# VALUATION RELATIONSHIPS UNDER GROWTH

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Fecha de recepción: 16-6-2003
Fecha de aceptación: 9-10-2003

**SUMARIO**

Uno de los tópicos más importantes en valoración es la relación apropiada entre flujos de caja y tasas de retorno. Yo reviso esta relación con la premisa, por Myers (1974), de que el costo de la deuda es la tasa de descuento apropiada para el escudo fiscal. Diferentes hipótesis han sido estudiadas para el riesgo del escudo fiscal; cada una de ellas produce diferentes resultados de valoración, especialmente cuando el crecimiento está presente. Una diferencia entre los resultados que yo obtengo y los resultados de otros es la presencia del crecimiento en las expresiones para las tasas de descuento, lo cual puede ser utilizado para estimar la validez empírica de cada uno de los métodos propuestos.

**PALABRAS CLAVES:**

Costo de capital, tasa de descuento sobre el patrimonio, valor del escudo fiscal, beta apalancado.

**Clasificación: A**

**ABSTRACT**

One of the most important topics on valuation is the appropriate relationships between cash flows and rate of returns. I review those relationships under the premise, by Myers (1974), of the cost of debt as the right discount for the tax shield. Different hypotheses have been advanced for the tax shield risk, each one producing different valuation results, especially when growth is present. The consequences of some common mistakes on valuation are explored. One difference between the results I obtain and results by others is the presence of growth in the expressions for the discount rates, which can be used to asses the empirical validity of each of the approaches.

**KEYWORDS:**

Cost of capital, return on equity, tax shield value, levered beta.

I review the calculations of the appropriate rates of return for free cash flows under alternative assumptions. One of the most contested assertions on this issue is the appropriate rate of return for the tax shield. Different assumptions led to differences on valuation. The seminal contributions of Modigliani and Miller (M&M) (1958 & 1963) generated tractable ways to deal with cash flows and rates of return. In their 1963 correction M&M discounted the sure tax shield of a perpetuity with the risk free rate, which was the debt interest rate, and established an enduring paradigm for this term. Myers (1974) argue that the appropriate rate of return for the tax shield is the debt rate, taking distance of M&M but producing a similar result for perpetuities. Harris and Pringle (1985) suggest, instead, that the tax shield bears the operational risk, which means that the appropriate discount rate is $k_0$, the discount rate for the firm's assets. Fernandez (2003) define the Tax shield as the difference in taxes paid by the unlevered firm and the levered firm, and for the case of unlevered firms arrive to the same answer of M&M and Myers.

I go through the valuation relationships for the case of growing perpetuities and finish the paper with some suggestions of how to solve the ongoing debate. Growing perpetuities are more realistic models of firm's cash flows, firms always grow, or at least they always forecast grow. I derive somewhat modified versions of the relationship between the weighted average cost of capital (kWACC) and the cost of equity (kS) and the valuation consequences of the modified assumptions. The results critically depend on the appropriate rate of return for the tax shield. The predicted effects on the betas can be used to shed some light on the ongoing controversy about the appropriate rate of return for the tax shields.

I finish my discussion with the most general case with the relevant relationships solved period by period.

The basic assumptions I use are:

1. The capital structure is constant: $\frac{D}{V_L} = K, \frac{S}{V_L} = 1 - \frac{D}{V_L}$

2. The tax rate T is constant

I begin with the most fundamental equations:

Let $EBITDA_i$ be earnings before interests, taxes, depreciation and amortization for period i, Dep= Depreciation, D=Debt, $k_D$=Cost of Debt, T=Tax rate, $\Delta$Nwc= Increment in net working capital and $\Delta$FA=Increment in fixed assets.

Then $ECF_i = (EBITDA_i - Dep_i - D_i k_D)(1-T) + Dep_i + gD_i - \Delta Nwc_i - \Delta FA_i$ is the equity cash flow and $FCF_i = (EBITDA_i - Dep_i)(1-T) + Dep_i - \Delta Nwc_i - \Delta FA_i$ is the free cash flow. The relationship between both is $FCF_i = ECF_i + D_i\, k_D\, (1 - T) - gD_i$

To simplify things I suppose $\Delta Nwc_i = k_w EBITDA_i$; $\Delta FA_i = k_{FA} EBITDA_i$. The free cash flow becomes $FCF_i = (EBITDA_i)(1 - T - k_w - k_{FA}) + TDep_i$

Under this approach $Dep_i$ also becomes proportional to $EBITDA_i$:

Let $D_r$=1/y (y=years for full depreciation), suppose $D_r$ is constant over the years (for 10 years depreciation, $D_r$=10%); TGA= Total gross assets, FDA= Fully depreciated assets, then: $Dep_i=[TGA_{i-1}-FDA_{i-1}]D_r$

For i>y,

$$
\begin{aligned}
Dep_i &= \left[\sum_{i=0}^{i-1}\Delta FA_i - \sum_{i=0}^{i-1-y}\Delta FA_i\right] D_r \\
&= \left[\sum_{j=i-y}^{i-1}\Delta FA_j\right] D_r \\
&= \left[k_{FA}\sum_{j=i-y}^{i-1} EBITDA_j\right] D_r \\
&= \left[\sum_{i=1}^{y}\frac{1}{(1+g)^i}\right] k_{FA}\, D_r EBITDA_i \\
&= \delta_y\, k_{FA}\, D_r EBITDA_i
\end{aligned}
$$

$$\text{Where } \delta_y = \sum_{i=1}^{y}\frac{1}{(1+g)^i}$$

The $FCF_i$ becomes $FCF_i = EBITDA_i (1 - T - k_w - k_{FA} (1 - \delta_y TD_r))$. If g is the EBITDA growing percentage, we have $FCF_{i+1} = FCF_i(1+g)$. In this scenario (an infinite growing perpetuity) the unlevered firm value, when $k_0$ is less than g, is:

$$V_u = \frac{FCF_1}{k_0 - g} \qquad (1)$$

$k_0$ is the discount rate for the firm assets, under a cero leverage policy (more on this rate follows). When leverage is greater than cero, the firm value results from the combined effects on the cash flows to debt holders and shareholders. As per assumption 1, the debt increases at the same rate (g) that the cash flows, then $D_{i+1}=D_i(1+g)$.[1]

The cash flow to the debt holders is: $DCF_i = -gD_i + k_D D_i$

Then cash flow to the investors, shareholders (ECF) and debt holders (DCF) is:

$$CF(V_L)_i = ECF_i + DCF_i$$

$$= (EBITDA_i - Dep_i - D_i k_D) (1-T)+ Dep_i + gD_i - \Delta Nwc_i - \Delta FA_i -gD_i + k_D D_i$$

$$= FCF_i + k_D D_i T^2$$

$CF(V_L) = FCF_1 + k_D D_0 T$, discounting the flows at the appropriate[3] rates yields:

$$V_L = \frac{FCF_1}{k_0 - g} + \frac{k_D D_0 T}{k_D - g} = V_u + D_0\, T^*,$$

$$\text{where } T^* = \frac{k_D}{k_D - g}\; T \qquad (2)$$

1. To check this assertion is enough to note that $VL_{i+1}=VL_i(1+g)$, without $FCF_{i+1.}$ Given that the debt proportion is constant, it follows that D grows at the same rate. Interest payments are due at the end of period.

2. Under Fernandez (2003) approach:

Tax Shield=$T_{xU} - T_{xL} = EBITDA(1-\delta_y k_{FA} D_r)T - [EBITDA(1-\delta_y k_{FA} D_r) + k_D D]T = k_D DT$.

The result is the same. The key difference is that for Fernandez $Tx_U$ and $Tx_L$ have different risk and should be discounted independently, under the assumptions of this paper that doesn't hold.

3. As I said, the discount rate for the tax shield is an unsolved issue on valuation, here I assume that this rate is the debt rate as Myers (1974).

Fernández (2003) arrives to a different expression for the Tax Shield:

$\frac{k_0 D_0 T.}{k_0 - g}$ He avoids cash flows and employs valuation equi valences.

$k_D$ is the interest rate for the firm debt, here I assume that this rate is the same rate that the debt holders are receiving. Under certain conditions these rates differ. The convergence condition is more severe, it requires that $g < k_D$.[4] The second term $D_0 T^*$, is known as the tax shield, except that the effective tax rate is higher, yielding a higher firm value.

A market balance at t=0 follows

**Market Balance**

| Assets | Liabilities |
|---|---|
| $V_u$ | $D_0$ |
| $D_0 T \frac{k_D}{k_D - g}$ | S |

Solving the accounting identity for $V_U$, we find an additional definition for the unlevered firm:

$$V_U = S + D_0 (1-T^*) \qquad (3)$$

### $k_S$, the equity cost for the levered firm

Now we have enough tools to find $k_S$. The cash flows produced by the assets and by the liabilities should be the same, then it must be that

$V_U k_0 + D_0 T^* k_D = S k_s + D_0 k_D$,[5] replacing $V_U$ with equation 3 and solving for $k_S$, we obtain:

$$k_s = k_0 + \frac{D_0}{S} - (1-T^*)(k_0 - k_D) \qquad (4)$$

The above result is the familiar definition of $k_S$, modified by the new effective tax rate. Interestingly, increasing flows reduce the required rate of return for the shareholders (Figure 1), when $k_0 > k_D$. We can also express this result as a combination of the standard equity cost with no growth $k_{S\,ng}$ and the growth effect.

4. It is not difficult to conceive firms with $g > k_D$, the only attenuant is that it is difficult to maintain indefinite growth rates higher than the cost of debt.

5. The result is the same if the equation is written for increasing flows

$V_U (k_0 - g) + D_0 T^* (k_D - g) = S(k_s - g) + D_0 (k_D - g)$.

$$k_s = k_0 + \frac{D_0}{S}\left(\frac{1 - T\,k_D}{k_D - g)}\right)(k_0 - k_D)$$

$$k_s = k_0 + \frac{D_0}{S}\left(1 - T\frac{k_D}{k_D - g} + T\frac{g}{k_D - g}\right)(k_0 - k_D) - \frac{D_0}{S}\left(T\frac{g}{k_D - g}\right)(k_0 - k_D)$$

$$k_s = k_0 + \frac{D_0}{S}(1 - T)(k_0 - k_D) - \frac{D_0}{S}\left(\Gamma\frac{g}{(k_D - g}\right)(k_0 - k_D)$$

$$k_s = k_{s\,ng} - \frac{gD_0T}{S(k_D - g)}(k_0 - k_D)$$

As long as $k_0 > k_{D,}$ the growth effect is negative. Lets proceed to check if under these conditions continue to hold another basic financial result: That discounting the unlevered flows at the weighted average cost of capital yields the same number that discounting the unlevered flows at the required rate of return for the firm assets plus the increased tax shield. To do this, is enough to find what definition of $k_{WACC}$ solves the following equality:

$$V_L = \frac{FCF_1}{k_{WACC} - g} = \frac{FCF_1}{k_0 - g} + D_0T^*$$

First multiply both sides of the equality by $\frac{k_0 - g}{FC_1}$, the result is

$$\frac{k_0 - g}{k_{WACC} - g} = 1 + \frac{D_0\,T^*}{\frac{FCF_1}{k_0 - g}}$$

The last result is the same (by equation 1) that $\frac{k_0 - g}{k_{WACC} - g} = 1 + \frac{D_0T^*}{V_U}$

Replacing $V_U$ by equation 3 yields

$$\frac{k_0 - g}{k_{WACC} - g} = 1 + \frac{D_0 T^*}{S + D_0 (1 - T^*)} \qquad (5)$$

Solving equation 4 for $k_0$ results in

$$k_0 = \frac{k_s S + k_D D_0 (1\text{-}T^*)}{S + D_0 (1\text{-}T^*)} \qquad (6)$$

We use this result in the equation 5 and solve for $k_{WACC}$:

$$k_{WACC} = \frac{k_s S + k_D D_0 (1 - T)}{S+D_0} = k_s \frac{S}{V_L} + k_D (1\text{-}T) \frac{D_0}{V_L} \qquad (7)$$

Here we find that the old expression for $k_{WACC}$ continues to hold (which means the results are coherent). Please note that here the tax rate is not the modified expression we defined above; the change confines to the calculation of $k_s$. The result is that growing firms have lower $k_{WACC}$. To see what are the effects on $k_{WACC}$ of the growing perpetuity let express it as a function of $k_0$

$$k_{WACC} = k_0 \left( 1 - \frac{D_0}{S + D_0} T \right) + g \frac{D_0}{S + D_0} T^* \qquad (8)$$

The previous equation shows that the effects of the constant growth are two fold. First, is a decreasing effect caused by the interaction of $k_0$ and $T^*$ and, second, an increasing effect through the interaction of g and $T^*$. Under no growth we have

$k_{WACC\,ng} = k_0 \left( 1 - \frac{D_0}{S + D_0} T \right)$. With that in mind, modifying equation 8 yields[6]

6. A simpler approach is to note that.

$$k_{WACC} = \left( k_{s\,ng} - \frac{gD_0T}{S(k_D\text{-}g)} (k_0\text{-}k_D) \right) \frac{S}{V_L} + k_D (1\text{-}T) \frac{D_0}{V_L} = k_{WACC\,ng} - \frac{gD_0T}{V_L (k_D\text{-}g)} (k_0 - K_D)$$

$$k_{WACC} = k_0 \left( 1 - \frac{D_0}{S + D_0} \; T \; \frac{k_D}{k_D - g} \right) + g \; \frac{D_0}{S + D_0} \; T \; \frac{k_D}{k_D - g}$$

$$k_{WACC} = k_0 \left( - \frac{D_0}{S + D_0} \; T \; \frac{k_D}{k_D - g} + \frac{D_0}{S + D_0} \; T \; \frac{g}{k_D - g} \right) + g \frac{D_0}{S - D_0} \; T \; \frac{k_D}{k_D - g} - k_0 \frac{D_0}{S + D_0} \; T \; \frac{g}{k_D - g}$$

$$k_{WACC} = k_0 \left( 1 - \frac{D_0}{S + D_0} T \right) - \frac{g D_0 T}{(S + D_0)(k_D - g)} \; (k_0 - k_0)$$

$$k_{WACC} = k_{WACC\,ng} - \frac{g D_0 T}{(S + D_0)\,(k_D - g)} \quad (k_0 - k_D)$$

As we saw before, under normal conditions ($k_0 > k_D$) the decreasing effect dominates (Figure 1).

The effects of leverage on the different required rates of return (Figure 2) shows how the $k_{WACC}$ decreases at a higher rate with constant growth and the $k_s$ increases at a lower rate. Again the benefits of growth are significant.

It is noteworthy to understand that only with the corrections here developed continue to hold the equality

$$V_L = \frac{FCF_1}{k_{WACC} - g} = \frac{FCF_1}{k_o - g} + D_o T^*$$

**Parameters**

| | |
|---|---|
| T | 40.00% |
| g | x-variable |
| kd | 12.00% |
| kf | 10.00% |
| km-kf | 15.00% |
| Bass | 1.20 |
| Bdebt | 13.33% |
| $k_o$ | 28.00% |
| $D/V_L$ | 40.00% |

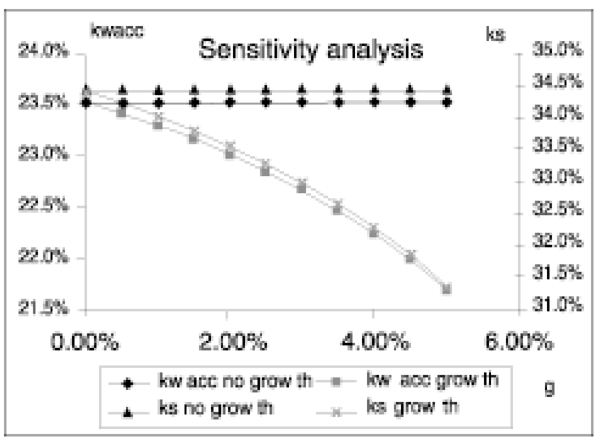


**Figure 1:** Required rates of return sensitivity to growth

| | |
|---|---|
| **Parameters** | |
| **ng: no growth** | |
| T | 40.00% |
| g | 6.00% |
| kd | 12.00% |
| kf | 10.00% |
| km-kf | 15.00% |
| Bass | 1.20 |
| Bdebt | 13.33% |
| $K_o$ | 28.00% |
| $D/V_L$ | x-variable |

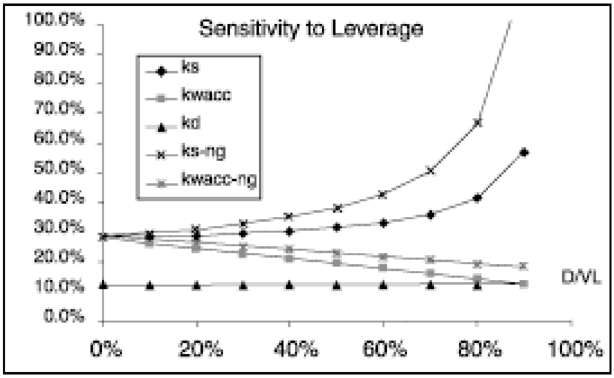


**Figure 2:** Required rates of return sensitivity to leverage

## MODIFIED BETA CALCULATIONS

The fundamental equation of CAPM permit us to find some additional equivalences. By the CAPM we have $k_s = k_f + \beta_s (k_m - k_f)$and $k_o = k_f + \beta_o (k_m - k_f)$, where the meaning of the different terms correspond to the usual ones. Rewriting equation 4 yields:

$$k_s = k_o \left(1 + \frac{D_o}{S}(1- T^*)\right) - k_D \frac{D_o}{S}(1 - T^*),$$

combining it with the former CAPM equations produces:

$$k_f + \beta_s (k_m-k_f) = (k_f+\beta_0 (k_m-k_f)) \left(1 + \frac{D_0}{S}(1-T^*)\right) -k_D \frac{D_0}{S}(1-T^*),$$

reordering terms gives the following result:

$$k_f + \beta_s (k_m-k_f) = k_f + k_f \frac{D_0}{S}(1-T^*) + \beta_0 (k_m-k_f) \left(1+ \frac{D_0}{S}(1 - T^*)\right) - k_D \frac{D_0}{S}(1-T^*)$$

$$\beta_s (k_m-k_f) = \beta_0 (k_m-k_f) \left(1+ \frac{D_0}{S}(1-T^*)\right) - k_D - k_f) \frac{D_0}{S}(1-T^*)$$

$$ß_s = ß_0 \left( 1+ \frac{D_0}{S} (1-T^*) \right) - \frac{k_D - k_f}{k_m - k_f} \frac{D_0}{S} (1-T^*)$$[7]

By CAPM $ß_D = \frac{k_D - k_f}{k_m - k_f}$, then

$$ß_s = ß_0 \left( 1+ \frac{D_0}{S} (1-T^*) \right) - ß_D \frac{D_0}{S} (1-T^*) \qquad (7)$$

or

$$ß_0 = ß_s \frac{S}{V_U} + ß_D \frac{D_0 (1-T^*)}{V_U} \qquad (7a)$$

$$ß_0 = \frac{ß_s}{1 + \frac{D_0}{S} (1-T^*)} + \frac{ß_D}{1 + \frac{S}{D_0} (1-T^*)} \qquad (7b)$$

The previous equations also shows that we have to reformulate our beta calculations when considering constant growth. Figures 3 and 4 illustrate the consequences of ignoring the corrections here contemplated. Note how the practice of ignoring $\beta_D$ increases the gap.

7. Most of the time the second term of this equation is ignored, the unique occasion when this practice is right is for $k_D=k_F$, which implies that $\beta_D=0$.

Parameters
ng/nBD: no corrections for growth and $\beta_D$

| | |
|---|---|
| T | 40.00% |
| g | x-variable |
| kd | 12.00% |
| kf | 10.00% |
| km-kf | 15.00% |
| Bass | 1.20 |
| Bdebt | 13.33% |
| $k_o$ | 28.00% |
| $D/V_L$ | 40.00% |

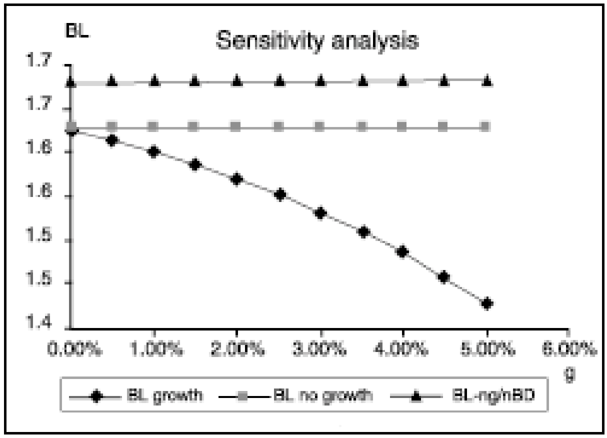


**Figure 3:** Levered Beta (BL) sensitivity to growth

Parameters
ng/nBD: no corrections for growth and $\beta_D$

| | |
|---|---|
| T | 40.00% |
| g | 6.00% |
| kd | 12.00% |
| kf | 10.00% |
| km-kf | 15.00% |
| Bass | 1.20 |
| Bdebt | 13.33% |
| $k_o$ | 28.00% |
| $D/V_L$ | x-variable |

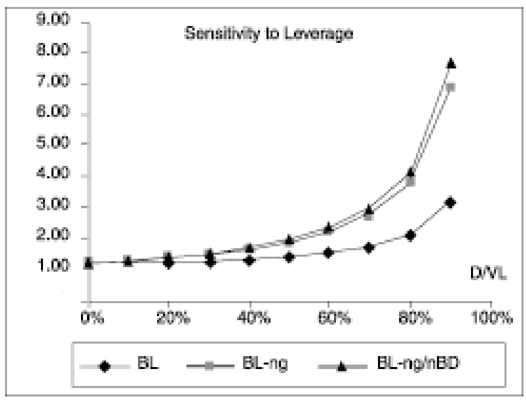


**Figure 4:** Levered Beta (BL) sensitivity to leverage

## OPERATIONAL REMARKS

The next paragraphs explore different approaches that use the concepts developed above. In particular they cover:

1. The valuation consequences of ignoring $T^*$
2. Tax shield estimation for constant debt but increasing CF, which means a variable capital structure.
3. How to use market betas.
4. A valuation approach.

## 1. Valuation Consequences

Here I performed sensitivity analysis to growth rates and leverage similar to those performed for the required rates of return and betas. Not surprisingly the consequences of ignoring the adjustments lead to undervaluation, that increases with growth and leverage.

The valuation formula we use is

$$V_L = \frac{FCF_1}{k_{WACC} - g}$$ an infinite growing perpetuity.

Parameters
ng/nBD: no corrections for growth and $\beta_D$

| | |
|---|---|
| T | 40.00% |
| g | x-variable |
| kd | 12.00% |
| kf | 10.00% |
| km-kf | 15.00% |
| Basset | 1.20 |
| Bdebt | 13.33% |
| $k_o$ | 28.00% |
| $D/V_L$ | 40.00% |
| $FC_1$ | 200 |

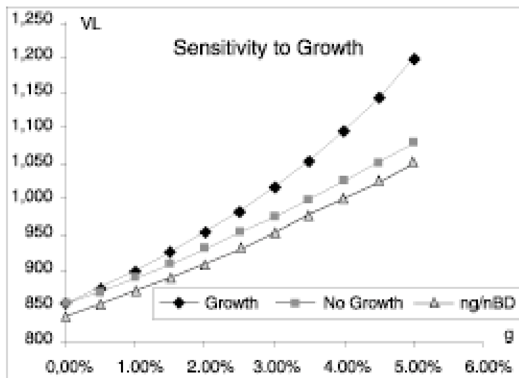


**Figure 5:** Levered firm value (VL) sensitivity to growth

Parameters
ng/nBD: no corrections for growth and $\beta_D$

| | |
|---|---|
| T | 40.00% |
| g | 4.00% |
| kd | 12.00% |
| kf | 10.00% |
| km-kf | 15.00% |
| Basset | 1.20 |
| Bdebt | 13.33% |
| $k_o$ | 28.00% |
| $D/V_L$ | x-variable |
| $FC_1$ | 200 |

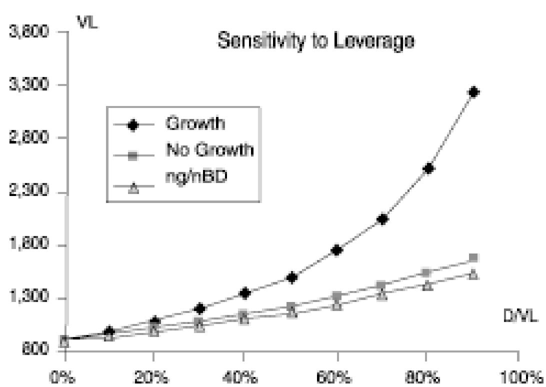


**Figure 6:** Levered firm value (VL) sensitivity to leverage;

## 2. Tax shield estimation for constant debt

Under this scenario the basic assumptions does not hold and we cannot use a unique $k_{WACC}$ to discount the cash flows, because it is changing each period. The only option left is to estimate the tax shield directly. If the debt is increasing but a different rate ($g_1$), it is still possible to use the same technique. For the estimation of the tax shield in the general case see the valuation example.

## 3. Market Betas

The use of market betas is implicitly explained in the previous section. Here the lesson it is do not forget the correction for $\beta_D$ or its proxy $(k_D-k_f)/(k_m-k_f)$. Measuring $\beta_0$ correctly implies to adjust for the cost of debt.

## 4. Valuation Example

How we implement a working model of these developments. The answer is that real calculations should use expressions that very each period; then the firm value needs to be solved backwards. Suppose the estimations of FC cover period 1 to period m, after that a constant growth $g_L$ is expected.[8] For any period $j \leq m$ holds

$$V_{Uj} = \sum_{i=j+1}^{m} \frac{FC_i}{(1+k_0)^i} + \frac{1}{(1+k_0)^{m-j}} \frac{FC_m (1+g_L)}{k_0 - g_L}$$

The Tax Shield is

$$TxSh_j = \sum_{i=j+1}^{m} \frac{D_{i-1} k_D T}{(1+k_D)^i} + \frac{1}{(1+k_D)^{m-j}} \frac{D_m k_D T}{k_D - g_L}$$

$V_{Lj} = V_{Uj} + TxSh_j = S_j + D_j$, which gives us an expression for the unlevered firm for the period j:

$$V_{Uj} = S_j + D_j - TxSh_j$$

The cash flows produced by the assets and the liabilities should be the same, then

$$S_j k_{sj} + D_j k_D = V_{Uj} k_0 + TxSh_j k_D = (S_j + D_j - TxSh_j) k_0 + TxSh_j k_D$$

8. The idea here is to present a methodology where the calculations are applied for the different periods.

Even though we suppose that $k_D$, T and $K_0$ are constant over time, this is not required and a subindex can be incorporated for a complete generalization. The assumption 1 (A constant leverage) is also relaxed for periods less than m (which applies to m+1 cash flows). Solving for $k_{sj}$ yields

$$k_{sj} = k_0 + (k_0 - k_D)\frac{D_j - TxSh_j}{S_j}$$

Now for each period holds:

$$V_{Lj} = V_{Uj} + TxSh_j = S_j + D_j = \sum_{i=j+1}^{m} \frac{FC_i}{\prod_{r=j}^{i-1}(1+k_{WACC\,r})} + \frac{1}{\prod_{r=j}^{m-1}(1+k_{WACC\,r})}\frac{FC_m(1+g_L)}{k_{WACC\,m} - g_L}$$

Again the expression for $k_{WACCj}$ is

$$k_{WACC\,j} = k_{s\,j}\frac{S_j}{V_{L\,j}} + k_D(1-T)\frac{D_j}{V_{L\,j}}$$

Each period has a set of simultaneous equations; implementing those equations in a spreadsheet produces circularities, which are solved through iterations.[9]

The implementation, which is illustrated in Table 1, begins when the forecasted flows begin to growth at a constant rate. Here holds all the equations we deduced in the above paragraphs:[10]

$$V_{L\,m} = \frac{FC_m(1+g_L)}{k_{WACC\,m} - g_L}; \quad TxSh_m = \frac{D_m k_D T;}{k_D - g_L}$$

$$k_{sm} = k_0 + (k_0 - k_D)\frac{D_m - TxSh_m}{S_m}; \quad k_{WACC\,m} = k_{s\,m}\frac{S_m}{V_{L\,m}} + k_D(1-T)\frac{D_m}{V_{L\,m}}$$

$$k_{WACC\,m} = k_0\left(1 - \frac{TxSh_m}{V_{L\,m}}\right) + k_D\left(\frac{TxSh_m - TD_m}{V_{L\,m}}\right)$$

9. To activate that feature in Excel go to Tools, choose Options, then Calculations; check in the Iterations box. See Velez and Tham (2001).

10. Implicitly we went back to assumption 1 (constant leverage).

To solve the equations system, some inputs are required:

$FC_m$, T, $k_D$,$k_0$,$g_L$ and either $D_m$ or the target leverage $D_m/S_m$.

Now we go backwards to solve the equations for the period m-1:

$$V_{L\,m-1} = \frac{V_{L\,m} + FC_m;}{1+k_{WACC\,m-1}} \quad TxSh_{m-1} = \frac{TxSh_m + D_{m-1}\,k_D T;}{1+k_D}$$

$$k_{s\,m-1} = k_0 + (k_0 - k_D)\frac{D_{m-1}\ TxSh_{m-1}}{S_{m-1}}\ ;\ k_{WACC\,m-1} = k_{s\,m-1}\frac{S_{m-1}}{V_{L\,m-1}} + k_D\,(1-T)\ \frac{D_{m-1};}{V_{L\,m-1}}$$

$$k_{WACC\,m-1} = k_0\left(1 - \frac{TxSh_{m-1}}{V_{L\,m-1}}\right) + k_D\left(\frac{TxSh_{m-1} - TD_{m-1}}{V_{L\,m-1}}\right)$$

The same formulas apply for the periods m-2 to 0. The algorithm stops when we reach the period 0.

Having sketched the approach, the numerical example is worked.

The Table 1 shows how this technique produces similar valuations (period by period): (1) through the direct discount of the free cash flows with $k_{WACCj}$; and (2) through the discount of free cash flows with $k_0$ plus the Tax Shield (The Adjusted Present Value proposed by Myers). The result holds for $k_D$, $k_0$ and T not constant.

With this methodology the effect on $k_s$ of the growing perpetuity only affects the period m. Given that the terminal value is not a negligible part of the firm value, the economic effects of this correction continue to be significant.

Finally, following the same procedure outlined before, we have:

$$ß_{sj} = ß_0\left( + \frac{D_j - TxSh_j}{S_j}\right) - \frac{k_D - k_f}{k_m - k_f}\ \frac{D_j - TxSh_j}{S_j} \quad \text{or}$$

$$ß_0 = \frac{ß_{s\,j}}{1+\dfrac{D_j - TxSh_j}{S_j}} + \frac{\left(\dfrac{k_D - k_f}{k_m - k_f}\right)}{\left(\dfrac{S_j}{D_j - TxSh_j} + 1\right)} \quad \text{or } ß_0 = ß_{sj}\frac{S_j}{V_{Uj}} + ß_D\frac{D_j - TxSh_j}{V_{Uj}}\ ;\ \text{for the}$$

operational model we develop.

Another history results if we accept the correction to the M&M model proposed by Harris and Pringle (1985). For them the tax shield bears the assets risk ($k_0$) not the debt risk. In this universe $k_{sj} = k_0 + (k_0 - k_D) D_j S_j$ and the effect of growth is not explicitly incorporated to the expression of $k_{sj}$, here the effect is indirect and only present when the leverage is not constant (or the amount of debt is constant). As expected the valuation results are lower and the difference increases with the distance between $k_0$ and $k_D$. The corresponding expressions for the betas are:

$$ß_{sj} = ß_0 \left( 1+ \frac{D_j}{S_j} \right) - \frac{k_D - k_f D_j}{k_m - k_f S_j} \text{ or } ß_0 = ß_{sj} \frac{S_j}{V_{Lj}} + ß_D \frac{D_j}{V_{Lj}}$$

The Fernandez (2003) model also doesn't incorporates the effect of growth in their cost of capital or beta, the equations are:

$$k_{sj} = k_0 + (k_0 - k_D) \frac{D_j}{S_j} (1\text{-}T) \text{ and}$$

$$ß_{sj} = ß_0 \left( 1+ \frac{D_j (1\text{-}T}{S_j} \right) - ß_D \frac{D_j (1\text{-}T)}{S_j}$$

He critics M&M (1963) and Myers (1974) on the grounds of $k_s<k_0$ for some values of g, but all the models of growing perpetuities (including Fernandez) depend critically of this measure. The question if the expected growth should reduce the cost of capital is important, here we differentiate the operational risk and its required return from expected growth. If the Myers (1974) approach is correct, growth should reduce the cost of capital: Are investors more prone to invest in firms with high growth, other things equal (specially assets risk)? If the answer is yes (which sounds reasonable) the empirical data should confirm it.

**Table 1**
**Valuation of the Levered Firm**
Variable Leverage

| | | | 0 | 1 | 2 | 3 | 4 | 5 | 6-ss:$g_L$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| Growth | $g_j$ | =Input data | | | 7.00% | 9.00% | 6.00% | 5.50% | 5.00% | |
| Free cashflow | $FC_j$ | $=FC_{j-1}(1-g_j)$ \|=Input data | | $100.00 | $107.00 | $116.63 | $123.63 | $130.43 | $136.95 | |
| MV of Debt | $D_j$ | =Input data | $600.00 | $620.00 | $640.00 | $660.00 | $680.00 | $700.00 | | |
| MV of Equity | $S_j$ | $=V_{LJ} - D_j$ | $964.38 | $1,027.02 | $1,092.83 | $1,163.51 | $1,232.19 | $1,311.65 | | |
| Leverage | $D_j/V_{LJ}$ | =Direct calculation | 38.35% | 37.64% | 36.93% | 36.19% | 35.56% | 34.80% | | |
| | $S_j/V_{LJ}$ | $=1 - D_j/V_{LJ}$ | 61.65% | 62.36% | 63.07% | 63.81% | 64.44% | 65.20% | | |
| Tax rate | T | =Input data | 35.00% | 35.00% | 35.00% | 35.00% | 35.00% | 35.00% | | |
| Periodic TS | $TS^*_j$ | $=D_{j-1}\ Tk_D$ | | 21.00 | 21.70 | 26.88 | 20.79 | 28.56 | 24.50 | |
| Cummulated TS | $TS_j$ | $=(TS^*_{j+a} + TS_{j+1})/(1+k_D)$ | 384.37 | 401.81 | 420.29 | 443.84 | 463.00 | 490.00 | <---= $D_m\ Tk_D/(k_D - g_L)$ | |
| Cost of Assets | $k_0$ | =Input dat | 14.00% | 14.00% | 14.00% | 14.00% | 14.00% | 14.00% | | |
| Cost of Equity | $k_{sj}$ | $=k_0+(k_0-k_D)(D_j-TS_j)/S_j$ | 14.89% | 14.85% | 14.40% | 14.93% | 14.35% | 14.64% | | |
| Cost of Debt | $k_D$ | =Input data | 10.00% | 10.00% | 12.00% | 9.00% | 12.00% | 10.00% | | |
| Average Cost | $k_{WACCj}$ | $=k_s(S_j/V_{LJ})+k_D(1-T)(D_j/V_{LJ})$ | 11.67% | 11.71% | 11.96% | 11.64% | 12.02% | 11.81% | | |
| Levered Firm | $V_{LJ}$ | $=(FC_{j+1}+V_{Lj+1})/(1+k_{WACCj})$ | $1,564.38 | $1.647.02 | $1,732.83 | $1.823.51 | $1,912.19 | $2,011.65 | <---= $FC_m(1+g_L)/(k_{WACCm}-g_L)$ | |
| | | | | | | | | | | |
| **Valuation of the Unlevered Firm plus the Tax Shield** | | | | | | | | | | |
| Terminal Value | $TV_j$ | | | | | | | $1,521.65 | <--- $TV_m=FC_m(1+g_L)/(k_0-g_L)$ | |
| | $FC_j+TV_j$ | =Direct calculation | | $100.00 | $107.00 | $116.63 | $123.63 | $1,652.08 | | |
| Unlevered Firm | $V_{uj}$ | $=(FC_{j+1}+ V_{uj+1})/(1+k_0)$ | $1,180.01 | $1,245.21 | $1,312.54 | $1,379.67 | $1,449.19 | $1,521.65 | | |
| | $+TS_j$ | =From above | $384.37 | $401.81 | $420.29 | $443.84 | $463.00 | $490.00 | | |
| Levered Firm | $=V_{LJ}$ | =Direct calculation | $1.564.38 | $1.647.02 | $1.732.83 | $1.823.51 | $1.912.19 | $2.011.65 | | |

## CONCLUSIONS

The equations we have worked here present a coherent system that preserves under all conditions the equality $V_L = V_u + TS$. They also show that a higher continuing and constant growth produces a lower cost of equity. If this conclusion is true, the empirical data should confirm it. Among firms working in the same business (similar $k_0$), those with higher growth should have lower equity risk. On the other side, if the corrections by Harris and Pringle (1985) or Fernandez (2003) holds, the cost of equity shouldn't be affected by growth. As it has been stated before, the tax shield becomes more risky when leverage increases or when the firm size does not isolate the firm of market adjustments. The tax shield also depends of the firm's ability to collect it, even after continuing losses.

A empirical test seems appropriate; after all, corporations always forecast growth. That test is feasible. Firms with higher growth should have a lower $k_{WACC}$. Measuring $k_{WACC}$ does not depend of how $k_s$ is stated. We can estimate $k_s$ through its CAPM definition $k_s = k_f + \beta_s(k_m - k_f)$. The cost of debt does not change and can be ignored. Controlling for industry and size should be enough to see how the predictions deals with reality.

## BIBLIOGRAPHY


Fernández, Pablo. 2003. *The value of tax shields is NOT equal to the present value of tax shields.* Journal of Financial Economics, forthcoming.

Harris, Robert S., John J. Pringle. 1985. *Risk-adjusted discount rates-extensions from the average-risk case.* The Journal of Financial Research 8, 237-244.

Modigliani, Merton, Merton Miler. 1958. *The cost of capital, corporation finance and the theory of investment.* American Economic Review 48, 261-297.

________,________. 1963. *Corporate income taxes and the cost of capital: a correction.* American Economic Review 48, 261-297.

Myers, Stewart C. 1974. *Interactions of corporate financing qand investment decision-Implications for capital budgeting.* The Journal of Finance 29, 1-25.

Ross, Stephen A., Randolph W. Westerfield and Jeffery Jaffe. 1999. *Corporate Finance*, 5th Edition, Irwin McGraw-Hill