $p_- \le p \le p_+$. Hence, the inverse function of *S(p)* exists for this range. It is denoted *p(ε)*:

$$p(\varepsilon) \equiv S^{-1}(\varepsilon). \tag{4}$$

In terms of demand, the studied range is given by $\varepsilon_- \le \varepsilon \le \varepsilon_+$, where $\varepsilon_- = \hat{S}(p_-)$ and $\varepsilon_+ = \hat{S}(p_+)$. Due to the assumptions for the bid curves of the competitors, it will also be true that $\varepsilon_- = S(p_-)$ and $\varepsilon_+ = S(p_+)$ for all possible deviations by producer *i* in the set $\mathbb{S}$. Hence, controlling the aggregated supply function, producer *i* effectively determines the price for each outcome in the range $\varepsilon_- \le \varepsilon \le \varepsilon_+$, under the constraints $p_- = p(\varepsilon_-)$ and $p_+ = p(\varepsilon_+)$. The optimal $p(\varepsilon)$ for this range can be calculated by solving an optimal control problem. The control variable is defined as $u = p'(\varepsilon)$, i.e. the rate of change in the price. It is a tool to choose *p(ε)* optimally in the range $\varepsilon_- \le \varepsilon \le \varepsilon_+$. Thus

$$u(\varepsilon_+) = p_l'(\varepsilon_+) \text{ and } u(\varepsilon_-) = p_r'(\varepsilon_-), \tag{5}$$

where the indexes *l* and *r* denote the left- and right-hand derivatives, respectively. The derivative of the inverse function in (4) can be calculated to be:

$$u = p'(\varepsilon) = \frac{dp}{d\varepsilon} = \left(\frac{d\varepsilon}{dp}\right)^{-1} = \frac{1}{S'(p(\varepsilon))} \tag{6}$$

Depending on $S_i(p)$, *S(p)* is not necessarily differentiable at very price, but it will be differentiable on the left and right at every price. Thus *u* is piece-wise continuous. Further, $u<\infty$, which is required for optimal control problems [6], as $\hat{S}'_{-i}(p) > 0$. It is required that all supply functions fulfill $0 \le S_i'(\cdot)$, thus the control variable is constrained by:

$$0 \le u \le \frac{1}{\hat{S}_{-i}'(p(\varepsilon))}. \tag{7}$$

In equilibrium, producer *i* submits his best allowed supply function, given the bids of the competitors. $\hat{S}_i(p)$ belongs to the set $\mathbb{S}$. Accordingly if it is the best response, it must also be the best response in $\mathbb{S}$. Thus given the bids of the competitors, it is necessary — but not sufficient — that the following optimal control problem returns the equilibrium candidate.

$$\begin{aligned} &\underset{p(\varepsilon)}{\text{Max}} \int_{\varepsilon_-}^{\varepsilon_+} \left\{\left[\varepsilon - \hat{S}_{-i}(p(\varepsilon))\right]p(\varepsilon) - C\left(\varepsilon - \hat{S}_{-i}(p(\varepsilon))\right)\right\} f(\varepsilon)d\varepsilon \\ &\text{s.t. } u = p'(\varepsilon) \quad 0 \le u \le \frac{1}{\hat{S}_{-i}'(p(\varepsilon))} \qquad p(\varepsilon_-) = p_- \qquad p(\varepsilon_+) = p_+ \end{aligned} \tag{8}$$

Hence, it is necessary that the contribution to the expected profit, when demand is in the interval $[\varepsilon_-, \varepsilon_+]$, is maximized in equilibrium, given $p(\varepsilon_+) = p_+$, $p(\varepsilon_-) = p_-$, and the bids of the competitors. The integrand of an optimal control problem should be continuously differentiable in the state variable, i.e. *p* [14]. This sufficient condition is fulfilled as all the competitors supply functions are twice continuously differentiable in $[p_-, p_+]$ and the cost function is twice continuously differentiable.

The slope constraint $0 \le u \le \frac{1}{\hat{S}_{-i}'(p(\varepsilon))}$ might bind, if the there is a profitable deviation from $\hat{S}_i(p)$, i.e. $\hat{S}_i(p)$ is not an equilibrium. However, if, as assumed, $\hat{S}_i(p)$ is to be an symmetric equilibrium with a monotonically increasing and smooth segment, i.e. $0 < \hat{S}_i'(p) < \infty$ for $p \in [p_-, p_+]$, then the slope constraints cannot bind in this interval. Hence, the Hamiltonian of the problem in (8) is [6,11]:

$$H(u, p, \lambda, \varepsilon) = \left\{\left[\varepsilon - \hat{S}_{-i}(p(\varepsilon))\right]p(\varepsilon) - C\left(\varepsilon - \hat{S}_{-i}(p(\varepsilon))\right)\right\} f(\varepsilon) + \lambda(\varepsilon)u(\varepsilon), \tag{9}$$

where λ is a costate or auxiliary variable of the optimal control problem [6]. The control variable u should be chosen such that the Hamiltonian is maximized for every ε [6]. Hence,

$$\frac{\partial H}{\partial u} = 0 = \lambda(\varepsilon) \tag{10}$$

and thus

$$\lambda(\varepsilon) \equiv \lambda'(\varepsilon) \equiv 0 \text{ for } \varepsilon \in [\varepsilon_-, \varepsilon_+]. \tag{11}$$

Further, the following equations of motion conditions are necessary for the optimal solution [6]:

$$p'(\varepsilon) = \frac{\partial H}{\partial \lambda} = u(\varepsilon) \tag{12}$$

and

$$\lambda'(\varepsilon) = -\frac{\partial H}{\partial p} = -\left\{\left[\varepsilon - \hat{S}_{-i}(p(\varepsilon))\right] + \left[C'\left(\varepsilon - \hat{S}_{-i}(p(\varepsilon))\right) - p(\varepsilon)\right]\left(\hat{S}_{-i}'(p(\varepsilon))\right)\right\} f(\varepsilon). \tag{13}$$

Combining (11) and (13) yields:

$$0 = \left[\varepsilon - \hat{S}_{-i}(p(\varepsilon))\right] + \left[C'\left(\varepsilon - \hat{S}_{-i}(p(\varepsilon))\right) - p(\varepsilon)\right]\hat{S}_{-i}'(p(\varepsilon)) \quad \forall \varepsilon \in (\varepsilon_-, \varepsilon_+)$$

We can now use (3) — supply equals demand — to simplify the equation above.

$$S_i(p(\varepsilon)) - \hat{S}_{-i}'(p(\varepsilon))\left[p(\varepsilon) - C'(S_i(p(\varepsilon)))\right] = 0 \quad \forall \varepsilon \in (\varepsilon_-, \varepsilon_+) \tag{14}$$

Before we continue with the analysis of this differential equation, note that by means of (1), (14) can be rewritten as:

$$\frac{p(\varepsilon) - C'[S_i(p(\varepsilon))]}{p(\varepsilon)} = \frac{S_i(p(\varepsilon))/\,p(\varepsilon)}{\hat{S}_{-i}'(p(\varepsilon))} = \frac{q_i(\varepsilon, p(\varepsilon))/\,p(\varepsilon)}{-\dfrac{\partial q_i(\varepsilon, p(\varepsilon))}{\partial p}} = -\frac{1}{\gamma_i^{res}} \quad . \tag{15}$$

A producer maximizes his profit for every outcome ε by following the monopoly mark-up rule with respect to the elasticity of his residual demand, γ_i^{res}.

The supply functions are monotonically increasing and twice continuously differentiable in the price range $p_- \le p \le p_+$. Thus the equilibrium price $p(\varepsilon)$ is continuous and montonically increasing in the demand range $\varepsilon_- \le \varepsilon \le \varepsilon_+$. Accordingly, if (14) is fulfilled for all $p \in (p_-, p_+)$, then it must also be fulfilled for all $\varepsilon \in (\varepsilon_-, \varepsilon_+)$. The implication holds in both directions. Thus (14) is equivalent to:

$$S_i(p) - \hat{S}_{-i}'(p)\left[p - C'(S_i(p))\right] = 0 \;\; \forall p \in (p_-, p_+) \; . \tag{16}$$

There is a similar differential equation for each producer. The system of differential equations corresponds to the system derived by Klemperer & Meyer [10], but their expression is more general, as it considers elastic demand. The considered equilibrium candidates are symmetric, so $S_i(p) \equiv S_j(p)$, and thus (16) can be written:

$$\widehat{S}_i(p) - (N-1)\widehat{S}_i'(p)\left[p - C'\left(\widehat{S}_i(p)\right)\right] = 0. \tag{17}$$

A solution to this differential equation is derived by Rudkevich et al. [13] and Anderson & Philpott [2]. For the boundary condition $p(\varepsilon_+) = p_+$, the solution is:

$$p(\varepsilon) = \frac{p_+ \varepsilon^{N-1}}{\varepsilon_+^{N-1}} + (N-1)\varepsilon^{N-1} \int_{\varepsilon}^{\varepsilon_+} \frac{C'(x/N)dx}{x^N} \quad \text{if } \varepsilon \in [\varepsilon_-, \varepsilon_+]. \tag{18}$$

Optimal control theory is only reliable when $u = p' < \infty$. Accordingly, equilibrium candidates with smooth symmetric transitions to inelastic elastic supply cannot be analyzed with optimal control theory. Nevertheless, the first-order condition in (17) can be used indirectly. Lemma 1 below rules out smooth transitions to inelastic supply. This also excludes isolated points in the (p,S) space, where $S_i'(p) = 0$.

Lemma 1: There are no symmetric equilibria that for a finite positive supply bounded away from zero have smooth symmetric transitions to an inelastic supply.

Proof: See Appendix.

3.2. Symmetric SFE with perfectly elastic segments do not exist

Now consider symmetric equilibrium candidates, where all producers have segments with perfectly elastic supply at some price p_0, i.e. the constraint $0 \le u = p' = \frac{1}{S'}$ binds. Then supply rationing is needed for some demand outcomes. In this section, I will show that any producer will find it profitable to deviate from the equilibrium candidate. He increases his expected profit by undercutting p_0 with his units that, for the equilibrium candidate, are offered at p_0 (see Fig. 3). The intuition is the same as for the Bertrand equilibrium, where producers undercut each others horizontal bids down to the marginal cost. As marginal costs are monotonically increasing, Bertrand equilibria can be ruled out in all price intervals. A formal proof using optimal control theory follows in Proposition 2. Optimal control theory is not applicable when the total supply is inelastic just below p_0, as the control variable $u = p'$ must be finite. This case is analyzed separately in Proposition 3. Note that one implication of

Proposition 2 and 3 is that symmetric SFE with perfectly elastic segments at the price cap can be excluded.

Negative mark-ups are ruled out in Proposition 1. This obvious result is useful when proving Proposition 2 and 3.

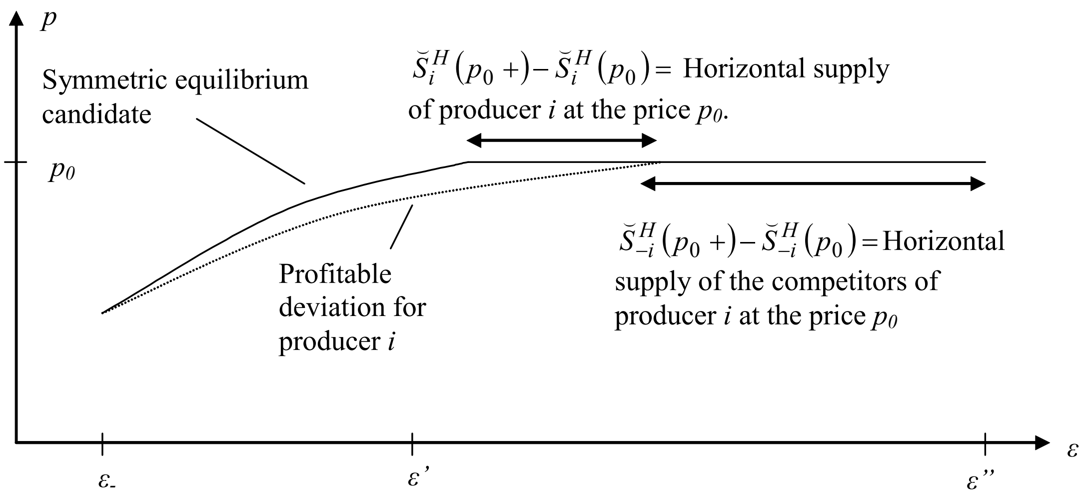

Fig. 3. *Symmetric equilibria with perfectly elastic segments can be ruled out. All producers will find it profitable to slightly undercut horizontal supply of the competitors.*

Proposition 1: In equilibrium no units are offered below their marginal cost.

Proof: See Appendix.

Proposition 2: For positive demand, there are no symmetric SFE with perfectly elastic segements at p_0, when the market supply is elastic just below p_0.

Proof: Consider a symmetric SFE candidate with perfectly elastic segments at $p_0 \leq \overline{p}$. Denote supply functions following the equilibrium candidate by $\breve{S}_i$. Thus $\Delta\breve{S}_i(p_0)=\breve{S}_i(p_0+)-\breve{S}_i(p_0)>0$ and $\Delta\breve{S}_{-i}(p_0)>0$. All considered supply functions are twice continuously differentiable in some price interval $[p_-, p_0]$, see Section 2. In this Proposition it is also assumed that the market supply is elastic just below p_0. Thus $0<\breve{S}_{il}'(p_0)<\infty$. Further, a p_- can be found such that $0<\breve{S}_i'(p)<\infty$ for all $p\in[p_-, p_0)$. Let $\varepsilon_- = \breve{S}(p_-)$. Assume that $p(\varepsilon)=p_0$, if and only if $\varepsilon\in[\varepsilon', \varepsilon'']$, where $0<\varepsilon'<\varepsilon''$.

For demand outcomes $\varepsilon \in (\varepsilon', \varepsilon'')$, the supply at p_0 has to be rationed somehow. The accepted ration of the perfectly elastic supply of producer i is given by $R\left(\varepsilon - \varepsilon', \Delta S_i\,(p_0), \Delta \breve{S}_{-i}(p_0)\right)$, where $\Delta S_i\,(p_0) = \varepsilon'' - \varepsilon' - \Delta \breve{S}_{-i}(p_0)$.

Now consider unilateral deviations of player i, where all his bids above p_0 and below p_- are unchanged. To keep the equilibrium candidate, $\breve{S}_i(p_0)$ must be the best response out of the considered deviation strategies. The best response can be derived from:

$$
\begin{aligned}
&\underset{p(\varepsilon)}{\text{Max}} \int_{\varepsilon_-}^{\varepsilon'} \left\{\left[\varepsilon - \breve{S}_{-i}(p(\varepsilon))\right]p(\varepsilon) - C\left(\varepsilon - \breve{S}_{-i}(p(\varepsilon))\right)\right\} f(\varepsilon)d\varepsilon + F(\varepsilon') \\
&\text{s.t. } u = p'(\varepsilon) \quad 0 \le u \le \frac{1}{\breve{S}_{-i}{}'(p(\varepsilon))} \\
&p(\varepsilon_-) = p_- \qquad p(\varepsilon') = p_0
\end{aligned}
\tag{19}
$$

The final value of the optimal control problem, *F(),* returns the contribution to the expected profit from the rationed supply at p_0.

$$
\begin{aligned}
F(\varepsilon') = \int_{\varepsilon'}^{\varepsilon''} &\left\{\left[\varepsilon' - \breve{S}_{-i}(p_0) + R\left(\varepsilon - \varepsilon', \varepsilon'' - \varepsilon' - \Delta\breve{S}_{-i}, \Delta\breve{S}_{-i}\right)\right]p_0 + \right. \\
&\left. - C\left[\varepsilon' - \breve{S}_{-i}(p_0) + R\left(\varepsilon - \varepsilon', \varepsilon'' - \varepsilon' - \Delta\breve{S}_{-i}, \Delta\breve{S}_{-i}\right)\right]\right\} f(\varepsilon)d\varepsilon
\end{aligned}
\tag{20}
$$

The slope constraints $0 \le u \le \frac{1}{\breve{S}_{-i}{}'(p(\varepsilon))}$ might bind for $\varepsilon \in [\varepsilon', \varepsilon_-]$, if there is a profitable deviation from $\breve{S}_i(p)$, i.e. $\breve{S}_i(p)$ is not an equilibrium. However, as in Section 3.1, the slope constraints can be disregarded when a necessary condition for $\breve{S}_i(p)$ is derived under the assumption that $\breve{S}_i(p)$ is a SFE.

The Hamiltonian, the Max H condition and the equations of motion are the same as for the optimal control problem in (8) [11]. In particular $\lambda(\varepsilon) \equiv 0$ for $\varepsilon \in [\varepsilon_-, \varepsilon']$. The transversality condition associated with the terminal constraints at the right end-point is [11]:

$$
H(u, p, \lambda, \varepsilon') + \frac{\partial F(\varepsilon')}{\partial \varepsilon'} = 0. \tag{21}
$$

The first term is the marginal value of increasing ε'. The second term, which is negative, represents the marginal loss in the final value. It is known that $p(\varepsilon') = p_0$ and

$R\left(0, \Delta S_i, \Delta \breve{S}_{-i}\right) = 0.$ These relations combined with (9), (20) and (21) imply:

$$H(u, p, \lambda, \varepsilon') + \frac{\partial F(\varepsilon')}{\partial \varepsilon'} = \lambda(\varepsilon')u(\varepsilon') + \\ + \int_{\varepsilon'}^{\varepsilon''} \left\{ (p_0 - C'(\cdot)) \left[1 + \frac{dR\left(\varepsilon - \varepsilon', \varepsilon'' - \varepsilon' - \Delta \breve{S}_{-i}, \Delta \breve{S}_{-i}\right)}{d\varepsilon'} \right] \right\} f(\varepsilon) d\varepsilon = \\ = \int_{\varepsilon'}^{\varepsilon''} \{ (p_0 - C'(\cdot)) [1 - R_1 - R_2] \} f(\varepsilon) d\varepsilon \tag{22}$$

Costs are strictly convex and there are no equilibria with negative mark-ups. Thus $p_0 > C'$ for $\varepsilon \in [\varepsilon', \varepsilon'')$ and $p_0 \geq C'$ for $\varepsilon = \varepsilon''$. Thus the combination of (2) and (22) implies that:

$$H(u, p, \lambda, \varepsilon') + \frac{\partial F(\varepsilon')}{\partial \varepsilon'} > 0. \tag{23}$$

For the equilibrium candidate, the marginal value of continuing is larger than the marginal loss in the final value. The reason is that by slightly undercutting p_0 as in Fig. 3, producer *i* can sell significantly more. The relation in (23) is true as long as producer *i* has a perfectly elastic supply left at p_0. Hence, equilibria of the type $\breve{S}_i(p)$ can be excluded.

Proposition 3: For positive demand, there are no symmetric SFE with perfectly inelastic segements at p_0, when the market supply is inelastic just below p_0.

Proof: See Appendix. Same intuition as in Proposition 2.

One can also rule out perfectly elastic segments starting at ε=0. Perfectly elastic segments at or below $C'(0)$ can be ruled out by Proposition 1, as the cost function is strictly convex. Perfectly elastic segments starting at ε=0 and that are above $C'(0)$ are ruled out by Proposition 3. Thus SFE with perfectly elastic segments can be completely ruled out.

3.3. The equilibrium price is not discontinuous

Assume that there is a discontinuity in the price at $\varepsilon_L \leq \bar{\varepsilon}$, where the price jumps from p_L to p_U. This means that all producers have an inelastic supply in the interval (p_L, p_U), i.e. the slope constraint $0 \leq S_i'(p)$ binds in this price interval. Then it turns out that any producer with bids just below p_L can increase his expected profit by deviating. He can increase the price for some units offered at and slightly below p_L and offer them slightly below p_U instead, see Fig. 4. This will significantly increase the price for realizations just below ε_L, while the sales reduction for the same realizations is small. Thus the deviation increases expected profit. This intuition is verified in Proposition 4. This proposition also rules out discontinuities in the equilibrium price at the demand outcome, for which all submitted bids of all firms are accepted. Thus in a symmetric equilibrium all supply functions must be elastic up to the price cap.

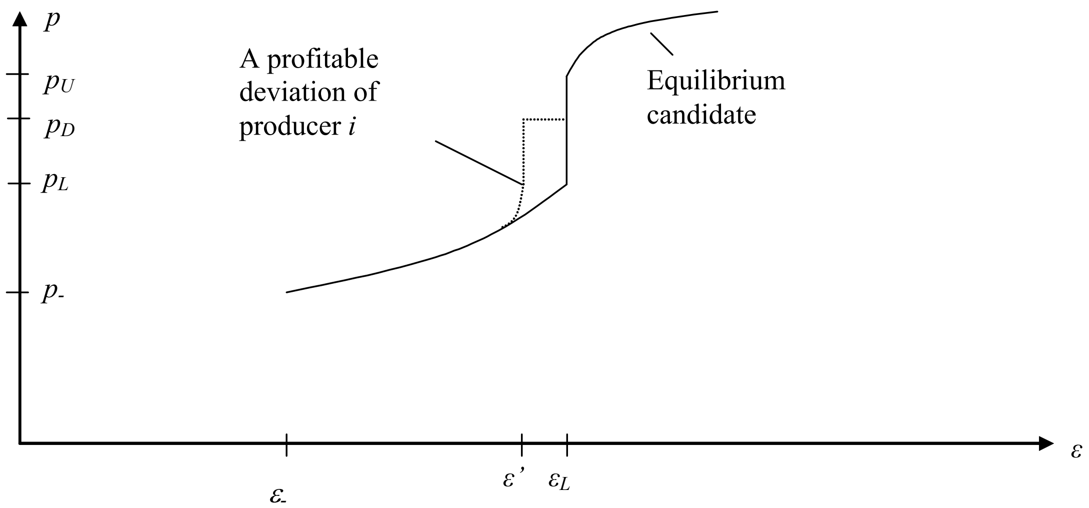

Fig. 4. *Discontinuities in the equilibrium price do not exist. All producers, e.g. producer i, will find it profitable to deviate.*

Proposition 4: For symmetric equilibria there are no discontinuities in the equilibrium price.

Proof: Consider a symmetric equilibrium candidate with a discontinuity in the price at $\varepsilon_L > 0$. Denote its upper price by p_U and its lower by p_L. Denote the equilibrium candidate by $\tilde{S}_i$. All considered supply functions are twice continuously differentiable in some price interval $[p_-, p_L]$, see Section 2. SFE with perfectly elastic segments are ruled out in Section 3.2. Further, smooth transitions to an inelastic supply are ruled out in Lemma 1. Thus $0 < \tilde{S}_{il}'(p_0) < \infty$. Further, a p_- can be found such that $0 < \tilde{S}_i'(p) < \infty$ for all $p \in [p_-, p_L)$, i.e. neither of the slope constraints bind just below p_L. Now, consider the following deviation strategy for producer i. Leave the supply above p_U and below p_- unchanged. Increase the bids for the last units in the range $p \in [p_-, p_L)$, and offer them at a price $p_D \in (p_L, p_U)$ instead. If it is optimal to change the bids for a positive amount of units, the deviation is more profitable than the equilibrium strategy and the equilibrium can be knocked out. Whether this is true or not can be investigated by an optimal control problem similar to (8), but with an added final value. The final value considers the contribution to the expected profit from the units sold at the price p_D.

$$\underset{p(\varepsilon)}{\text{Max}} \int_{\varepsilon_-}^{\varepsilon'} \left\{ \left[\varepsilon - \tilde{S}_{-i}(p(\varepsilon))\right] p(\varepsilon) - C\left(\varepsilon - \tilde{S}_{-i}(p(\varepsilon))\right)\right\} f(\varepsilon) d\varepsilon + F(\varepsilon')$$

$$\text{s.t. } u = p'(\varepsilon) \quad 0 \le u \le \frac{1}{\tilde{S}_{-i}{}'(p(\varepsilon))} \tag{24}$$

$$p(\varepsilon_-) = p_- \quad p(\varepsilon') = p_L$$

where

$$F(\varepsilon') = \int_{\varepsilon'}^{\varepsilon_L} \left\{ \left(\varepsilon - \tilde{S}_{-i}(p_L)\right) p_D - C\left(\varepsilon - \tilde{S}_{-i}(p_L)\right)\right\} f(\varepsilon) d\varepsilon. \tag{25}$$

The slope constraints $0 \le u \le \frac{1}{\tilde{S}_{-i}{}'(p(\varepsilon))}$ might bind for $\varepsilon \in [\varepsilon', \varepsilon_-]$, if there is a profitable deviation from $\tilde{S}_i(p)$, i.e. $\tilde{S}_i(p)$ is not an equilibrium. However, as in Section 3.1, the slope constraints can be disregarded when a necessary condition for $\tilde{S}_i(p)$ is derived under the assumption that $\tilde{S}_i(p)$ is a SFE.

The Hamiltonian, the Max H condition and the equations of motion are the same as for the optimal control problem in (8) [11]. In particular $\lambda(\varepsilon) \equiv 0$ for $\varepsilon \in [\varepsilon_-, \varepsilon']$. The transversality condition associated with the terminal constraint at the right end-point is [11]:

$$H(u, p, \lambda, \varepsilon') + \frac{\partial F(\varepsilon')}{\partial \varepsilon'} = 0.$$

From (9) and (25) we get:

$$H(u, p, \lambda, \varepsilon') + \frac{\partial F(\varepsilon')}{\partial \varepsilon'} = \left\{\left[\varepsilon' - \tilde{S}_{-i}(p_L)\right] p(\varepsilon') - C\left(\varepsilon' - \tilde{S}_{-i}(p_L)\right)\right\} f(\varepsilon') +$$
$$- \left\{\left(\varepsilon' - \tilde{S}_{-i}(p_L)\right) p_D - C\left(\varepsilon' - \tilde{S}_{-i}(p_L)\right)\right\} f(\varepsilon').$$

The relation must be true for $\varepsilon' = \varepsilon_L$, otherwise $\tilde{S}_i(p)$ cannot be part of an equilibrium. Thus

$$H(u, p, \lambda, \varepsilon_L) + \left.\frac{\partial F(\varepsilon')}{\partial \varepsilon'}\right|_{\varepsilon' = \varepsilon_L} = \underbrace{\left[\varepsilon_L - \tilde{S}_{-i}(p_L)\right]}_{+} \underbrace{(p_L - p_D)}_{-} f(\varepsilon_L) < 0.$$

Thus the transversality condition cannot be fulfilled for equilibrium candidates with a discontinuity in the price. The marginal value of continuing is less than the marginal loss in the final value. Thus, as in Fig. 4, all producers will find it profitable to unilaterally rise the price for some units offered just below p_L

3.4 There are no symmetric SFE with kinks

In this section, symmetric equilibria with supply functions $S_i(p)$ that have kinks at a price p are ruled out. In a symmetric equilibrium, the first-order condition in (17) must be fulfilled just below and just above p, as SFE with vertical and horizontal segments have been ruled out. $S_i(p)$ is continuous at p and the cost function is twice continuously differentiable. Thus the first-order condition implies that $S'_i(p)$ must also be continuous at p, i.e. there is no kink at p.

3.5 No capacity is withheld in equilibrium

Producers are not required by law to make bids to the procurement auction with all of their capacity. Will firms withhold capacity in equilibrium? Proposition 5 ensures that they do not. Instead of withholding some units, it is always better to offer these units at the price cap. Thus the bids of the producers will be exhausted exactly when the total capacity constraint binds, i.e. at $\varepsilon = \bar{\varepsilon}$.

Proposition 5: If $\bar{p} > C'\left(\frac{\bar{\varepsilon}}{N}\right)$ no capacity is withheld from the supply in equilibrium.

Proof: Consider a unit that is withheld from the supply by producer i in a potential equilibrium. Then there is a profitable deviation for producer i. He can offer the unit at a price equal to the price cap. If producer i has a horizontal supply at the price cap, assume that the previously withheld unit is offered as the last unit at this price. This deviation strategy will not negatively affect the sales of other units or their equilibrium price. Further, as $\bar{p} > C'\left(\frac{\bar{\varepsilon}}{N}\right)$ and as there is a positive probability that the demand exceeds or equals the total up-regulation capacity of all producers, the expected profit from the previously withheld unit will be positive. Accordingly, the deviation increases the expected profit of producer i. Thus there are no equilibria where units are withheld from the supply.

3.6 There is a unique equilibrium candidate fulfilling the necessary conditions

Proposition 5 ensures that no capacity is withheld from the procurement auction in equilibrium. Section 3.2-3.4 rule out all irregular symmetric equilibrium candidates. Thus symmetric equilibria must fulfill the first-order condition in (17) for $\varepsilon \in [0, \bar{\varepsilon}]$.

For $\varepsilon > \bar{\varepsilon}$, demand rationing is needed and the price equals $\bar{p}$. According to Proposition 4 there are no discontinuities in the price for symmetric equilibria. Thus the equilibrium price must reach the price cap at $\varepsilon \leq \bar{\varepsilon}$. Otherwise, there will be a discontinuity in the price at $\varepsilon = \bar{\varepsilon}$. Further, Proposition 2 and 3 ensure that the equilibrium price cannot be horizontal at the price cap. Thus the equilibrium price must reach the price cap exactly when the total capacity constraint binds. This is a necessary terminal condition for all symmetric SFE.

The solution of the first-order condition is given in (18).There is exactly one solution that fulfills the terminal condition, i.e. $p(\bar{\varepsilon}) = \bar{p}$:

$$p(\varepsilon) = \frac{\bar{p}\varepsilon^{N-1}}{\bar{\varepsilon}^{N-1}} + (N-1)\varepsilon^{N-1} \int_{\varepsilon}^{\bar{\varepsilon}} \frac{C'(x/N)dx}{x^N} \quad \text{if } \varepsilon \geq 0 \tag{26}$$

3.7. The unique candidate is a SFE

In Section 3.6 it has been shown that there is a unique symmetric SFE candidate given by (26) that fulfills the necessary first-order condition and end-condition. This unique candidate is denoted by $S_j^X(p)$. In this section it will be verified that the unique candidate also fulfills a second order condition, i.e. given the bids of his competitors, the strategy implied by the unique candidate is a globally best response. It is sufficient, but not necessary, to show that, for every demand outcome, the equilibrium price $p^X(\varepsilon)$ globally maximizes the profit.

Given that the total supply of the competitors is $S_{-i}^X(p)$, the profit of producer *i* is, for the outcome *ε,* given by:

$$\pi_X(\varepsilon, p) = \left[\varepsilon - S_{-i}^X(p(\varepsilon))\right] p(\varepsilon) - C\left(\varepsilon - S_{-i}^X(p(\varepsilon))\right).$$

Does $p^X(\varepsilon)$ maximize this profit for every *ε*?

Klemperer & Meyer [10] show that $\frac{\partial^2 \pi_X(\varepsilon, p)}{\partial p^2} < 0.$ The equilibrium price fulfills

$\left. \frac{\partial \pi_X(\varepsilon, p)}{\partial p} \right|_{p=p^X(\varepsilon)} = 0,$ as this is the first-order condition. Thus it seems that $p^X(\varepsilon)$ globally maximizes the profit of producer *i* for every outcome. However, as has been observed by Genc & Reynolds [8], the derivation of Klemperer & Meyer does not consider the reservation price and capacity constraints. Hence, before any final conclusions can be drawn, the influence by the reservation price and capacity constraints has to be investigated.

When considering capacity constraints we do not have to consider outcomes $\varepsilon > \bar{\varepsilon},$ as for these outcomes all producers are selling all of their capacity at the maximum price. They cannot do better than that. For $\varepsilon < \frac{\bar{\varepsilon}}{N},$ neither the capacity constraint nor the price cap is binding for the unique SFE candidate. Thus for these outcomes the result of Klemperer & Meyer applies directly. For $\varepsilon \in \left[\frac{\bar{\varepsilon}}{N}, \bar{\varepsilon} \right],$ there is some highest price $\tilde{p}(\varepsilon)$, such that the capacity constraint of producer *i* binds, if his last unit is sold at $\tilde{p}(\varepsilon)$. It can never be profitable for producer *i* to push the price below $\tilde{p}(\varepsilon)$, as his supply cannot increase. Thus the best price must be in the range $p \in \left[\tilde{p}(\varepsilon), \bar{p}\right]$. No capacity constraints and the price cap will not bind in this price interval, except at the boundaries. Thus the result of Klemperer & Meyer can be used. The conclusion is that the equilibrium price globally maximizes the profit of producer *i* for every outcome. Thus the equilibrium candidate is a NE and SFE.

4. CHARACTERIZING THE UNIQUE SYMMETRIC SFE

It has been shown that with reservation prices and capacity constraints, there is a unique symmetric SFE. This is good news for comparative statics. Thus, for symmetric equilibria, the formula in (31) is valid, even if e.g. the number of firms changes, the marginal costs change, the reservation prices change or the capacity constraints change.

4.1. Mark-ups

In a competitive equilibrium, the price is set by the marginal cost of the marginal unit. The marginal costs of cheaper and more expensive generators have no influence on the price. In the unique SFE, the equilibrium price, see (31), is given by a term related to the price cap and a term weighting the marginal costs of generators more expensive than the marginal unit. As in the competitive case, generators cheaper than the marginal unit have no influence on the equilibrium price. Instead the price of the marginal unit of a producer is limited by the cost of the alternative, generators of the competitors with a higher marginal cost. Thus the marginal costs of generators more expensive than the marginal unit influence the size of the mark-ups and accordingly the bid of the marginal unit. It is evident that for the term with weighted marginal costs, the weight decreases, the higher the marginal cost gets. Further, all weights are positive and integrate/sum up to less than or equal to one. This is shown with the calculation below.

$$(N-1)\varepsilon^{N-1}\int_{\varepsilon}^{\bar{\varepsilon}}\frac{dx}{x^{N}} = \left[-\frac{\varepsilon^{N-1}}{x^{N-1}}\right]_{\varepsilon}^{\bar{\varepsilon}} = 1-\frac{\varepsilon^{N-1}}{\bar{\varepsilon}^{N-1}} \le 1$$

According to Proposition 1, the equilibrium price for positive demand will never go below the marginal cost of the marginal unit. It turns out that producers will have a positive mark-up for every positive demand. This can be shown by means of (31).

$$p(\varepsilon) = \varepsilon^{N-1}\left[\frac{\bar{p}}{\bar{\varepsilon}^{N-1}} + (N-1)\int_{\varepsilon}^{\bar{\varepsilon}}\frac{C'(x/N)dx}{x^{N}}\right] \ge \varepsilon^{N-1}\left[\frac{\bar{p}}{\bar{\varepsilon}^{N-1}} + (N-1)C'(\varepsilon/N)\int_{\varepsilon}^{\bar{\varepsilon}}\frac{dx}{x^{N}}\right] =$$

$$= \varepsilon^{N-1}\left\{\frac{\bar{p}}{\bar{\varepsilon}^{N-1}} - C'(\varepsilon/N)\left[\frac{1}{x^{N-1}}\right]_{\varepsilon}^{\bar{\varepsilon}}\right\} = \varepsilon^{N-1}\left[\frac{\bar{p}}{\bar{\varepsilon}^{N-1}} - C'(\varepsilon/N)\left(\frac{1}{\bar{\varepsilon}^{N-1}} - \frac{1}{\varepsilon^{N-1}}\right)\right] =$$

$$= \frac{\varepsilon^{N-1}}{\bar{\varepsilon}^{N-1}}\underbrace{\left[\bar{p} - C'(\varepsilon/N)\right]}_{>0} + C'(\varepsilon/N) > C'(\varepsilon/N)$$

4.2. Comparative statics

For any positive demand outcome, it is obvious from (31) that the equilibrium price will increase, if the price cap is increased. (31) can also be used to investigate the effect of a symmetric change in the capacity constraints.

$$\frac{\partial p(\varepsilon)}{\partial \bar{\varepsilon}} = \frac{-(N-1)\left[\bar{p} - C'(\bar{\varepsilon}/N)\right]\varepsilon^{N-1}}{\bar{\varepsilon}^{N}} < 0, \text{if } \varepsilon > 0.$$

Hence, an increased capacity constraint decreases the price for all positive demand outcomes.

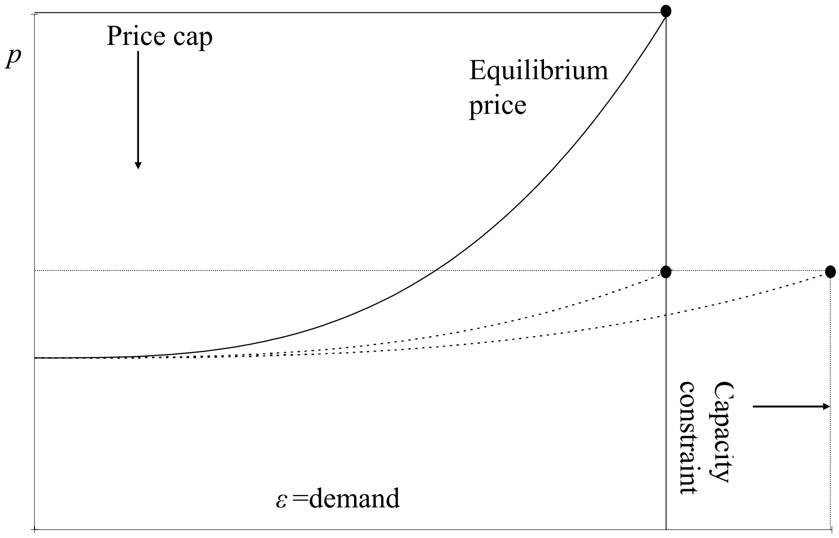

Fig. 5. *Reducing the price cap* $\bar{p}$ *and/or increasing the total capacity constraint of the market* $\bar{\varepsilon}$ *push down the equilibrium price for every demand.*

What happens if the number of producers increases? Assume that the total capacity and aggregated cost function is independent of the number of producers. Denote the total cost to meet the demand by $C_{tot}(\varepsilon)$. In the unique symmetric SFE, this total cost is N times the cost of each symmetric producer. Hence,

$$C_{tot}(\varepsilon) = NC(S_i) = NC(\varepsilon / N).$$

Thus

$$C_{tot}'(\varepsilon) = C'(\varepsilon / N). \tag{27}$$

Combining (31) and (27), the equilibrium price of the unique SFE can be written:

$$p(\varepsilon) = \varepsilon^{N-1}\left[\frac{\bar{p}}{\bar{\varepsilon}^{N-1}} + (N-1)\int_{\varepsilon}^{\bar{\varepsilon}} \frac{C_{tot}'(x)dx}{x^N}\right] \text{ if } \varepsilon \geq 0.$$

The cost function is twice continuously differentiable and strictly convex. Thus $C_{tot}''(\varepsilon) > 0$. Now, using integration by parts, the equilibrium price for positive demand can be rewritten:

$$\begin{aligned} p(\varepsilon) &= \bar{p}\left(\frac{\varepsilon}{\bar{\varepsilon}}\right)^{N-1} - \left[\left(\frac{\varepsilon}{x}\right)^{N-1} C_{tot}'(x)\right]_{\varepsilon}^{\bar{\varepsilon}} + \int_{\varepsilon}^{\bar{\varepsilon}} \left(\frac{\varepsilon}{x}\right)^{N-1} C_{tot}''(x)dx = \\ &= \underbrace{\left[\bar{p} - C_{tot}'(\bar{\varepsilon})\right]}_{>0} \underbrace{\left(\frac{\varepsilon}{\bar{\varepsilon}}\right)^{N-1}}_{\leq 1} + C_{tot}'(\varepsilon) + \int_{\varepsilon}^{\bar{\varepsilon}} \underbrace{\left(\frac{\varepsilon}{x}\right)^{N-1}}_{\leq 1} \underbrace{C_{tot}''(x)}_{>0} dx \end{aligned} \tag{28}$$

It is evident that all terms are positive and that the first and last term decreases with *N,* unless $\varepsilon = \bar{\varepsilon}$. The middle term is not influenced by *N* at all. Hence, for every positive demand below $\bar{\varepsilon}$, the equilibrium price of the unique SFE decreases, when the number of symmetric producers increases, while the total capacity and aggregated cost function is kept constant. By means of (28), it can also be noted that the equilibrium price approaches the marginal cost of the marginal unit as the number of symmetric producers goes to infinity.

What happens if entrants increase the total capacity? This can be seen as a combination of an increase in the number of producers and an increase in the total capacity. Both will decrease the equilibrium price for every positive demand.

With (28) it is also easy to verify that $p(0) = C_{tot}{}'(0) = C'(0)$. Hence, the equilibrium price equals the marginal cost of the marginal unit for zero demand.

4.3. The slope at zero demand

The unique SFE fulfills the first-order condition in (17) for all prices $p \in \left[C'(0), \bar{p}\right]$. Using (18) and integration by parts it can be shown that:

$$\frac{\partial p(\varepsilon)}{\partial \varepsilon} = \frac{(N-1)p_+ \varepsilon^{N-2}}{\varepsilon_+^{N-1}} + (N-1)^2 \varepsilon^{N-2} \int_{\varepsilon}^{\varepsilon_+} \frac{C'(x/N)dx}{x^N} - (N-1)\varepsilon^{N-1} \frac{C'(\varepsilon/N)}{\varepsilon^N} =$$
$$= \frac{(N-1)\varepsilon^{N-2}(p_+ - C'(\varepsilon_+/N))}{\varepsilon_+^{N-1}} + \frac{(N-1)\varepsilon^{N-2}}{N} \int_{\varepsilon}^{\varepsilon_+} \frac{C''(x/N)dx}{x^{N-1}}. \tag{29}$$

Thus $0 < \dfrac{\partial p(\varepsilon)}{\partial \varepsilon} < \infty$ for positive demand. What happens if $\varepsilon \to 0+$? For $N>2$, the limit is of the type $0 \cdot \infty$, as $C''(\varepsilon) > 0$ and $\displaystyle\int_0^{\bar{\varepsilon}} \frac{dx}{x^{N-1}} \to \infty$. The limit can be written on the form $\dfrac{\infty}{\infty}$. Thus it can be calculated by means of l'Hospitals rule [1].

$$\lim_{\varepsilon \to 0+} \frac{\partial p(\varepsilon)}{\partial \varepsilon} = \lim_{\varepsilon \to 0+} \frac{(N-1)\varepsilon^{N-2}}{N} \int_{\varepsilon}^{\bar{\varepsilon}} \frac{C''(x/N)dx}{x^{N-1}} = \lim_{\varepsilon \to 0+} \frac{(N-1)}{N\varepsilon^{2-N}} \int_{\varepsilon}^{\bar{\varepsilon}} \frac{C''(x/N)dx}{x^{N-1}} =$$
$$= \lim_{\varepsilon \to 0+} \frac{(N-1)\dfrac{-C''(\varepsilon/N)}{\varepsilon^{N-1}}}{(2-N)N\varepsilon^{1-N}} = \frac{(N-1)C''(\varepsilon/N)}{(N-2)N}. \tag{30}$$

Thus $0 < \frac{\partial p(\varepsilon)}{\partial \varepsilon} < \infty$ for N>2, as the cost function is twice continuously differentiable by assumption. What happens for N=2? The following can be proven by means of (29):

$$\frac{\partial p(\varepsilon)}{\partial \varepsilon} \geq \frac{(N-1)\varepsilon^{N-2}\left(\bar{p} - C'\left(\bar{\varepsilon}/N\right)\right)}{\bar{\varepsilon}^{N-1}} + \frac{(N-1)}{N}\int_{\varepsilon}^{\bar{\varepsilon}} \frac{C''(x/N)dx}{x}.$$

Thus $\lim_{\varepsilon \downarrow 0} \frac{\partial p(\varepsilon)}{\partial \varepsilon} = \infty$, as $C''(\varepsilon) > 0$ and $\int_0^{\bar{\varepsilon}} \frac{dx}{x} \to \infty$, when N=2.

The conclusion is that the equilibrium price increases monotonically and continuously in the demand, if N>2. For N=2, there is an exception at ε=0, where the symmetric supply functions are inelastic.

5. BALANCING MARKETS

The assumptions used in this paper: inelastic demand, price caps backed up with possible disconnection of consumers, risk of power shortage and a convex support of $f(\varepsilon)$ that includes both ε=0 and $\varepsilon = \bar{\varepsilon}$ are especially reasonable for balancing markets.

It is very expensive to store electric energy, compared to the production cost. Hence, in most power systems the stored electric energy is negligible; power consumption and production have to be roughly in balance, every single minute. Most of the electric power produced is traded on the day-ahead market or with long-term agreements. Neither consumption nor production is fully predictable, so to maintain the balance, adjustments have to be made in real-time. The balancing market is an important component in this process. It is an auction, where the independent system operator (ISO) can buy additional power (positive demand) from the producers or sell back power (negative demand) to the power producers. The latter is called down-regulation and occurs if total production in real-time, including inflexible production — which cannot be adjusted on short-notice—, exceeds the total demand. A producer can offer down-regulation, if he has less inflexible power running than the power he has sold on the day-ahead market or committed to in long-term contracts.

Somewhat different assumptions should be made for down-regulation bids: $S_i(p)$ should be right-hand continuous. Further, in case an inelastic segment of the total supply coincides with a negative inelastic demand, the highest price should be chosen, i.e. the best price of the ISO. The total down-regulation capacity is denoted by $\underline{\varepsilon}$, which is a negative number. Hence, the cost function $C(\varepsilon)$ can have negative arguments. This reflects flexible production — which can be adjusted on short-notice— that has already been sold, e.g. on the day-ahead market, but that can be bought back and turned off. The power will not be bought back, if the price exceeds the marginal cost. Instead producers will use their market power to push down the price below the marginal cost. Thus for down-regulation, there should be a price floor. It does not have to be positive and is denoted by $\underline{p}$. It is assumed that $\underline{p} \le C'\left(\frac{\underline{\varepsilon}}{N}\right)$. One can use arguments analogous to the up-regulation case to show that for down-regulation there is a unique symmetric SFE. Considering both up- and down-regulation, the equilibrium price is given by:

$$p(\varepsilon)=\begin{cases} \dfrac{\overline{p}\varepsilon^{N-1}}{\overline{\varepsilon}^{N-1}}+(N-1)\varepsilon^{N-1}\displaystyle\int_{\varepsilon}^{\overline{\varepsilon}}\frac{C'(x/N)dx}{x^N} & \text{if } \varepsilon \ge 0 \\ \dfrac{\underline{p}\varepsilon^{N-1}}{\underline{\varepsilon}^{N-1}}+(N-1)\varepsilon^{N-1}\displaystyle\int_{\varepsilon}^{\underline{\varepsilon}}\frac{C'(x/N)dx}{x^N} & \text{if } \varepsilon < 0 \end{cases} \tag{31}$$

Klemperer & Meyer [10] show that $p(0)=C'(0)$ for all symmetric SFE. This was also verified for (26) in Section 4.2. It is true for all combinations of $\overline{p}, \overline{\varepsilon}, \underline{p}$ and $\underline{\varepsilon}$. Thus the equilibrium price is continuous at $\varepsilon=0$.

6. A NUMERICAL ILLUSTRATION OF THE UNIQUE SFE

If the cost function is given by a polynomial, it is straightforward to analytically solve for the equilibrium price as a function of the demand by means of (31). Here the equilibrium is illustrated with a simple example with a quadratic cost function, i.e. marginal costs are linear.

$$C'_{tot}(x)=c_0+kx .$$

The result for $N>2$ is:

$$p(\varepsilon)=\begin{cases} c_0+(\overline{p}-c_0)\dfrac{\varepsilon^{N-1}}{\overline{\varepsilon}^{N-1}}+\dfrac{k(N-1)\varepsilon}{(N-2)}\left(1-\dfrac{\varepsilon^{N-2}}{\overline{\varepsilon}^{N-2}}\right) & \text{if } \varepsilon \ge 0 \\ c_0+(\underline{p}-c_0)\dfrac{\varepsilon^{N-1}}{\underline{\varepsilon}^{N-1}}+\dfrac{k(N-1)\varepsilon}{(N-2)}\left(1-\dfrac{\varepsilon^{N-2}}{\underline{\varepsilon}^{N-2}}\right) & \text{if } \varepsilon < 0. \end{cases} \tag{32}$$

Negative demand is relevant for normal auctions and positive demand for procurement auctions, i.e. electric power auctions. In case of balancing markets, both positive and negative demand is relevant. (32) is used in Fig. 6 to illustrate the influence the number of producers have on the equilibrium price. For positive (negative) demand, more producers reduce the mark-ups (mark-downs). In Section 4.2, this was proven for all strictly convex and twice continuously differentiable cost functions. For N=100, the market price is very close to the marginal cost, except near the capacity constraints. The residual demand is less elastic and mark-ups more extreme closer to the capacity constraints. This is in agreement with the monopoly mark-up rule in (15). For negative demand, the market price is below the marginal cost. The oligopoly producers use their market power to buy back power at a price below their marginal cost. Note also that in all cases the price equals the marginal cost at zero supply/demand.

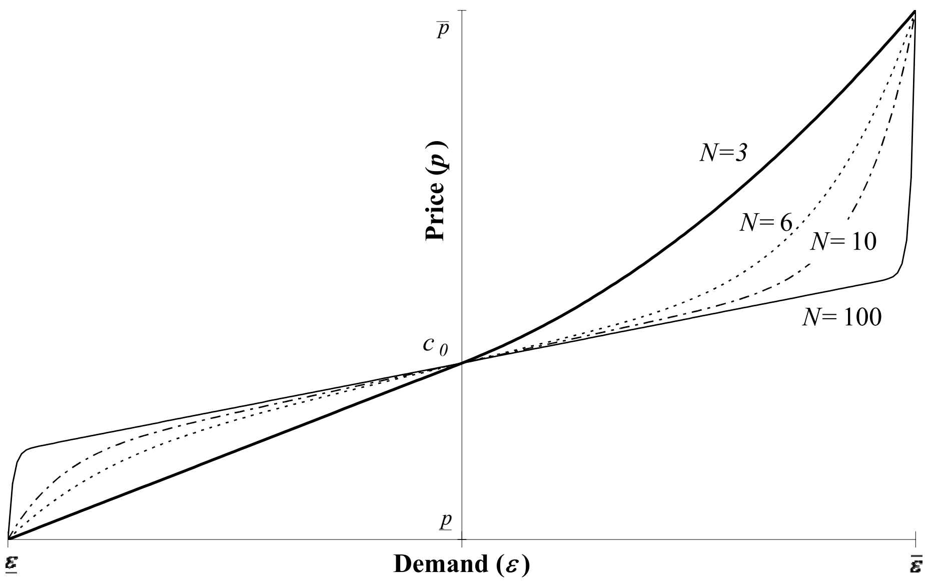

Fig. 6. *The unique symmetric supply function equilibrium for linear marginal costs. N= number of symmetric producers. Negative demand is relevant for auctions and positive demand for procurement auctions. Both are relevant for balancing markets.*

7. CONCLUSIONS

Inelastic demand and capacity constraints that bind with a positive probability are reasonable assumptions for electric power markets and balancing markets in particular. Under these conditions, it is shown that for symmetric producers there is a unique symmetric SFE.

A reservation price, i.e. price cap, is often used in electric power markets and other procurement auctions. The market price in the unique supply function equilibrium reaches the price cap exactly when the capacity constraints bind. The equilibrium does not depend on the probability density of the demand.

If the price cap is decreased or if the capacity constraint is increased, the equilibrium price decreases for each positive demand outcome. Thus changing these constraints affects the price, also for outcomes when the constraints are not binding. Increasing the number of producers decreases the equilibrium price for every level of positive demand. Mark-ups are zero at zero supply and positive for every positive supply.

No capacity will be withheld from the auction in equilibrium. However, this result depends on the assumption that nothing prevents the producers from bidding all the way to the price cap. If the bidding is constrained by e.g. competition law, morals or public relations, the offered capacity would probably decrease. If so, the risk of a capacity shortage would increase.

Supply functions with binding slope constraints, i.e. supply functions with horizontal or vertical segments, are not considered in the previous SFE literature. Here such supply functions are allowed, as they are useful deviation strategies when ruling out equilibria. Extending the space of valid supply functions complicates the analysis, as non-smooth SFE have to be ruled out as well. Optimal control theory with final values is a useful tool in this process. With more than two producers, all SFE with inelastic or perfectly elastic supply can be ruled out.

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APPENDIX

Proof of Lemma 1:

The result follows from the symmetric first-order condition in (17). If $\varepsilon \geq A > 0$, i.e. $\widehat{S}_i(p) > \frac{A}{N}$ then $0 < m \leq \widehat{S}_i{}'(p)$, where *m* is a number independent of *ε*. Thus $\frac{1}{\widehat{S}_i{}'(p)}$ is bounded for a positive supply bounded away from zero.

If there would have been a smooth transition to $\widehat{S}_{il}{}'(p_0) = 0$ from the left, then $\widehat{S}_i{}'(p)$ is twice continuously differentiable and monotonically increasing in some interval below, but arbitrarily close to p_0. From the argument above it follows that $\frac{1}{\widehat{S}_i{}'(p)}$ is bounded for *p* arbitrarily close to p_0. Thus a smooth transition to $\widehat{S}_i{}'(p_0) = 0$ from the left can be ruled out. With similar reasoning it can be shown that there are no smooth transitions from the right to an inelastic supply, if the positive supply is bounded away from zero.

Proof of Proposition 1:

Assume that there is an equilibrium, where producer *i* offers production below its marginal cost, i.e. there are some prices p, for which $C'\left[S_i^Z(p)\right] > p.$ Denote this set of prices by $\mathbb{P}$. Then there is a profitable deviation for producer *i*. Adjust the supply of producer *i* such that units previously offered below their marginal cost are now offered at their marginal cost. Formally, $C'[S_i(p)] = p$ for all $p \in \mathbb{P}$, where $S_i(p)$ is the adjusted supply function. The supply is unchanged for all other *p*, i.e. $S_i(p) = S_i^Z(p)\ \forall p \notin \mathbb{P}.$ $S_i(p)$ is non-decreasing like $S_i^Z(p),$ as *C()* is strictly convex and increasing. The contribution to expected profits from units that are offered at or above their marginal costs are not negatively affected by the deviation. Their contribution might even increase, as the equilibrium price increases for some imbalance outcomes. Now consider a unit that was previously offered below its marginal cost c_0. Let ε_0 denote the imbalance, for which the market price reaches c_0 in the assumed equilibrium. After the deviation, the price will reach c_0 at an imbalance $\varepsilon \leq \varepsilon_0$. Further, market prices will not decrease for any positive imbalances. Thus the positive contribution of the considered unit to the expected profit is increased or unchanged. Further, demand outcomes for which the considered unit has a negative contribution to the profit are avoided after the deviation. The same reasoning is true for all units offered below their marginal cost. Thus the deviation increases the expected profit of producer *i.* Thus in equilibrium no production is offered below its marginal cost.

Proof of Proposition 3

Use the same notation as in Proposition 2, but now the aggregate supply is inelastic just below p_0. Denote supply functions following the equilibrium candidate by the superscript *H*. Isolated inelastic points are ruled out by Lemma 1. Thus the supply must be inelastic in a price interval below p_0. Consider the following deviation. Producer *i* can offer $x = \varepsilon' - \varepsilon_0'$ units previously offered at p_0 to the price p_0-η, where η is positive and infinitesimally small. As in Proposition 2, the perfectly elastic aggregate supply starts at ε'. In the potential equilibrium, $\varepsilon'=\varepsilon'_0$. The optimal $\varepsilon'\geq\varepsilon'_0$ is given by:

$$\begin{aligned}\operatorname*{Max}_{\varepsilon'} \Omega = &\int_{\varepsilon_0'}^{\varepsilon'} \left\{\left[\varepsilon - S_{-i}^H(p_0)\right](p_0-\eta) - C\left[\varepsilon - S_{-i}^H(p_0)\right]\right\} f(\varepsilon)d\varepsilon + \\ &+\int_{\varepsilon'}^{\varepsilon''}\left\{\left[\varepsilon' - S_{-i}^H(p_0) + R_i\left(\varepsilon-\varepsilon', \varepsilon''-\varepsilon' - \Delta S_{-i}^H, \Delta S_{-i}^H\right)\right]p_0 + \right. \\ &\left. - C\left[\varepsilon' - S_{-i}^H(p_0) + R_i\left(\varepsilon-\varepsilon', \varepsilon''-\varepsilon' - \Delta S_{-i}^H, \Delta S_{-i}^H\right)\right]\right\} f(\varepsilon)d\varepsilon +\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial\Omega}{\partial\varepsilon'} = &-\left\{\left[\varepsilon' - S_{-i}^H(p_0)\right]p_0 - C\left[\varepsilon' - S_{-i}^H(p_0)\right]\right\} f(\varepsilon') + \\ &+\int_{\varepsilon'}^{\varepsilon''}\left[1+\frac{dR_i\left(\varepsilon-\varepsilon', \varepsilon''-\varepsilon' - \Delta S_{-i}^H, \Delta S_{-i}^H\right)}{d\varepsilon'}\right]\left[p_0 - C'(\cdot)\right] f(\varepsilon)d\varepsilon + \\ &+\left\{\left[\varepsilon' - S_{-i}^H(p_0)\right](p_0-\eta) - C\left(\varepsilon' - S_{-i}^H(p_0)\right)\right\} f(\varepsilon')\end{aligned}$$

In order to keep the potential equilibrium with discontinuous supply functions at p_0, $\varepsilon'=\varepsilon'_0$ must be optimal.

$$\left.\frac{\partial\Omega}{\partial\varepsilon'}\right|_{\varepsilon'=\varepsilon_0'} = -\left\{\left[\varepsilon_0' - S_{-i}^H(p_0)\right]\eta f(\varepsilon_0')\right\} + \int_{\varepsilon_0'}^{\varepsilon''}\left\{\left[1 - R_1 - R_2\right]\left[p_0 - C'(\cdot)\right] f(\varepsilon)d\varepsilon\right. \quad (33)$$

The first term is negative, but infinitesimally small, as η is infinitesimally small. It is known from the proof of Proposition 2 that the second term is positive and bounded away from zero.

Thus $\left.\frac{\partial\Omega}{\partial\varepsilon'}\right|_{\varepsilon'=\varepsilon_0{}'}$ > 0. Hence, producer *i* will find it profitable to deviate by slightly reducing the price of his perfectly elastic supply at p_0, i.e. $\varepsilon' > \varepsilon'_0$ for the optimal ε'. Accordingly, symmetric equilibria where supply functions are perfectly elastic can be ruled out, also if supply functions are inelastic just below p_0.

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