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Abstract

This paper discusses results concerning multivariate normal distributions that are subject to truncation by a hyperplane and how such results can be applied to uncertainty analysis in the environmental sciences. We present a suite of results concerning truncated multivariate normal distributions, some of which already appear in the mathematical literature. The focus here is to make these types of results more accesible to the environmental science community and to this end we include a conceptually simple alternative derivation of an important result. We illustrate how the theory of truncated multivariate normal distributions can be employed in the environmental sciences by means of an example from the economics of climate change control.

Keywords: Truncated normal distribution; expectation; uncertainty analysis; environmental economics; greenhouse gas control

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1. Introduction

A problem that arises in a number of applications in environmental modelling is the need to find the expectation of a function of many uncertain variables, subject to some kind of sharp cut-off, threshold or truncation limit. For example, beyond a certain climatic limit, snowfall may suddenly become zero; beyond geographic limits like coastlines, rivers or mountain ranges, the population densities of many species suddenly become zero; and beyond some economic limits, firms or countries may suddenly choose to opt out of some voluntary environmental scheme, because it is unprofitable to them.

The functions in these applications will usually in fact be non-linear, and also the underlying probability distributions will usually be non-normal. However, within an acceptable margin of error, one can often linearise the functions and assume a multivariate normal (multinormal) distribution of random variables. For example, the latter assumption is often employed when the random variables represent errors in a model or uncertanties in a process.

In environmental modelling, uncertainty is typically dealt with by solving a deterministic model under a number of alternative realisations of the random variables, using Monte Carlo experiments or other procedures such as Gaussian quadrature (see for example Abler, et al., 1999; Krajewski, et al., 1991), where expectations of functions of random variables are approximated by averaging over the results of a large number of model runs. While such methods can accommodate truncated as well as non-normal distributions of random variables, they have some serious downsides. For larger models, computational requirements to achieve reliable approximations may be excessive; it may not be possible to solve the model for the optimal parameter settings; and since model uncertainty is dealt with only numerically, there can be no analytical insights. The method we propose and illustrate suffers from none of these limitations.

The particular example which motivates us comes from an environmental economics analysis of greenhouse gas control (Jotzo and Pezzey, 2004). In this analysis there arises the need to calculate a country's perceived payoff from signing up to a global treaty that limits future emissions of greenhouse gases, where that country's emissions target is "non-binding" (i.e. optional), and all countries' future emissions are uncertain. Jotzo and Pezzey assume that once the uncertain future actually arrives, a country enacts a non-binding traget only when its perceived net payoff (from selling spare emission permits) is positive. Since the payoff is a linearised function of the uncertainties, this condition holds on one side of a truncating hyperplane that bisects probability space.

The problem of deriving expectations with respect to a multinormal distribution subject to various truncating conditions has been considered by many authors in the mathematical literature. Below we consider a single planar truncation, which excludes all n-vectors $\mathbf{x}$ that are not part of the set

$$\left\{\mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbb{R}^n : \boldsymbol{\alpha}^T\mathbf{x} \geq c,\ \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)^T \in \mathbb{R}^n,\ c \in \mathbb{R}\right\}, \tag{1}$$

and derive expectation results for multinormal distributions truncated by this hyperplane.

The most investigated case has actually been that of rectangular truncation[1] (Birnbaum and Meyer, 1953; Tallis, 1961; Horrace and Hernandez, 2001; Horrace, 2004), but there

[1] A rectangular truncation excludes all n-vectors that are not part of the set

$$\{\mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbb{R}^n : x_1 \geq c_1, \ldots, x_n \geq c_n\}.$$

is also the paper of Tallis (1965) that deals with the theory of planar truncations. This derived results for multinormal distributions truncated by a number of hyperplanes, and our results overlap Tallis's to some extent, though they also include some minor extensions. However, Tallis used the method of moment-generating functions (also used in Tallis, 1961), whereas we use direct integration methods. Also he treated any truncated distribution as a probability distribution in its own right (i.e. with total probability of 1); whereas we treat it as part of a distribution of probability 1 over all of $\mathbb{R}^n$. This is because in the motivating example, even though values on both sides of the truncating hyperplane occur with non-zero probability, calculating the expected payoff only requires consideration of values on the side of the hyperplane corresponding to profitable outcomes. This difference in approach means our results do not contain Tallis's normalising factors.

Our paper therefore has three purposes: to obtain Tallis's results using an alternative, more direct method, and to give some modest extention to his results, as well as presenting some related truncated multinormal expectation results; to show how all the results can be used in the context of a climate change control treaty; and most importantly, to make all the results more accessible to environmental modellers. In section 2 we use a well-known theorem on multinormal distributions to prove a specialised truncated expectation result, and then derive the truncated multinormal expectation of both a linear scalar function, and the exponential of a linear scalar function, with respect to an arbitrary truncating hyperplane. Such functions are commonly employed in enviromental modelling applications, particularly in hydrology (Keig and McAlpine, 1974; Makhlouf and Michel, 1994; Xiong and Guo, 1999; Ferdowsian, et al., 2001;) and ecology (Ratnieks, 1996; Pacala, et al., 1996; Baguette, 2003). The results follow as corollaries of Tallis's results, but are inobvious and therefore hitherto unavailable to a typical environmental modeller. In section 3 we show how our results can be used in the motivating application of Jotzo and Pezzey (2004) of a climate change control treaty, where a country would make a future, and therefore uncertain, dollar-valued net gain of $\widetilde{G}$ from joining the treaty. However, its psychologically perceived "payoff" (used to determine whether or not it chooses to join the treaty) is modelled as

$$U(\widetilde{G}) = \widetilde{G} + s\left(1 - \exp(-r\widetilde{G})\right), \tag{2}$$

with s, r positive constants. This is a strictly increasing, concave function of $\widetilde{G}$ which allows for the diminishing marginal value of money gains. Our results allow the expected payoff $E[U(\widetilde{G})]$, to be calculated and compared across countries, which will help to gauge the political feasibility of various types and levels of emission targets.

2. Expectations with respect to truncated multinormal distributions

2.1 Notation and definitions

We use bold, lower-case Greek characters such as $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)^T$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \boldsymbol{\beta}_n)^T$ to denote vectors of fixed parameters, and bold, lower-case Latin characters such as $\mathbf{x} = (x_1, \ldots, x_n)^T$ and $\mathbf{y} = (y_1, \ldots, y_n)^T$ to denote multinormal random variables in the n-dimensional Euclidean space $\mathbb{R}^n$.

The standard Euclidean inner product is

$$\boldsymbol{\alpha}^T\boldsymbol{\beta} = \sum_{i=1}^{n} \alpha_i \beta_i$$

and the square of the standard Euclidean norm is $\|\boldsymbol{\alpha}\|^2 = \boldsymbol{\alpha}^T\boldsymbol{\alpha}$.

We will assume $\mathbf{x}$ has zero mean so that its probability density function is given by

$$\phi_n(\mathbf{x}, M) = \frac{\exp\left(-\frac{1}{2}\mathbf{x}^T M^{-1}\mathbf{x}\right)}{(2\pi)^{n/2}|M|^{1/2}}.$$

Here M is the dispersion (variance-covariance) matrix of the multinormal distribution and $|M|$ is its determinant. Although we restrict our attention to zero mean multinormal variables, our results could easily be generalised to accommodate a nonzero vector of means. Given a function f of the random variable $\mathbf{x}$, its expectation, or expected value (over all of $\mathbb{R}^n$), is defined as

$$E[f(\mathbf{x})] = \int_{\mathbb{R}^n} f(\mathbf{x})\,\phi_n(\mathbf{x}, M)\,d\mathbf{x}.$$

If the multinormal distribution is truncated then the integral is taken over the subset of $\mathbb{R}^n$ defined by the truncating condition.

A scalar function, or functional, is a function whose range lies in the scalar field of real numbers $\mathbb{R}$. An important mathematical theorem (Kreyszig, 1978; p188) asserts that if $\ell : \mathbb{R}^n \to \mathbb{R}$ is a bounded linear functional acting on $\mathbb{R}^n$ then the action of ℓ on an element of $\mathbb{R}^n$ may be realised via the inner product as

$$\ell(\mathbf{x}) = \boldsymbol{\alpha}^T\mathbf{x}$$

for some uniquely suitable $\boldsymbol{\alpha} \in \mathbb{R}^n$. Hence every bounded linear scalar function acting on $\mathbb{R}^n$ corresponds uniquely to some element of $\mathbb{R}^n$.

2.2 Standard results for multinormal distributions

A characterising feature of the multinormal distribution is its amenability with respect to linear functionals. This is apparent, for example, in the following well known result (Rohatgi and Saleh, 2001; Section 5.4,Theorem 7).

Theorem 1 *Let* $\mathbf{x} = (x_1, \ldots, x_n)^T$. *Then* $\mathbf{x}$ *has an* n*-dimensional normal distribution with zero mean and dispersion matrix* M *if and only if every linear combination* $\boldsymbol{\alpha}^T\mathbf{x}$ *has a univariate normal distribution with zero mean and variance* $\sigma^2 = \boldsymbol{\alpha}^T M \boldsymbol{\alpha}$.

In particular, the above Theorem immediately implies the following Corollary concerning the multinormal expectation of a continuous function whose argument is of the form $(\boldsymbol{\alpha}^T\mathbf{x} - c)$, where $\mathbf{x}$ is a multinormal variable.

Corollary 2 *Let* U *be a continuous function and* $\mathbf{x} = (x_1, \ldots, x_n)^T$ *be a multinormal variable with zero mean and dispersion matrix* M. *Then*

$$\int_{\mathbb{R}^n} U(\boldsymbol{\alpha}^T\mathbf{x} - c)\ \phi_n(\mathbf{x}, M)\, d\mathbf{x} \ = \ \int_{-\infty}^{\infty} U(\eta - c)\frac{\exp\left(-\frac{1}{2}\frac{\eta^2}{\sigma^2}\right)}{\sigma\sqrt{2\pi}}\, d\eta \tag{3}$$

where $\sigma^2 = \boldsymbol{\alpha}^T M \boldsymbol{\alpha}$.

In essence the result states that, because $\boldsymbol{\alpha}^T\mathbf{x}$ is linear in $\mathbf{x}$, the multinormal probability density has been concentrated along a single canonical direction, i.e. the direction of $\boldsymbol{\alpha}$. The multivariate integral has been reduced to a univariate integral, which is much easier to deal with.

2.3 Results for canonical half-spaces

If $U : \mathbb{R} \to \mathbb{R}$ is a strictly increasing, continuous function and there exists $\zeta \in \mathbb{R}$ such that $U(\zeta - c) = 0$ then the expression $U(\boldsymbol{\alpha}^T\mathbf{x} - c)$ has associated with it a canonical splitting of $\mathbb{R}^n$ into two half-spaces,

$$\begin{aligned} \mathcal{U}_n^+ &= \left\{\mathbf{x} \in \mathbb{R}^n : U(\boldsymbol{\alpha}^T\mathbf{x} - c) \geq 0\right\} \\ \mathcal{U}_n^- &= \left\{\mathbf{x} \in \mathbb{R}^n : U(\boldsymbol{\alpha}^T\mathbf{x} - c) < 0\right\}. \end{aligned}$$

The separating hyperplane is defined by $\boldsymbol{\alpha}^T\mathbf{x} = \zeta$. Applying the previous Corollary to the subset of continuous functions that are strictly increasing, we obtain the following result. The proof is in Appendix 1.

Theorem 3 *Suppose that U is continuous, strictly increasing and that $U(\zeta - c) = 0$. Suppose also that $\boldsymbol{\alpha} \in \mathbb{R}^n$ is an arbitrary, but fixed, vector and that $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is a multinormal random variable with mean zero and dispersion matrix M. If $\sigma^2 = \boldsymbol{\alpha}^T M \boldsymbol{\alpha}$ then*

$$\int_{\mathcal{U}_n^+} U(\boldsymbol{\alpha}^T\mathbf{x} - c)\ \phi_n(\mathbf{x}, M)\, d\mathbf{x} \ = \ \int_{\zeta}^{\infty} U(\eta - c)\frac{\exp\left(-\frac{1}{2}\frac{\eta^2}{\sigma^2}\right)}{\sigma\sqrt{2\pi}}\, d\eta. \tag{4}$$

Note that if U is a continuous function and there is no $\zeta \in \mathbb{R}$ for which $U(\zeta - c) = 0$, then either $U > 0$, which implies that $\mathcal{U}_n^+ = \mathbb{R}^n$, or $U < 0$, which implies that $\mathcal{U}_n^- = \mathbb{R}^n$. In each of these cases Theorem 3 reverts to Corollary 2.

For a non-binding target for greenhouse gas emissions, the function U would be a country's payoff from signing up to the target, with the zero-payoff hyperplane being the boundary between $\mathcal{U}_n^+$, where the target is enacted, and $\mathcal{U}_n^-$, where the target is not enacted.

2.4 Results for arbitrary truncating hyperplanes

The uncertain payoff that one country (say B) will get as a result of country A enacting a non-binding emissions target is given (to a linear approximation) by some function of $\boldsymbol{\beta}^T\mathbf{x}$, for some parameter vector $\boldsymbol{\beta} \in \mathbb{R}^n$ and some random variable $\mathbf{x} \in \mathbb{R}^n$ which includes various uncertainties in emissions. Country A chooses whether or not to enact the target depending on whether $\boldsymbol{\alpha}^T\mathbf{x} \geq c$ or not (and hence whether or not $U(\boldsymbol{\alpha}^T\mathbf{x} - c) \geq 0$, if we define payoff $U(\cdot)$ as a strictly increasing function with $U(0) = 0$, as for example in

(2)). But $\boldsymbol{\beta}^T\mathbf{x}$, the uncertain determinant of country B's payoff, will generally be quite different from $\boldsymbol{\alpha}^T\mathbf{x}$.

The question thus arises whether a result similar to Theorem 3 holds above (or below) some arbitrary hyperplane that is not associated with the argument of the function U. The answer is unfortunately that no simple general expression exists. However, restricting our attention to linear functionals of real multinormal variables, it is possible to derive a simple expression. The following Lemma establishes such a result for zero mean, independent, unit variance multinormal variables whose dispersion matrix M is thus the identity matrix. The general case will then follow from the Lemma. The Lemma's proof relies only on simple changes of coordinates and integration by parts, but is quite lengthy and is therefore left to Appendix 2. The Lemma and the subsequent Theorem make use of the following notation. We denote the half-space above the hyperplane $\boldsymbol{\alpha}^T\mathbf{x} = c$, ($\boldsymbol{\alpha} \neq \mathbf{0}$) as

$$\mathcal{H}_c^\alpha = \{\mathbf{x} \in \mathbb{R}^n : \boldsymbol{\alpha}^T\mathbf{x} \geq c\}.$$

We may assume without loss of generality that $\|\boldsymbol{\alpha}\| = 1$.

Lemma 4 *Suppose that $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^n$ are fixed arbitrary vectors, with $\|\boldsymbol{\alpha}\| = 1$, and $\mathbf{x} \in \mathbb{R}^n$ is a multinormal random variable such that each x_i, $i = 1, \ldots, n$ is an independent, normal random variable with zero mean and unit variance. Then*

$$\int_{\mathcal{H}_c^\alpha} \boldsymbol{\beta}^T\mathbf{x}\, \phi_n(\mathbf{x}, I)\, d\mathbf{x} = \exp\left(-\frac{c^2}{2}\right) \frac{\boldsymbol{\alpha}^T\boldsymbol{\beta}}{\sqrt{2\pi}}. \tag{5}$$

We note here that Lemma 4 and the ensuing Theorem 5 could also be established using the mean vector $\boldsymbol{\mu}$ derived by Tallis (1965), after taking into account the fact that we do not require the truncated distribution to be a probability distribution as Tallis did. As such, the proofs of Lemma 4 and Theorem 5 serve as a conceptually more simple, alternative derivation of Tallis's mean vector that avoids the need for the moment-generating function. The proof of Theorem 5 is in Appendix 3.

Theorem 5 *Suppose that $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^n$ are fixed, arbitrary vectors and $\mathbf{x} \in \mathbb{R}^n$ is a zero mean multivariate normal random variable with dispersion matrix M. Then*

$$\int_{\mathcal{H}_c^\alpha} \boldsymbol{\beta}^T \mathbf{x}\, \phi_n(\mathbf{x}, M)\, d\mathbf{x} = \exp\left(-\frac{1}{2}\frac{c^2}{\sigma^2}\right) \frac{\boldsymbol{\alpha}^T M \boldsymbol{\beta}}{\sigma\sqrt{2\pi}} \tag{6}$$

where $\sigma^2 = \boldsymbol{\alpha}^T M \boldsymbol{\alpha}$.

We may also obtain a simple expression for the expectation of the exponential of a linear functional. Such an expectation is, modulo a normalising factor, the moment generating function, and hence the result is in Tallis (1965), but for completeness we include the result here and provide the proof in Appendix 4.

Theorem 6 *Suppose that $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^n$ are fixed, arbitrary vectors and $\mathbf{x} \in \mathbb{R}^n$ is a zero mean multivariate normal random variable with dispersion matrix M. Then*

$$\int_{\mathcal{H}_c^\alpha} \exp\left(r\boldsymbol{\beta}^T \mathbf{x}\right) \phi_n(\mathbf{x}, M)\, d\mathbf{x} = \exp\left(\tfrac{1}{2} r^2 \boldsymbol{\beta}^T M \boldsymbol{\beta}\right)\, \Phi\left[\frac{r\boldsymbol{\alpha}^T M\boldsymbol{\beta} - c}{\sigma}\right], \tag{7}$$

where $\Phi[z] := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left(-\frac{1}{2}s^2\right) ds$ is the cumulative distribution function of the univariate normal distribution, and $\sigma^2 = \boldsymbol{\alpha}^T M \boldsymbol{\alpha}$.

3. Example from a climate change treaty with uncertainty in emissions: quantifying risk aversion

Here we calculate the expected payoffs for two different countries, A and B, that result from country A signing up to a climate change treaty by accepting a non-binding target for its future greehouse gas emissions, given that the payoff function is $U(\widetilde{G}) = \widetilde{G} + s\left(1 - \exp(-r\widetilde{G})\right)$ as in (2). Because the target is non-binding, signing up to the target now does not mean that it will be enacted when an uncertain future arrives. It is precisely because of country A's option to enact the target when the future arrives that the expected payoff to both countries are defined over truncated half-spaces. The simplified linear model of Jotzo and Pezzey (2004) shows how the expected payoffs can be defined in terms of vectors $\boldsymbol{\alpha}^T$ and $\boldsymbol{\beta}^T$ and parameters s_A, c_A, s_B, c_B and r as follows, and we show in appendices how to calculate the expected payoffs stated using the above Theorems.

3.1 Expected payoff to country A from A signing up to a non-binding target

Assuming that the dollar-valued net gain, $\widetilde{G}_A$, of country A is a linearised function of the multinormal random variable $\mathbf{x}$, we have $\widetilde{G}_A = \boldsymbol{\alpha}^T\mathbf{x} - c_A$. According to equation (2) the payoff to country A is given by

$$U(\widetilde{G}_A) = \widetilde{G}_A + s_A\left(1 - \exp(-r\widetilde{G}_A)\right)$$

where we suppose that $s_A > 0$, and $r > 0$. The expected payoff to country A from A signing up to a non-binding emissions target is therefore defined as

$$E[U(\widetilde{G}_A)|_A] = \int_{\widetilde{G}_A \geq 0} \left[\widetilde{G}_A + s_A\left(1 - \exp\left(-r\widetilde{G}_A\right)\right)\right] \phi_n(\mathbf{x}, M)\, d\mathbf{x}$$

Applying Theorem 3 we obtain the result (see Appendix 5 for proof):

$$\begin{aligned} E[U(\widetilde{G}_A)|_A] &= (s_A - c_A)\Phi\left[\frac{-c_A}{\sigma}\right] - s_A \exp\left(rc_A + \tfrac{1}{2}r^2\sigma^2\right)\Phi\left[\frac{-(c_A + r\sigma^2)}{\sigma}\right] \\ &\quad + \frac{\sigma}{\sqrt{2\pi}} \exp\left(\frac{-c_A^2}{2\sigma^2}\right) \end{aligned} \tag{8}$$

with $\sigma^2 = \boldsymbol{\alpha}^T M \boldsymbol{\alpha}$.

3.2 Expected payoff to country B from A signing up to a non-binding target

The expected payoff to country B from A signing up to a non-binding emissions target is defined as

$$E[U(\widetilde{G}_B)|_A] = \int_{\widetilde{G}_A \geq 0} \left[\widetilde{G}_B + s_B\left(1 - \exp\left(-r\widetilde{G}_B\right)\right)\right] \phi_n(\mathbf{x}, M)\, d\mathbf{x}$$

where $\widetilde{G}_A$ and r are as before, and now $\widetilde{G}_B = \boldsymbol{\beta}^T\mathbf{x} - c_B$ and $s_B > 0$.

Theorems 5 and 6 can be utilised to obtain the result (see Appendix 6 for proof):

$$\begin{aligned} E[U(\widetilde{G}_B)|_A] &= (s_B - c_B)\Phi\left[\frac{-c_A}{\sigma}\right] - s_B \exp\left(rc_B + \tfrac{1}{2}r^2\boldsymbol{\beta}^T M\boldsymbol{\beta}\right)\Phi\left[\frac{-(c_A + r\boldsymbol{\alpha}^T M\boldsymbol{\beta})}{\sigma}\right] \\ &\quad + \frac{\boldsymbol{\alpha}^T M\boldsymbol{\beta}}{\sqrt{2\pi}\sigma}\exp\left(\frac{-c_A^2}{2\sigma^2}\right) \end{aligned} \tag{9}$$

with $\sigma^2 = \boldsymbol{\alpha}^T M\boldsymbol{\alpha}$.

These results highlight the effect of risk aversion that results from the strict concavity of the payoff function (2). The parameter r describes the sharpness of a country's risk aversion, and s_A and s_B its respective importance in comparison to the dollar valued gains $\widetilde{G}_A$ and $\widetilde{G}_B$ themselves. So setting $r = s_A = s_B = 0$ in the above results, and subtracting the outcomes from the original results reveals the net effects of risk aversion.

4. Conclusion

The results presented here provide a way of calculating the expected values of some of the functions commonly used in environmental modelling, when the underlying distribution is taken to be multivariate normal, subject to a truncation or cut-off condition. Cut-off conditions arise naturally in many instances in environmental and other sciences and the associated variables can often be characterised as being approximately normally distributed. The formulae presented in this paper should therefore be widely applicable. As well as allowing efficient estimation of mean values of functions encountered in modelling applications, the results also provide theoretical insight into the nature of uncertainties associated with particular modelling approaches. It is our hope that by enhancing the accessibility of such results and illustrating their use with a concrete example, environmental modellers will find it easier to use them in their own applications when appropriate.

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Mathematical Appendices

In this section we present mathematical proofs for the results cited above.

Appendix 1. Proof of Theorem 3

Define the continuous function $\widetilde{U}$ as

$$\widetilde{U}(z) = \begin{cases} U(z) & \text{if } z \geq \zeta - c \\ 0 & \text{if } z < \zeta - c \end{cases}$$

Note that since U is strictly increasing, $\widetilde{U} \geq 0$. From Corollary 2 we therefore have

$$\begin{aligned} \int_{\mathcal{U}_n^+} U(\boldsymbol{\alpha}^T \mathbf{x} - c)\ \phi_n(\mathbf{x}, M)\, d\mathbf{x} &= \int_{\mathbb{R}^n} \widetilde{U}(\boldsymbol{\alpha}^T \mathbf{x} - c)\ \phi_n(\mathbf{x}, M)\, d\mathbf{x} \\ &= \int_{-\infty}^{\infty} \widetilde{U}(\eta - c) \frac{\exp\left(-\frac{1}{2}\frac{\eta^2}{\sigma^2}\right)}{\sigma\sqrt{2\pi}}\, d\eta \\ &= \int_{\zeta}^{\infty} U(\eta - c) \frac{\exp\left(-\frac{1}{2}\frac{\eta^2}{\sigma^2}\right)}{\sigma\sqrt{2\pi}}\, d\eta \end{aligned}$$

with $\sigma^2 = \boldsymbol{\alpha}^T M \boldsymbol{\alpha}$. ∎

Appendix 2. Proof of Lemma 4

For convenience we let

$$J = \int_{\mathcal{H}_c^\alpha} \boldsymbol{\beta}^T \mathbf{x}\, \phi_n(\mathbf{x}, I)\, d\mathbf{x}.$$

Employing the change of coordinates $\mathbf{y} = \mathbf{x} - c\boldsymbol{\alpha}$ we find that

$$\begin{aligned} (2\pi)^{n/2} J &= \exp\left(-\frac{c^2}{2}\right) \int_{\mathcal{H}_0^\alpha} \boldsymbol{\beta}^T (\mathbf{y} + c\boldsymbol{\alpha}) \exp\left[-\frac{1}{2}\left(\mathbf{y}^T\mathbf{y} + 2c\boldsymbol{\alpha}^T\mathbf{y}\right)\right] d\mathbf{y} \\ &= \exp\left(-\frac{c^2}{2}\right) \sum_{j=1}^{n} J_j \end{aligned}$$

where

$$J_j = \int_{\mathcal{H}_0^\alpha} \beta_j (y_j + c\alpha_j) \exp\left[-\frac{1}{2}\sum_{i=1}^{n}\left(y_i^2 + 2c\alpha_i y_i\right)\right] dy_1 \cdots dy_n$$

$$= \int_{\mathbb{R}^{n-1}} \int_{\xi}^{\infty} \beta_j \left(y_j + c\alpha_j\right) \exp\left[-\frac{1}{2}\sum_{i=1}^{n}\left(y_i^2 + 2c\alpha_i y_i\right)\right] dy_1 \cdots dy_n,$$

and where we have set $\xi = -\frac{1}{\alpha_n}\sum_{i=1}^{n-1}\alpha_i y_i$.

Hence if $j \neq n$ then

$$J_j = \int_{\mathbb{R}^{n-1}} A_n \beta_j \left(y_j + c\alpha_j\right) \exp\left[-\frac{1}{2}\sum_{i=1}^{n-1}\left(y_i^2 + 2c\alpha_i y_i\right)\right] dy_1 \cdots dy_{n-1}$$

where

$$A_n = \int_{\xi}^{\infty} \exp\left[-\frac{1}{2}\left(y_n^2 + 2c\alpha_n y_n\right)\right] dy_n.$$

To simplify the expression for J_j we perform an integration by parts. By the Fundamental Theorem of Calculus we have

$$dA_n = \frac{\alpha_j}{\alpha_n} \exp\left\{-\frac{1}{2}\left[\frac{1}{\alpha_n^2}\left(\sum_{i=1}^{n-1}\alpha_i y_i\right)^2 - 2c\sum_{i=1}^{n-1}\alpha_i y_i\right]\right\} dy_j$$

and that if

$$dv = \beta_j \left(y_j + c\alpha_j\right) \exp\left[-\frac{1}{2}\left(y_j^2 + 2c\alpha_j y_j\right)\right] dy_j$$

then

$$v = -\beta_j \exp\left[-\frac{1}{2}\left(y_j^2 + 2c\alpha_j y_j\right)\right].$$

The integration by parts formula reads

$$\int_{\mathbb{R}} A_n\, dv = \left[A_n v\right]_{-\infty}^{\infty} - \int_{\mathbb{R}} v\, dA_n$$

and so integrating by parts, noting that $A_n v \to 0$ as $y_j \to \pm\infty$, we get

$$\begin{aligned} J_j &= \int_{\mathbb{R}^{n-1}} \frac{\alpha_j \beta_j}{\alpha_n} \exp\left[-\frac{1}{2}\left(\sum_{i=1}^{n-1} y_i^2 + \frac{1}{\alpha_n^2}\left(\sum_{i=1}^{n-1}\alpha_i y_i\right)^2\right)\right] dy_1 \cdots dy_{n-1} \\ &= \frac{\alpha_j \beta_j}{\alpha_n} J^* \end{aligned}$$

where

$$J^* = \int_{\mathbb{R}^{n-1}} \exp\left\{ -\frac{1}{2}\left[\sum_{i=1}^{n-1} y_i^2 + \frac{1}{\alpha_n^2}\left(\sum_{i=1}^{n-1}\alpha_i y_i\right)^2 \right] \right\} dy_1 \cdots dy_{n-1}.$$

Conversely, if $j = n$ then

$$J_n = \int_{\mathbb{R}^{n-1}} \beta_n\, C_n \exp\left[-\frac{1}{2}\sum_{i=1}^{n-1}\left(y_i^2 + 2c\alpha_i y_i\right) \right] dy_1 \cdots dy_{n-1},$$

where

$$\begin{aligned} C_n &= \int_{\xi}^{\infty} (y_n + c\alpha_n) \exp\left[-\frac{1}{2}\left(y_n^2 + 2c\alpha_n y_n\right) \right] dy_n \\ &= \exp\left\{ -\frac{1}{2}\left[\frac{1}{\alpha_n^2}\left(\sum_{i=1}^{n-1}\alpha_i y_i\right)^2 - 2c\sum_{i=1}^{n-1}\alpha_i y_i \right] \right\}. \end{aligned}$$

Substituting this expression for C_n into the expression for J_n gives $J_n = \beta_n J^*$, which in turn implies that

$$(2\pi)^{n/2}\, J = \exp\left(-\frac{c^2}{2}\right) \sum_{j=1}^{n} \frac{\alpha_j \beta_j}{\alpha_n}\, J^*.$$

The integral J^* can now be reduced to a simple expression by means of a reduction formula. Considering the exponent in the integrand we may deduce that

$$\begin{aligned} \sum_{i=1}^{n-1} y_i^2 + \frac{1}{\alpha_n^2}\left(\sum_{i=1}^{n-1}\alpha_i y_i\right)^2 &= \frac{1}{\alpha_n^2}\left[\left(y_1\sqrt{\alpha_n^2+\alpha_1^2} + \frac{\alpha_1 \sum_{i=2}^{n-1}\alpha_i y_i}{\sqrt{\alpha_n^2+\alpha_1^2}} \right)^2 \right] \\ &\quad + \sum_{i=2}^{n-1} y_i^2 + \frac{1}{\alpha_n^2+\alpha_1^2}\left(\sum_{i=2}^{n-1}\alpha_i y_i\right)^2. \end{aligned}$$

Letting

$$t = y_1\sqrt{\alpha_n^2+\alpha_1^2} + \frac{\alpha_1 \sum_{i=2}^{n-1}\alpha_i y_i}{\sqrt{\alpha_n^2+\alpha_1^2}}$$

we have $dt = \sqrt{\alpha_n^2+\alpha_1^2}\; dy_1$ and so

$$\begin{aligned} J^* &= \int_{\mathbb{R}^{n-2}} \exp\left\{ -\frac{1}{2}\left[\sum_{i=2}^{n-1} y_i^2 + \frac{1}{\alpha_n^2+\alpha_1^2}\left(\sum_{i=2}^{n-1}\alpha_i y_i\right)^2 \right] \right\} \int_{-\infty}^{\infty} \frac{\exp\left(-\frac{1}{2}\frac{t^2}{\alpha_n^2}\right)}{\sqrt{\alpha_n^2+\alpha_1^2}} dt\; dy_2 \cdots dy_{n-1} \\ &= \frac{\sqrt{2\pi}\;\alpha_n}{\sqrt{\alpha_n^2+\alpha_1^2}} \int_{\mathbb{R}^{n-2}} \exp\left\{ -\frac{1}{2}\left[\sum_{i=2}^{n-1} y_i^2 + \frac{1}{\alpha_n^2+\alpha_1^2}\left(\sum_{i=2}^{n-1}\alpha_i y_i\right)^2 \right] \right\} dy_2 \cdots dy_{n-1}. \end{aligned}$$

Iterating this result we obtain

$$J^* = \alpha_n \, (2\pi)^{(n-1)/2}.$$

Substituting this back into the expression for $(2\pi)^{n/2} \, J$, cancelling like terms, we find

$$J = \exp\left(-\frac{c^2}{2}\right) \frac{\boldsymbol{\alpha}^T \boldsymbol{\beta}}{\sqrt{2\pi}}$$

and the proof is complete. ■

Appendix 3. Proof of Theorem 5

For convenience let

$$J = \int_{\mathcal{H}_c^\alpha} \boldsymbol{\beta}^T \mathbf{x} \, \phi_n(\mathbf{x}, M) \, d\mathbf{x}$$

so that

$$(2\pi)^{n/2} |M|^{1/2} \, J = \int_{\mathcal{H}_c^\alpha} \boldsymbol{\beta}^T \mathbf{x} \, \exp\left(-\frac{1}{2} \mathbf{x}^T M^{-1} \mathbf{x}\right) dx_1 \cdots dx_n.$$

The matrix M^{-1} is by definition a symmetric, positive definite matrix and so possesses a Cholesky decomposition $M^{-1} = \Psi^T \Psi$, where Ψ, the Cholesky factor, is a nonsingular, upper triangular matrix (Gentle, 1998). Transforming the variables to $\mathbf{y} = \Psi \mathbf{x}$ we find that $d\mathbf{y} = |\Psi| \, d\mathbf{x}$. Note that $M^{-1} = \Psi^T \Psi$ implies that $|M| = |\Psi|^{-2}$ and so we have

$$(2\pi)^{n/2} \, J \quad = \quad \int_{\widehat{\mathcal{H}}_c^\alpha} \widehat{\boldsymbol{\beta}}^T \mathbf{y} \, \exp\left(-\frac{1}{2} \mathbf{y}^T \mathbf{y}\right) d\mathbf{y}$$

where we have set $\widehat{\boldsymbol{\beta}} = \Psi^{-T} \boldsymbol{\beta}$ and

$$\widehat{\mathcal{H}}_c^\alpha = \Psi \left(\mathcal{H}_c^\alpha\right) = \left\{ \mathbf{y} \in \mathbb{R}^n : \widehat{\boldsymbol{\alpha}}^T \mathbf{y} \geq c \right\} \tag{10}$$

where $\widehat{\boldsymbol{\alpha}} = \Psi^{-T} \boldsymbol{\alpha}$. Applying Lemma 4 we find that

$$J = \exp\left(-\frac{1}{2} \frac{c^2}{\|\widehat{\boldsymbol{\alpha}}\|^2}\right) \frac{\widehat{\boldsymbol{\alpha}}^T \widehat{\boldsymbol{\beta}}}{\sqrt{2\pi} \, \|\widehat{\boldsymbol{\alpha}}\|} = \exp\left(-\frac{1}{2} \frac{c^2}{\sigma^2}\right) \frac{\boldsymbol{\alpha}^T M \boldsymbol{\beta}}{\sigma \sqrt{2\pi}}.$$

with $\sigma^2 = \boldsymbol{\alpha}^T M \boldsymbol{\alpha}$ ■

Appendix 4. Proof of Theorem 6

Let

$$J = \int_{\mathcal{H}_c^\alpha} \exp\left(r\boldsymbol{\beta}^T\mathbf{x}\right) \phi_n(\mathbf{x}, M)\, d\mathbf{x}$$

Introducing the Cholesky decomposition of $M^{-1} = \Psi^T\Psi$ and transforming the variables to $\mathbf{y} = \Psi\mathbf{x}$, as was done in Appendix 3, we find that

$$(2\pi)^{n/2}|M|^{1/2}|\Psi|\, J = \int_{\widehat{\mathcal{H}}_c^\alpha} \exp\left[-\tfrac{1}{2}(\mathbf{y}^T\mathbf{y} - 2\widehat{\boldsymbol{\beta}}^T\mathbf{y})\right] d\mathbf{y},$$

where $\widehat{\mathcal{H}}_c^\alpha$ is defined in Appendix 3 and $\widehat{\boldsymbol{\beta}} = r\Psi^{-1}\boldsymbol{\beta}$.

Setting $\mathbf{z} = \mathbf{y} - \widehat{\boldsymbol{\beta}}$ it follows that $\mathbf{y}^T\mathbf{y} - 2\widehat{\boldsymbol{\beta}}^T\mathbf{y} = \mathbf{z}^T\mathbf{z} - \|\widehat{\boldsymbol{\beta}}\|^2$. Hence if we define $p = c - \widehat{\boldsymbol{\alpha}}^T\widehat{\boldsymbol{\beta}}$ we have

$$(2\pi)^{n/2}|M|^{1/2}|\Psi|\, J = \exp\left(\tfrac{1}{2}\|\widehat{\boldsymbol{\beta}}\|^2\right) \int_{\widehat{\mathcal{H}}_p^\alpha} \exp\left(-\tfrac{1}{2}\mathbf{z}^T\mathbf{z}\right) d\mathbf{z}$$

Following the method of Tallis (1965), we let $\mathbf{z} = B\mathbf{w}$, where the first column of the matrix B is the unit vector $\widehat{\boldsymbol{\alpha}}/\|\widehat{\boldsymbol{\alpha}}\|$ and the other columns of B form an orthonormal basis for the hyperplane defined by $\widehat{\boldsymbol{\alpha}}^T\mathbf{z} = p$. Then

$$\widehat{\mathcal{H}}_p^\alpha = \{\mathbf{w} \in \mathbb{R}^n : \ \widehat{\boldsymbol{\alpha}}^T B\mathbf{w} \geq p\} = \{\mathbf{w} = (w_1, \ldots, w_n)^T \in \mathbb{R}^n : \ w_1 \geq \hat{p}\}, \qquad (11)$$

where $\hat{p} = p/\|\widehat{\boldsymbol{\alpha}}\|$.

Therefore, separating the integral accordingly, we have

$$\begin{aligned}(2\pi)^{n/2}|M|^{1/2}|\Psi|\, J &= \exp\left(\tfrac{1}{2}\|\widehat{\boldsymbol{\beta}}\|^2\right) \int_{\mathbb{R}^{n-1}} \exp\left(-\tfrac{1}{2}\sum_{i=2}^{n} w_i^2\right) \int_{\hat{p}}^{\infty} \exp\left(-\tfrac{1}{2}w_1^2\right) dw_1\, dw_2 \cdots dw_n \\ &= \exp\left(\tfrac{1}{2}\|\widehat{\boldsymbol{\beta}}\|^2\right) (2\pi)^{(n-1)/2} \int_{\hat{p}}^{\infty} \exp\left(-\tfrac{1}{2}w_1^2\right) dw_1 \\ &= \exp\left(\tfrac{1}{2}\|\widehat{\boldsymbol{\beta}}\|^2\right) (2\pi)^{n/2}\, \Phi\left[-\hat{p}\right]\end{aligned}$$

Cancelling like terms and substituting for $\widehat{\boldsymbol{\alpha}}$, $\widehat{\boldsymbol{\beta}}$ and $\hat{p}$ we obtain (7) as required. ■

Appendix 5. Derivation of equation (8)

The result follows from Theorem 3. We note that $U(z) = z + s_A(1 - \exp(-rz)$ is a continuous function that is increasing with z, satisfying $U(0) = 0$. Noting also that $\widetilde{G}_A \geq 0$ defines the half-space

$$\mathcal{U}_n^+ = \{\mathbf{x} \in \mathbb{R}^n : U(\boldsymbol{\alpha}^T\mathbf{x} - c_A) \geq 0\},$$

Theorem 3 implies

$$\begin{aligned} E[U(\widetilde{G}_A|_A)] &= \int_{c_A}^{\infty} [\eta - c_A + s_A(1 - \exp(-r(\eta - c_A))] \frac{\exp\left(-\frac{1}{2}\frac{\eta^2}{\sigma^2}\right)}{\sigma\sqrt{2\pi}}\, d\eta \\ &= (s_A - c_A)I_{A1} + I_{A2} - s_A \exp(rc_A) I_{A3}, \end{aligned}$$

where $\sigma^2 = \boldsymbol{\alpha}^T M \boldsymbol{\alpha}$ and

$$\begin{aligned} I_{A1} &= \int_{c_A}^{\infty} \frac{\exp\left(-\frac{1}{2}\frac{\eta^2}{\sigma^2}\right)}{\sigma\sqrt{2\pi}}\, d\eta \\ &= \int_{c_A/\sigma}^{\infty} \frac{\exp\left(-\frac{1}{2}v^2\right)}{\sqrt{2\pi}}\, dv, \quad \text{by substituting } v = \eta/\sigma \\ &= \Phi\left[\frac{-c_A}{\sigma}\right], \quad \text{by symmetry of the normal distribution.} \\ I_{A2} &= \int_{c_A}^{\infty} \eta \frac{\exp\left(-\frac{1}{2}\frac{\eta^2}{\sigma^2}\right)}{\sigma\sqrt{2\pi}}\, d\eta \\ &= \frac{1}{\sqrt{2\pi}}\left[-\sigma \exp\left(-\frac{1}{2}\frac{\eta^2}{\sigma^2}\right)\right]_{c_A}^{\infty} \\ &= \frac{\sigma}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{c_A^2}{\sigma^2}\right). \\ I_{A3} &= \int_{c_A}^{\infty} \frac{\exp\left(-\frac{1}{2}\frac{\eta^2}{\sigma^2} + r\eta\right)}{\sigma\sqrt{2\pi}}\, d\eta \\ &= \int_{c_A}^{\infty} \frac{\exp\left(-\frac{1}{2}\frac{(\eta + r\sigma^2)^2}{\sigma^2} + \frac{1}{2}r^2\sigma^2\right)}{\sigma\sqrt{2\pi}}\, d\eta \\ &= \exp\left(\tfrac{1}{2}r^2\sigma^2\right) \int_{(c_A + r\sigma^2)/\sigma}^{\infty} \frac{\exp\left(-\frac{1}{2}v^2\right)}{\sqrt{2\pi}}\, dv, \quad \text{after substituting } v = (\eta + r\sigma^2)/\sigma \end{aligned}$$

$$= \exp\left(\tfrac{1}{2}r^2\sigma^2\right)\Phi\left[\frac{-(c_A + r\sigma^2)}{\sigma}\right].$$

Combining the above expressions we obtain equation (8). ■

Appendix 6. Derivation of equation (9)

Noting that $\widetilde{G}_A = \boldsymbol{\alpha}^T\mathbf{x} - c_A \geq 0$ defines the half-space

$$\mathcal{H}^{\alpha}_{c_A} = \{\mathbf{x} \in \mathbb{R}^n : \boldsymbol{\alpha}^T\mathbf{x} \geq c_A\},$$

we wish to calculate

$$E[U(\widetilde{G}_B|_A)] = \int_{\mathcal{H}^{\alpha}_{c_A}} \left[\widetilde{G}_B + s_B\left(1 - \exp\left(-r\widetilde{G}_B\right)\right)\right]\ \phi_n(\mathbf{x}, M)\, d\mathbf{x}$$

with $\widetilde{G}_B = \boldsymbol{\beta}^T\mathbf{x} - c_B$.

We can write $E[U(\widetilde{G}_B|_A)] = (s_B - c_B)I_{B1} + I_{B2} - s_B\exp(rc_B)I_{B3}$, where

$$\begin{aligned} I_{B1} &= \int_{\mathcal{H}^{\alpha}_{c_A}} \phi_n(\mathbf{x}, M)\, d\mathbf{x} \\ I_{B2} &= \int_{\mathcal{H}^{\alpha}_{c_A}} \boldsymbol{\beta}^T\mathbf{x}\, \phi_n(\mathbf{x}, M)\, d\mathbf{x} \\ I_{B3} &= \int_{\mathcal{H}^{\alpha}_{c_A}} \exp\left(-r\boldsymbol{\beta}^T\mathbf{x}\right) \phi_n(\mathbf{x}, M)\, d\mathbf{x} \end{aligned}$$

I_{B1} may be evaluated by first completing the square and then using the method of Tallis (1965)to transform the coordinates, as was done in appendix 4. This results in

$$I_{B1} = \Phi\left[\frac{-c_A}{\sigma}\right]$$

with $\sigma^2 = \boldsymbol{\alpha}^T M\boldsymbol{\alpha}$. Theorems 5 and 6, respectively, imply that

$$\begin{aligned} I_{B2} &= \frac{\boldsymbol{\alpha}^T M\boldsymbol{\beta}}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{c_A^2}{\sigma^2}\right) \\ I_{B3} &= \exp\left(rc_B + \tfrac{1}{2}r^2\boldsymbol{\beta}^T M\boldsymbol{\beta}\right)\Phi\left[\frac{-(c_A + r\boldsymbol{\alpha}^T M\boldsymbol{\beta})}{\sigma}\right]. \end{aligned}$$

Combining the above expressions for I_{B1}, I_{B2} and I_{B3} we obtain equation (9). ■

References

Abler, D.G., Rodrguez, A.G. and Shortle, J.S. (1999) Parameter uncertainty in CGE modeling of the environmental impacts of economic policies. *Environmental and Resource Economics*, **14**(1), 75-94.

Baguette, M. (2003) Long distance dispersal and landscape occupancy in a meta population of the cranberry fritillary butterfly. *Ecography*, **26c**, 153-160.

Birnbaum, Z.W., Meyer, P.L. (1953) On the effect of truncation in some or all co-ordinates of a multi-normal population. *Journal of the Indian Society of Agricultural Statistics*, **5**, 17-28.

Ferdowsian, R., Pannell, D.J., McCarron, C., Ryder, A. and Crossing, L. (2001) Explaining groundwater trends: separating atypical rainfall events from time trends. *Australian Journal of Soil Research*, **39**, 861-875.

Gentle, J.E. (1998) *Numerical Linear Algebra for Applications in Statistics.* Springer-Verlag, Berlin.

Horrace, W.C., Hernandez, C. (2001) The characteristic function from a family of truncated normal distributions. *Econometric Theory: Problems and Solutions*, **17**, 1026.

Horrace, W.C. (2004) On ranking and selection from independent truncated normal distributions. Revised and resubmitted to *Journal of Econometrics.*

Jotzo, F., Pezzey, J.C.V. (2004) Flexible targets for greenhouse gas emissions under uncertainty. Typescript, *Centre for Resource and Environmental Studies, Australian National University.*

Keig, G. and McAlpine, J.R. (1974) WATBAL: A computer system for the estimation and analysis of soil moisture regimes from simple climatic data. CSIRO Division of Land Use Research. Technical Report 74/4.

Krajewski, W.F., Lakshmi, V., Georgakakos, K.P. and Jain, S.C. (1991) A Monte Carlo study of rainfall sampling effect on a distributed catchment model. *Water Resources Research*, **27**(1), 119-128.

Kreyszig, E. (1978) *Introductory Functional Analysis with Applications.* Wiley, New York.

Makhlouf, Z. and Michel, C. (1994) A two-parameter monthly water balance model for French watersheds. *Journal of Hydrology*, **162**(3-4), 299-318.

Pacala, S.W., Canham, C.D., Saponara, J., Silander, J.A., Kobe, R.K. and Ribbens, E. (1996) Forest models defined by field measurements: estimation, error analysis and dynamics. *Ecological Monographs*, **66**(1), 1-43.

Ratnieks, F.L.W. (1996) Evolution of unstable and stable biparental care. *Behavioral Ecology*, **7**(4), 490-493.

Rohatgi, V.K., Saleh, A.K. Md. Ehsanes. (2001) *An Introduction to Probability and Statistics.* Wiley, New York.

Tallis, G.M. (1961) The moment generating function of the truncated multi-normal distribution. *Journal of the Royal Statistical Society, Series B*, **23**, 223-229.

Tallis, G.M. (1965) Plane truncation in normal populations. *Journal of the Royal Statistical Society, Series B*, **27**, 301-307.

Xiong, L. and Guo, S. (1999) A two-parameter monthly water balance model and its application. *Journal of Hydrology*, **216**, 111-123.